OXFORD
UNIVERSITY PRESS

ASPIRE
SUCCEED
PROGRESS

Complete Additional Mathematics for Cambridge IGCSE® & O Level

For the updated syllabus

Tony Beadsworth

Oxford excellence for Cambridge IGCSE® & O Level

OXFORD

OXFORD
UNIVERSITY PRESS

Great Clarendon Street, Oxford OX2 6DP

Oxford University Press is a department of the University of Oxford. It furthers the University's objective of excellence in research, scholarship, and education by publishing worldwide. Oxford is a registered trade mark of Oxford University Press in the UK and in certain other countries

British Library Cataloguing in Publication Data

Data available

ISBN: 978-0-19-837670-5
10 9 8 7 6 5 4 3 2 1

Printed in China by Golden Cup

The publishers would like to thank Cambridge International Examinations for their kind permission to reproduce past paper questions. Cambridge International Examinations bears no responsibility for the example answers to questions taken from its past question papers which are contained in this publication.

The questions, example answers, marks awarded and comments that appear in this book were written by the author. In examination, the way marks would be awarded to answers like these may be different.

Acknowledgements

The publisher would like to thank the following for their permission to reproduce photographs:

p2: bingdian/iStock; **p26:** VanderWolf Images/Shutterstock; **p43:** hans engbers/Shutterstock; **p60:** Aleksandar Todorovic/Shutterstock; **p80:** Malgorzata Gluc/Shutterstock; **p92:** Tim Draper/Getty Images; **p105:** Sergey Dzyuba/Shutterstock; **p119:** VIEW Pictures Ltd/Alamy Stock Photo; **p133:** T photography/Shutterstock; **p148:** HildaWeges Photography/Shutterstock; **p173:** Pigprox/Shutterstock; **p192:** han871111/Shutterstock; **p208:** Elnur/Shutterstock; **p231:** Gunter Nezhoda/Shutterstock; **p258:** Andrew Twort/Alamy Stock Photo; **p281:** JRP Studio/Shutterstock; **p303:** Werner Dieterich/Alamy Stock Photo; **p319:** Allan Baxter/Getty Images; **p338:** Jo Chambers/Shutterstock; **p355:** GuoZhongHua/Shutterstock; **p379:** Steve Allen Travel Photography/Alamy Stock Photo; **p403:** Josef Hanus/Shutterstock

Cover image: Thanapol Pratueangpong/Dreamstime.com

Introduction

About this book

This book has been written to cover the **Cambridge IGCSE® Additional Mathematics (0606)** course and is fully aligned with the syllabus.

The syllabus was designed for more able students at this level and it was intended that they should have already completed the 0580 syllabus or equivalent before starting this course.

It is possible to teach the 0580 and 0606 courses in parallel, but this requires very careful design of a scheme of work to ensure that students do not meet topics covered in this course before they have completed preliminary work in 0580. An alternative is to teach more able students the 0580 syllabus at a faster rate and then begin the 0606 course, covering it at the same rate.

At the start of each chapter there is a list of the objectives drawn from the 0606 syllabus that are covered in the chapter. They are not all in syllabus order, rather they are selected to follow a suitable scheme of work which may allow some of the parallel teaching mentioned earlier.

The book has been designed to assist teachers in preparing their students for the examination, and thus contains many worked examples and a host of rigorous exercises. The examples show the important techniques required to tackle questions.

Each chapter contains a summative exercise, a one-hour timed test, and in most cases, a set of past examination questions relevant to the topic, including questions which may involve material from previous chapters. The examination questions are taken from pre-2010 papers so that the most recent papers can be used in a complete format by teachers for final revision.

At the end of chapters 4, 8, 11, 15, 19 and 22 there is a timed exercise covering mixed topics that can be used as a term test. There are more of these terms tests available online at: www.oxfordsecondary.com/9780198376705

Finally, to assist in preparing students for the final examination, there is a collection of revision tests and practice papers that can be used as required. Practice paper 1 can be found at the end of this book; the remaining practice papers and all revision tests are also available online at: www.oxfordsecondary.com/9780198376705

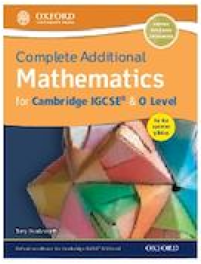
About the author

Tony Beadsworth has graduated from four universities, three in the UK plus Makerere University, Kampala, Uganda. He has taught Mathematics at O-level, IGCSE, A-level (including Further Mathematics) at a range of secondary, tertiary and Sixth Form institutions in East Africa, the UK and Malaysia for over 50 years. He has also been a tutor-counsellor for the UK's Open University, tutoring first and second year university students, and has had extensive experience in examining. Currently semi-retired and living in Kuala Lumpur, he spends his time teaching, writing, and helping Malaysian students prepare for entrance to UK universities.

OXFORD
UNIVERSITY PRESS

Student book & Cambridge syllabus matching grid

ASPIRE
SUCCEED
PROGRESS



Student Book: Complete Additional Mathematics for Cambridge IGCSE & O Level

Syllabus: Cambridge IGCSE Additional Mathematics 0606

1 Set language and notation	
• use set language and notation, and Venn diagrams to describe sets and represent relationships between sets as follows: $A = \{x : x \text{ is a natural number}\}$ $B = \{(x, y) : y = mx + c\}$ $C = \{x : a < x < b\}$ $D = \{a, b, c, \ldots\}$	pages 2–17
• understand and use the following notation: Union of A and B — $A \cup B$ Intersection of A and B — $A \cap B$ Number of elements in set A — $\mathrm{n}(A)$ "… is an element of …" — $\in$ "… is not an element of …" — $\notin$ Complement of set A — A' The empty set — $\varnothing$ Universal set — $\mathscr{E}$ A is a subset of B — $A \subseteq B$ A is a proper subset of B — $A \subset B$ A is not a subset of B — $A \nsubseteq B$ A is not a proper subset of B — $A \not\subset B$	pages 2–17

2 Functions	
• understand the terms: function, domain, range (image set), one-one function, inverse function and composition of functions	pages 60–71
• use the notation $\mathrm{f}(x) = \sin x$, $\mathrm{f} : x \mapsto \lg x \ (x > 0)$, $\mathrm{f}^{-1}(x)$ and $\mathrm{f}^2(x)$ $[= \mathrm{f}(\mathrm{f}(x))]$	pages 61–73
• understand the relationship between $y = \mathrm{f}(x)$ and $y = \lvert \mathrm{f}(x) \rvert$, where $\mathrm{f}(x)$ may be linear, quadratic or trigonometric	pages 71–73
• explain in words why a given function is a function or why it does not have an inverse	pages 65–71
• find the inverse of a one-one function and form composite functions	pages 65–71
• use sketch graphs to show the relationship between a function and its inverse	pages 69–76

3 Quadratic functions	
• find the maximum or minimum value of the quadratic function $f : x \mapsto ax^2 + bx + c$ by any method	pages 82–88
• use the maximum or minimum value of $f(x)$ to sketch the graph or determine the range for a given domain	pages 83–90
• know the conditions for $f(x) = 0$ to have: (i) two real roots (ii) two equal roots (iii) no real roots and the related conditions for a given line to: (i) intersect a given curve (ii) be a tangent to a given curve (iii) not intersect a given curve	pages 87–90 pages 94–97
• solve quadratic equations for real roots and find the solution set for quadratic inequalities	pages 98–102
4 Indices and surds	
• perform simple operations with indices and with surds, including rationalising the denominator	pages 26–39
5 Factors of polynomials	
• know and use the remainder and factor theorems	pages 122–125
• find factors of polynomials	pages 125–129
• solve cubic equations	pages 123–129
6 Simultaneous equations	
• solve simultaneous equations in two unknowns with at least one linear equation	pages 93–97
7 Logarithmic and exponential functions	
• know simple properties and graphs of the logarithmic and exponential functions including $\ln x$ and e^x (series expansions are not required) and graphs of $ke^{nx} + a$ and $k\ln(ax + b)$ where n, k, a and b are integers	pages 320–332
• know and use the laws of logarithms (including change of base of logarithms)	pages 321–325
• solve equations of the form $a^x = b$	pages 326–332
8 Straight line graphs	
• interpret the equation of a straight line graph in the form $y = mx + c$	pages 136–140
• transform given relationships, including $y = ax^n$ and $y = Ab^x$, to straight line form and hence determine unknown constants by calculating the gradient or intercept of the transformed graph	pages 326–332
• solve questions involving mid-point and length of a line	pages 134–136
• know and use the condition for two lines to be parallel or perpendicular	pages 136–140
9 Circular measure	
• solve problems involving the arc length and sector area of a circle, including knowledge and use of radian measure	pages 192–200

10 Trigonometry

• know the six trigonometric functions of angles of any magnitude sine, cosine, tangent secant, cosecant, cotangent	pages 174–178
• understand amplitude and periodicity and the relationship between graphs of e.g. sin x and sin $2x$	pages 179–183
• draw and use the graphs of $y = a\sin(bx) + c$ $y = a\cos(bx) + c$ $y = a\tan(bx) + c$ where a and b are positive integers and c is an integer	pages 291–294
• use the relationships $\frac{\sin A}{\cos A} = \tan A, \frac{\cos A}{\sin A} = \cot A,$ $\sin^2 A + \cos^2 A = 1, \sec^2 A = 1 + \tan^2 A, \operatorname{cosec}^2 A = 1 + \cot^2 A$ and solve simple trigonometric equations involving the six trigonometric functions and the above relationships (not including general solution of trigonometric equations) sine, cosine, tangent secant, cosecant, cotangent	pages 285–290
• prove simple trigonometric identities	pages 284–290

11 Permutations and combinations

• recognise and distinguish between a permutation case and a combination case	pages 49–53
• know and use the notation $n!$ (with $0! = 1$), and the expressions for permutations and combinations of n items taken r at a time	pages 51–53
• answer simple problems on arrangement and selection (cases with repetition of objects, or with objects arranged in a circle or involving both permutations and combinations, are excluded)	pages 44–53

12 Binomial expansions

• use the binomial theorem for expansion of $(a + b)^n$ for positive integral n	pages 107–112
• use the general term $\binom{n}{r}a^{n-r}b^r$, $0 \leqslant r \leqslant n$ (knowledge of the greatest term and properties of the coefficients is not required)	pages 107–112

13 Vectors in 2 dimensions

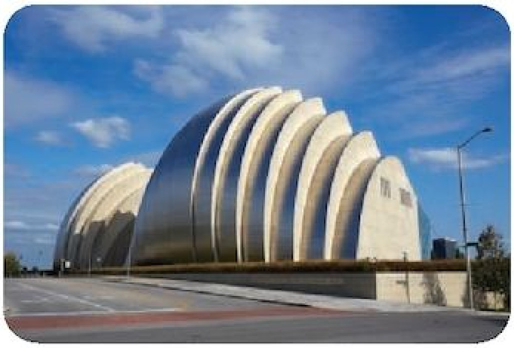
• use vectors in any form, e.g. $\begin{pmatrix} a \\ b \end{pmatrix}$, $\overrightarrow{AB}$, $\mathbf{p}$, $a\mathbf{i} + b\mathbf{j}$	pages 380–381
• know and use position vectors and unit vectors	pages 381–385
• find the magnitude of a vector; add and subtract vectors and multiply vectors by scalars	pages 381–388
• compose and resolve velocities	pages 389–391
• use relative velocity, including solving problems on interception (but not closest approach)	pages 391–395

14 Matrices	
• display information in the form of a matrix of any order and interpret the data in a given matrix	pages 232–248
• state the order of a given matrix	pages 232–233
• solve problems involving the calculation of the sum and product (where appropriate) of two matrices and interpret the results	pages 232–240
• calculate the product of a scalar quantity and a matrix	page 234
• use the algebra of 2 × 2 matrices (including the zero, **O**, and identity, **I**, matrix)	pages 234–240
• calculate the determinant and inverse, $\mathbf{A}^{-1}$, of a non-singular 2 × 2 matrix and solve simultaneous linear equations	pages 241–248

15 Differentiation and integration	
• understand the idea of a derived function	pages 151–168
• use the notations $f'(x)$, $f''(x)$, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$, $\left[-\frac{d}{dx}\left(\frac{dy}{dx}\right)\right]$	pages 152–157 pages 161–165
• use the derivatives of the standard functions together with constant multiples, sums of these x^n (for any rational n) $\sin x$, $\cos x$, $\tan x$ e^x, $\ln x$	pages 152–168
• differentiate composite functions	pages 219–221
• differentiate products and quotients of functions	pages 304–313
• apply differentiation to gradients, tangents and normals stationary points connected rates of change small increments and approximations practical maxima and minima problems	pages 159–168 pages 208–224
• use the first and second derivative tests to discriminate between maxima and minima	pages 164–167
• understand integration as the reverse process of differentiation	pages 258–259
• integrate sums of terms in powers of x, excluding $\frac{1}{x}$	pages 260–262
• integrate functions of the form $(ax+b)^n$ (excluding $n=-1$) e^{ax+b} $\sin(ax+b)$, $\cos(ax+b)$	pages 259–261 pages 341–343 pages 357–362
• evaluate definite integrals and apply integration to the evaluation of plane areas	pages 265–270
• apply differentiation and integration to kinematics problems that involve displacement, velocity and acceleration of a particle moving in a straight line with variable or constant acceleration, and the use of x-t and v-t graphs	pages 403–419

Contents

The inspiration for the photos that appear on the first page of each chapter came from the use of mathematics in modern architecture. Advanced digital tools such as parametric modelling allow designers and architects to experiment with certain features of a new building concept, with considerations for maximising energy efficiency, how wind blows around the building, acoustic properties, and so on.



With the aid of computers, just about every aspect of a building can be modelled, and any knock-on effects of changes to one aspect will be revealed in other areas.

Every photo in the book is of a real building somewhere in the world, apart from one which is a 3D graphical rendering; see if you can spot it!

1 Set language and notation

Syllabus statements

- use set language and notation, and Venn diagrams to describe sets and represent relationships between sets as follows:
  $A = \{x : x \text{ is a natural number}\}$
  $B = \{(x, y) : y = mx + c\}$
  $C = \{x : a \leqslant x \leqslant b\}$
  $D = \{a, b, c, \ldots\}$
- understand and use the following notation:

Union of A and B	$A \cup B$
Intersection of sets A and B	$A \cap B$
Number of elements in set A	$\mathrm{n}(A)$
"…is an element of…"	$\in$
"…is not an element of…"	$\notin$
Complement of set A	A'
The empty set	$\varnothing$
Universal set	$\mathscr{E}$
A is a subset of B	$A \subseteq B$
A is a proper subset of B	$A \subset B$
A is not a subset of B	$A \nsubseteq B$
A is not a proper subset of B	$A \not\subset B$

1.1 Describing sets

Mathematics is a sort of shorthand. Instead of writing lots of words, we use symbols instead.

Set Theory and the ideas it produces is fundamental to the development of mathematical language.

This chapter develops that language and we will use it throughout the course (and beyond) in studying different areas of mathematics.

Format	Example
$A = \{$<list>$\}$	$A = \{1, 2, 3, 4, 5\}$ $A = \{\text{lion, tiger, giraffe}\}$
$A = \{x :$ <description>$\}$	$A = \{x : x \text{ is a natural number}\}$ $A = \{x : 2 < x < 5.2\}$
$A = \{(x, y) :$ <expression>$\}$	$A = \{(x, y) : y = 2x - 3\}$ $A = \{(x, y) : x^2 + y^2 \leqslant 25\}$

The brackets {...} mean "the set whose members are".

The colon " : " means "such that".

1.2 Sets of numbers

Notation	Number set	Example
$\mathbb{N}$	The Natural numbers	$\{1, 2, 3, 4, \ldots\}$
$\mathbb{Z}$	The set of Integers	$\{0, \pm1, \pm2, \pm3, \ldots\}$
$\mathbb{Z}^+$	The set of positive Integers	$\{1, 2, 3, 4, \ldots\}$
$\mathbb{Q}$	The set of Rational numbers	$\frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0$
$\mathbb{Q}^+$	The set of positive Rational numbers	$\{x : x \in \mathbb{Q}, x > 0\}$
$\mathbb{R}$	The set of Real numbers	

1.3 Set properties

Notation	Meaning	Example
		$A = \{1, 2, 3, 4, 5\}$
$\in$	is a member of; belongs to	$2 \in A$, $4 \in \{1, 2, 3, 4, 5\}$
$\notin$	is not a member of; does not belong to	$9 \notin A$
n(..)	the number of members of	$\text{n}(A) = 5$
$\varnothing$ or { }	the empty set	$\text{n}(\varnothing) = 0$

Sometimes the word "**element**" is used instead of "member".

Exercise 1.1

1 Use set notation to list the members of the following sets:

> It does not matter what order you write them in, but do not repeat anything in your list.

- **a** The things in your pocket or purse
- **b** Five people nearest you
- **c** Your six favourite foods
- **d** Your siblings
- **e** The prime numbers less than 20
- **f** The factors of 36
- **g** The square numbers between 200 and 410

2 Use set notation to describe the following sets (e.g. $A = \{x :$ <description>$\}$):

- **a** $A : \{2, 4, 6, 8, 10\}$
- **b** $B : \{1, 2, 3, 4, 6, 12\}$
- **c** $C : \{1, 3, 6, 10, 15\}$
- **d** $D : \{1, 4, 9, 16, \ldots\}$
- **e** $E : \{15, 13, 19, 11, 17\}$
- **f** $F : \{(0, 5), (1, 4), (2, 3), (3, 2), (4, 1), (5, 0)\}$
- **g** $G : \{(3, 4, 5), (5, 12, 13), (6, 8, 10), (7, 24, 25), (8, 15, 17)\}$

3 Identify these sets of numbers:

> Think about it: what distinguishes whether a number has an odd or even number of factors?

- **a** $A : \{x : x \text{ has only 1 factor}\}$
- **b** $B : \{x : x \text{ has only 2 factors}\}$
- **c** $C : \{x : x \text{ has an odd number of factors}\}$

4 Use set notation to list the members of the following sets:

- **a** $A : \{x : \frac{x}{2} \in \mathbb{N} \text{ and } x \text{ is a factor of } 36\}$
- **b** $B : \{m : m \in \mathbb{Z}; m^2 < 20\}$
- **c** $C : \{p : p \text{ is an odd number and } p \text{ is the number of factors of an integer smaller than } 150\}$
- **d** $D : \{q : q = 3n + 2; n \in \mathbb{Z}; 5 \leqslant n \leqslant 9\}$

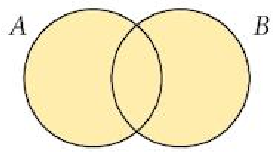
5 For the sets in question **2**, say whether these statements are true or false:

- **a** $10 \in C$
- **b** $\mathrm{n}(B) = 7$
- **c** $\mathrm{n}(F) = \mathrm{n}(B)$
- **d** $7 \notin C$
- **e** $8 \notin A$

1.4 Combining sets

1.4.1 Union $\cup$

$A \cup B$ is shaded

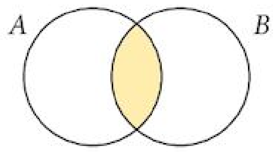


The important word here is "**or**".

If $x \in A \cup B$ then either $x \in A$ **or** $x \in B$ (including when $x \in A$ and $x \in B$).

1.4.2 Intersection $\cap$

$A \cap B$ is shaded

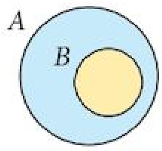


The important word here is "**and**".

If $x \in A \cap B$ then $x \in A$ **and** $x \in B$.

1.4.3 Is a subset of $\subseteq$

$A \subseteq B$ if all the members of A are also members of B.



The definition of a subset also means that $A \subseteq A$ and $\varnothing \subseteq A$. These two are special cases that are called "**improper**" subsets. All other subsets are called "**proper**". Because of this we have two subset symbols:

$\subseteq$ is a subset of

$\subset$ is a proper subset of

Most subsets will be proper ones, but if they are improper you must use the correct symbol.

We can also negate these symbols by crossing through them.

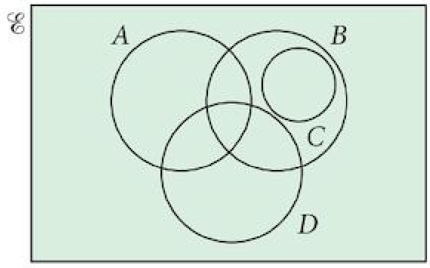
$\not\subseteq$ is not a subset of

$\not\subset$ is not a proper subset of

1.4.4 The Universal set $\mathscr{E}$

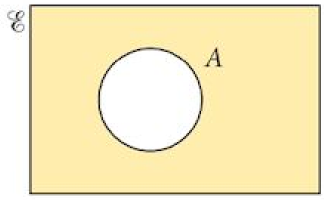
The Universal set (usually drawn as a rectangle) is large enough to contain every element involved in a problem.

$A \subseteq \mathscr{E}$, $B \subseteq \mathscr{E}$, $C \subseteq \mathscr{E}$ and $D \subseteq \mathscr{E}$.



1.4.5 The complement A'

The complement of A (shaded) is the set of all things **not in** A.

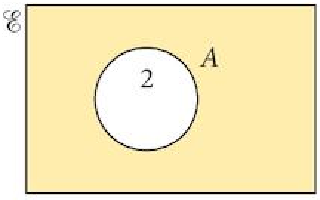


The important words here are "**not in**" or "**outside**".

1.5 Putting elements in sets

When we draw Venn diagrams, we sometimes put the elements in the drawing and sometimes we put the **number** of elements in the drawing. There is no accepted way of distinguishing between these two cases.

When you are drawing Venn diagrams, you must be careful not to get confused. We usually manage without making problems for ourselves, but you could always make a note in the margin to indicate what you mean.



The Venn diagram as drawn here could mean:

$2 \in A$

or $n(A) = 2$

Example 1.1

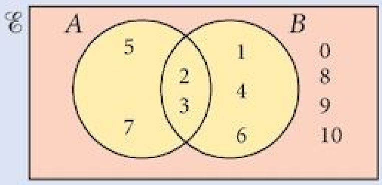
If $\mathscr{E} = \{x : x \in \mathbb{Z}; 0 \leqslant x \leqslant 10\}$, $A = \{x : x \text{ is a prime number}\}$, $B = \{x : x \text{ is a factor of } 12\}$:

a Draw a Venn diagram showing this information.

b Find **(i)** $n(A \cup B)$ **(ii)** $n(A \cap B)$ **(iii)** $n(A')$

Solution:

a



b **(i)** $n(A \cup B) = 7$

(ii) $n(A \cap B) = 2$

(iii) $n(A') = 7$

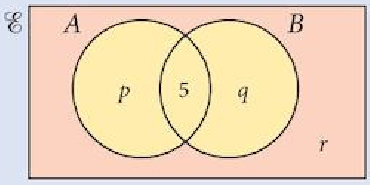
Example 1.2

If $n(\mathscr{E}) = 30$, $n(A) = 16$, $n(B) = 15$ and $n(A \cap B) = 5$:

a Draw a Venn diagram showing this information.

b Find **(i)** $n(A')$ **(ii)** $n(A \cup B)$ **(iii)** $n([A \cup B]')$

Solution:

a 

Step 1: Draw the Venn diagram.

This diagram has 4 regions.

Step 2: Put in the values that you know.

We only know about the 5 in $(A \cap B)$.

Step 3: Put variables p, q, r in the other regions.

Step 4: Write down the equations:

$$p + 5 = 16$$
$$5 + q = 15$$
$$p + 5 + q + r = 30$$

You could solve these in your head.

Step 5: Solve the equations.

$$p = 11$$
$$q = 10$$
$$r = 4$$

b **(i)** $n(A') = 14$

(ii) $n(A \cup B) = 26$

(iii) $n([A \cup B]') = 4$

Exercise 1.2

1 In each of these questions, draw a Venn diagram showing the information and find:

(i) $n(A \cup B)$ **(ii)** $n(A \cap B)$ **(iii)** $n(A')$ **(iv)** $n(B')$

a $\mathscr{E} = \{x : x \in \mathbb{Z};\ 0 \leqslant x \leqslant 20\}$, $A = \{2, 4, 6, 8, 10, 12, 14, 16, 18, 20\}$, $B = \{1, 4, 9, 16\}$

b $\mathscr{E} = \{x : x \in \mathbb{Z};\ 0 \leqslant x \leqslant 20\}$, $A = \{x : x \text{ is a prime number}\}$, $B = \{x : x \text{ is a factor of } 36\}$

c $\mathscr{E} = \{a, b, c, d, e, f, g, h, i\}$, $A = \{c, a, b, g, e\}$, $B = \{c, a, g, e\}$

d $\mathscr{E} = \{x : x \in \mathbb{Z};\ 1 \leqslant x \leqslant 10\}$, $A = \{x : x \text{ is an odd number}\}$, $B = \{x : x \text{ is a square number}\}$

e $\mathscr{E} = \{x : x \in \mathbb{Z};\ 0 \leqslant x \leqslant 20\}$, $A = \{x : x \text{ is a multiple of } 2\}$, $B = \{x : x \text{ is a multiple of } 3\}$

2 In each row of the table there is some information about sets A, B, the Universal set, but some information is missing. For each row, find the missing values and draw a Venn diagram.

	$n(\mathscr{E})$	$n(A)$	$n(B)$	$n(A \cap B)$	$n(A \cup B)$	$n([A \cup B]')$
a	14	5	6	2		
b	15	3	5		8	
c	24	14		9		6
d			16	10	24	4
e		24		9	44	4
f		12	9		21	9
g	40		24	6		6
h	32	12		5	26	

3 **a** Using the results from question **2**, formulate a relationship between $n(A)$, $n(B)$, $n(A \cap B)$, and $n(A \cup B)$.

b Your result in part **a** was derived from specific examples and this would not constitute a valid proof.

Using algebra, formulate a general proof of the result.

1.6 Converting from English to Mathematics

One of the problems you will face is taking a statement in English and converting it to mathematical symbols so that you can solve a problem. Mathematical notation is really just a shorthand way of writing things down.

Step 1: Define your sets. Use a single capital letter that relates to the objects.

e.g. P = {students who study Physics}

Step 2: Look for important words like AND, OR, ALL, etc.

Step 3: Translate each statement into mathematical symbols.

Example 1.3

A There are 20 students in a homeroom class.
Of these:

B 12 students study Physics.

C 10 students study Biology.

D 18 students study one or the other or both of these subjects.

E All students who study Physics also study Mathematics.

Write each of the statements above in set notation.

Solution:

Step 1: Define the sets.

$\mathscr{E} = \{\text{all students in the homeroom class}\}$

$P = \{\text{students who study Physics}\}$

$B = \{\text{students who study Biology}\}$

$M = \{\text{students who study Mathematics}\}$

Step 2: Check for important words.

None in statements **A**–**C**.

D 18 students study one **OR** the other **OR** both of these subjects.

E **ALL** students who study Physics also study Mathematics.

Step 3: **A** $n(\mathscr{E}) = 20$

B $n(P) = 12$

C $n(B) = 10$

D $n(P \cup B) = 18$

E $P \subset M$

Example 1.4

A The 30 students in a class play sports as follows:

B 17 students play tennis.

C 20 students play badminton.

D 22 students swim.

E 13 students play both tennis and badminton.

F 14 students play tennis and also swim.

G 5 students play badminton and swim but do not play tennis.

H 2 students do not play any sport.

Write each of the statements above in set notation.

Solution:

Step 1: Define the sets.

$\mathscr{E}$ = {all students in the class}

T = {students who play tennis}

B = {students who play badminton}

S = {students who swim}

Step 2: Check for important words.

None in statements **A–D**.

E 13 students play both tennis **AND** badminton.

F 14 students play tennis **AND** also swim.

G 5 students play badminton **AND** swim but do **NOT** play tennis.

H 2 students do **NOT** play any sport.

Step 3: **A** $n(\mathscr{E}) = 30$

B $n(T) = 17$

C $n(B) = 20$

D $n(S) = 22$

E $n(T \cap B) = 17$

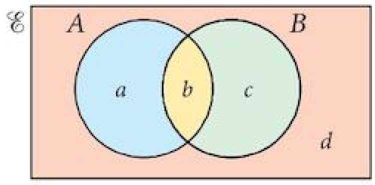
F $n(T \cap S) = 14$

G $n(T' \cap B \cap S) = 5$

H $n([T \cup B \cup S]') = 2$

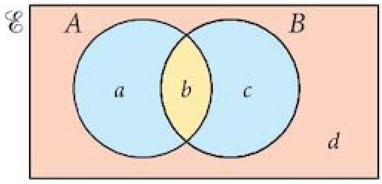
1.7 Identifying regions

1.7.1 Simple regions



Region containing element	Set		Region definition	Think:
	A	*B*		
a	in	not in	$A \cap B'$	*A* and not *B*
b	in	in	$A \cap B$	*A* and *B*
c	not in	in	$A' \cap B$	not *A* and *B*
d	not in	not in	$A' \cap B'$	not *A* and not *B*

1.7.2 Combining regions



The blue region is a combination of two simple regions.

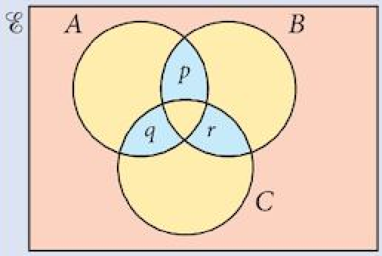
It is $(A \cap B') \cup (A' \cap B)$.

Notice, however, that $(A \cap B') \cup (A \cap B) \cup (A' \cap B)$ is the same as $(A \cup B)$.

This should be obvious, but proving it uses Boolean algebra which is beyond the syllabus (and the A-level syllabus!).

Example 1.5

Identify the region shaded in blue in the Venn diagram.



Solution:

Simple region containing element	Think	Definition
p	A and B and not C	$A \cap B \cap C'$
q	A and not B and C	$A \cap B' \cap C$
r	not A and B and C	$A' \cap B \cap C$

So the blue region is

$(A \cap B \cap C') \cup (A \cap B' \cap C) \cup (A' \cap B \cap C)$

Example 1.6

ℰ is the set of integers between 1 and 20 inclusive.

X is the set of integers between 4 and 15, excluding 4 and 15.

Y is the set of integers greater than or equal to 9, and less than or equal to 20.

Z is the set of multiples of 5 between 1 and 20, inclusive.

a Write the statements above in set notation.

b List the elements of $X \cap Y$.

c List the elements of $X \cup Y$.

d Find $(X \cup Y)' \cap Z$.

Solution:

a $\mathscr{E} = \{n : n \in \mathbb{Z}; 1 \leqslant n \leqslant 20\}$

$X = \{x : x \in \mathbb{Z}; 4 < x < 15\}$

$Y = \{y : y \in \mathbb{Z}; 9 \leqslant y \leqslant 20\}$

$Z = \{z : z, k \in \mathbb{Z}; z = 5k; 1 \leqslant k \leqslant 4\}$

b $X \cap Y = \{9, 10, 11, 12, 13, 14\}$

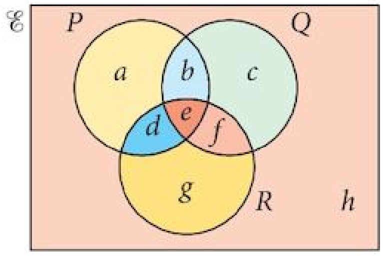
c $X \cup Y = \{5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$

d $(X \cup Y)' \cap Z = \{\ \} = \varnothing$

Note that you can choose what variable you want to use.

Exercise 1.3

1 The general Venn diagram for three sets is shown. Define each of the eight regions (a to h) in terms of P, Q and R.



2 $\mathscr{E}$ = {all students in a class}

M = {students studying Mathematics}

H = {students studying History}

P = {students studying Physics}

Draw a Venn diagram showing the sets if all students studying Physics also study Mathematics. (Note: there is more than one possible arrangement of the sets.)

3 $\mathscr{E}$ = {all students in a class}

M = {students studying Mathematics}

H = {students studying History}

E = {students studying Economics}

Express each of the following statements in set notation. Note that they do not all apply at the same time.

a All students studying History also study Economics.

b No student studying History also studies Mathematics.

c The number of students studying both Mathematics and Economics is 32.

Describe in words the students belonging to these sets:

d $M' \cap E \cap H'$

e $(M \cup E) \cap H'$

4 $\mathscr{E} = \{x : x \text{ is a shopper in a survey}\}$
$D = \{d : d \text{ is a dog owner}\}$
$J = \{j : j \text{ is a jogger}\}$
$C = \{c : c \text{ is a shopper who drives a car}\}$

Express each of the following statements in set notation.
Note that they do not all apply at the same time.

a All dog-owning joggers drive a car.

b No shoppers who drive a car are dog owners.

c The number of car drivers is greater than the number of joggers who own a dog.

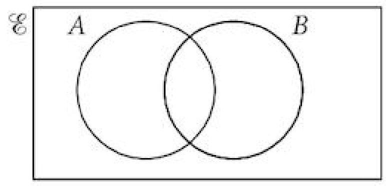
Describe in words the shoppers belonging to these sets:

d $C' \cap D \cap J'$

e $(C \cap J) \cup D$

f $C' \subset J$

5 Make copies of the Venn diagram opposite and use them to shade the regions representing these sets:



a **(i)** $A \cup B$ **(ii)** $(A \cup B)'$

b **(i)** A' **(ii)** B' **(iii)** $A' \cap B'$

c What do you notice about the shaded regions in **a(ii)** and **b(iii)**?

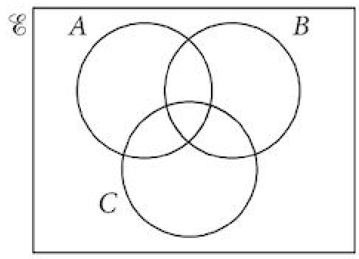
6 Make more copies of the diagram in question **5** and use them to shade the regions representing these sets:

a (i) $A \cap B$ (ii) $(A \cap B)'$

b (i) A' (ii) B' (iii) $A' \cup B'$

c What do you notice about the shaded regions in **a(ii)** and **b(iii)**?

7 Make copies of the Venn diagram opposite and use them to shade the regions representing these sets:



a (i) $B \cup C$ (ii) $A \cap (B \cup C)$

b (i) $A \cap B$ (ii) $A \cap C$
(iii) $(A \cap B) \cup (A \cap C)$

c What do you notice about the shaded regions in **a(ii)** and **b(iii)**?

8 Make more copies of the Venn diagram in question 7 and use them to shade the regions representing these sets:

a (i) $B \cap C$ (ii) $A \cup (B \cap C)$

b (i) $A \cup B$ (ii) $A \cup C$ **(iii)** $(A \cup B) \cap (A \cup C)$

c What do you notice about the shaded regions in **a(ii)** and **b(iii)**?

9 Use Venn diagrams and shading as in questions **5** to **8** to prove the following results:

a $A \cap (A \cup B) = A$

b $A \cup (A \cap B) = A$

c $A \cup (B \cup C) = (A \cup B) \cup C$

d $A \cap (B \cap C) = (A \cap B) \cap C$

1.8 Using sets to solve problems

Example 1.7

There are 30 students in a class. Of these, 15 take History, 18 take Geography, and 5 take neither of these subjects. How many students take both subjects?

Solution:

Step 1: Define your sets:

$\mathscr{E}$ = {students in the class}

H = {students who take History}

G = {students who take Geography}

Step 2: Translate the question statements into mathematics:

$n(\mathscr{E}) = 30$

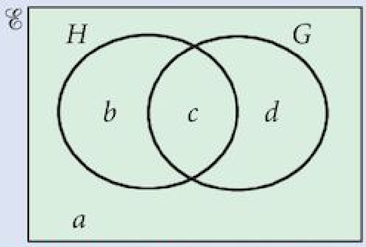
$n(H) = 15$

$n(G) = 18$

$n(H' \cap G') = 5$

You need to find $H \cap G$.

Step 3: Draw the Venn diagram and put dummy values in the regions.



Note that these variables represent the **number** of elements in each region.

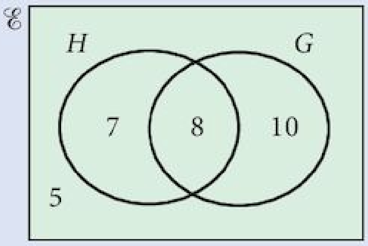
Step 4: Replace the dummy values by calculation.

The only dummy value you can replace initially is a, which is 5.

The other regions must total 25.

$H = (b + c) = 15$, so $d = 10$.

Then c must be 8 and b must be 7.



You could translate your equations into simultaneous ones and solve them.

$$a + b + c + d = 30$$
$$b + c = 15$$
$$c + d = 18$$
$$a = 5$$

It is often the case that the only value you can put in at the start is the last one given. You will need to search the list.

Step 5: Answer the question.

$H \cap G = 8$. Eight students take both History and Geography.

Exercise 1.4

1 In a class of 25 students, 8 are taking Physics and 15 are taking Biology.
6 students take both subjects. How many students take neither subject?

2 A group of 10 students was asked to draw a parallelogram.
3 students drew a rhombus while 4 drew a rectangle.
If 5 students drew a parallelogram with no special properties, how many of them drew a square?

3 A group of 50 travellers visiting an island are asked about their flight habits.
20 said that they would never fly with Coconut Airways.
8 said that they would only fly with Mangojet.
20 use both airlines.
How many passengers would only fly with Coconut Airways?

4 A group of 40 shoppers were asked about their preferences for washing powder.
17 said that they would use Whizbright.
18 said that they would use Sudso.
10 said that they would never use either of these products.
How many of the shoppers would use only one of the products?

5 A group of 30 volunteer students were challenged to eat either a whole hot chili pepper or a 6 cm length of pickled ginger.
4 students refused to take part in the challenge.
20 of them were willing to eat the ginger of whom 4 were also brave enough to eat the chilli.
How many of the students accepted the challenge and ate the chili pepper?

6 In a class of 20 students, all those who study Physics also study Mathematics.

9 students study Biology.

5 students study all three subjects.

7 students studying Physics do not study Biology.

1 Biologist studies Mathematics but not Physics.

Everyone in the class studies at least one of these subjects.

How many students study Mathematics?

7 50 customers at a supermarket are asked about washing powder.

20 used Ozo sometimes, 20 used Brightsuds sometimes and 25 used Bluebright sometimes.

5 used none of these three products while 3 used all three products at different times.

8 were exclusively Ozo, 9 exclusively Brightsuds and 9 exclusively Bluebright.

How many customers used more than one product?

8 There are 100 cars in a car park.

60 of them are black.

40 of them are SUV's.

20 of the SUV's are automatic, the rest are manual.

15 of the manual SUV's are not black.

10 of the manual cars are not black, nor are they an SUV.

25 of the automatic cars are not an SUV.

41 of the black cars are not an SUV.

a How many automatic cars are there?

b How many black SUV's are there?

c How many black automatic cars are there?

9 A group of 120 patients with a particular disease agree to take part in a trial involving three new drugs. Patients were allergic to combinations of the drugs in the following ways:

Allergic to drug A	60
Allergic to drug B	40
Allergic to drug C	60
Allergic to both drugs A and B	30
Allergic to both drugs A and C	20
Allergic to both drugs B and C	15
Allergic to all three drugs	10

Find **a** the number of patients showing no allergic reaction of any sort

b the number of patients who are allergic to any pair of drugs but not all three

c the number of patients who are allergic to either drug A or drug B but not drug C.

10 Given that $n(\mathscr{E}) = 30$, $n(A) = 16$ and $n(B) = 20$, find the greatest and least values of:

a $n(A \cap B)$

b $n([A \cup B]')$

11 Given that $n(\mathscr{E}) = 40$, $n(A) = 28$, $n(B) = 25$, $n(A \cap B) = x$ and $n(A' \cap B') = y$.

a Express y in terms of x.

b Find the greatest and least values of x xnd y.

12 You are given the following information about the three sets A, B and C:

$n(A \cap B \cap C) = x$; $n(A \cap B) = 20$; $n(A \cap C) = 18$; $n(B \cap C) = 16$;

$n(A \cap B' \cap C') = 4$; $n(A' \cap B \cap C') = 16$; $n(A' \cap B' \cap C) = 26$;

$n(\mathscr{E}) = 100$ and $n(A' \cap B' \cap C') = y$.

a Express y in terms of x.

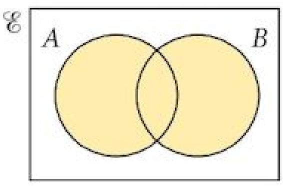
b Find the greatest and least values of $n(A)$.

Summary

Set properties

Notation	Meaning	Example
		$A = \{1, 2, 3, 4, 5\}$
$\in$	is a member of; belongs to	$2 \in A$ $4, \in \{1, 2, 3, 4, 5\}$
$\notin$	is not a member of; does not belong to	$9 \notin A$
n(..)	the number of members of	$n(A) = 5$
$\varnothing$ or { }	the empty set	$n(\varnothing) = 0$

Venn diagrams

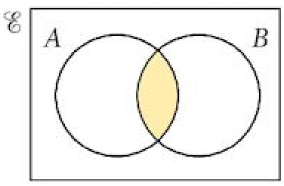


$A \cup B$

A union B

$x \in A \cup B$

$x \in A$ **or** B

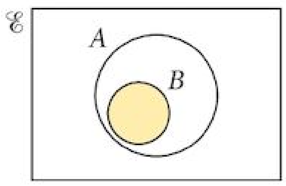


$A \cap B$

A intersection B

$x \in A \cap B$

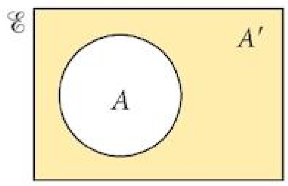
$x \in A$ **and** B



$B \subset A$

B is a subset of A

$x \in B \rightarrow x \in A$



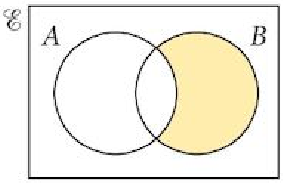
A'

Complement of A

$x \notin A$

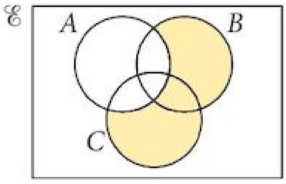
not A

Identifying regions



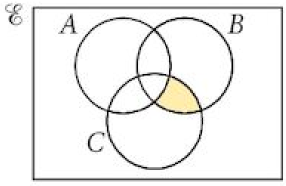
$A' \cap B$

(not A) and B



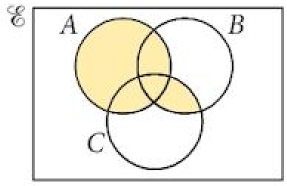
$A' \cap (B \cup C)$

(B or C) and not A



$A' \cap B \cap C$

(not A) and B and C



$A \cup (B \cap C)$

A or (B and C)

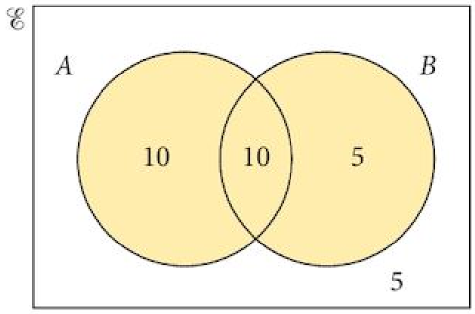
Counting elements

Example: $n(\mathscr{E}) = 30$

$n(A) = 20$

$n(B) = 15$

$n(A \cap B) = 10$



Chapter 1 Summative Exercise

1 List the members of the following sets:

a $P = \{\text{prime numbers between 20 and 40}\}$

b $Q = \{\text{square numbers between 80 and 200}\}$

c $R = \{\text{factors of 36}\}$

2 Set $M = \{m \in \mathbb{Z} : 6 \leqslant m \leqslant 15\}$

a List the elements of M.

b Find $n(M)$.

Set $P \subset M$. If $p \in P$, then p is a prime number.

c List the members of P.

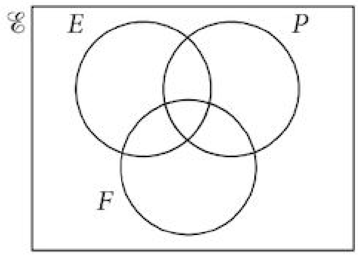
3 In the diagram: $\mathscr{E} = \{x \in \mathbb{Z} : 0 < x \leqslant 12\}$

$E = \{\text{even numbers}\}$

$P = \{\text{prime numbers}\}$

$F = \{\text{factors of 24}\}$

Copy the diagram and insert all the elements of the Universal set in the appropriate areas.



4 The sets $\mathscr{E}$, A, B and C are defined as follows:

$$\mathscr{E} = \{\text{all car drivers in a town}\}$$
$$A = \{\text{drivers of German-made cars}\}$$
$$B = \{\text{female drivers}\}$$
$$C = \{\text{drivers of automatic cars}\}$$

Write each of these statements in set notation:

a Female drivers drive only automatic cars.

b No female drivers drive German-made cars.

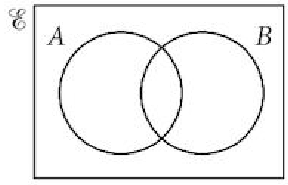
c Male drivers do not drive automatic cars.

5 Give an example of each of these situations:

a $A \subset B$ **b** $C \cap D = \varnothing$

c $E \cup F = \mathscr{E}$ **d** $n(G') > n(T)$

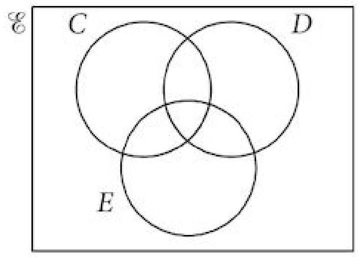
6 Make copies of the Venn diagram and shade the regions representing the sets:



a $A' \cap B$

b $(A \cup B)'$

7 Make copies of the Venn diagram and shade the regions representing the sets:



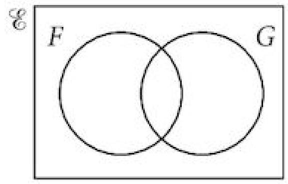
a $C' \cap (D \cup E)$

b $C \cup (D \cap E)$

c $(C' \cap D \cap E) \cup (C \cap D' \cap E)$

d $(C' \cap D \cap E') \cup (C \cap D' \cap E')$

8 In the Venn diagram, $n(\mathscr{E}) = 20$, $n(F) = 12$, $n(G) = 7$ and $n(F' \cup G') = 17$.
Find:



a $n(F \cap G)$

b $n(F' \cap G)$

c $n(F' \cap G')$

9 A company organised a celebratory lunch for its 50 employees on the occasion of the Managing Director's birthday. In this multicultural group, 8 members were fasting and so they did not eat anything. Of the rest, 10 were strict vegetarians. Everyone who wasn't fasting had the vegetables. There were 22 members who had lamb, and 6 members had both lamb and fish.

Draw a Venn diagram illustrating the situation, eliminating any empty regions, and find:

a the number of people who ate lamb, fish and vegetables

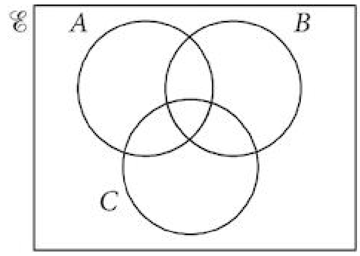
b the number of people who ate fish and vegetables

c the number of people who ate only the lamb and the vegetables but not the fish.

Chapter 1 Test

1 hour

1 Copy the Venn diagram twice and shade the regions.



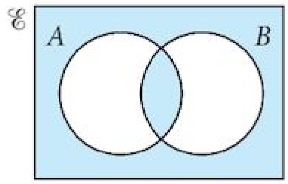
a $A \cap (B \cup C)$ [1]

b $B' \cap C'$ [1]

2 If $\mathscr{E} = \{x : 50 \leqslant x \leqslant 70; x \in \mathbb{Z}\}$,
$A = \{x : x \text{ is a multiple of } 2\}$ and
$B = \{x : x \text{ is a multiple of } 3\}$,
find:

a $A \cap B$ b $A \cup B$ c $n(A \cap B')$ [3]

3 Express, in set notation, the set represented by the shaded region in the Venn diagram to the right. [2]

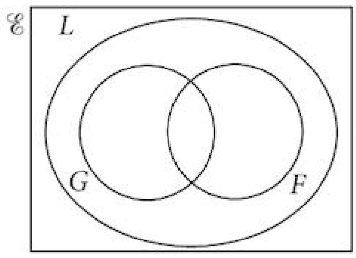


4 A question on set notation states: "In a Mathematics class, there are 20 students. Of this group, 12 also study Physics and 14 study English. If 3 students study neither subject, find the number who study both."

a Translate each of the statements in the question into set notation, using the sets $\mathscr{E}$, P and E. [4]

b Solve the problem. [1]

5 The Venn diagram at right shows the sets:
$\mathscr{E}$ = {the students in the upper years of a school}
L = {the members of the Language Club at the school}
F = {the students who speak French}
G = {the students who speak German}.



Given that $n(\mathscr{E}) = 375$, $n(L) = 135$, $n(F) = 75$, $n(G) = 55$ and $n(L \cap F' \cap G') = 20$, find:

a $n(G \cup F)$ [1]

b $n(G' \cap F)$ [1]

c $n(F' \cap G)$ [1]

d $n((G \cup F)')$ [1]

6 If P = {prime numbers}, S = {square numbers} and N = {all integers between 40 and 80}, express each of the following in set notation.

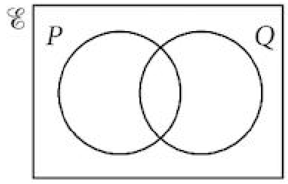
a 43 is a prime number. [1]

b 65 is not a square number. [1]

c There are no square numbers that are prime. [1]

d There are 10 prime numbers between 40 and 80. [1]

7 In the Venn diagram, $n(P) = 14$, $n(Q) = 16$, $n(P' \cap Q) = 7$ and $n(\mathscr{E}) = 30$.



a Copy the Venn diagram and insert the number of elements in each region. [3]

b Find $n[(P \cap Q) \cup (P \cup Q)']$. [1]

8 The school Science Club has members who study Biology, Chemistry or Physics or a combination of these subjects.

$\mathscr{E}$ = {the students in the Science Club}

B = {the members who study Biology}

C = {the members who study Chemistry}

P = {the members who study Physics}.

Write each of the following statements in set notation.

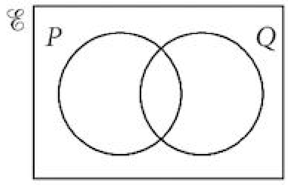
a The Science Club has 86 members. [1]

b All Chemists also study either Biology or Physics. [1]

c 30 students study all three subjects. [1]

d 15 Physics students do not study Biology. [1]

9 **a** Copy the Venn diagram and shade the region $P' \cup Q$. [1]



b Use your diagram to help you write $P' \cup Q$ in an alternative way. [2]

10 In year 10 of a Kuala Lumpur International School, there are 188 students.

C = {students who like chilli}

D = {students who like durian}

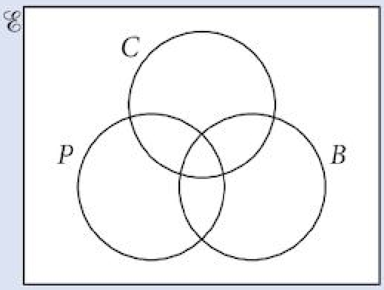
90 like durian and 130 like chilli. The number who like both is x and the number who like neither is $212 - 2x$. Find the value of x. [3]

11 You are given the following information about three sets, A, B and C and their universal set $\mathscr{E}$:

$n(\mathscr{E}) = 44$, $n(A) = 23$, $n(B) = 18$, $n(C) = 14$, $n(A \cup B) = 32$, $n(B \cup C) = 27$,
$n(A \cup C) = 29$ and $n(A' \cap B' \cap C') = 8$.

Draw a Venn diagram and put the number of elements in each region of the diagram. [6]

Examination Questions

1 **a** 

The Venn diagram above represents the universal set ℰ of all teachers in a college. The sets C, B and P represent teachers who teach Chemistry, Biology and Physics respectively. Sketch the diagram twice.

(i) On the first diagram shade the region which represents those teachers who teach Physics and Chemistry but not Biology. [1]

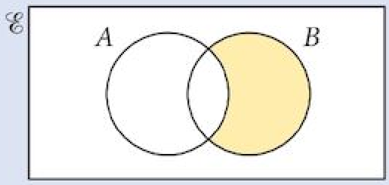
(ii) On the second diagram shade the region which represents those teachers who teach either Biology or Chemistry or both, but not Physics. [1]

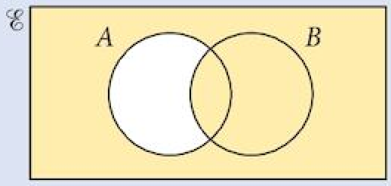
b In a group of 20 language teachers, F is the set of teachers who teach French and S is the set of teachers who teach Spanish. Given that $n(F) = 16$ and $n(S) = 10$, state the maximum and minimum possible values of

(i) $n(F \cap S)$,

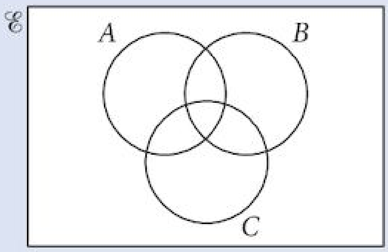
(ii) $n(F \cup S)$. [4]

[Cambridge IGCSE Additional Mathematics 0606, June 2006, P2, Qu 6]

2 **a** **(i)** 

(ii) 

For each of the Venn diagrams above, express the shaded region in set notation. [2]

b 

(i) Copy the Venn diagram above and shade the region that represents $A \cap B \cap C'$. [1]

(ii) Copy the Venn diagram above and shade the region that represents $A' \cap (B \cup C)$. [1]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2005, P1, Qu 2]

3 Given that $\mathscr{E}$ = {students in a college},

A = {students who are over 180 cm tall},

B = {students who are vegetarian},

C = {students who are cyclists},

express in words each of the following

(i) $A \cap B \neq \varnothing$, **(ii)** $A \subset C'$. [2]

Express in set notation the statement

(iii) all students who are both vegetarians and cyclists are not over 180 cm tall. [2]

[Cambridge IGCSE Additional Mathematics 0606, June 2004, P2, Qu 3]

4 The universal set $\mathscr{E}$ and the sets O, P and S are given by

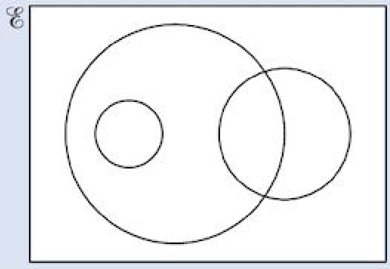
$\mathscr{E} = \{x : x \text{ is an integer such that } 3 \leq x \leq 100\}$,

$O = \{x : x \text{ is an odd number}\}$,

$P = \{x : x \text{ is a prime number}\}$,

$S = \{x : x \text{ is a perfect square}\}$.

In the Venn diagram below, each of the sets O, P and S is represented by a circle.



(i) Copy the Venn diagram and label each circle with the appropriate letter. [2]

(ii) Place each of the numbers 34, 35, 36 and 37 in the appropriate part of your diagram. [2]

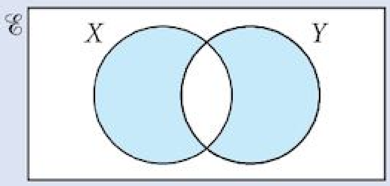
(iii) State the value of $n(O \cap S)$ and of $n(O \cup S)$. [3]

[Cambridge IGCSE Additional Mathematics 0606, June 2003, P1, Qu 8]

5 **a** Illustrate the following statements using a separate Venn diagram for each.

(i) $A \cap B = \varnothing$, **(ii)** $(C \cup D) \subset E$. [2]

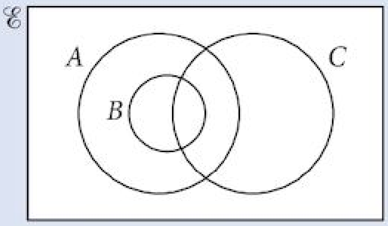
b



Express, in set notation, the set represented by the shaded region. [2]

[Cambridge IGCSE Additional Mathematics 0606, June 2008, P2, Qu 2]

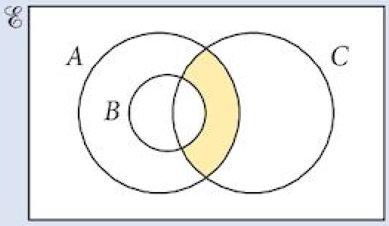
6 **a**



The diagram above shows a universal set ℰ and the three sets A, B and C.

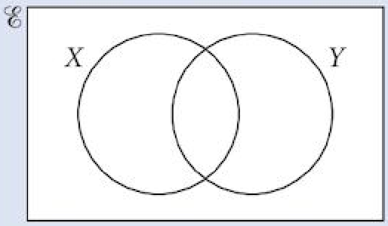
(i) Copy the diagram and shade the region representing $(A \cap C') \cup B$. [1]

(ii)



Express, in set notation, the set represented by the shaded region in the diagram above. [1]

b



The diagram shows a universal set ℰ and the sets X and Y. Show, by means of two diagrams, that the set $(X \cup Y)'$ is not the same as the set $X' \cup Y'$. [2]

[Cambridge IGCSE Additional Mathematics 0606, June 2007, P1, Qu 1]

7 A youth club has facilities for members to play pool, darts and table-tennis. Every member plays at least one of the three games. P, D and T represent the sets of members who play pool, darts and table-tennis respectively. Express each of the following in set language and illustrate each by means of a Venn diagram.

(i) The set of members who only play pool. [2]

(ii) The set of members who play exactly 2 games, neither of which is darts. [2]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2004, P1 Qu 2]

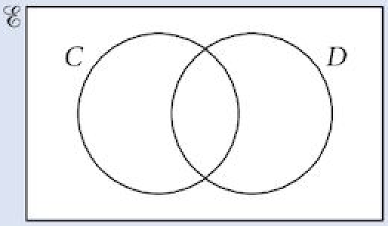
8 Express each of the following statements in appropriate set notation.

(i) x is not an element of set A.

(ii) The number of elements not in set B is 16.

(iii) Sets C and D have no common element. [3]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2006, P1 Qu 1]

9 

The Venn diagram above represents the sets

$\mathscr{E} = \{$homes in a certain town$\}$,

$C = \{$homes with a computer$\}$,

$D = \{$homes with a dishwasher$\}$.

It is given that

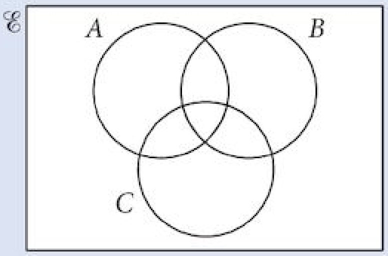
$$\mathrm{n}(C \cap D) = k,$$
$$\mathrm{n}(C) = 7 \times \mathrm{n}(C \cap D),$$
$$\mathrm{n}(D) = 4 \times \mathrm{n}(C \cap D),$$

and $$\mathrm{n}(\mathscr{E}) = 6 \times \mathrm{n}(C' \cap D').$$

(i) Copy the Venn diagram above and insert, in each of its four regions, the number, in terms of k, of homes represented by that region. [5]

(ii) Given that there are 165 000 homes which do not have both a computer and a dishwasher, calculate the number of homes in the town. [2]

[Cambridge IGCSE Additional Mathematics 0606, June 2005, P2, Qu 8]

10 

(i) Copy the Venn diagram above and shade the region that represents $A \cup (B \cap C)$. [1]

(ii) Copy the Venn diagram above and shade the region that represents $A \cap (B \cup C)$. [1]

(iii) Copy the Venn diagram above and shade the region that represents $(A \cup B \cup C)'$. [1]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2008, P1, Qu 1]

2 Indices and surds

Syllabus statements

- perform simple operations with indices and with surds, including rationalising the denominator

2.1 Introduction

You have already used indices in your study of Mathematics.
In this chapter, we extend their use to cover negative and rational indices. We also cover surds. This is just a way of writing down some irrational numbers that could not be done accurately using decimal notation.

2.2 Laws of indices

You will already have met indices. They are the powers of numbers or of variables. You should be familiar with the following results for positive integers m and n:

The first law of indices: $a^m \times a^n = a^{m+n}$ [1]
The second law of indices: $a^m \div a^n = a^{m-n}$ [2]
The third law of indices: $(a^m)^n = a^{mn}$ [3]

The terms 'index' and 'power' mean the same thing.

In fact, these laws are true for all values of m and n.

We can use them to develop additional laws.

Zero index: $a^0 = 1$	[4]	**Law [4] is derived from law [2], when $m = n$.**
Negative index: $a^{-n} = \dfrac{1}{a^n}$	[5]	**Law [5] is derived from law [2], when $m = 0$.**
Fractional index: $a^{\frac{1}{n}} = \sqrt[n]{(a)}$	[6]	**Law [6] is derived from law [2], when $m = \frac{1}{n}$.**
Rational index: $a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$	[7]	You can use either version of Law [7], but the first version is usually better in practice because the numbers are smaller.

Example 2.1

Simplify each of these. **a** $16^{\frac{1}{4}}$ **b** $\left(\dfrac{4}{9}\right)^{-\frac{3}{2}}$

a $16^{\frac{1}{4}} = \sqrt[4]{16}$

$= 2$

b $\left(\dfrac{4}{9}\right)^{-\frac{3}{2}} = \left(\dfrac{9}{4}\right)^{\frac{3}{2}} = \left(\left(\dfrac{9}{4}\right)^{\frac{1}{2}}\right)^3 = \left(\dfrac{3}{2}\right)^3 = \dfrac{27}{8}$

2.3 Powers of small numbers

You will find it very useful to know the following powers of small numbers.

	Power								
Number (a)	a^2	a^3	a^4	a^5	a^6	a^7	a^8	a^9	a^{10}
2	4	8	16	32	64	128	256	512	1024
3	9	27	81	243					
4	16	64	256						
5	25	125	625						
6	36	216							
7	49	343							
8	64	512							
9	81	729							
10	100	1000							
11	121								
12	144								
15	225								
16	256								
20	400								
25	625								

Example 2.2

Simplify the expression $25^x \times 5^{3x-4}$.

Solution:

$$25^x \times 5^{3x-4} = (5^2)^x \times 5^{3x-4}$$
$$= 5^{2x} \times 5^{3x-4}$$
$$= 5^{5x-4}$$

Examples 2.2 and 2.3 use the first and third laws of indices.

Example 2.3

Find the value of x, given that: $1024 = 8^{2x} \times 16^{3x-2}$

Solution:

$$2^{10} = (2^3)^{2x} \times (2^4)^{3x-2}$$
$$2^{10} = 2^{6x} \times 2^{12x-8}$$
$$2^{10} = 2^{18x-8}$$

Thus powers must be equal, so $10 = 18x - 8$

Therefore $x = 1$

Express each quantity as the power of the same number, if possible. Use the table on page 26 to help you.

Example 2.4

Find the values of p and q in the equation $\dfrac{32^4 \times 625^3}{8^6 \times 25^4} = 2^p 5^q$

Solution:

$$\frac{32^4 \times 625^3}{8^6 \times 25^4}$$

Step 1: Write each element as a power of 2 or 5 $= \dfrac{2^{20} \times 5^{12}}{2^{18} \times 5^8}$

Step 2: Simplify the expression $= 2^2 \times 5^4$

Step 3: Equate the elements $p = 2$ and $q = 4$

Example 2.5

a Simplify $\dfrac{2^{3x+3} + 20(8^x)}{3(2^{3x+2})}$

b Solve the equation $\dfrac{3^{8x+1}}{3^{2x-5}} = \dfrac{243^{x+1}}{27^{x-1}}$

Solution:

a $$\frac{2^{3x+3}+20(8^x)}{3(2^{3x+2})}$$

$$=\frac{8\times 2^{3x}+20(2^{3x})}{3(4\times 2^{3x})}$$

Seperate the powers and covert numbers to powers of the same base (in this case 2).

Factorise $$=\frac{28(2^{3x})}{12(2^{3x})}$$

Simplify $$=\frac{28}{12}$$

$$=\frac{7}{3}$$

b $$\frac{3^{8x+1}}{3^{2x-5}}=\frac{243^{x+1}}{27^{x-1}}$$

Convert numbers to powers of the same base.

$$\frac{3^{8x+1}}{3^{2x+5}}=\frac{3^{5x+5}}{3^{3x-3}}$$

$$3^{6x-4}=3^{2x+8}$$

Simplify using the laws of indices.

$$6x-4=2x+8$$

$$x=3$$

Example 2.6

Solve the following simultaneous equations.

$4^x\times 2^y=256;$ [1] $81^x\div 27^y=729$ [2]

Solution:

[1] $2^{2x}\times 2^y=2^8$

[2] $3^{4x}\div 3^{3y}=3^6$

Express each term as a power of the same base number.

From which $2^{2x+y}=2^8$

and $3^{4x-3y}=3^6$

giving $2x+y=8$

and $4x-3y=6$

We solve these to give $x=3$ and $y=2$

Example 2.7

Solve the following simultaneous equations:

$$\frac{9^x \times 27^y}{27} = \frac{243^4}{3^{x-4} \times 3^{y-1}} \quad [1]$$

$$\frac{64^x \times 16^y}{16} = 64^y 4^x \quad [2]$$

Solution:

Step 1: Simplify each equation [1] [2]

$$\frac{3^{2x} \times 3^{3y}}{3^3} = \frac{3^{20}}{3^{x-4} \times 3^{y-1}} \qquad \frac{2^{6x} \times 2^{4y}}{2^4} = 2^{6y}2^{2x}$$

$$3^{(2x+3y-3)} = 3^{(25-x-y)} \qquad 2^{(6x+4y-4)} = 2^{6y+2x}$$

So $2x + 3y - 3 = 25 - x - y$ and $6x + 4y - 4 = 2x + 6y$

giving $3x + 4y = 25$ and $4x - 2y = 4$

Step 2: Solve the simultaneous linear equations. $x = 3$ and $y = 4$

You should already know how to do this.

Exercise 2.1

1 Simplify the following expressions.

a $25^{\frac{1}{2}}$ **b** $64^{\frac{1}{3}}$ **c** $81^{\frac{1}{4}}$ **d** $125^{\frac{1}{3}}$ **e** $128^{\frac{1}{7}}$ **f** $625^{\frac{1}{4}}$

2 Simplify the following expressions.

a $81^{-\frac{1}{2}}$ **b** $27^{-\frac{1}{3}}$ **c** $32^{-\frac{1}{5}}$ **d** $125^{-\frac{1}{3}}$ **e** $343^{-\frac{1}{3}}$ **f** $512^{-\frac{1}{3}}$

3 Simplify the following expressions.

a $16^{\frac{3}{4}}$ **b** $125^{\frac{2}{3}}$ **c** $343^{\frac{2}{3}}$ **d** $36^{-\frac{3}{2}}$ **e** $243^{-\frac{3}{5}}$ **f** $128^{-\frac{4}{7}}$

4 Simplify the following expressions.

a $\left(\frac{64}{125}\right)^{\frac{2}{3}}$ **b** $\left(\frac{125}{216}\right)^{\frac{2}{3}}$ **c** $\left(\frac{81}{625}\right)^{\frac{3}{4}}$ **d** $\left(\frac{16}{49}\right)^{\frac{3}{2}}$ **e** $\left(\frac{16}{81}\right)^{\frac{5}{4}}$ **f** $\left(\frac{36}{25}\right)^{\frac{3}{2}}$

5 Simplify the following expressions.

a $\left(\frac{27}{64}\right)^{-\frac{4}{3}}$ **b** $\left(\frac{25}{64}\right)^{-\frac{3}{2}}$ **c** $\left(\frac{81}{256}\right)^{-\frac{3}{4}}$ **d** $\left(\frac{343}{729}\right)^{-\frac{2}{3}}$ **e** $\left(\frac{81}{49}\right)^{-\frac{3}{2}}$ **f** $\left(\frac{32}{243}\right)^{-\frac{3}{5}}$

6 Simplify the following expressions, writing your answer as a power of the number in brackets.

a $4^3 \times 8^2$ [2] **b** $27^2 \times 9^3$ [3] **c** $625^2 \times 25^3$ [5]

d $16^{-3} \times 4^2$ [2] **e** $9^2 \times 81^2$ [3] **f** $125^2 \times 25^2$ [5]

g $32^2 \div 4^2$ [2] **h** $243^3 \div 81^2$ [3] **i** $125^3 \div 25^5$ [5]

j $16^{-3} \div 4^2$ [2] **k** $9^{-4} \div 81^3$ [3] **l** $125^2 \div 25^{-3}$ [5]

7 Simplify the following expressions, writing your answer as a power of the number in brackets.

a $4^{3x} \times 8^{1-x}$ [2] **b** $27^{2x} \times 9^{3+x}$ [3] **c** $625^{2+2x} \times 25^{3-x}$ [5]

d $16^{-3x+2} \times 4^{2x-4}$ [2] **e** $9^{2x+3} \times 81^{2x-1}$ [3] **f** $625^{2x+3} \times 125^{2x-2}$ [5]

g $64^{-3x-1} \div 16^{2x+2}$ [2] **h** $243^{3x-2} \div 81^{2-3x}$ [3] **i** $125^{3+2x} \div 625^{5x-3}$ [5]

j $128^{-3-3x} \div 64^{2x+5}$ [2] **k** $9^{-4x+2} \div 81^{3x-4}$ [3] **l** $125^{2+3x} \div 25^{-3x+1}$ [5]

8 Solve the following equations for x.

a $4^{3x} = 512$ **b** $9^{3+x} = 243$ **c** $25^{2+2x} = 625$

d $16^{-3x+2} = 4^{2x-4}$ **e** $9^{2x+3} = 81^{2x-1}$ **f** $125^{2x+3} = 25^{2x-2}$

g $16^{-3x-1} = 64^{2x+2}$ **h** $243^{3x-2} = 81^{2-3x}$ **i** $125^{3-2x} = 625^{5x-3}$

j $4^{-3-3x} = 16^{2x+5}$ **k** $9^{-4x+2} = 81^{3x-4}$ **l** $125^{2+3x} = 25^{-3x+1}$

m $7^{-2+3x} = 1$ **n** $11^{-4x+2} - 1 = 0$ **o** $25^{x^2} = 5^{6-4x}$

9 Solve the following equations, expressing the values of p and q in term of x and/or a constant.

a $\dfrac{4^3 \times 125^3}{25^3 \times 16^2} = 2^p 5^q$ **b** $\dfrac{9^3 \times 25^3}{125^2 \times 81^3} = 3^p 5^q$ **c** $\dfrac{16^2 \times 243^2}{81^4 \times 64^3} = 2^p 3^q$

d $\dfrac{16^{-3x+2} \times 25^{3x}}{125^{2x+3} \times 4^{2x}} = 2^p 5^q$ **e** $\dfrac{9^{2x} \times 625^{5x-3}}{125^{2x} \times 9^{-4x+2}} = 3^p 5^q$ **f** $\dfrac{16^{-3x} \times 243^{2x+3}}{81^{2x-1} \times 4^{2x}} = 2^p 3^q$

g $\dfrac{16^{-3x-1} \div 25^{-3x+1}}{125^{2x-2} \times 4^{2x+2}} = 2^p 5^q$ **h** $\dfrac{243^{3x-2} \times 25^{3x-1}}{125^{2+3x} \div 81^{2x-1}} = 3^p 5^q$ **i** $\dfrac{64^{3-2x} \div 81^{2-3x}}{9^{2x+3} \times 4^{2x+5}} = 2^p 3^q$

10 Solve the following pairs of simultaneous equations:

a $1331^{x+y} = 121^{x-y}$

$25^{x+y} = \dfrac{125}{5^{y-4}}$

b $256^{x-1} \times 16^y = 8^{x+y+2}$

$3125^x \times 25^{y+1} = 5 \times 25^x \times 125^{y+1}$

c $2401^{2x+y} \times 49 = 49^{x+6}$

$729^x \times 81^{y-1} = 9^{x+3} \times 729^{y+1}$

d $25^{2x+1} = 5^x \times 5^y$

$81^x \times 9^y = 9 \times 81^3$

11 Solve the following pairs of simultaneous equations:

a $8^x \div 2^y = 32$

$9^x \times 27^y = 39$

b $5^x \div 25^y = 5$

$9^x \div 27^y = 81$

c $125^x = 625^y$

$32^x \div 16^y = 256$

2.4 Surds

A surd is just a root that is irrational. Examples are $\sqrt{2}$, $\sqrt{3}$, $\sqrt{5}$, etc.

When we write $\sqrt{2}$ what we have written is 100% accurate. The value given by your calculator, 1.414213562 is not, because it has been rounded off. In a similar way, we write an answer of say 2π because that is an accurate statement, whereas 6.283185307 is not.

2.5 The arithmetic of surds

The following results apply when you are manipulating surds:

The product rule	$\sqrt{a}\times\sqrt{b}=\sqrt{ab}$
The division rule	$\dfrac{\sqrt{a}}{\sqrt{b}}=\sqrt{\dfrac{a}{b}}$
The distributive rule (i)	$a\sqrt{c}+b\sqrt{c}=(a+b)\sqrt{c}$
The distributive rule (ii)	$a\sqrt{b}+a\sqrt{c}=a\left(\sqrt{b}+\sqrt{c}\right)$

Example 2.8

a $\sqrt{2}\times\sqrt{2}=\sqrt{4}=2$

b $\sqrt{2}\times\sqrt{3}=\sqrt{6}$

c $\sqrt{32}=\sqrt{16}\times\sqrt{2}=4\sqrt{2}$

d $\dfrac{\sqrt{24}}{\sqrt{6}}=\sqrt{\left(\dfrac{24}{6}\right)}=\sqrt{4}=2$

e $3\sqrt{2}+2\sqrt{2}=5\sqrt{2}$

f $\sqrt{32}+\sqrt{18}=4\sqrt{2}+3\sqrt{2}=7\sqrt{2}$

If a surd contains a square number as a factor (eg 4, 9, 16, etc.) then it can be simplified.

2.6 Mixed numbers

We cannot simplify an expression such as $2+3\sqrt{2}$ or $\sqrt{2}+\sqrt{3}$ because there are no common terms. In the same way, we cannot simplify $2y+x^2$ or $3a-2b$.

When we multiply numbers like this together, we multiply just as we do in ordinary algebra.

Example 2.9

a $\left(2+3\sqrt{2}\right)\left(1+\sqrt{3}\right)=2+3\sqrt{2}+2\sqrt{3}+3\sqrt{6}$

b $\left(2+3\sqrt{2}\right)\left(1+\sqrt{2}\right)=2+3\sqrt{2}+2\sqrt{2}+6$
$=8+5\sqrt{2}$

c $\left(2+3\sqrt{2}\right)^2=4+6\sqrt{2}+6\sqrt{2}+18$
$=22+12\sqrt{2}$

Part **a** cannot be simplified as there are no like terms.

2.7 The conjugate

Problem 2.1

Calculate

a $\left(6+3\sqrt{3}\right)\left(5-2\sqrt{3}\right)$ **b** $\left(6+3\sqrt{3}\right)\left(6-3\sqrt{3}\right)$ **c** $\left(5+2\sqrt{3}\right)\left(5-2\sqrt{3}\right)$

Notice the difference in your answers to these questions.

In **a**, you obtained another mixed number. $\left(6+3\sqrt{3}\right)\left(5-2\sqrt{3}\right)=12+3\sqrt{3}$

However, in **b** and **c**, the answers were integers. The surd part of the number had vanished.

$$\left(6+3\sqrt{3}\right)\left(6-3\sqrt{3}\right)=9$$

and

$$\left(5+2\sqrt{3}\right)\left(5-2\sqrt{3}\right)=13$$

When we have a mixed number, $a+b\sqrt{c}$, we can create another mixed number by changing the sign.

In this case, we would get $a-b\sqrt{c}$.

The product of these two numbers is

$$\left(a+b\sqrt{c}\right)\left(a-b\sqrt{c}\right)=a^2-b^2c$$

This process uses the factorisation of the difference of two squares.

This is an integer.

Two numbers like this are called **conjugate**. They are conjugates of each other.

Multiplying a number by its conjugate provides us with a means of simplifying fractions involving surds.

2.8 Rationalising the denominator

When we simplify expressions involving surds, we prefer to finish with no surds in the denominator. This process is called **rationalising the denominator**.

In order to eliminate surds from the denominator, multiply the numerator and denominator by a number that will rationalise the denominator to find an equivalent fraction.

Example 2.10

a $\dfrac{4}{\sqrt{6}}=\dfrac{4\times\sqrt{6}}{\sqrt{6}\times\sqrt{6}}=\dfrac{4\sqrt{6}}{6}=\dfrac{2\sqrt{6}}{3}$

Notice what happens when multiplying the top and bottom by $\sqrt{6}$.

b $\dfrac{2+\sqrt{5}}{3-\sqrt{5}}=\dfrac{\left(2+\sqrt{5}\right)\left(3+\sqrt{5}\right)}{\left(3-\sqrt{5}\right)\left(3+\sqrt{5}\right)}$

$=\dfrac{11+5\sqrt{5}}{4}$

This time we multiply top and bottom by $3+\sqrt{5}$, the **conjugate** of the denominator.

2.9 Square roots of mixed numbers

Look back at Example 2.9c. When we squared a mixed number of the type $\left(a+b\sqrt{c}\right)$ the result was another number of the same type.

We can sometimes use this idea to reverse the process and find square roots of mixed numbers.

Example 2.11

Find the square roots of $\left(31+12\sqrt{3}\right)$.

We start by assuming that the answer is of the same form $\left(a+b\sqrt{3}\right)$ and square it.

$$\left(a+b\sqrt{3}\right)^2 = a^2+2ab\sqrt{3}+3b^2$$

If $\quad 31+12\sqrt{3} = a^2+2ab\sqrt{3}+3b^2$

then, the integer parts give $\quad 31 = a^2+3b^2 \quad$ [1]

and the coefficients of $\sqrt{3}$ give $\quad 6 = ab \quad$ [2]

Note that this technique does not always work! However, it is worth trying.

Now substitute [2] into [1]: $\quad 31 = a^2+\dfrac{108}{a^2}$

From which
$$\left(a^2\right)^2-31a^2+108=0$$
$$\left(a^2-4\right)\left(a^2-27\right)=0$$
$$a^2=4$$

$a^2 \neq 27$, since a is an integer.

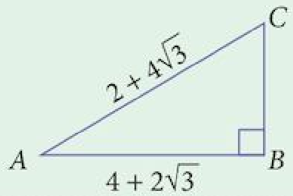
So either $\quad a = 2 \text{ or } -2$

and, from [2], $\quad b = 3 \text{ or } -3$

So the square roots of $\left(31+12\sqrt{3}\right)$ are $\left(2+3\sqrt{3}\right)$ and $\left(-2-3\sqrt{3}\right)$.

Example 2.12

The diagram shows a right-angled triangle, ABC, in which $AB = 4+2\sqrt{3}$ and $AC = 2+4\sqrt{3}$. Find the length of BC.



Solution:

Using Pythagoras' theorem,
$$\begin{aligned} BC^2 &= AC^2 - AB^2 \\ &= \left(2+4\sqrt{3}\right)^2 - \left(4+2\sqrt{3}\right)^2 \\ &= \left(4+16\sqrt{3}+48\right) - \left(16+16\sqrt{3}+12\right) \\ &= 24 \end{aligned}$$

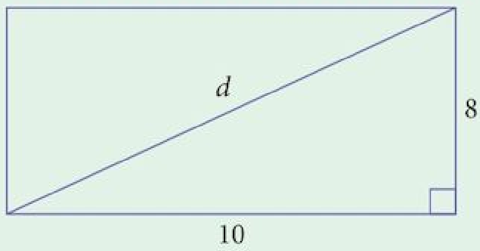
So
$$\begin{aligned} BC &= \sqrt{24} \\ &= 2\sqrt{6} \end{aligned}$$

2.10 Geometric applications

Example 2.13

A rectangle is 10 cm long and 8 cm wide. Find the length of its diagonal.

Solution:



Step 1: Draw a diagram

Step 2: By Pythagoras' theorem, $d^2 = 10^2 + 8^2$

$$d^2 = 164$$

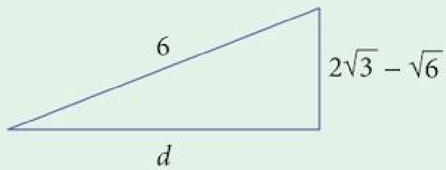
So $d = \sqrt{164}$

$$= 2\sqrt{41}$$

Example 2.14

A right-angled triangle has a hypotenuse that is 6 cm long.
Its shortest side is $(2\sqrt{3} - \sqrt{6})$ cm long.
Find the length of the third side of the triangle.

Solution:



Step 1: Draw a diagram.

Step 2: By Pythagoras' theorem, $d^2 = 6^2 - (2\sqrt{3} - \sqrt{6})^2$

$$d^2 = 36 - (12 - 4\sqrt{18} + 6)$$

So $d^2 = 36 - (18 - 4\sqrt{18})$

$$d^2 = 18 + 4\sqrt{18} \qquad [1]$$

$$d^2 = 12 + 4\sqrt{18} + 6$$

So $d = 2\sqrt{3} + \sqrt{6}$

This kind of relationship is sometimes not easy to spot.

Alternatively, from [1], we could use the techniques of section 2.9.

$$d^2 = 18 + 4\sqrt{18} \qquad [1]$$

$$= 18 + 12\sqrt{2}$$

$$= 3\left(6 + 4\sqrt{2}\right)$$

So we need to find $\sqrt{6 + 4\sqrt{2}}$.

If the solution is $\left(a + b\sqrt{2}\right)$, then $\left(a + b\sqrt{2}\right)^2 = a^2 + 2b^2 + 2ab\sqrt{2}$

Thus, we need $a^2 + 2b^2 = 6$

and $2ab = 4$

Substituting for b gives $a^4 - 6a^2 - 8 = 0$

or $(a^2 - 4)(a^2 - 2) = 0$

giving $a^2 = 4$ or $a^2 = 2$

so $a = -2$ or $a = 2$

$b = -1 \quad b = 1$

So our solution is $d = \pm\sqrt{3}\left(2+\sqrt{2}\right)$

d must be positive, so $d = \sqrt{3}\left(2+\sqrt{2}\right)$

$= 2\sqrt{3}+\sqrt{6}$ as before.

Don't forget, *a* is supposed to be an integer.

Remember to include the two sqaure roots of 3.

Exercise 2.2

1 Simplify each of these.

a $\sqrt{24}$ **b** $\sqrt{75}$ **c** $\sqrt{98}$ **d** $\sqrt{108}$

e $\sqrt{192}$ **f** $\sqrt{405}$ **g** $\sqrt{768}$ **h** $\sqrt{1125}$

2 Simplify each of these.

a $\sqrt{24}+\sqrt{486}$ **b** $2\sqrt{75}-\sqrt{75}$ **c** $\sqrt{32}+\sqrt{98}$ **d** $\sqrt{27}+\sqrt{108}$

e $\sqrt{192}+\sqrt{432}$ **f** $\sqrt{405}-2\sqrt{180}$ **g** $\sqrt{768}+\sqrt{48}$ **h** $\sqrt{112}-\sqrt{63}$

3 Rationalise these surds, writing them in the form $\frac{a\sqrt{b}}{c}$ where a, b and $c \in \mathbb{Z}$.

a $\frac{\sqrt{5}}{\sqrt{3}}$ **b** $\frac{1}{\sqrt{2}}$ **c** $\frac{\sqrt{24}}{\sqrt{5}}$ **d** $\frac{2}{\sqrt{7}}$

e $\frac{3\sqrt{5}}{\sqrt{3}}$ **f** $\frac{12\sqrt{12}}{\sqrt{24}}$ **g** $\frac{15\sqrt{32}}{\sqrt{72}}$ **h** $\frac{18\sqrt{21}}{\sqrt{108}}$

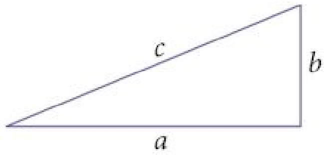
4 Rationalise these surds, writing them in the form $\frac{a+b\sqrt{c}}{d}$ where a, b, c and $d \in \mathbb{Z}$.

a $\frac{1}{1-\sqrt{3}}$ **b** $\frac{1}{3+\sqrt{2}}$ **c** $\frac{1}{2-\sqrt{5}}$ **d** $\frac{1}{3-\sqrt{7}}$

e $\frac{1+3\sqrt{3}}{2-2\sqrt{3}}$ **f** $\frac{3-\sqrt{2}}{2+3\sqrt{2}}$ **g** $\frac{1+3\sqrt{3}}{3-2\sqrt{3}}$ **h** $\frac{2-\sqrt{7}}{3+\sqrt{7}}$

5 A right-angled triangle has sides a, b and c as shown.
In the following questions, work out the missing values in the table.
Express your answers in accurate surd format.

	a	b	c
a	$1+\sqrt{3}$	$\sqrt{3}-1$	
b	$1+4\sqrt{5}$		$2\sqrt{15}$
c		$\sqrt{6}$	$4-\sqrt{2}$
d	$6-\sqrt{5}$	$2\sqrt{2}$	
e		$8+6\sqrt{2}$	$9+6\sqrt{2}$



6 A rectangle is 5 cm long and 3 cm wide. What is the exact length of the diagonal?

7 An equilateral triangle has sides of 2 cm. What is the exact height of the triangle?

8 What is the exact length of the longest rod that will fit into a box measuring 3m by 4m by 2m?

9 A rectangle has a length of $2+\sqrt{3}$ and a height of $2\sqrt{3}-1$.

a What is the exact perimeter of the rectangle?

b What is the exact area of the rectangle?

10 The number $\Phi = \frac{1}{2}(\sqrt{5}+1)$ is known as the **Golden ratio** (or Golden number).

a Find an accurate expression for its reciprocal $\frac{1}{\Phi}$.

b Use your calculator to find Φ correct to 9 decimal places.

c Use your calculator to find $\frac{1}{\Phi}$ correct to 9 decimal places.
What do you notice?

The Golden ratio can be found in architecture, painting, botany, and even the dimensions of a credit card.

Summary

Laws of indices

The first law of indices $\quad a^m \times a^n = a^{m+n}$

The second law of indices $\quad a^m \div a^n = a^{m-n}$

The third law of indices $\quad (a^m)^n = a^{mn}$

Some results of the laws of indices

$a^0 = 1 \qquad a^{\frac{1}{n}} = \sqrt[n]{a}$

$a^{-n} = \dfrac{1}{a^n} \qquad a^{\frac{m}{n}} = \left(\sqrt[n]{a}\right)^m$

The arithmetic of surds

The product rule $\quad \sqrt{a}\times\sqrt{b}=\sqrt{ab}$

The division rule $\quad \dfrac{\sqrt{a}}{\sqrt{b}}=\sqrt{\dfrac{a}{b}}$

The distributive rule (i) $\quad a\sqrt{c}+b\sqrt{c}=(a+b)\sqrt{c}$

The distributive rule (ii) $\quad a\sqrt{b}+a\sqrt{c}=a(\sqrt{b}+\sqrt{c})$

Rationalising the denominator

Remove all surds from the denominator of fractions.

Simple cases: $\dfrac{a}{\sqrt{b}} = \dfrac{a\sqrt{b}}{\sqrt{b}\times\sqrt{b}} = \dfrac{a\sqrt{b}}{b}$

More complicated cases: see below.

Mixed numbers

Product $\quad (a+b\sqrt{n})(c+d\sqrt{n}) = (ac+bdn)+(ad+bc)\sqrt{n}$

Quotient $\quad \dfrac{(a+b\sqrt{n})}{(c+d\sqrt{n})} = \dfrac{(a+b\sqrt{n})(c-d\sqrt{n})}{(c+d\sqrt{n})(c-d\sqrt{n})} = \dfrac{(ac-bdn)+(bc-ad)\sqrt{n}}{(c^2-d^2n)}$

Chapter 2 Summative Exercise

1 Simplify the following, writing your answer as a power of the number in brackets.

a $16^3\times4^4$ [2] **b** $27^{-3}\times9^4$ [3] **c** $625^{-2}\times125^3$ [5]

d $64^{-2}\div8^{-3}$ [2] **e** $81^4\div243^5$ [3] **f** $25^4\div125^6$ [5]

2 Solve the following equations for x.

a $16^x\div32=128$ **b** $81^x\div3^3=27^9\div9^x$ **c** $125^{2x}\div25=25^x\times125^2$

3 In the following equations, find the value of p and the value of q.

a $\dfrac{2^{12x+4}\times81^{x+2}}{27^{2x}\times8^{3x+1}}=2^p3^q$ **b** $\dfrac{125^{2x-2}\times8^{2x}}{512^2\times625^x}=2^p5^q$ **c** $\dfrac{625^{2x-2}\div243^{-x}}{125^{2x}\div27^{2x}}=3^p5^q$

4 Write these in the form $a\sqrt{b}$, where a and b are integers.

a $\sqrt{8}$ **b** $\sqrt{27}$ **c** $\sqrt{80}$ **d** $\sqrt{147}$

5 Simplify these by rationalising the denominator.

a $\dfrac{1}{\sqrt{2}}$ **b** $\dfrac{\sqrt{80}}{\sqrt{5}}$ **c** $\dfrac{\sqrt{729}}{\sqrt{3}}$ **d** $\dfrac{\sqrt{784}}{\sqrt{7}}$

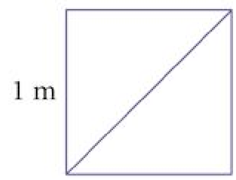
6 Simplify these by rationalising the denominator.

a $\dfrac{1}{2+\sqrt{3}}$ **b** $\dfrac{1}{2-\sqrt{7}}$ **c** $\dfrac{1+\sqrt{5}}{1-\sqrt{5}}$ **d** $\dfrac{2-\sqrt{3}}{3+2\sqrt{3}}$

7 The square in the diagram has side length of 1 m.

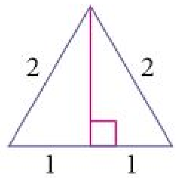
a Find the length of the diagonal.

b Using the diagram, find sin 45°.

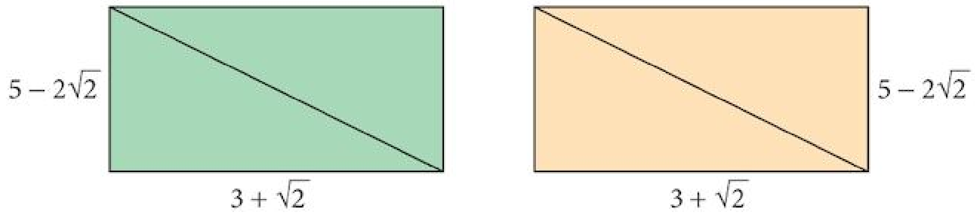


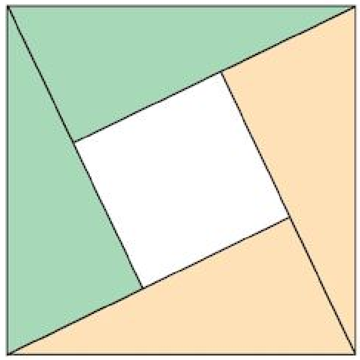
8 An equilateral triangle has sides of length 2 units.
From the diagram, find the exact values of:

a sin 60° **b** tan 30°



9





Two congruent rectangles of length $3+\sqrt{2}$ and width $5-2\sqrt{2}$ are cut in half along the diagonal and rearranged to form a square as shown.

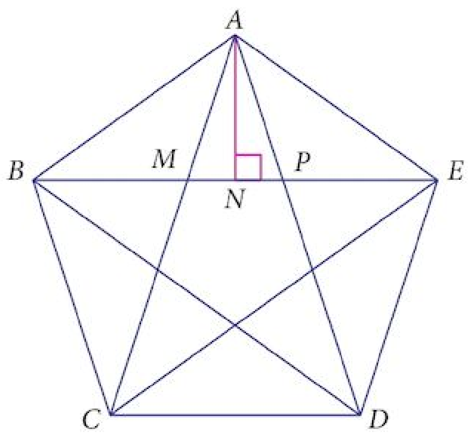
a Find the area of the central square.

b Find the total area of the triangles.

c Find the area of the large square.

d Use the diagram to verify Pythagoras' theorem.

10 A stellated pentagon (a pentagram) is drawn inside a regular convex pentagon as shown. The length $MP = 2$ units. A perpendicular is drawn from vertex A to meet BE at N.



a Show that the angle $NAP = 18°$.

b Use your calculator to verify that $\sin 18° = \frac{1}{1+\sqrt{5}}$.

c Find an expression for the length of AD.

d Triangle ACD is an enlargement of the triangle AMP. Find the scale factor of the enlargement and hence show that the length CD is of the form $a+b\sqrt{5}$, where a and b are integers.

e Hence find an accurate expression for $\cos 36°$ in the form $\frac{c+d\sqrt{5}}{e}$, where c, d and e are integers.

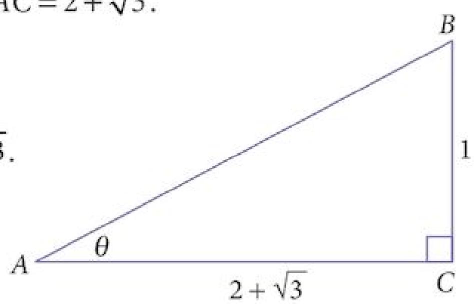
Chapter 2 Test

1 hour

1 Solve the equation $(9^{x-3})(3^{2x-3}) = 27$. [3]

2 Simplify $\frac{\sqrt{3}-2}{\sqrt{3}+2}$, giving your answer in the form $a+b\sqrt{3}$, where a and b are integers. [2]

3 ABC is a right-angled triangle, with sides $BC = 1$ and $AC = 2+\sqrt{3}$.



a Show that $\left[\frac{\sqrt{2}}{2}\left(1+\sqrt{3}\right)\right]^2 = 2+\sqrt{3}$. [2]

b Hence find the length of AB in terms of $\sqrt{2}$ and $\sqrt{3}$. [3]

c Hence express $\cos\theta$ in terms of $\sqrt{2}$ and $\sqrt{3}$, rationalising all surds in the denominator. [2]

4 Express $\dfrac{a\times(3a^2)^3}{\sqrt{27a^3}}$ in the form $3^n a^m$ where n and m are rational numbers. [3]

5 Solve the simultaneous equations:

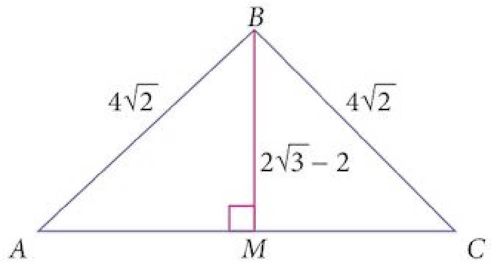
$$\frac{16^x}{8^y}=\frac{2\times 8^x}{4^y} \qquad \frac{243^x}{9\times 243^y}=\frac{81\times 9^x}{27^y}$$ [6]

6 A box in the shape of a cuboid has a square base of side length $(5-\sqrt{3})$ and a volume of $(82-12\sqrt{3})$. Find the height of the box in the form $a+b\sqrt{3}$. [5]

7 An isosceles triangle ABC has $AB=BC=4\sqrt{2}$.
M is the mid-point of AC, and $BM=\sqrt{12-2}$.

a Show that $AM=2+2\sqrt{3}$.

b Find the area of the triangle. [5]



8 Express $\dfrac{(4-\sqrt{3})^2}{2+\sqrt{3}}$ in the form $a+b\sqrt{3}$, where a and b are integers. [4]

9 Solve the simultaneous equations:

$$3^{3x}\times 9^y=81 \qquad 5^x\times 25^{2y}=\frac{1}{25}$$ [5]

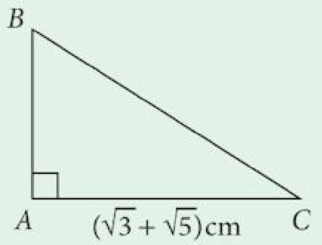
Examination Questions

1 The area of a rectangle is $\left(1+\sqrt{6}\right)\text{m}^2$. The length of one side is $\left(\sqrt{3}+\sqrt{2}\right)\text{m}$. Find, without using a calculator, the length of the other side in the form $\sqrt{a}-\sqrt{b}$, where a and b are integers. [4]

[Cambridge IGCSE Additional Mathematics, June 2003, P 1, Qu 2]

2 Simplify $\dfrac{16^{x+1}+20\left(4^{2x}\right)}{2^{x-3}8^{x+2}}$. [4]

[Cambridge IGCSE Additional Mathematics, June 2003, P 2, Qu 4]

3 

The diagram shows a right-angled triangle ABC in which the length of AC is $\left(\sqrt{3}+\sqrt{5}\right)$cm.

The area of triangle ABC is $\left(1+\sqrt{15}\right)\text{cm}^2$.

(i) Find the length of AB in the form $\left(a\sqrt{3}+b\sqrt{5}\right)$cm, where a and b are integers. [3]

(ii) Express $(BC)^2$ in the form $\left(c+d\sqrt{15}\right)\text{cm}^2$, where c and d are integers. [3]

[Cambridge IGCSE Additional Mathematics, June 2006, P 2, Qu 5]

4 Express $\dfrac{8-3\sqrt{2}}{4+3\sqrt{2}}$ in the form $a+b\sqrt{2}$, where a and b are integers. [3]

[Cambridge IGCSE Additional Mathematics, June 2008, P 1, Qu 1]

5 Solve the equation $\dfrac{5^{4y-1}}{25^y}=\dfrac{125^{y+3}}{25^{2-y}}$. [4]

[Cambridge IGCSE Additional Mathematics, June 2008, P 2, Qu 8 (part)]

6 **a** Solve the equation $9^{2x-1}=27^x$. [3]

b Given that $\dfrac{a^{-\frac{1}{2}}b^{\frac{2}{3}}}{\sqrt{a^3b^{-\frac{2}{3}}}}=a^pb^q$, find the value of p and of q. [2]

[Cambridge IGCSE Additional Mathematics, June 2009, P 1, Qu 5]

7 Without using a calculator, solve the equation $\dfrac{2^{x-3}}{8^{-x}}=\dfrac{32}{4^{\frac{1}{2}x}}$. [4]

[Cambridge IGCSE Additional Mathematics, Nov 2003, P 1, Qu 2]

8 A rectangular block has a square base. The length of each side of the base is $\left(\sqrt{3}-\sqrt{2}\right)$m and the volume of the block is $\left(4\sqrt{2}-3\sqrt{3}\right)\text{m}^3$. Find, without using a calculator, the height of the block in the form $\left(a\sqrt{2}+b\sqrt{3}\right)$m, where a and b are integers. [5]

[Cambridge IGCSE Additional Mathematics, Nov 2003, P 1, Qu 4]

9 Without using a calculator, solve, for x and y, the simultaneous equations

$$8^x \div 2^y = 64,$$

$$3^{4x} \times \left(\frac{1}{9}\right)^{y-1} = 81. \qquad [5]$$

[Cambridge IGCSE Additional Mathematics, Nov 2004, P 1. Qu 3]

10 Given that $k = \frac{1}{\sqrt{3}}$ and that $p = \frac{1+k}{1-k}$, express in its simplest surd form

(i) p,

(ii) $p - \frac{1}{p}$. [5]

[Cambridge IGCSE Additional Mathematics, Nov 2002, P 2, Qu 3]

11 a Find, in its simplest form, the product of $a^{\frac{1}{3}} + b^{\frac{2}{3}}$ and $a^{\frac{2}{3}} - a^{\frac{1}{3}}b^{\frac{2}{3}} + b^{\frac{4}{3}}$. [3]

b Given that $2^{2x+2} \times 5^{x-1} = 8^x \times 5^{2x}$, evaluate 10^x. [4]

[Cambridge IGCSE Additional Mathematics, Nov 2002, P 2, Qu 9]

12 (i) Express 9^{x+1} as a power of 3. [1]

(ii) Express $\sqrt[3]{27^{2x}}$ as a power of 3. [1]

(iii) Express $\dfrac{54 \times \sqrt[3]{27^{2x}}}{9^{x+1} + 216\left(3^{2x-1}\right)}$ as a fraction in its simplest form. [3]

[Cambridge IGCSE Additional Mathematics, Nov 2007, P 2, Qu 3]

13 Given that $\sqrt{a + b\sqrt{3}} = \frac{13}{4+\sqrt{3}}$, where a and b are integers, find, without using a calculator, the value of a and of b. [4]

[Cambridge IGCSE Additional Mathematics, Nov 2004, P 2, Qu 2]

14 A cuboid has a square base of side $\left(2-\sqrt{3}\right)$m and a volume of $\left(2\sqrt{3}-3\right)\text{m}^3$. Find the height of the cuboid in the form $\left(a+b\sqrt{3}\right)$m, where a and b are integers. [4]

[Cambridge IGCSE Additional Mathematics, Nov 2005, P 1, Qu 4]

15 Solve the equation $\frac{4^x}{2^{5-x}} = \frac{2^{4x}}{8^{x-3}}$. [3]

[Cambridge IGCSE Additional Mathematics, Nov 2008, P 2, Qu 5 (part)]

3 Permutations and combinations

Syllabus statements

- recognise and distinguish between a permutation case and a combination case
- know and use the notation $n!$ (with $0! = 1$), and the expressions for permutations and combinations of n terms taken r at a time.
- answer simple problems on arrangement and selection (cases with repetition of objects, or with objects arranged in a circle, or involving both permutations and combinations are excluded).

3.1 Introduction

This topic is all about counting. The counting process is important in probability theory and also in expanding powers of brackets using the binomial theorem.

3.2 Orderings

Example 3.1

How many different arrangements can you make from the letters A, B and C without repetition?

Solution:

											Choices:
The first letter could be:	A		or	B		or	C				3
The second letter could be:	B	C	or	A	C	or	A	B			2 for each
The third letter is:	C	B		C	A		B	A			1 left for each
											$3 \times 2 \times 1 = 6$

So, there are 6 possible arrangements: ABC BAC CAB

ACB BCA CBA

Example 3.2

a How many different numbers can be made using the digits 5, 6, 7, 8 and 9, using each digit only once?

b How many even numbers can be made using the digits 5, 6, 7, 8 and 9, using each digit only once?

Solution:

a

Position	1	2	3	4	5	**Total**
Choices	5	4	3	2	1	$5 \times 4 \times 3 \times 2 \times 1 = 120$

So there are 120 different numbers.

b In this question we have a **restriction**.

The number has to be even, so the final digit must be either 6 or 8.
We have 2 choices for the last place, then 4 choices for the first digit, three choices for the second digit, and so on.

Position	1	2	3	4	5	**Total**
Choices	4	3	2	1	2	$4 \times 3 \times 2 \times 1 \times 2 = 48$

So there are 48 different even numbers.

3.3 Factorial notation

$2!$	$=$	2×1	$=$	2
$3!$	$=$	$3 \times 2 \times 1$	$=$	6
$4!$	$=$	$4 \times 3 \times 2 \times 1$	$=$	24
$5!$	$=$	$5 \times 4 \times 3 \times 2 \times 1$	$=$	120

And so on.

Calculations like this keep appearing when we are counting, and this notation makes it easier to write them down.

Note that $5! = 5 \times 4!$ In general, $(n+1)! = (n+1) \times n!$

Example 3.3

Calculate **a** $4! \times 5!$ **b** $\dfrac{8!}{6!}$ **c** $\dfrac{9!}{5!\,4!}$

Solution:

a $4! \times 5! = 24 \times 120 = 2880$

b $$\frac{8!}{6!} = \frac{8 \times 7 \times \cancel{6} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}{\cancel{6} \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}} = 8 \times 7 = 56$$

c $$\frac{9!}{5!\,4!} = \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{(5 \times 4 \times 3 \times 2 \times 1) \times (4 \times 3 \times 2 \times 1)} = 126$$

There are many ways to cancel here.

3.4 Restrictions

Restrictions in orderings can take many forms and in any given question there might be more than one of them. One problem you have is to decide what order to impose the restrictions on your calculations.

Sometimes you break up the problem into smaller ones that you solve and then combine the results. On other ocassions it is easier to calculate the number of orderings that you *do not* want, and then subtract that from the total.

Often there is more than one way to work out the answer.

Example 3.4

How many numbers greater than 1000 can be made with the digits 0, 1, 2, 3, 4, 5 without repetition?

Solution:

Here we have two restrictions:
1. The first digit cannot be 0.
2. There can be 4, 5 or 6 digits.

Split the problem up into 4, 5 or 6 digit numbers.

	Position	10^5	10^4	10^3	10^2	10^1	10^0	Total
4 digits:	**Choices**			**5**	5	4	3	300
5 digits:			**5**	5	4	3	2	600
6 digits:		**5**	5	4	3	2	1	600
							Total	1500

Since the first digit cannot be 0, there are only 5 choices for the first digit, then they are all available.

There are a total of 1500 numbers greater than 1000.

Example 3.5

In Example 3.4, how many of the numbers are even?

Solution:

For the even numbers, the last digit could be 0, 2 or 4 but the first digit cannot be 0.

We have to split this up into two cases, one with 0 as the last digit and one where the last digit is 2 or 4.

(i) First case: The last digit is 0.

	Position	10^5	10^4	10^3	10^2	10^1	10^0	Total
4 digits:	**Choices**			5	4	3	**1**	60
5 digits:			5	4	3	2	**1**	120
6 digits:		5	4	3	2	1	**1**	120
							Total	300

(ii) Second case: The last digit is 2 or 4.

	Position	10^5	10^4	10^3	10^2	10^1	10^0	Total
4 digits:	**Choices**			**4**	4	3	**2**	96
5 digits:			**4**	4	3	2	**2**	192
6 digits:		**4**	4	3	2	1	**2**	192
							Total	480

So, there are 300 + 480 = 780 even numbers.

An alternative solution to Example 3.5:

You could calculate how many **odd** numbers there are.

Split the problem up again into 4, 5 or 6 digit numbers.

	Position	10^5	10^4	10^3	10^2	10^1	10^0	Total
4 digits:	Choices			**4**	4	3	**3**	144
5 digits:			**4**	4	3	2	**3**	288
6 digits:		**4**	4	3	2	1	**3**	288
							Total	720

There are 720 odd numbers.

So the total number of even numbers is $1500 - 720 = 780$, the same as the first solution.

Example 3.6

The letters of the word CHEMISTRY are written out on cards, one letter to each card. They are arranged in a line. Find:

a the number of different arrangements there are,

b the number of these arrangements that end in a vowel,

c the number of these arrangements that begin and end with a vowel.

Solution:

a There are 9 different letters in the word.

Position	1	2	3	4	5	6	7	8	9
Choices	9	8	7	6	5	4	3	2	1

The total number of arrangements is $9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 362\,880$

b In this part, we have a restriction so we need to deal with that first.

There are only 2 vowels in the word, so we have 2 choices for the last position.

Once we have chosen one of those, there are 8 letters left to arrange in the remaining positions.

Position	1	2	3	4	5	6	7	8	9
Choices	8	7	6	5	4	3	2	1	2

Deal with the restriction first.

The total number of arrangements is $8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 \times 2 = 80\,640$

c In this part, we have a stronger restriction that must be dealt with first.

There are only 2 vowels in the word, so we have 2 choices for the first position and then only 1 choice for the last position.

Once we have chosen these, there are 7 letters left to arrange in the remaining positions.

Position	1	2	3	4	5	6	7	8	9
Choices	2	7	6	5	4	3	2	1	1

Deal with the restriction first.

The total number of arrangements is $2 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 \times 1 = 10\,080$

Exercise 3.1

1 Find out how your calculator works out factorials. Some have a function button, others use a menu selection. Use your calculator to calculate the following:

a $7!$ **b** $10!$ **c** $15!$ **d** $20!$ **e** $30!$

f $\frac{10!}{5!}$ **g** $\frac{8!}{3!}$ **h** $\frac{6!}{2!}$ **i** $\frac{11!}{7!}$ **j** $\frac{15!}{12!}$

k $\frac{7!}{4!\,3!}$ **l** $\frac{9!}{2!\,7!}$ **m** $\frac{12!}{6!\,6!}$ **n** $\frac{14!}{5!\,9!}$ **o** $\frac{15!}{3!\,12!}$

2 **a** What is the largest factorial number that your calculator can display without using standard form?

b What is the largest factorial number that your calculator can display using standard form?

3 **a** Why does 6! end in the digit 0?

b How many zeros are there at the end of 20!?

c How many zeros are there at the end of 30!?

4 How many letter arrangements can be made from the letters in the word SPACE using each letter once and:

a containing 3 letters

b containing 4 letters

c containing 5 letters?

5 How many letter arrangements can be made from the letters of the word PORTUGAL using each letter once and:

a containing 5 letters

b containing 6 letters

c containing 7 letters?

6 How many registration codes can be made using two letters from the word DUBAI followed by 3 digits from the set {1, 2, 3, 4, 5}?

7 You are given the set of digits {1, 2, 3, 4, 5, 6, 7}. Using each digit only once,

 a how many 3 digit numbers can be made from them?,

 b how many 4 digit numbers greater than 3000 can be made from them?,

 c how many 6 digit even numbers can be made from them?

8 Three friends go to a restaurant for dinner. The restaurant has 4 choices of starter, 5 choices of main course and 3 choices of dessert.

 How many different meal arrangements could the three friends have if no two of them had the same dish for any course?

9 Three girls, Aisha, Bhavya and Caroline, and two boys, Danial and Eng stand in a line.

 a How many different ways can they do this if Danial and Eng always stand next to each other?

 b How many different ways can they do this if Aisha does not stand next to Bhavya?

10 A piece of wood 12 cm long is to be cut into 3 pieces so that the length of each piece is a whole number of cm.

 a In how many ways can this be done?

 b In how many of these ways is it possible to combine 2 of the 3 pieces to make a length of 6 cm?

3.5 Permutations and combinations

Example 3.7

How many 4 digit numbers can be made from the set {1, 2, 3, 4, 5, 6}?

Solution:

This can be written as

$$6\times5\times4\times3=\frac{6\times5\times4\times3\times2\times1}{2\times1}$$

$$=\frac{6!}{2!}$$

$$=\frac{6!}{(6-4)!}$$

$$=360$$

There are 6 digits available and you have been asked to make numbers containing 4 digits.

In Example 3.7, the order is important. If you change the order of the chosen digits, you have a different number. This is called a **permutation**.

Notation	Formula	Meaning
${}_nP_r$	$\dfrac{n!}{(n-r)!}$	The number of permutations of r things chosen from n things.

Example 3.8

In how many ways can 4 students be chosen from a group of 6 students?

Solution:

This is almost the same question as in Example 3.7, with one big difference.

This time it does not matter in what position each student is chosen. What is important is just whether or not each is chosen. Start with the solution of Example 3.7. There are 360 lines of students that could be chosen if they stood in order.

However, there are 4! (= 24) ways that the same group could be chosen.

The total number of different groups that could be chosen is therefore $360 \div 24 = 15$

$$= \frac{6!}{4!\,2!}$$

In Example 3.8, the order is **not** important. If you change the order in which the students are chosen, you still have the same group. This is called a **combination**.

Notation	Formula	Meaning
${}_nC_r$	$\dfrac{n!}{r!(n-r)!}$	The number of combinations of r things chosen from n things.

Remember: The difference between a permutation and a combination is the question of **order**.

Example 3.9

A committee of 5 students is to be chosen from 6 boys and 9 girls.

Find the number of ways that this can be done if:

a there are no restrictions

b there are to be 3 boys and 2 girls on the committee

c there must be more girls than boys on the committee.

Solution:

a This question is about **combinations**. There is no order implied.

If there are no restrictions, then there will be ${}^{15}C_5 = 3003$ ways to form the committee.

b The number of boys is restricted and the number of girls is restricted.

There are ${}^5C_3$ ways to choose the boys and ${}^9C_2$ ways to choose the girls.

In total there will be ${}^5C_3 \times {}^9C_2 = 360$ ways to form the committee.

c This question must be approached by splitting it up into different cases.

There are 3 ways in which we can have more girls than boys on the committee.

	Girls	Boys	giving			
(i)	3	2	${}^9C_3 \times {}^5C_2$	=	840	ways
(ii)	4	1	${}^9C_4 \times {}^5C_1$	=	630	ways
(iii)	5	0	${}^9C_5$	=	126	ways
Making a total of					1596	ways to form the committee.

3.6 0!

You might ask the question "How many ways are there of arranging no (zero) things in a line?" The answer, which may seem strange, is 1.

An alternative argument is "How many combinations of n things chosen from n things are there?" The answer is obviously 1.

The formula gives $\dfrac{n!}{n!(n-n)!} = \dfrac{1}{0!} = 1$

This makes sense only if $0! = 1$

In Example 3.9, when we had 5 girls and 0 boys, we could have calculated ${}^9C_5 \times {}^5C_0$. It would have given us the same answer.

Note: Sometimes alternative notations are used:

${}_nP_r \quad {}^nP_r$	${}_nC_r \quad {}^nC_r \quad \dbinom{n}{r}$

Exercise 3.2

1 Write down the formula for each of the following and use your calculator to work out the answer:

a $^{5}P_{3}$ b $^{8}P_{4}$ c $^{10}P_{6}$ d $^{9}P_{5}$ e $^{7}P_{2}$ f $^{4}P_{2}$ g $^{12}P_{8}$

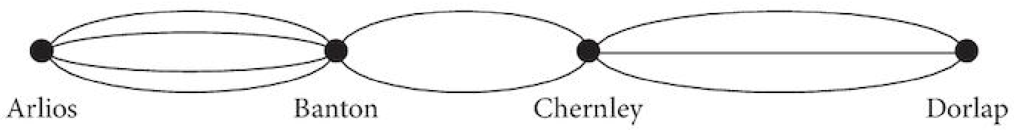
2 In the Olympic 100m final, there are 8 runners. In how many ways can the Gold, Silver and Bronze medals be distributed among the runners?

3 A sporting event between two teams could finish in one of three ways: home win, draw, away win. In how many ways could the results of a competition round between 12 teams (6 matches) finish?

4 In a horserace there are 6 runners. In how many different ways can the six horses finish the race?

5 I have eight books on a shelf in my library. In how many different ways can they be put onto the shelf?

6 In a journey from Arlios to Dorlap via Banton and Chernley, there are four possible routes from Arlios to Banton, two from Banton to Chernley and three from Chernley to Dorlap. If you want to travel a different single journey from Arlios to Dorlap each day, how many days will it be before you have to repeat a route?



7 a How many different 4 digit numbers can be formed from the set $\{2, 3, 5, 6, 7, 9\}$, if no digit may be repeated?

b How many of them are odd?

c How many of them are greater than 4000?

8 A group of 8 female students and 6 male students are standing for election to a Club Committee. Three officers are required: Chairperson, Secretary and Treasurer.

a In how many different ways could the Club officers be elected?

b In how many different ways could the Club officers be elected if the Chairperson must be a female student?

c How many ways are there of choosing all male club officers?

d How many ways of choosing the officers are there such that there is a majority of females (excluding an all female selection)?

9 Write down the formula for each of the following and use your calculator to work out the answer:

a ${}^{5}C_{3}$ **b** $\binom{8}{2}$ **c** ${}^{10}C_{6}$ **d** $\binom{9}{5}$ **e** ${}^{7}C_{2}$ **f** $\binom{4}{2}$ **g** ${}^{12}C_{8}$

10 A company produces Chummy Chocolate Bars in 10 different flavours. They are sold in packs of 4 so that, in each pack, each bar is a different flavour. How many different packs are possible?

11 For a school trip, there are 10 seats in the minibus. Applications are received from 8 boys and 7 girls for the trip.

a In how many ways could the group that goes on the trip be selected?

b How many of the groups would contain equal numbers of boys and girls?

c How many of the groups would contain more girls than boys?

12 Three boys and three girls are flying to Madrid together. They all sit in the same row on the airplane, where there are three seats on each side of the aisle.

a If the seats are allocated at random, in how many ways could the group be seated?

b If the girls insist on sitting together, in how many ways could the group be seated?

When the group arrives, they arrange for two taxis to take them to their hotel, with three people to each taxi.

c One of the girls refuses to travel in the same taxi as the boy who teased her during the plane journey. In how many ways could the groups be allocated to the taxis?

13 Lucas has 3 blue tiles and 5 green tiles to put in a line on a wall.

a In how many different ways can he attach them to the wall?

b In how many ways will the 3 blue tiles be next to each other?

14 A group of friends are keen cricketers. Among them are 8 bowlers, 10 batsmen and 3 wicket-keepers. For a particular match, the coach decides that he needs 5 batsmen, 5 bowlers and a wicket-keeper. In how many ways could the team be chosen?

Summary

Permutations	The order is important.
Combinations	The order does not matter.
Orderings	The number of different ways something can be done. These are permutations.
Groupings	The number of different groups that can be made. These are combinations.
Factorial notation	$n! = n \times (n-1) \times (n-2) \times (n-3) \times \cdots \times 3 \times 2 \times 1$ $(n+1)! = (n+1) \times n!$ $0! = 1$
Formulae	$_nP_r = \frac{n!}{(n-r)!}$ The number of permutations of r things chosen from n things. $_nC_r = \frac{n!}{r!(n-r)!}$ The number of combinations of r things chosen from n things. $\binom{n}{r}$ is also used.

Chapter 3 Summative Exercise

1 Use your calculator to find:

a $6!$ **b** $8!$ **c** $9!$ **d** $12!$ **e** $13!$

2 Use your calculator to find:

a $\frac{9!}{3!}$ **b** $\frac{10!}{4!}$ **c** $\frac{12!}{7!}$ **d** $\frac{18!}{10!}$ **e** $\frac{20!}{14!}$

f $_9P_6$ **g** $_{10}P_6$ **h** $_{12}P_5$ **i** $_{18}P_8$ **j** $_{20}P_6$

3 Use your calculator to find:

a $\frac{8!}{3!\,5!}$ **b** $\frac{10!}{6!\,4!}$ **c** $\frac{13!}{6!\,7!}$ **d** $\frac{20!}{8!\,12!}$ **e** $\frac{30!}{12!\,18!}$

f $_8C_3$ **g** $_{10}C_6$ **h** $_{13}C_6$ **i** $_{20}C_8$ **j** $_{30}C_{12}$

4 The formulae for $_8C_3$ and $_8C_5$ are identical. Describe a situation which would illustrate why this should be so.

5 How many zeros are there at the end of $_{18}P_{10}$?

6 Find the number of arrangements that can be made from the letters of the word VELOCITY, using each letter only once, containing:

a 3 letters

b 4 letters

c 5 letters.

7 A children's book consists of cartoon pictures of people and animals cut into three sections so that there are 7 different heads, 9 different bodies and 5 different pairs of legs. By opening the pages of the book at the appropriate pages, you can create pictures of different, unlikely, cartoon characters such as the head of an old man with the body of a chicken and the legs of an alligator.

How many different characters could you create?

8 Identify whether the following situations are permutations or combinations.

- **a** Three children from a class of 20 are selected to form an organising committee.
- **b** A new style of car number plate is proposed because the authorities are running out of numbers in the old style.
- **c** A class has Mathematics lessons on three days in a week.

9 What is 0! (zero factorial)?

10 You are given the set of digits 1, 2, 3, 6, 7, 8. Find the number of different 4 digit numbers that can be formed if:

- **a** there is no restriction
- **b** the number must be larger than 4000
- **c** the number must be even
- **d** the number must be an odd number greater than 4000.

11 I have three books written by William Shakespeare, four written by Charles Dickens and three written by Geoffrey Chaucer. In how many ways can they be arranged on the shelves if:

- **a** there is no restriction
- **b** the books by each author must be kept together
- **c** the books must be put in alphabetical order by the author's name.

12 Four boys and four girls are going on a school outing, using 2 four-seater vehicles. Find how many ways they can travel in the vehicles if:

- **a** the boys must travel in one vehicle and the girls in the other
- **b** there is no restriction
- **c** Betty and Caroline will not travel in the same vehicle.

13 An icosahedron has 20 faces, each of which is an identically sized equilateral triangle.

- **a** The triangular faces are joined edge to edge.

  How many edges must the icosahedron have?
- **b** The triangular vertices are joined in groups of five around a vertex.

  How many vertices must an icosahedron have?
- **c** Repeat the exercise for a dodecahedron, which has 12 regular pentagonal faces.

Chapter 3 Test

1 hour

1 A student committee is to be made up of 6 people chosen from a group consisting of 8 girls and 10 boys. Find how many ways this can be done if:

- **a** there are no restrictions [2]
- **b** there must be the same number of boys and girls on the committee [3]
- **c** there are more boys than girls on the committee. [3]

2 A 4 digit number is to be formed from the digits 1, 3, 4, 7, 8 and 9. Each digit can be used only once. Find the number of 4 digit numbers that can be formed if:

- **a** there are no restrictions [2]
- **b** the 4 digit number must be even [1]
- **c** the 4 digit number must greater than 6000 [1]
- **d** the 4 digit number must be even and greater than 6000. [1]

3 Five girls and four boys are seated in a row. Calculate the number of ways this can be done if:

- **a** there are no restrictions [1]
- **b** they sit alternately (boy, girl, boy, girl, etc.) [2]
- **c** the girls sit together and the boys sit together [2]
- **d** Adam and Betty are not allowed to sit next to each other. [3]

4 A 3 digit number is to be formed from the digits 1, 2, 3, 4, 5 and 6, using each digit only once. Find the number of 3 digit numbers that can be formed if:

- **a** there are no restrictions [1]
- **b** the 3 digit number must be even [1]
- **c** the 3 digit number must be a multiple of 5 [2]
- **d** the 3 digit number must be a multiple of 3. [2]

5 Six students are selected to go on a trip from a class of 12 boys and 10 girls. Find how many ways this can be done if:

- **a** there are no restrictions [1]
- **b** the group cannot contain only one boy or only one girl [2]
- **c** Ahmad and Belinda, whose parents own the venue that they are due to visit, must be a part of the group, but they cannot be the only member of their gender. [2]

6 A Mathematics Club committee is to consist of 8 people chosen from 6 boys and 8 girls. Find the number of different committees that can be formed if:

- **a** there are no restrictions [2]
- **b** there must be an equal number of boys and girls on the committee [3]
- **c** there must be at least one girl and at least one boy. [3]

Examination Questions

1 **(i)** Find the number of different arrangements of the letters of the word MEXICO.

Find the number of these arrangements

(ii) which begin with M,

(iii) which have the letter X at one end and the letter C at the other end. [5]

Four of the letters of the word MEXICO are selected at random. Find the number of different combinations if

(iv) there is no restriction on the letters selected,

(v) the letter M must be selected. [3]

Cambridge IGCSE Additional Mathematics 0606, Jun 2003, P2, Qu 7

2 A student has a collection of 9 CDs, of which 4 are by the Beatles, 3 are by Abba and 2 are by the Rolling Stones. She selects 4 of the CDs from her collection. Calculate the number of ways in which she can make her selection if

(i) her selection must contain her favourite Beatles CD, [2]

(ii) her selection must contain 2 CDs by one group and 2 CDs by another. [3]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2005, P1, Qu 2]

3 **a** 7 boys are to be seated in a row. Calculate the number of different ways in which this can be done if 2 particular boys, Andrew and Brian, have exactly 3 of the other boys between them. [4]

b A box contains sweets of 6 different flavours. There are at least 2 sweets of each flavour. A girl selects 3 sweets from the box. Given that these 3 sweets are not all the same flavour, calculate the number of different ways she can select her 3 sweets. [3]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2006, P2, Qu 7]

4 An artist has 6 watercolour paintings and 4 oil paintings. She wishes to select 4 of these 10 paintings for an exhibition.

(i) Find the number of different selections she can make. [2]

(ii) In how many of these selections will there be more watercolours than oil paintings? [3]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2007, P2, Qu 4]

5 A committee of 5 people is to be selected from 6 men and 4 women. Find

(i) the number of different ways in which the committee can be selected, [1]

(ii) the number of these selections with more women than men. [4]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2008, P1, Qu 2]

6 A musician has to play 4 pieces from a list of 9. Of these 9 pieces 4 were written by Beethoven, 3 by Handel and 2 by Sibelius. Calculate the number of ways the 4 pieces can be chosen if

(i) there are no restrictions, [2]

(ii) there must be 2 pieces by Beethoven, 1 by Handel and 1 by Sibelius, [3]

(iii) there must be at least one piece by each composer. [4]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2009, P2, Qu 9]

7 **a** The producer of a play requires a total cast of 5, of which 3 are actors and 2 are actresses. He auditions 5 actors and 4 actresses for the cast. Find the total number of ways in which the cast can be obtained. [3]

b Find how many different odd 4 digit numbers less than 4000 can be made from the digits 1, 2, 3, 4, 5, 6, 7 if no digit may be repeated. [3]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2002, P2, Qu 5]

8 A garden centre sells 10 different varieties of rose bush. A gardener wishes to buy 6 rose bushes, all of different varieties.

(i) Calculate the number of ways she can make her selection. [2]

Of the 10 varieties, 3 are pink, 5 are red and 2 are yellow. Calculate the number of ways in which her selection of 6 rose bushes could contain

(ii) no pink rose bush, [1]

(iii) at least one rose bush of each colour. [4]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2003, P2, Qu 8]

9 **a** Find the number of different arrangements of the 9 letters of the word SINGAPORE in which S does **not** occur as the first letter. [2]

b 3 students are selected to form a chess team from a group of 5 girls and 3 boys. Find the number of possible teams that can be selected in which there are more girls than boys. [4]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2004, P1, Qu 7]

10 **a** Each day a newsagent sells copies of 10 different newspapers, one of which is *The Times*. A customer buys 3 different newspapers. Calculate the number of ways the customer can select his newspapers

(i) if there is no restriction, [1]

(ii) if 1 of the 3 newspapers is *The Times*. [1]

b Calculate the number of different 5 digit numbers which can be formed using the digits 0, 1, 2, 3, 4 without repetition and assuming that a number cannot begin with 0. [2]

How many of these 5 digit numbers are even? [4]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2005, P2, Qu 11]

11 a How many different four digit numbers can be formed from the digits 1, 2, 3, 4, 5, 6, 7, 8, 9 if no digit may be repeated? [2]

b In a group of 13 entertainers, 8 are singers and 5 are comedians. A concert is to be given by 5 of these entertainers. In the concert there must be at least 1 comedian and there must be more singers than comedians. Find the number of different ways that the 5 entertainers can be selected. [6]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2006, P2, Qu 10]

12 A badminton team of 4 men and 4 women is to be selected from 9 men and 6 women.

(i) Find the total number of ways in which the team can be selected if there are no restrictions on the selection. [3]

Two of the men are twins.

(ii) Find the number of ways in which the team can be selected if exactly one of the twins is in the team. [3]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2007, P1, Qu 4]

13 a A sports team of 3 attackers, 2 centres and 4 defenders is to be chosen from a squad of 5 attackers, 3 centres and 6 defenders. Calculate the number of different ways in which this can be done. [3]

b How many different 4 digit numbers greater than 3000 can be formed using the six digits 1, 2, 3, 4, 5 and 6 if no digit can be used more than once? [3]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2008, P2, Qu 6]

14 A committee of 8 people is to be selected from 7 teachers and 6 students. Find the number of different ways in which the committee can be selected if

(i) there are no restrictions, [2]

(ii) there are to be more teachers than students on the committee. [4]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2009, P1, Qu 7]

4 Functions

Syllabus statements

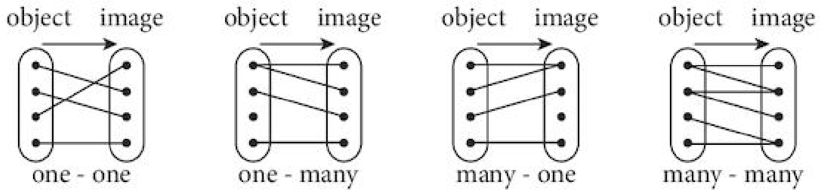
- understand the terms: function, domain, range (image set), one-one function, inverse function, and composition of functions
- use the notation $f(x) = \sin x$, $f : x \mapsto \lg x (x > 0)$, $f^{-1}(x)$; and $f^2(x)$ [$= f(f(x))$]
- understand the relationship between $y = f(x)$ and $y = |f(x)|$, where $f(x)$ may be linear, quadratic or trigonometric
- explain in words why a given function is a function or why it does not have an inverse
- find the inverse of a one-one function and form composite functions
- use sketch graphs to show the relationship between a function and its inverse

4.1 Introduction

While you have used functions in much of your work so far, you have done so without defining specifically what you were talking about. As in many areas of Mathematics, we need a precise definition of mathematical objects in order to be absolutely clear. This chapter seeks to do that and introduces new notations to aid our study. We also look at reversible processes and a new type of function.

4.2 Mappings

A mapping is a relationship between the members of two sets: a set of objects (or input values) and a set of images (or output values). We can generalise them into four types of mapping:



Notes: 1 Many means more than one.

2 It needs only one of the objects (or images) to have more than one image (or object) to qualify as "many". In most cases there will be more than one such object (or image).

4.3 Functions

A mapping is a **function** if and only if each object has only one image.

Thus, both **one-one** and **many - one** mappings represent functions.

Example 4.1

Which of the following numerical mappings is a function?

a Multiply by 2 and add 5. **b** Take the square root of the number.

Solution:

a This is a function because for whatever number you start with, you will always get a single answer.

b You must choose a positive number, but you will get two answers for each. For example, when you take the square root of 4, you get 2 or –2. So, this is not a function.

4.4 Defining functions

We must have
a a set of object (input) values;
b a rule telling you what to do with the object values;
c some idea of what the image (output) values might be;
d a name for the function.

In **Example 4.1 b**, you could not choose a negative number as the input value.

The set of object values is called the **domain**.

The set of output values we can obtain is called the **range**.

Historically, several ways of writing down the definition of a function have been used.

These are all equivalent:

(i) $\mathrm{f}: x \mapsto 2x+5\,;\, x \in \mathbb{R}$

(ii) $\mathrm{f}(x) = 2x+5\,;\, x \in \mathbb{R}$

(iii) $y = 2x+5\,;\, x \in \mathbb{R}$

Function f maps x onto $(2x + 5)$ where x is a real number.

Since it is possible to get any number out of the function by choosing a suitable input value, the range is also $\mathbb{R}$, the set of all real numbers.

4.5 Finding the value of a function

Example 4.2

You are given the function $f : x \mapsto 2x + 5\ ;\ x \in \mathbb{R}, -2 < x < 5$.

a Find $f(2)$.

b Find the range of the function.

Solution:

a $f(2) = 2(2) + 5 = 9$

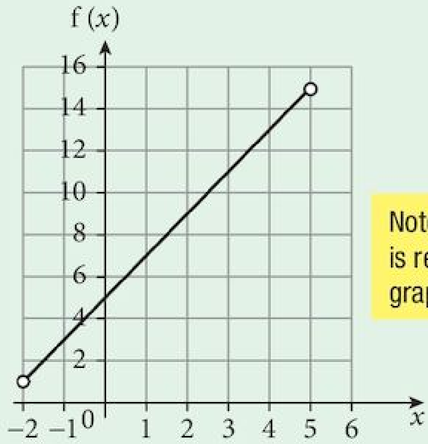
b The graph of $f(x)$ is shown.

It is linear and x can take any real value between -2 and 5.

The smallest value of $f(x)$ will be $f(-2) = 1$.

The largest value of $f(x)$ will be $f(5) = 15$.

Thus $1 < f(x) < 15$



Note that the extreme values, −2 and 5 are not in the domain so f(−2) and f(5) will not be in the range.

Note also that because the domain is restricted, there is no part of the graph beyond the end points.

Example 4.3

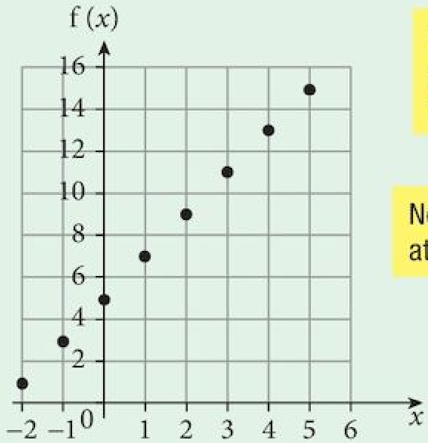
If $f : x \mapsto 2x + 5\ ;\ x \in \mathbb{Z}, -2 \leqslant x \leqslant 5$, find the range of the function.

Solution:

The graph of $f(x)$ is shown.
It is linear but x can take only integer values between -2 and 5 (inclusive).

The range,

$f(x) = \{1, 3, 5, 7, 9, 11, 13, 15\}$



The graph consists only of the points shown since the domain is restricted to integer values.

Notice this time that the points at $x = -2$ and $x = 5$ are included.

Exercise 4.1

1 Find the value of the functions for the input values given.

a $f : x \mapsto 5x - 3\ ;\ x = 4$

b $f : x \mapsto x^2 + 1\ ;\ x = 3$

c $g : x \mapsto 4 - x^2\ ;\ x = 5$

d $g : x \mapsto x^3 - x^2\ ;\ x = 6$

e $h : x \mapsto \dfrac{12}{x^2}\ ;\ x = 2$

f $h : x \mapsto \dfrac{2x+8}{x-2}\ ;\ x = 5$

2 Find the value of the functions for the input values given.

a $f(x) = 3x + 5\ ;\ x = 2$

b $f(x) = x^2 - 3\ ;\ x = 4$

c $g(x) = 81 - x^3\ ;\ x = 4$

d $g(x) = x(2 + x)^2\ ;\ x = 6$

e $h(x) = \dfrac{24}{x}\ ;\ x = 3$

f $h(x) = \dfrac{4x-2}{x+1}\ ;\ x = 5$

3 Find the value of the functions for the input values given.

a $y = 2x + 4\ ;\ x = 3$

b $y = x^2 + x\ ;\ x = 2$

c $y = x^3 - x\ ;\ x = 1$

d $y = (x - 2)(3 + x)\ ;\ x = 4$

e $y = \dfrac{6}{x}\ ;\ x = 2$

f $y = \dfrac{x+6}{x-2}\ ;\ x = 3$

4 Find the range of these functions.

a $f : x \mapsto 3x - 2\ ;\ x \in \mathbb{R}$

b $f : x \mapsto 3x - 2\ ;\ x \in \mathbb{R}, -3 \leqslant x \leqslant 3$

c $g(x) = 3x - 2\ ;\ x \in \mathbb{Z}$

d $g(x) = 3x - 2\ ;\ x \in \mathbb{Z}, -3 \leqslant x \leqslant 3$

e $y = x^3\ ;\ x \in \{1, 3, 5\}$

f $y = (x - 2)(3 + x)\ ;\ x \in \mathbb{R}, -5 \leqslant x \leqslant -1$

g $y = \dfrac{12}{x}\ ;\ x \in \{1, 2, 3, 4, 6\}$

h $y = \dfrac{x+6}{x-2}\ ;\ x \in \mathbb{R}, 3 \leqslant x \leqslant 10$

i $h(x) = 4 - x^2\ ;\ x \in \{-2, -1, 0, 1, 2\}$

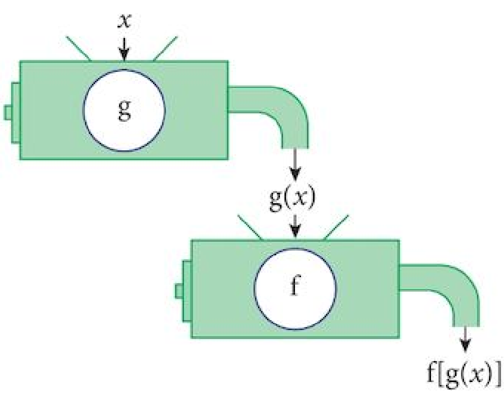
j $h(x) = (x + 2)(x - 4)\ ;\ x \in \mathbb{R}, -2 \leqslant x \leqslant 6$

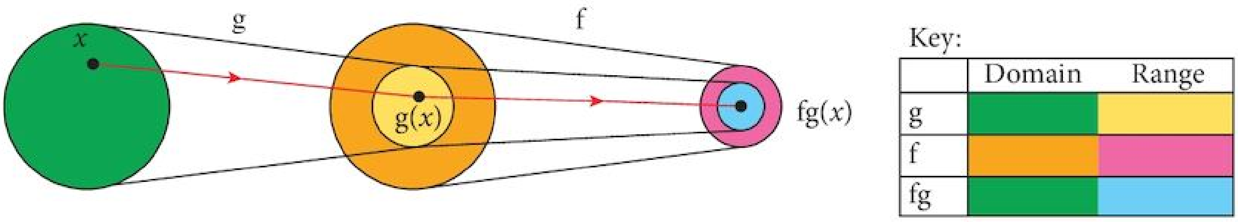
4.6 Composite functions

A **composite function** is formed when the output from one function is fed into a second function, as shown.

In this case, the output from function g is fed into the function f. The final output is the value $f[g(x)]$.

Remember: $f[g(x)]$ means g first, then f.
We often shorten $f[g(x)]$ by writing $fg(x)$.





In order for this composition to be successful, the range of g must be a subset of the domain of f. If that is not true, the function fg cannot exist.

The domain of the composite is the domain of the first (inner) function.

Example 4.4

The functions f and g are defined as follows:

$\mathrm{f}: x \mapsto 2x + 5\ ;\ x \in \mathbb{R}$ and $\mathrm{g}: x \mapsto 3x - 2\ ;\ x \in \mathbb{R}, -2 \leqslant x \leqslant 5$

a Form the functions **(i)** fg and **(ii)** $\mathrm{f}^2(=\mathrm{ff})$ and find the range of each function.

b Why do the functions **(i)** gf and **(ii)** g^2 not exist?

Solution:

a **(i)** $\mathrm{f}[\mathrm{g}(x)] = \mathrm{f}[3x - 2]$

$= 2[3x - 2] + 5$

$\mathrm{f}[\mathrm{g}(x)] = 6x + 1\ ;\ x \in \mathbb{R}, -2 \leqslant x \leqslant 5$

(ii) $\mathrm{f}^2(x) = \mathrm{f}[\mathrm{f}(x)]$

$= \mathrm{f}[2x + 5]$

$= 2[2x + 5] + 5$

$\mathrm{f}^2(x) = 4x + 15\ ;\ x \in \mathbb{R}$

b **(i)** $\mathrm{gf}(1) = \mathrm{g}(7)$

7 is not in the domain of g so this composite does not exist.

We could have chosen any value such that f(x) is not in the domian of g.

(ii) $\mathrm{g}^2(3) = \mathrm{g}(7)$

As before, 7 is not in the domain of g so this composite does not exist.

$\mathrm{g}(1) = -1$.

This is in the domain of g but that does not matter.

Exercise 4.2

The functions f, g and h are defined as follows in the box at the right.

$f : x \mapsto 3x - 2 \; ; x \in \mathbb{R}$

$g : x \mapsto 4 - 2x \; ; x \in \mathbb{R}$

$h : x \mapsto \dfrac{12}{x} \; ; x \in \mathbb{R} \; ; x \neq 0$

1 For each of the following composite functions:

(i) find the image of the input value given

(ii) define the functions fully.

a $fg \; ; x = 3$	**b** $f^2 \; ; x = -2$	**c** $fh \; ; x = 6$
d $f^2g \; ; x = 2$	**e** $gf \; ; x = -4$	**f** $g^2 \; ; x = -2$
g $gh \; ; x = 6$	**h** $h^2 \; ; x = 4$	

2 Show that the composites **(i)** hf and **(ii)** hg cannot exist.

3 Show that $(fg)^2$ is not the same as f^2g^2:

a by choosing a specific input value

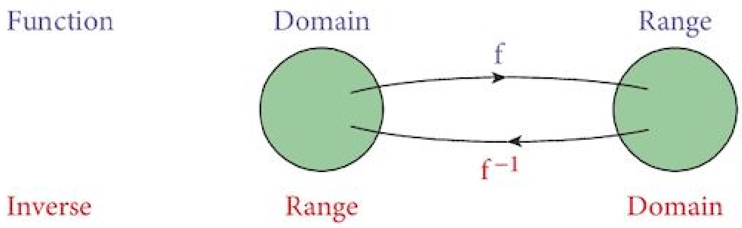
b by finding the definition of each function.

4 Solve the equations:

a $fg(x) = -20$ **b** $fh(x) = 10$ **c** $g^2(x) = 24$ **d** $gh(x) = 0$

4.7 Inverse functions

An **inverse function** (or just **inverse**) is another function that will reverse the effect of the first function. The inverse of a function f is written f^{-1}.



The **Domain** of the inverse is the **Range** of the function.

The **Range** of the inverse is the **Domain** of the function.

The inverse of a one-one mapping is another one-one mapping and that is a function. However, the inverse of a many-one mapping would be one-many and that is not a function.

Thus, only one-one functions have inverses.

Linear functions are 1:1 and so they all have inverses.

In all cases, $ff^{-1}(x) = f^{-1}f(x) = x$.

Example 4.5

The function f is defined as follows:

$f : x \mapsto 2x + 5 \; ; x \in \mathbb{R}$.

a Find the inverse of the function.

b Sketch the graph of the function and its inverse.

c Describe the relationship between the graph of the function and the graph of its inverse.

Solution:

a The function is 1 : 1 and so it has an inverse.

Its domain is $\mathbb{R}$ and so its range is also $\mathbb{R}$.

Writing $y = 2x + 5$ to find the inverse we make x the subject of the equation:

$$x = \frac{1}{2}(y - 5)$$

Then we swap the x and y:

$$y = \frac{1}{2}(x - 5)$$

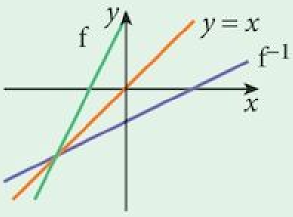
So $f^{-1} : x \mapsto \frac{1}{2}(x - 5)\ ;\ x \in \mathbb{R}$.

The domain of the inverse is the range of the function.

The functions

$f : x \mapsto 3x - 2$
and $g : y \mapsto 3y - 2$
are identical.

b

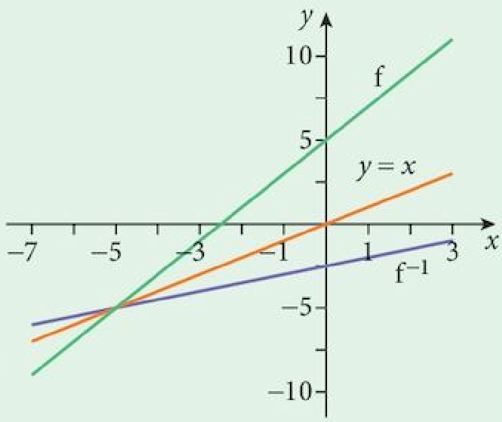


c The graph of the inverse is a reflection of the graph in the line $y = x$.

Note: When you are sketching graphs of functions and inverses, always keep the axis scales the same.
Then the relationship will be clear.

Always include the line $y = x$ drawn at 45° to aid your description.

This is the same graph with different scales and it does not look like a reflection. The line $y = x$ is not at 45°.



Exercise 4.3

For each function given below:

a Find the inverse of the function.

b Sketch the graph of the function and its inverse.

c Describe the relationship between the graph of the function and the graph of its inverse.

1 $f : x \mapsto 2x + 1\ ;\ x \in \mathbb{R}$

2 $f : x \mapsto 2x - 4\ ;\ x \in \mathbb{R}$

3 $g : x \mapsto 6 - 2x\ ;\ x \in \mathbb{R}$

4 $g : x \mapsto 4 - \frac{1}{2}x\ ;\ x \in \mathbb{R}$

5 $h : x \mapsto 2x + 1\ ;\ x \in \mathbb{Z}$

6 $h : x \mapsto 4 - \frac{1}{2}x\ ;\ x \in \mathbb{Z}$

7 $f : x \mapsto 2x - 1\ ;\ x \in \mathbb{R},\ 0 \leqslant x \leqslant 3$

8 $f : x \mapsto 4 - \frac{1}{2}x\ ;\ x \in \mathbb{R},\ 1 \leqslant x \leqslant 4$

9 $g : x \mapsto 3x - 2\ ;\ x \in \mathbb{R}$

10 $g : x \mapsto 3x - 2\ ;\ x \in \mathbb{R},\ -3 \leqslant x \leqslant 3$

11 $h : x \mapsto 3x - 2\ ;\ x \in \mathbb{Z}$

12 $h : x \mapsto 3x - 2\ ;\ x \in \mathbb{Z},\ -3 \leqslant x \leqslant 3$

13 $y = 1 + \frac{12}{x}\ ;\ x \in \mathbb{R},\ 1 \leqslant x \leqslant 6$

14 $y = \frac{16}{x^2}\ ;\ x \in \mathbb{R},\ 1 \leqslant x \leqslant 4$

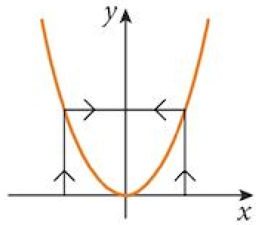
15 $g : x \mapsto \frac{1}{x}\ ;\ x \in \mathbb{R}$

16 $g : x \mapsto 4 - x\ ;\ x \in \mathbb{R}$

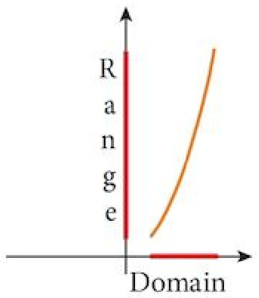
4.8 Many - one functions

The function $y = x^2\ ;\ x \in \mathbb{R}$, is not 1 : 1. Hence it does not have an inverse. There are many useful functions like this. How, then, do we make a calculator find a square root of a number?

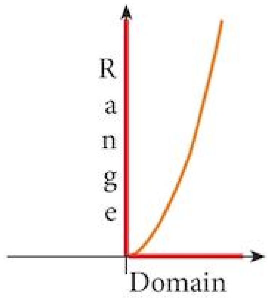
The answer is to restrict the domain to a section for which the function is 1 : 1.



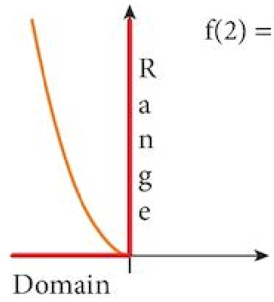
$f(2) = f(-2) = 4$



$x \in \mathbb{R},\ 1 \leqslant x \leqslant 3$
This is one possible restriction (but not a very useful one).

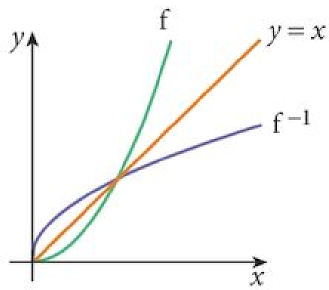


$x \in \mathbb{R},\ x \geqslant 0$
This is the maximum size we can make the domain and still retain a 1 : 1 function.
This is the one used in a calculator.



$x \in \mathbb{R},\ x \leqslant 0$
Another possibility that maintains a maximum size for the domain.

Using the standard restriction, $x \in \mathbb{R},\ x \geqslant 0$, the graphs of $f : x \mapsto x^2$ and its inverse, $f^{-1} : x \mapsto \sqrt{x}$ are shown on the right.

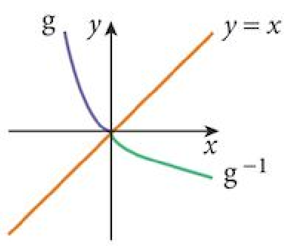


Notice that the graph of f intersects the graph of f^{-1} when they both intersect with $y = x$ (in this particular case, at (0, 0) and (1, 1)). This will always happen.

For completeness, we show

$g : x \mapsto x^2;\ x \in \mathbb{R},\ x \leqslant 0$

$g^{-1} : x \mapsto -\sqrt{x}\ ;\ x \in \mathbb{R},\ x \geqslant 0$



Note that when you are finding the formula for an inverse involving a square root, you must be careful to choose the correct root, either positive or negative.

Exercise 4.4

1 **a** Show that the function $\mathrm{f}: x \mapsto x^2 - 4x + 3$; $x \in \mathbb{R}$, is not 1 : 1.

b Find the range of the function f.

c The function $\mathrm{g}: x \mapsto x^2 - 4x + 3$; $x \in \mathbb{R}$, $x \geqslant a$ is a 1 : 1 function.
Find the smallest possible value of a for this to be true.

d For this value of a, sketch the graphs of $y = \mathrm{g}(x)$ and $y = \mathrm{g}^{-1}(x)$ on the same axes.

e State the relationship between the two graphs sketched in **d**.

2 **a** Show that the function $\mathrm{f}: x \mapsto x^2 - 4$; $x \in \mathbb{R}$, is not 1 : 1.

b Find the range of the function.

c The function $\mathrm{g}: x \mapsto x^2 - 4$; $x \in \mathbb{R}$, $x \leqslant a$ is a 1 : 1 function.
Find the largest possible value of a for this to be true.

d For this value of a, sketch the graphs of $y = \mathrm{g}(x)$ and $y = g^{-1}(x)$ on the same axes.

e State the relationship between the two graphs sketched in **d**.

3 The function $\mathrm{h}: x \mapsto x^2 - 4$; $x \in \mathbb{R}$, $x \geqslant 0$ is a 1 : 1 function.

a Find the formula for the inverse function h^{-1}.

b Sketch the graphs of $y = \mathrm{h}(x)$ and $y = \mathrm{h}^{-1}(x)$ on the same axes, stating the relationship between them.

4 The function $\mathrm{f}: x \mapsto 4 - x^2$; $x \in \mathbb{R}$, $x \geqslant 0$ is a 1 : 1 function.

a Find the formula for the inverse function f^{-1}.

b Sketch the graphs of $y = \mathrm{f}(x)$ and $y = \mathrm{f}^{-1}(x)$ on the same axes, stating the relationship between them.

4.9 The inverse of a composite function

Example 4.6

Functions f and g are defined as

$\mathrm{f}: x \mapsto x^2$; $x \in \mathbb{R}$, $x \geqslant 0$

$\mathrm{g}: x \mapsto 2x + 5$; $x \in \mathbb{R}$, $x \geqslant -2.5$.

Find: **a** fg **b** f^{-1} **c** g^{-1} **d** $\mathrm{f}^{-1}\mathrm{g}^{-1}$ **e** $\mathrm{g}^{-1}\mathrm{f}^{-1}$

f $(\mathrm{f}^{-1}\mathrm{g}^{-1})(\mathrm{fg})$ **g** $(\mathrm{g}^{-1}\mathrm{f}^{-1})(\mathrm{fg})$

Solutions:

a $\text{fg} : x \mapsto (2x+5)^2;\ x \in \mathbb{R},\ x \geqslant -2.5.$

b $\text{f}^{-1} : x \mapsto \sqrt{x}\ ;\ x \in \mathbb{R},\ x \geqslant 0.$

c $\text{g}^{-1} : x \mapsto \frac{1}{2}(x-5)\ ;\ x \in \mathbb{R},\ x \geqslant 0.$

d $\text{f}^{-1}\text{g}^{-1} : x \mapsto \sqrt{\frac{1}{2}(x-5)}\ ;\ x \in \mathbb{R},\ x \geqslant 0.$

e $g^{-1}\text{f}^{-1} : x \mapsto \frac{1}{2}(\sqrt{x}-5)\ ;\ x \in \mathbb{R},\ x \geqslant 0.$

f $(\text{f}^{-1}\text{g}^{-1})(\text{fg}) : x \mapsto \sqrt{\frac{1}{2}((2x+5)^2-5)}\ ;\ x \in \mathbb{R},\ x \geqslant 0.$

g $(\text{g}^{-1}\text{f}^{-1})(\text{fg}) : x \mapsto \frac{1}{2}\left(\sqrt{(2x+5)^2}-5\right);\ x \in \mathbb{R},\ x \geqslant 0.$

$(\text{g}^{-1}\text{f}^{-1})(\text{fg}) : x \mapsto x$

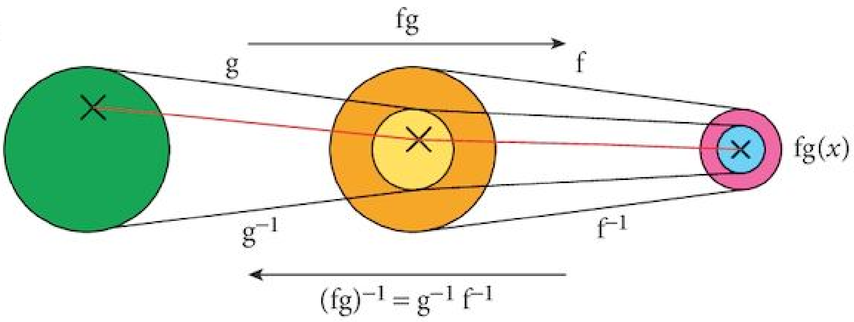
Notice that since $(\text{g}^{-1}\text{f}^{-1})(\text{fg}) : x \mapsto x$, $(\text{fg})^{-1} = \text{g}^{-1}\text{f}^{-1}$.

To find the inverse of a composite:

1. Find the inverse of each component,
2. Reverse the order of the components.

This is illustrated here:

Function:



Inverse:

Example 4.7

Functions f and g are defined as

$\text{f} : x \mapsto x^2$

$\text{g} : x \mapsto 2x+3$

Solve the equation $\text{fg}(x) = 121$.

Solution:

There are three ways you could solve this problem

(i) Put $y = \mathrm{g}(x)$

Solve $\mathrm{f}(y) = 121$

$y^2 = 121$

$y = 11$

Then solve $\mathrm{g}(x) = 11$

$2x + 3 = 11$

giving $x = 4$

(ii) Find a formula for $\mathrm{fg}(x)$

$$\mathrm{fg}(x) = \mathrm{f}(2x + 3)$$

$$= (2x + 3)^2$$

Then solve $(2x + 3)^2 = 121$

giving $x = 4$

(iii) Find a formula for $[\mathrm{fg}(x)]^{-1}$

$$[\mathrm{fg}(x)]^{-1} = \frac{\sqrt{x} - 3}{2}$$

then find $[\mathrm{fg}(121)]^{-1} = \dfrac{\sqrt{121} - 3}{2}$

$$= 4$$

Notice that these three approaches all lead to the same calculations.

Exercise 4.5

In this exercise, functions are defined as follows:

$\mathrm{f} : x \mapsto 3x - 1$ $\qquad$ $\mathrm{g} : x \mapsto 4 - 2x$

$\mathrm{h} : x \mapsto \dfrac{1}{x - 2}$ $\qquad$ $\mathrm{j} : x \mapsto x^2 + 3$

Note that the domains of the functions are not specified.

1 Find $(\mathrm{fg})^{-1}$ and show that $(\mathrm{fg})^{-1}(\mathrm{fg}) : x \mapsto x$.

2 **a** Find y if $y = \mathrm{jf}(1)$.

b Find the function jf, expanding the brackets and simplifying it.

c If the domain of f is $x \in \mathbb{R}$, what is the domain of $(\mathrm{jf})^{-1}$?

d By writing $(\mathrm{jf})^{-1}$ as $\mathrm{f}^{-1}\mathrm{j}^{-1}$, show that $(\mathrm{jf})^{-1}(y) = 1$.

3 **a** In order to form the composite hf, what is the largest possible domain of f?

b With this domain, what is the range of the function hf?

c Find the function $(\mathrm{hf})^{-1}$, stating its domain and range.

4 a What is the largest possible domain of the function h?

b With this domain, what is the range of the function h?

c What is the range of the function hj?

d If $x > 0$, find the inverse of the function hj.

5 If $x > 2$, find the inverse of the composite function jh.

6 a If the domain of g is $x \geqslant 2$, find y if $y = \mathrm{jg}(2)$.

b If the domain of j is $x \geqslant 0$, find g^{-1} and j^{-1}.

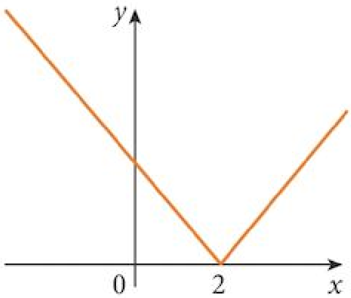
c Show that $\mathrm{g}^{-1}\mathrm{j}^{-1}(y) = 2$.

4.10 The modulus function

The function $y = |x|$ is called the **modulus function**.

The graph $y = |x - 2|$ is shown. It has a vertex at (2, 0).

The part of the graph $y = x - 2$ below the x-axis has been reflected in the axis.



Examples:
$|4| = |-4| = 4 \qquad |x| = +\sqrt{(x^2)}$

In practice, the modulus function eliminates negative values.

The modulus function is not 1 : 1. Hence it has no inverse.

Example 4.8

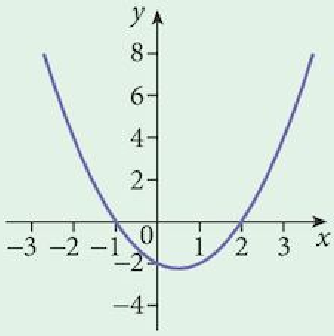
Sketch the function $y = |(x + 1)(x - 2)|$.

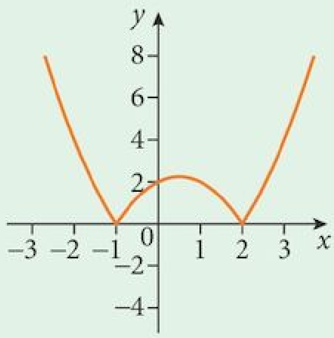
Solution: Start by drawing the graph of $y = (x + 1)(x - 2)$ (shown in blue).

This has x-intercepts of -1 and 2 and the y-intercept is -2.

The modulus function will reflect all parts of the graph below the x-axis in the axis. All sections of the graph above the x-axis remain in the same place.

This results in the graph shown in orange.





Example 4.9

a Sketch, on the same diagram, the graphs of

$$y = x + 4 \quad \text{and} \quad y = |2x - 1|$$

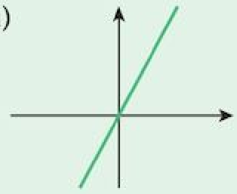
b From your sketch determine the number of real roots of the equation

$$x = |2x - 1| - 4$$

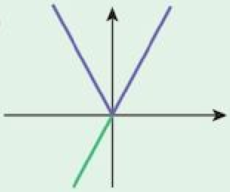
c Solve the equation $x = |2x - 1| - 4$

Solution:

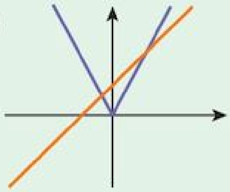
(i)



(ii)



(iii)



a Start with the graph of $y = 2x - 1$ **(i)**

Convert it into the graph of $y = |2x - 1|$ **(ii)**

Finally, add the graph of $y = x + 4$ **(iii)**

b The equation $x = |2x - 1| - 4$

is equivalent to $x + 4 = |2x - 1|$

The solutions to this equation are represented by the points where the graphs intersect.

There are 2 real solutions to this equation.

c The positive solution is given by

$$x + 4 = 2x - 1$$

or $x = 5$

The negative solution is given by

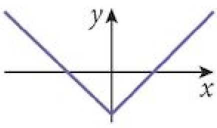
$$x + 4 = -(2x - 1)$$

or $x = -1$

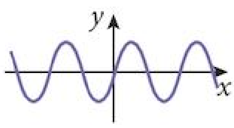
Exercise 4.6

1 For each of the following functions $y = \mathrm{f}(x)$, copy the graph and sketch the graph of $y = |\mathrm{f}(x)|$.

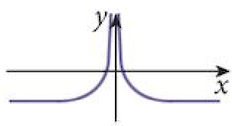
a



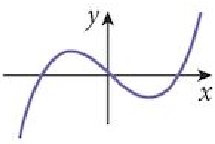
b



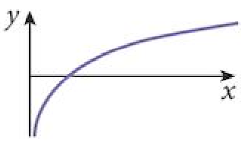
c



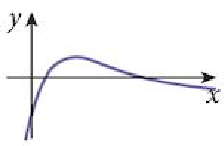
d



e



f



2 Sketch the graphs of these functions.

a $f : x \mapsto |2x - 4|\ ; x \in \mathbb{R}$

b $f : x \mapsto |x| - 2\ ; x \in \mathbb{R}$

c $f : x \mapsto |(x+2)(x-3)|\ ; x \in \mathbb{R}$

d $f : x \mapsto |4 - x^2|\ ; x \in \mathbb{R}$

3 Given the function $f : x \mapsto x\ ; x \in \mathbb{R}$, sketch the graphs of these functions.

a $y = f(x)$

b $y = |f(x)|$

c $y = |f(x)| - 2$

d $y = ||f(x)| - 2|$

4 Draw the graphs of the functions $f : x \mapsto |3x - 6|\ ; x \in \mathbb{R}$, and $g : x \mapsto |x|\ ; x \in \mathbb{R}$.

How many solutions are there to the equation $|3x - 6| = |x|$?

Use your graph to find any solutions.

Summary

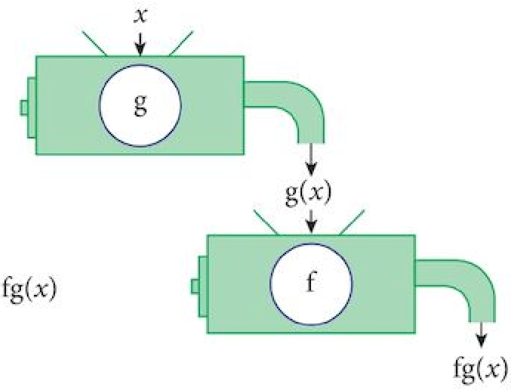
Definition A function is a mapping in which each input value can generate only one output value.

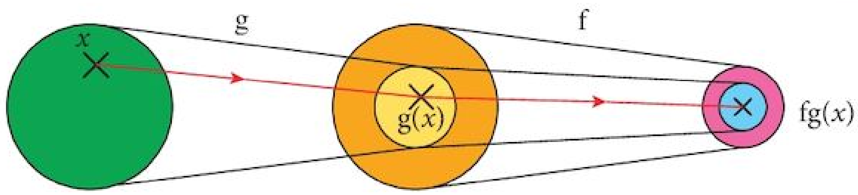
Domain The set of input values.

Range The set of output values attained.

Composite functions fg means "g first, followed by f".

The range of g must be a subset of the domain of f.



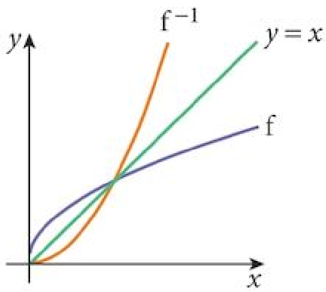


Inverse function f^{-1} exists only if f is 1 : 1.

Then $f f^{-1}(x) = f^{-1} f(x) = x$.

The **domain** of the inverse is the **range** of the function.

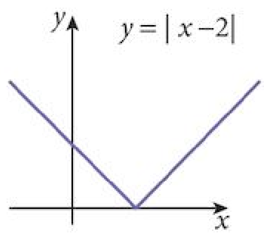
The **range** of the inverse is the **domain** of the function.



Graph of the inverse The graphs of the function and its inverse are reflections of each other in the line $y = x$.

The modulus function $|x| = +\sqrt{x^2}$

Makes all values positive.



Chapter 4 Summative Exercise

1 Find the value of the following functions for the input values given.

a $\mathrm{f}: x \mapsto (x-2)^2 - 4$; $\quad x = 5$

b $\mathrm{f}: x \mapsto 3x - 2$; $\quad x = 3$

c $\mathrm{f}: x \mapsto \dfrac{3x+2}{2x-2}$; $\quad x = 6$

d $\mathrm{f}: x \mapsto x + \dfrac{1}{x^2}$; $\quad x = -2$

2 Find the range of these functions for the domain of real numbers.

a $\mathrm{f}: x \mapsto (x-2)^2 - 4$

b $\mathrm{f}: x \mapsto 3x - 2$

3 You are given the functions $\quad \mathrm{f}: x \mapsto \dfrac{3x+2}{2x-2} \quad$ and $\quad \mathrm{g}: x \mapsto x + \dfrac{1}{x^2}$.

a Which real number cannot be in the domain of each of the functions?

b Find the range of each function if the domain of each is as large as possible.

4 You are given the functions $\quad \mathrm{f}: x \mapsto 2x - 1, \quad \mathrm{g}: x \mapsto \dfrac{1}{2-x}\, x \neq 2 \quad$ and $\quad \mathrm{h}: x \mapsto x^2 + 1$.

a Form the composite functions:

(i) fh **(ii)** fg **(iii)** hg **(iv)** hf

State the range of each composite function.

b Explain why the following composites cannot be formed.

(i) gf **(ii)** gh

c Form the composite functions:

(i) f^2 **(ii)** h^2

d **(i)** In order to form the composite function g^2, what extra value must be eliminated from the domain of g?

(ii) Form the composite g^2 and state its range.

5 Given the functions $\quad \mathrm{f}: x \mapsto 3 - x, x \neq 1 \quad$ and $\quad \mathrm{g}: x \mapsto \dfrac{12}{x+2}, x \neq -2$, solve the equations:

a $\mathrm{fg}(x) = 1$

b $\mathrm{gf}(x) = 4$

6 When can you form the inverse of a function?

7 Prove that the function $\mathrm{f}: x \mapsto (x+2)^2 - 1$ has no inverse.

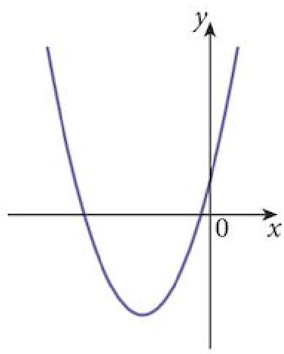
8 The graph shows the function $\mathrm{g}: x \mapsto (x+2)^2 - 3$.

The function $\mathrm{f}: x \mapsto (x+2)^2 - 3, x \geqslant k$ is a 1 : 1 function.

a What is the smallest possible value of k?

b For this value of k, find the inverse f^{-1}.

c Sketch the graph of the function and its inverse, describing the relationship between them.



9 The function $h : x \mapsto (x+2)^2 - 3$, $x \leqslant k$, where k is the value found in question **8**.

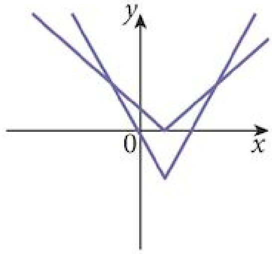
a Find the inverse h^{-1}.

b Sketch the graph of the function and its inverse, describing the relationship between them.

10 On the same axes, sketch the graphs of the following functions:

a $y = |x-3|$ **b** $y = |2x-3|$ **c** $y = |2(x-3)|$

11 The diagram shows the graphs of $y = |2(x-1)| - 2$ and $y = |x-1|$.



a Use the graphs to solve the equation $|2(x-1)| - 2 = 4$.

b Use the graphs to solve the inequality $|2(x-1)| - 2 < |x-1|$.

12 a Sketch the graph of the function $y = |(x+2)(x-4)|$.

b The equation $|(x+2)(x-4)| = k$ has four solutions.

State the range of vaues of k for this to be possible.

13 Given the function $f : x \mapsto \sqrt{(2x-1)}$, $x > \frac{1}{2}$:

a find f(13)

b solve the equation $f(x) = 7$

c find f^{-1}, stating its domain and range.

Chapter 4 Test

1 hour

1 The function f is defined by $f(x) = 2(x-3)^2 + 2$ for $x \leqslant 3$.

a Sketch the function f and show that it is 1 : 1. [1]

b State the range of f. [1]

c Find an expression for f^{-1}. [3]

d Add the function f^{-1} to your sketch, showing clearly the relationship between the two curves. [2]

2 The functions f and g are defined by

$$f(x) = 2x - 1 \quad , \quad \text{for } x \in \mathbb{R}$$

$$g(x) = \frac{1}{x-3} \quad , \quad \text{for } x \in \mathbb{R}, x \neq 3$$

a Explain why it is not possible to form the composite gf. [1]

b Find an expression for the composite fg. [1]

c Find the range of g. [1]

d Find an expression for g^{-1}, stating its domain and range. [2]

3 a Sketch the graph of $y = |\,4x - 2\,|$ for $-3 \leqslant x \leqslant 4$. [2]

On your graph write in the coordinates of the points where the graph meets the axes.

b On the same diagram, sketch the graph of $2x + y = 4$. [1]

c Solve the equation $|\,4x - 2\,| = 4 - 2x$. [3]

4 Functions f and g are defined as follows:

$$\mathrm{f} : x \mapsto \frac{x}{3 - x},\ x \neq 3.$$

$\mathrm{g} : x \mapsto ax + b$, where a and b are constants.

a Find an expression for f^{-1}. [4]

b Sketch the graphs of f and f^{-1} showing the relationship between them. [3]

c Given that $\mathrm{g}(3) = 9$ and $\mathrm{gf}(6) = -11$, find the values of a and of b. [3]

5 a Sketch the graphs of the functions $y = |x - 4|$ and $y = |2x - 2|$. [3]

b Use your graphs to solve the inequality $|2x - 2| < |x - 4|$. [3]

6 A function f is defined by $\mathrm{f} : x \mapsto \dfrac{4x}{x - 2}$, $x \neq 2$, $x \neq a$, where the value of a is to be found later.

a Find an expression for f^2. [2]

b Find the value of a necessary to enable f^2 to be formulated. [2]

c State the range of f^2. [2]

Examination Questions

1 (i) Sketch on the same diagram the graphs of $y = |2x + 3|$ and $y = 1 - x$. [3]

(ii) Find the values of x for which $x + |2x + 3| = 1$. [3]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2004, P1, Qu 3]

2 The function f is defined, for $x \in \mathbb{R}$, by

$$\mathrm{f} : x \mapsto \frac{3x + 11}{x - 3},\ x \neq 3.$$

(i) Find f^{-1} in terms of x and explain what this implies about the symmetry of the graph of $y = \mathrm{f}(x)$. [3]

The function g is defined, for $x \in \mathbb{R}$, by

$$\mathrm{g} : x \mapsto \frac{x - 3}{2}.$$

(ii) Find the values of x for which $\mathrm{f}(x) = \mathrm{g}^{-1}(x)$. [3]

(iii) State the value of x for which $\mathrm{gf}(x) = -2$. [1]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2004, P1, Qu 8]

3 Given that each of the following functions is defined for the domain $-2 \leqslant x \leqslant 3$, find the range of

(i) $f: x \mapsto 2 - 3x$, [1]

(ii) $g: x \mapsto |2 - 3x|$, [2]

(iii) $h: x \mapsto 2 - |3x|$. [2]

State which of the functions f, g and h has an inverse. [2]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2005, P1, Qu 6]

4 A function f is defined by $f: x \mapsto |2x - 3| - 4$, for $-2 \leqslant x \leqslant 3$.

(i) Sketch the graph of $y = f(x)$. [2]

(ii) State the range of f. [2]

(iii) Solve the equation $f(x) = -2$. [3]

The function g is defined by $g: x \mapsto |2x - 3| - 4$, for $-2 \leqslant x \leqslant k$.

(iv) State the largest value of k for which g has an inverse. [1]

(v) Given that g has an inverse, express g in the form $g: x \mapsto ax + b$, where a and b are constants. [2]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2005, P1, Qu 11]

5 Functions f and g are defined for $x \in \mathbb{R}$ by

$f: x \mapsto x^3$,

$g: x \mapsto x + 2$.

Express each of the following as a composite function, using only f, g, f^{-1} and/or g^{-1}:

(i) $x \mapsto x^3 + 2$ [1]

(ii) $x \mapsto x^3 - 2$ [1]

(iii) $x \mapsto (x + 2)^{\frac{1}{3}}$. [1]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2006, P2, Qu 1]

6 Functions f and g are defined for $x \in \mathbb{R}$ by

$f: x \mapsto 3x - 2$,

$g: x \mapsto \dfrac{7x - a}{x + 1}$, where $x \neq -1$ and a is a positive constant.

(i) Obtain expressions for f^{-1} and g^{-1}. [3]

(ii) Determine the value of a for which $f^{-1}g(4) = 2$. [3]

(iii) If $a = 9$, show that there is only one value of x for which $g(x) = g^{-1}(x)$. [3]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2007, P2, Qu 10]

7 **(i)** Sketch, on the same diagram, the graphs of $y = x - 3$ and $y = |2x - 9|$. [3]

(ii) Solve the equation $|2x - 9| = x - 3$. [2]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2008, P2, Qu 4]

8 **a** Functions f and g are defined for $x \in \mathbb{R}$ by

$f(x) = 3 - x,$

$g(x) = \frac{x}{x+2}$, where $x \neq -2$.

(i) Find $fg(x)$. [2]

(ii) Hence find the value of x for which $fg(x) = 10$. [2]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2008, P1, Qu 10]

9 The function f is defined by $f(x) = 2 + \sqrt{x-3}$ for $x \geqslant 3$. Find

(i) The range of f. [1]

(ii) An expression for $f^{-1}(x)$. [2]

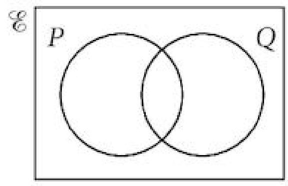
The function g is defined by $g(x) = \frac{12}{x} + 2$ for $x > 0$. Find

(iii) $gf(12)$. [2]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2009, P2, Qu 6]

Term test 1A (Chapters 1–4) 1 hour

1 Copy the Venn diagram three times and shade the region that represents the set:



a $(A' \cap B)$ [1]

b $(A' \cap B')$ [1]

c $(A' \cap B) \cup (A \cap B')$. [1]

2 Express $\frac{8}{\sqrt{5}-1}$ in the form $a + b\sqrt{5}$, where a and b are integers. [3]

3 A Prefects committee of 7 students is to be formed from 6 boys and 5 girls. Calculate the number of ways this can be done if:

a there are no restrictions [1]

b there must be 3 girls and 3 boys plus one other on the committee [3]

c the Head Girl or the Head Boy, but not both, must be a member. [2]

4 Functions f and g are defined, for $x \in \mathbb{R}$ as follows:

$$f(x) = x^2 - 1$$

$$g(x) = 5 - 2x$$

a Find the range of f. [1]

b Find an expression for the function $fg(x)$. [1]

c Solve the equation $fg(x) = 48$. [2]

d Give a reason why $f^{-1}(x)$ does not exist. [1]

e Find $g^{-1}(x)$, the inverse of g. [4]

f Sketch the functions g and g^{-1}, clearly showing the relationship between them. [3]

5 In a survey of preferences about two brands of washing powder, Sudso and Washbrite, the following results were obtained:

A 50 shoppers were in the survey.

B 5 shoppers used Sudso only.

C 35 shoppers used Washbrite.

D 20 shoppers had no preference and used either product.

a Write each of the statements above in set notation, using sets S and W. [4]

b Draw a Venn diagram and put the number of elements into each region. [3]

6 If $9^x \times 3^{2y} \times 9^{2x+y} = 1$ and $5^{x-y} = \frac{1}{5}$, find the value of x and the value of y. [4]

7 An organising committee of 5 students is to be selected from 6 girls and 5 boys. Find the number of ways that this can be done if:

a there are no restrictions [1]

b there must be more girls than boys on the committee. [4]

5 Quadratic functions

Syllabus statements

- find the maximum or minimum value of the quadratic function $x \mapsto ax^2 + bx + c$ by any method
- use the maximum or minimum value of $f(x)$ to sketch the graph or determine the range for a given domain
- know the conditions for $f(x) = 0$ to have:
  (i) two real roots
  (ii) two equal roots
  (iii) no real roots
- solve quadratic equations for real roots

5.1 Introduction

In your IGCSE Mathematics course, you have already met quadratic expressions, expanding and factorising them, and solving quadratic equations.

This chapter reminds you of these techniques and extends the theory of quadratic functions to cover additional useful techniques.

5.2 Completing the square

You should know these results:

$(x+a)^2 = x^2 + 2ax + a^2$ **[1]**

$(x-a)^2 = x^2 - 2ax + a^2$ **[2]**

You can use them in a process called **completing the square**.

Example 5.1

Complete the square for the expression $x^2 + 6x + 8$.

Solution:

Step 1: Move the 8 out of the way. $x^2 + 6x \qquad + 8$

Step 2: Compare with **[1]**: $(x+a)^2 = x^2 + 2ax + a^2$

$a = 3 \rightarrow a^2 = 9$

Step 3: Complete the square and correct for the addition. $= (x^2 + 6x + 9) - 9 + 8$

$$x^2 + 6x + 8 = (x+3)^2 - 1$$

Example 5.2

Complete the square for the expression $2x^2 - 16x + 3$.

Solution:

Step 1: Move the 3 out of the way and factorise out the 2. $2(x^2 - 8x) \qquad + 3$

Step 2: Compare with **[2]**: $(x-a)^2 = x^2 - 2ax + a^2$

$a = 4 \rightarrow a^2 = 16$

Step 3: Complete the square and correct for the addition. $= 2(x^2 - 8x + 16) - 32 + 3$

$$2x^2 - 16x + 3 = 2(x-4)^2 - 29$$

Note: There is a 2 outside the brackets, so we've effectively added $2 \times 16 = 32$ which we must then correct for, outside the brackets.

Exercise 5.1

Complete the square for each of the following quadratic expressions:

1 $x^2 + 4x + 6$	**2** $x^2 + 8x - 3$	**3** $x^2 + 6x - 7$
4 $x^2 - 6x + 2$	**5** $x^2 - 10x - 15$	**6** $x^2 - 2x + 5$
7 $x^2 - 4x + 9$	**8** $x^2 + 12x + 3$	**9** $x^2 - 16x + 70$
10 $2x^2 + 12x - 9$	**11** $3x^2 + 12x + 15$	**12** $4x^2 + 16x - 3$
13 $2x^2 - 8x + 5$	**14** $3x^2 - 18x + 29$	**15** $4x^2 - 16x + 12$
16 $-x^2 - 4x + 3$	**17** $-x^2 + 2x - 2$	**18** $-x^2 - 6x + 5$
19 $-2x^2 + 4x - 11$	**20** $-3x^2 + 18x - 20$	**21** $-5x^2 - 60x - 75$

5.3 Interpreting the expression

Any quadratic expression can be written in the form $a(x + b)^2 + c$.

The perfect square $(x + b)^2$ is always greater than zero, i.e. $(x + b)^2 \geqslant 0$.

It is equal to zero when $x = -b$.

Thus if $a > 0$, $\quad a(x + b)^2 + c \geqslant c$.

and if $a < 0$, $\quad a(x + b)^2 + c \leqslant c$.

Example 5.3

Find the maximum or minimum of the following expressions and state the value of x for which the expressions attains this extreme value.

a $x^2 + 6x + 8$

b $2x^2 - 16x + 3$

c $-3x^2 + 18x + 25$

Solution:

a $x^2 + 6x + 8 = (x + 3)^2 - 1$ has a minimum value of -1 when $x = -3$.

b $2x^2 - 16x + 3 = 2(x - 4)^2 - 29$ has a minimum value of -29 when $x = 4$.

c $-3x^2 + 18x + 25 = -3(x - 3)^2 + 52$ has a maximum value of 52 when $x = 3$.

Exercise 5.2

For each of these functions (taken from Exercise 5.1) find the maximum or minimum value of the function, as appropriate, and state the value of x for which this extreme is obtained.
In each case $x \in \mathbb{R}$.

1 $f(x) = x^2 + 4x + 6$	**2** $f(x) = x^2 + 8x - 3$	**3** $f(x) = x^2 + 6x - 7$
4 $f(x) = x^2 - 6x + 2$	**5** $f(x) = x^2 - 10x - 15$	**6** $f(x) = x^2 - 2x + 5$
7 $f(x) = x^2 - 4x + 9$	**8** $f(x) = x^2 + 12x + 3$	**9** $f(x) = x^2 - 16x + 70$
10 $f(x) = 2x^2 + 12x - 9$	**11** $f(x) = 3x^2 + 12x + 15$	**12** $f(x) = 4x^2 + 16x - 3$
13 $f(x) = 2x^2 - 8x + 5$	**14** $f(x) = 3x^2 - 18x + 29$	**15** $f(x) = 4x^2 - 16x + 12$
16 $f(x) = -x^2 - 4x + 3$	**17** $f(x) = -x^2 + 2x - 2$	**18** $f(x) = -x^2 - 6x + 5$
19 $f(x) = -2x^2 + 4x - 11$	**20** $f(x) = -3x^2 + 18x - 20$	**21** $f(x) = -5x^2 - 60x - 75$

5.4 Sketching the graph of a quadratic function

An equation in the format $y = Px^2 + Qx + R$ can be converted into the format $y = a(x+b)^2 + c$.

Note: $P = a$.

Value of P $(= a)$	Max/Min	Vertex	y-intercept
$P > 0$	Minimum	$(-b, c)$	R
$P < 0$	Maximum	$(-b, c)$	R

The y-intercept is usually easy to find.

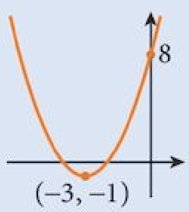
Example 5.4

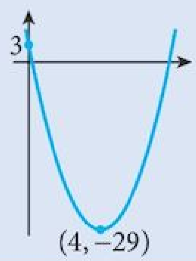
Sketch these curves:

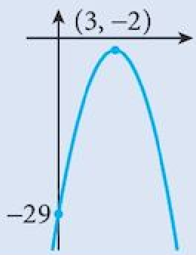
a $y = x^2 + 6x + 8$ **b** $y = 2x^2 - 16x + 3$ **c** $y = -3x^2 + 18x - 29$

Solution:

	a	**b**	**c**
Curve	$y = x^2 + 6x + 8$	$y = 2x^2 - 16x + 3$	$y = -3x^2 + 18x - 29$
Completed square	$y = (x+3)^2 - 1$	$y = 2(x-4)^2 - 29$	$y = -3(x-3)^2 - 2$
Vertex	$(-3, -1)$	$(4, -29)$	$(3, -2)$
y-intercept	8	3	-29
Shape	$\cup$	$\cup$	$\cap$







Exercise 5.3

Use your results from Exercise 5.2 to sketch the graphs of these functions:

1 $y = x^2 + 4x + 6$

2 $y = x^2 + 8x - 3$

3 $y = x^2 + 6x - 7$

4 $y = x^2 - 6x + 2$

5 $y = x^2 - 10x - 15$

6 $y = x^2 - 2x + 5$

7 $y = x^2 - 4x + 9$

8 $y = x^2 + 12x + 3$

9 $y = x^2 - 16x + 70$

10 $y = 2x^2 + 12x - 9$

11 $y = 3x^2 + 12x + 15$

12 $y = 4x^2 + 16x - 3$

13 $y = 2x^2 - 8x + 5$

14 $y = 3x^2 - 18x + 29$

15 $y = 4x^2 - 16x + 12$

16 $y = -x^2 - 4x + 3$

17 $y = -x^2 + 2x - 2$

18 $y = -x^2 - 6x + 5$

19 $y = -2x^2 + 4x - 11$

20 $y = -3x^2 + 18x - 20$

21 $y = -5x^2 - 60x - 75$

22 The one-one function $y = 2x^2 + 12x + 9$ is defined for all $x \geqslant k$ where x is real. By completing the square, find the minimum value of k and the range of the function.

23 When a ball is thrown vertically upwards with a speed of 9.8 ms^{-1}, the formula for its height, h, is given by $h = 9.8t - 4.9t^2$ where t is the time after release.

a By completing the square, find the maximum height of the ball and the time it takes to reach this maximum height.

b How long does it take for the ball to land?

5.5 Solving a quadratic equation

5.5.1 Solution by factorisation

Some quadratic equations can be factorised, but the vast majority cannot. The factorisation technique will work only if the solutions are simple rational numbers.

Example 5.5

Solve the equation $x^2 - 5x + 6 = 0$ by factorisation.

Solution:

Factorise: $(x - 2)(x - 3) = 0$

So, either $x - 2 = 0$ or $x - 3 = 0$

which gives $x = 2$ or $x = 3$

The solution set is $x \in \{2, 3\}$.

5.5.2 Solution by completing the square

Any quadratic equation can be solved by completing the square.

Example 5.6

Solve the equation $x^2 - 8x - 9 = 0$ by completing the square.

Solution:

Complete the square: $(x-4)^2 - 25 = 0$

$$(x-4)^2 = 25$$

$$x - 4 = \pm 5$$

$$x = 4 \pm 5$$

which gives $x = -1$ or $x = 9$

The solution set is $x \in \{-1, 9\}$.

Example 5.7

Solve the equation $2x^2 + 12x + 13 = 0$ by completing the square.

Solution:

Complete the square: $2(x+3)^2 - 5 = 0$

$$2(x+3)^2 = 5$$

$$x + 3 = \pm\frac{\sqrt{5}}{\sqrt{2}}$$

$$x = -3 \pm \frac{\sqrt{5}}{\sqrt{2}}$$

which gives $x = -4.58$ or $x = -1.42$

The solution set is $x \in \{-4.58, -1.42\}$.

5.5.3 Solution by formula

Completing the square leads to a formula that can be used for all quadratic equations.

Example 5.8

Solve the equation $ax^2 + bx + c = 0$ by completing the square.

Solution:

Complete the square: $a\left(x+\frac{b}{2a}\right)^2 + c - \frac{b^2}{4a} = 0$

$$a\left(x+\frac{b}{2a}\right)^2 = \frac{b^2}{4a} - c$$

$$\left(x+\frac{b}{2a}\right)^2 = \frac{b^2-4ac}{4a^2}$$

$$x+\frac{b}{2a} = \pm\sqrt{\left(\frac{b^2-4ac}{4a^2}\right)}$$

$$x+\frac{b}{2a} = \pm\frac{\sqrt{(b^2-4ac)}}{2a}$$

$$x = \frac{-b\pm\sqrt{(b^2-4ac)}}{2a}$$

This is the quadratic equation formula; you should memorise this.

Example 5.9

Solve the equation $3x^2 + 7x + 2 = 0$ using the quadratic equation formula.

Solution:

Comparing with $ax^2+bx+c=0$: $a=3, b=7, c=2$

Substitute in the formula: $x = \frac{-7\pm\sqrt{(7^2-4\times3\times2)}}{2\times3}$

$x = -2$ or $x = -0.33$

Exercise 5.4

Solve these quadratic equations by factorisation.

1 $x^2 + 7x + 12 = 0$ **2** $x^2 + 9x + 20 = 0$ **3** $x^2 + 11x + 30 = 0$

4 $x^2 + 2x - 15 = 0$ **5** $x^2 + 4x - 12 = 0$ **6** $x^2 + 2x - 8 = 0$

7 $x^2 - 6x + 8 = 0$ **8** $x^2 - 8x + 15 = 0$ **9** $x^2 - 9x + 20 = 0$

10 $x^2 - 2x - 15 = 0$ **11** $x^2 - 5x - 6 = 0$ **12** $x^2 - 4x - 12 = 0$

13 $2x^2 + 7x + 3 = 0$ **14** $6x^2 + 7x + 2 = 0$ **15** $3x^2 + 16x + 16 = 0$

16 $3x^2 - 13x + 4 = 0$ **17** $6x^2 - 7x + 2 = 0$ **18** $5x^2 - 12x + 4 = 0$

Solve these quadratic equations by completing the square:

19 $x^2 + 6x + 5 = 0$ **20** $x^2 + 8x + 15 = 0$ **21** $x^2 + 7x + 10 = 0$

22 $x^2 + 3x - 10 = 0$ **23** $x^2 + x - 12 = 0$ **24** $x^2 + 2x - 24 = 0$

25 $x^2 - 7x + 12 = 0$ **26** $x^2 - 6x + 5 = 0$ **27** $x^2 - 9x + 18 = 0$

28 $x^2 - 3x - 10 = 0$ **29** $x^2 - 2x - 8 = 0$ **30** $x^2 - 3x - 4 = 0$

31 $3x^2 + 7x + 2 = 0$ **32** $2x^2 + 7x + 6 = 0$ **33** $4x^2 + 5x + 1 = 0$

34 $2x^2 - 7x + 3 = 0$ **35** $3x^2 - 10x + 8 = 0$ **36** $5x^2 - 13x - 6 = 0$

Solve these quadratic equations by using the quadratic formula:

37 $x^2 + 6x + 5 = 0$ **38** $x^2 + 8x + 15 = 0$ **39** $x^2 + 7x + 10 = 0$

40 $x^2 + 3x - 10 = 0$ **41** $x^2 + x - 12 = 0$ **42** $x^2 + 2x - 24 = 0$

43 $x^2 - 7x + 12 = 0$ **44** $x^2 - 6x + 5 = 0$ **45** $x^2 - 9x + 18 = 0$

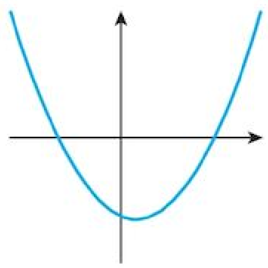
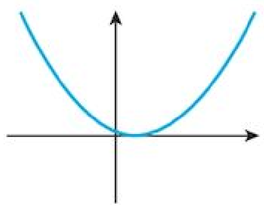
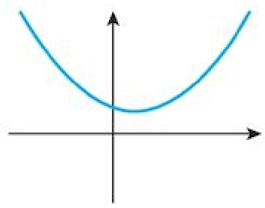
46 $x^2 - 3x - 10 = 0$ **47** $x^2 - 2x - 8 = 0$ **48** $x^2 - 3x - 4 = 0$

49 $3x^2 + 7x + 2 = 0$ **50** $2x^2 + 7x + 6 = 0$ **51** $4x^2 + 5x + 1 = 0$

52 $2x^2 - 7x + 3 = 0$ **53** $3x^2 - 10x + 8 = 0$ **54** $5x^2 - 13x - 6 = 0$

5.6 The discriminant

In the quadratic formula, the expression $(b^2 - 4ac)$ is called the **discriminant**.

Discriminant	Positive	Zero	Negative
Number of real roots	2 distinct real roots	2 equal real roots	No real roots
Graph			

Exercise 5.5

For each of these functions, find the discriminant of the quadratic expression and state the number of real roots of the equation $f(x) = 0$.

1 $f(x) = x^2 + 4x + 6$ **2** $f(x) = x^2 + 2x + 1$ **3** $f(x) = x^2 + 8x - 3$

4 $f(x) = x^2 + 6x - 7$ **5** $f(x) = x^2 - 6x + 2$ **6** $f(x) = x^2 - 6x + 9$

7 $f(x) = x^2 - 8x + 16$ **8** $f(x) = x^2 - 10x - 15$ **9** $f(x) = x^2 - 2x + 5$

10 $f(x) = x^2 - 4x + 9$

11 $f(x) = 4x^2 + 4x + 1$

12 $f(x) = x^2 + 12x + 3$

13 $f(x) = 9x^2 - 6x + 1$

14 $f(x) = x^2 - 16x + 70$

15 $f(x) = 2x^2 + 12x - 9$

16 $f(x) = 16x^2 + 8x + 1$

17 $f(x) = 3x^2 + 12x + 15$

18 $f(x) = 4x^2 + 16x - 3$

19 $f(x) = 2x^2 - 8x + 5$

20 $f(x) = 4x^2 + 12x + 9$

21 $f(x) = 3x^2 - 18x + 29$

22 $f(x) = 4x^2 - 16x + 12$

23 $f(x) = -x^2 - 4x + 3$

24 $f(x) = -x^2 + 2x - 2$

25 $f(x) = -16x^2 + 24x - 9$

26 $f(x) = -x^2 - 6x + 5$

27 $f(x) = -2x^2 + 4x - 11$

28 $f(x) = -3x^2 + 18x - 20$

29 $f(x) = -25x^2 + 20x + 4$

30 $f(x) = -5x^2 - 60x - 75$

Summary

Definition

A quadratic function is one in which the highest power of x is 2.

Completing the square

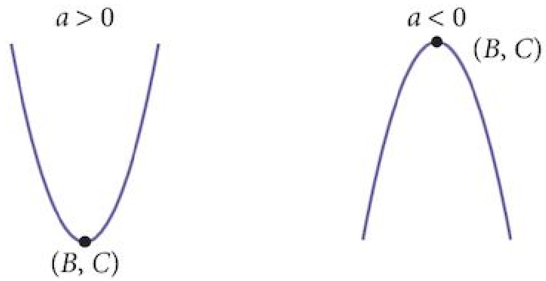
Any quadratic function can be written in completed square format.

$ax^2 + bx + c = A(x - B)^2 + C$

One way to find A, B and C is to multiply out the completed square format and then compare the coefficients.

The graph of a quadratic function

$y = ax^2 + bx + c$



Minimum at (B, C) Maximum at (B, C)

Solving quadratic equations

1 Factorising

Create the format $(ax + b)(cx + d) = 0$

then, either $(ax + b) = 0$ or $(cx + d) = 0$

then solve the two linear equations.

(Can be used for specially selected equations)

2 Completing the square

This leads to: $(x + a)^2 \pm b$

3 Formula

$$x = \frac{-b \pm \sqrt{(b^2 - 4ac)}}{2a}$$

(Can be used in all cases.)

The discriminant

If $(b^2 - 4ac)$ $>$ 0, there are 2 distinct real roots.

$=$ 0, there are 2 equal real roots.

$<$ 0, there are no real roots.

Chapter 5 Summative Exercise

1 Write each of the following in completed square form: $a(x+b)^2+c$, where a, b and c are real numbers.

a x^2-8x+5 **b** $x^2+12x+9$ **c** $x^2-10x-7$

d $2x^2-12x+11$ **e** $3x^2+18x+8$ **f** $5x^2-10x-12$

g $11-12x-2x^2$ **h** $3x^2+12x+7$ **i** $4-6x-3x^2$

2 Find the extrema of the following quadratic functions, stating whether the extreme represents a maximum or a minimum and the value of x at which this extreme is attained.

a $f(x)=x^2+6x-3$ **b** $f(x)=6-8x-2x^2$ **c** $f(x)=4x^2-6x-10$

d $f(x)=8-8x-3x^2$ **e** $f(x)=2x^2+9x+5$ **f** $f(x)=3x^2-16x-6$

g $f(x)=4x-3-2x^2$ **h** $f(x)=5+6x-3x^2$ **i** $f(x)=4x+1-3x^2$

3 Sketch the following quadratic functions, indicating the coordinates of the vertex.

a $f(x)=2x^2-12x-5$ **b** $f(x)=3x^2+12x+7$ **c** $f(x)=5x^2-10x+9$

d $f(x)=7+4x-x^2$ **e** $f(x)=4x-5-3x^2$ **f** $f(x)=3x^2-12x+5$

g $f(x)=4x-2-2x^2$ **h** $f(x)=3x+2x^2-5$ **i** $f(x)=4+3x-x^2$

4 Find the discriminant of the following quadratic functions and state the number of real roots of the equation $f(x)=0$.

a $f(x)=3x^2-5x+1$ **b** $f(x)=3x^2+6x+3$ **c** $f(x)=5x^2-9x+4$

d $f(x)=16+8x-x^2$ **e** $f(x)=8x-3-3x^2$ **f** $f(x)=2x^2-3x+2$

g $f(x)=5x-3-2x^2$ **h** $f(x)=7x+2x^2-3$ **i** $f(x)=4+3x-5x^2$

5 Solve these quadratic equations by factorisation.

a $x^2+5x+6=0$ **b** $2x^2+9x+9=0$ **c** $3x^2+x-10=0$

d $3x^2-2x-21=0$ **e** $2x^2-3x-5=0$ **f** $2x^2-5x+3=0$

g $4x^2+7x-15=0$ **h** $5x^2+x-6=0$ **i** $4x^2+21x+20=0$

6 Solve the quadratic equations by completing the square.

a $x^2-6x+5=0$ **b** $2x^2-x-6=0$ **c** $3x^2+11x+10=0$

d $6x^2-7x+2=0$ **e** $4x^2+4x-3=0$ **f** $4x^2-16x+7=0$

g $9x^2-36x+20=0$ **h** $4x^2-24x+11=0$ **i** $x^2+6x+7=0$

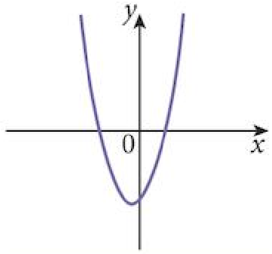
7 Solve the quadratic equations by using the formula.

a $x^2-4x+1=0$ **b** $3x^2+2x-5=0$ **c** $3x^2+4x-2=0$

d $6x^2-4x-1=0$ **e** $5x^2+4x-4=0$ **f** $2x^2-16x+5=0$

g $2x^2+7x+2=0$ **h** $4x^2-5x-3=0$ **i** $9x^2+20x+5=0$

8 The diagram shows the graph of the quadratic function $y = ax^2 + bx + c$. In each of the following, sketch the graph of the quadratic function and use your sketch to solve the quadratic inequality.

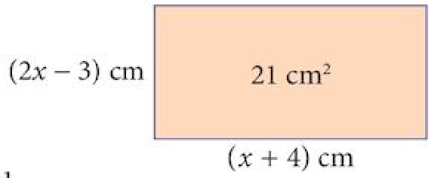


a $(x+3)(x-2) < 0$ **b** $x^2 - x - 12 > 0$

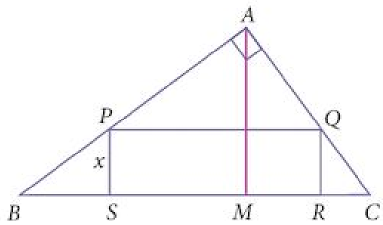
c $10 + 11x - 6x^2 \geqslant 0$ **d** $4x^2 + 4x - 15 \geqslant 0$

9 A rectangle has sides of length $(2x-3)$ cm and $(x+4)$ cm. It has an area of 21 cm^2.

Find the dimensions of the rectangle.



10 Triangle ABC is such that angle $BAC = 90°$ and $AC = 5$ cm. The perpendicular from A to BC meets BC at M. $AM = 4$ cm, and $MC = 3$ cm. Rectangle $PQRS$ is drawn within the triangle.



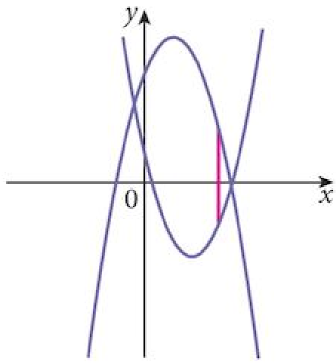
a If the height of the rectangle is x cm:

(i) show that $SR = \dfrac{25}{12}(4 - x)$

(ii) find the area of the rectangle in terms of x.

b Complete the square and hence find the maximum area of the rectangle and the corresponding value of x.

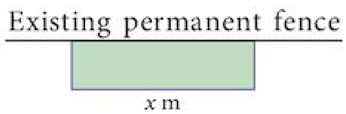
11 The diagram shows the graphs of $y = 6 + 3x - x^2$ and $y = x^2 - 5x + 2$.
The line drawn shows the vertical distance between the two graphs.



a Find the length of this line in terms of x.

b By completing the square, find the maximum value of the length of this line and the corresponding value of x.

12 A farmer has 200 m of fencing from which to build a temporary pen, using an existing permanent fence, as shown in the diagram. If the pen is a rectangle, of width x m:



a find the area of the pen in terms of x

b complete the square and find the maximum area of the pen and the corresponding value of x.

Chapter 5 Test

1 hour

1 Express $6-4x-x^2$ in completed square form $a-(x+b)^2$, and hence find the range of values of the function $\mathrm{f}: x \mapsto 6-4x-x^2, x \in \mathbb{R}$. [4]

2 You are given the sets: $X = \{x : x^2+7x+12=0\}$

$$Y = \{y : (y+3)(y-2)(y+1)=0\}$$
$$Z = \{z : z^2+z+2=0\}$$

a Find:

(i) $\mathrm{n}(X)$ [1]

(ii) $\mathrm{n}(Y)$ [1]

(iii) $\mathrm{n}(Z)$ [1]

b List the elements of the sets:

(i) Y [1]

(ii) $X \cup Y$ [1]

c Describe the set Z' in words. [1]

3 It is given that $\mathrm{f}(x) = 4x^2+4x-15$.

a Write the function $\mathrm{f}(x)$ in completed square form $a(x+b)^2+c$. [3]

b Sketch the graph of $|\mathrm{f}(x)|$ for $-4 \leqslant x \leqslant 4$. [4]

c Find the set of values for which the equation $|\mathrm{f}(x)| = k$ has four distinct roots. [2]

4 Find the set of values of k for which the function $x^2+(k+1)x+2k+2=0$ has no real roots. [4]

5 The function f is defined as $\mathrm{f}: x \mapsto \dfrac{6x}{x-2}, x \neq 2$.

a Find:

(i) f^2 [1]

(ii) f^{-1} [2]

b Solve the equation $\mathrm{f}^2(x) = \mathrm{f}^{-1}(x)$. [3]

6 Find the set of values of k for which the curve $y=-x^2+kx+k-2$ lies below the x-axis for all values of x. [5]

7 Find the set of values of k for which the function $\mathrm{f}(x) = 6x^2+kx-k$ has two distinct real roots. [4]

8 Sets X and Y are such that $X = \{x : x^2-4x=0\}$ and $Y = \{y : y^2+y+3=0\}$. Find:

a $\mathrm{n}(X)$ [1]

b $\mathrm{n}(Y)$ [1]

6 Simultaneous equations and inequalities

Syllabus statements

- know the conditions for a given line to
  - (i) intersect a given curve
  - (ii) be the tangent to a given curve
  - (iii) not intersect a given curve
- find the solution set for quadratic inequalities
- solve simultaneous equations in two unknowns with at least one linear equation

6.1 Introduction

In this chapter we extend the theory of solving simultaneous equations to cover the case where one of the equations is quadratic. We use the theory of quadratic equations from Chapter 5 to investigate a physical representation of this idea, that of a line intersecting with a curve, or not in some circumstances. We also look at what it means to solve quadratic inequalities.

6.2 Simultaneous linear equations

You should know how to solve a pair of simultaneous linear equations such as

$2x + 3y = 18$ [1]

$3x - y = 5$ [2]

There are two techniques that we use: substitution and elimination, shown respectively in Examples 6.1 and 6.2.

Solving means "to find values of the variables that make the equations true". **Simultaneous** means "at the same time".

Example 6.1

Solve the equations by substitution. $2x + 3y = 18$ [1]

$3x - y = 5$ [2]

Solution:

In [2] $y = 3x - 5$

Substitute for y in [1]: $2x + 3(3x - 5) = 18$

$2x + 9x - 15 = 18$

$11x = 33$

$x = 3$

Substitute for x in [2]: $y = 3 \times 3 - 5$

$y = 4$

The solution is (3, 4).

Example 6.2

Solve the equations by elimination. $2x + 3y = 18$ [1]

$3x - y = 5$ [2]

Solution:

[2] $\times 3$ $\qquad 9x - 3y = 15 \quad$ [3]

[1] + [3]: $\qquad 11x = 33$

$\qquad x = 3$

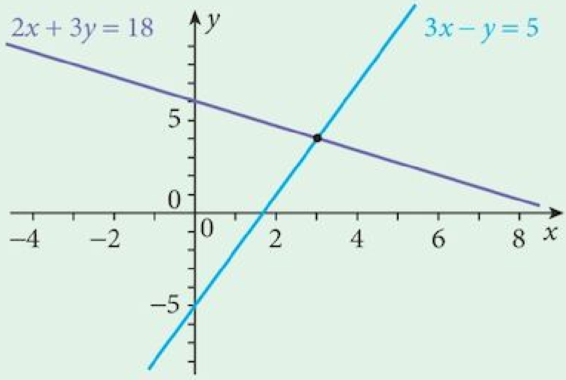
Substitute for x in [2]: $\quad 3 \times 3 - y = 5$

$\qquad y = 4$

The solution is (3, 4).

If we draw the graphs of the two equations, each point on a graph is a solution to that equation. The **simultaneous** solution is where the two graphs intersect, that is, at (3, 4).

The idea that the solution of simultaneous equations is where the graphs intersect can be used in many other situations.



6.3 Simultaneous equations: one linear, one non-linear

We would normally use the substitution technique to solve equations like these. This would give us an equation with one variable only, which we then try to solve. Sometimes, this equation is quadratic.

Example 6.3

Solve the equations and sketch the graphs. $\qquad y = x^2 + 5x - 9 \quad$ [1]

$\qquad y = 2x + 1 \quad$ [2]

Solution: Substitute [2] into [1].

$$2x + 1 = x^2 + 5x - 9$$

Simplify: $\qquad 0 = x^2 + 3x - 10 \quad$ [3]

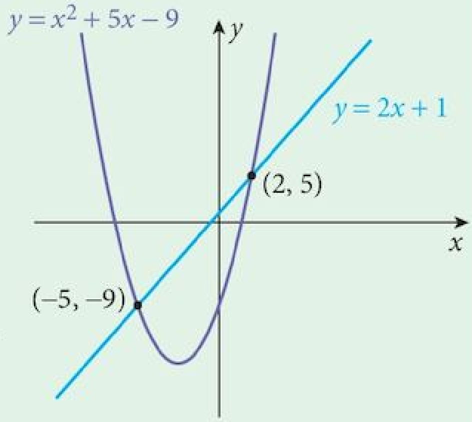
Solve the quadratic: $\qquad x = -5 \qquad$ or $\qquad x = 2$

Find the corresponding values of y:

$\qquad y = -9 \qquad\qquad y = 5$

Solutions are $(-5, -9)$ and $(2, 5)$.

Notice the discriminant of the quadratic equation [3] is 49:
$b^2 - 4ac = 3^2 - 4(1)(-10) = 49$
This is greater than 0, so there will be 2 distinct real solutions.
This also means that there will be 2 distinct points of intersection, as shown in the sketch.



Example 6.4

Solve the equations and sketch the graphs. $xy = 12$ [1]

$2y = 3x - 6$ [2]

Solution: Substitute [1] into [2].

$$\frac{24}{x} = 3x - 6$$

Simplify: $24 = 3x^2 - 6x$

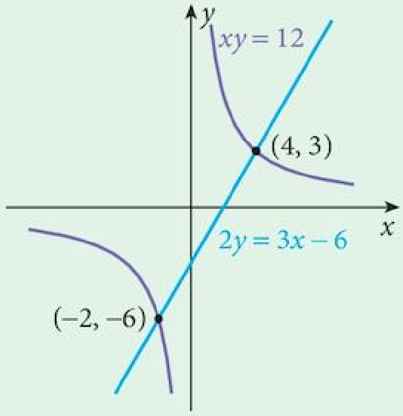
$0 = x^2 - 2x - 8$ [3]

Solve the quadratic: $x = -2$ or $x = 4$

Find the corresponding values of y:

$y = -6$ $y = 3$

Solutions are $(-2, -6)$ and $(4, 3)$.



Notice that the discriminant of the quadratic equation [3] is 36.
Once again this means that there will be 2 distinct points of intersection as shown in the sketch.

Example 6.5

Show that the line $y = 6x - 13$ [1] is a tangent to the curve $y = x^2 + 4x - 12$ [2] and find the coordinates of the point at which the line touches the curve.

Solution: Substitute [1] into [2].

$6x - 13 = x^2 + 4x - 12$

Simplify $0 = x^2 - 2x + 1$ [3]

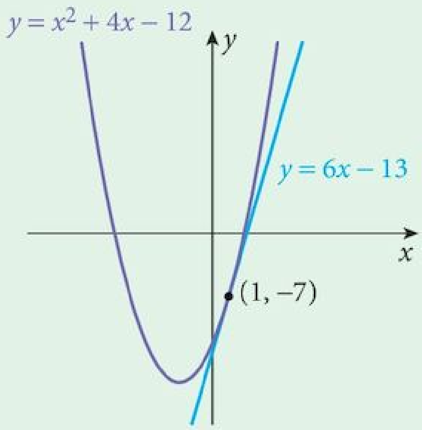
The discriminant of the quadratic equation [3] is 0.
Thus it has two equal real roots.
This means that there is only one point of intersection of the line and the curve.
Thus the line is a tangent to the curve.

Solve the quadratic $x^2 - 2x + 1 = 0$: $x = 1$ or $x = 1$

Find the corresponding value of y:

$y = -7$

The line touches the curve at the point $(1, -7)$.



Example 6.6

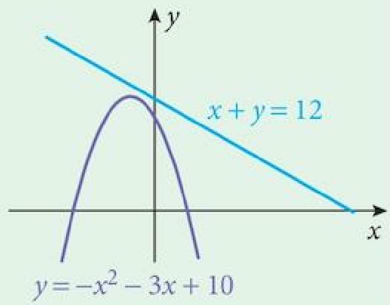
Show that the line $x + y = 12$ [1] does not intersect the curve $y = -x^2 - 3x + 10$ [2].
Sketch the graphs.

Solution: Substitute [1] into [2].

$$12 - x = -x^2 - 3x + 10$$

Simplify $\quad x^2 + 2x + 2 = 0 \qquad$ [3]

The discriminant of the quadratic equation [3] is -4.
Thus it has no real roots.
This means that there is no point of intersection of the line and the curve.



Exercise 6.1

In questions 1 to 12,

(i) determine whether or not the given line intersects or touches the curve

(ii) if there is an intersection, find the coordinates of the points of intersection; if a tangent, find the coordinates of the point at which the line touches the curve.

1 $y = 5x - 4$
$y = 2x^2 - 3x + 4$

2 $y = x + 13$
$y = 15 + 2x - x^2$

3 $x + y = 3$
$y = x^2 + 2x + 5$

4 $y = 2x + 2$
$y = x^2 + 3x - 4$

5 $2y = x - 12$
$y = 2x^2 + 3x - 4$

6 $7x + y + 22 = 0$
$y = x^2 + x - 6$

7 $y = x - 2$
$y = 2x - 3 - 3x^2$

8 $y = 7 - 3x$
$y = 6 - x - x^2$

9 $x + y = 3$
$y = 18 - 3x - x^2$

10 $2x + 3y = 20$
$xy = 6$

11 $y = 2x + 5$
$(x - 2)(y - 3) = 8$

12 $x + y = 3$
$xy = 9$

13 The line $x + y = k$ intersects the curve $y = x^2 - 5x + 9$. Find the minimum value of k for this to be true.

14 The line $4y = 3x + k$ does not intersect the curve $y = x^2 + 2x - 2$. Find the range of values of k for this to be true.

15 The line $y = x - 6$ is a tangent to the curve $y = x^2 + kx - 2$ for two values of k. Find these values of k.

16 Jason is at a window 10 m above the ground; Kamal is on the ground directly below the window, holding a tennis ball. Kamal throws the ball vertically upward to Jason. The height of the ball, h, is given by $h = ut - 4.9t^2$ where u is the speed at which Kamal throws the ball (in metres per second) and t is the time (in seconds) after which the ball has been thrown.

a Find the minimum value of u for which Jason can catch the ball.

b Find the time of flight if Jason does catch the ball when it is thrown at this speed.

17 Faisal and Amrit are cousins. The sum of their ages is 17 and the product of their ages is 72. Find their ages.

18 A rectangle has a perimeter of 50 cm and an area of 144 cm^2. Find the length of the sides of the rectangle.

For question 17, choose appropriate variables and then formulate two equations using the information given.

19 The sum of the roots, p and q, of a quadratic equation is 15 and their product is 36.

a Write down two equations showing the relationship between p and q.

b Solve your equations to find the values of p and q.

c What is the quadratic equation whose roots are p and q?

6.4 The sign diagram

Provided that an expression can be described as a product and/or quotient of linear factors, we can investigate the properties of the expression using a **sign diagram**. Factorised quadratic expressions are the simplest examples of this technique.

Example 6.7

Investigate the sign of the expression $(x+3)(x-4)$ for various values of x.

Solution:

Step 1: Find the critical values of x. $x+3=0$ so $x=-3$

$x-4=0$ so $x=4$

These are the values that make the expression zero.

Step 2: **a** Draw up the table.

		-3		4	
$x+3$					
$x-4$					

a Put in critical values

(i) The scale does not have to be linear.

(ii) Leave a space between each critical value.

b Put in the zeros.

		-3		4	
$x+3$		0			
$x-4$				0	

b eg. $(x+3)=0$ when $x=-3$

c Put in the + and − signs.

		-3		4	
$x+3$	−	0	+	+	+
$x-4$	−	−	−	0	+
Product	+	0	−	0	+

c Indicate whether the functions are +ve or −ve in each section.

Step 3: Work out the products of the values of the expression in each section. These go in the last row of the table.

The expression $(x+3)(x-4)=0$ when $x=-3$ or $x=4$.
The expression $(x+3)(x-4)>0$ when $x<-3$ or $x>4$.
The expression $(x+3)(x-4)<0$ when $-3<x<4$.

Example 6.8

Investigate the sign of the expression $\dfrac{(x+2)(x-3)}{(x+4)(5-x)}$ for various values of x.

Questions involving 4 brackets will not appear on the exam.

Solution:

Step 1: Find the critical values of x.

$x+4=0$ so $x=-4$

$x+2=0$ so $x=-2$

$x-3=0$ so $x=3$

$5-x=0$ so $x=5$

Step 2: Draw up the table.
Put in the zeros.
Put in the + and − signs.

		-4		-2		3		5	
$x+2$	$-$	$-$	$-$	0	$+$	$+$	$+$	$+$	$+$
$x-3$	$-$	$-$	$-$	$-$	$-$	0	$+$	$+$	$+$
$x+4$	$-$	0	$+$	$+$	$+$	$+$	$+$	$+$	$+$
$5-x$	$+$	$+$	$+$	$+$	$+$	$+$	$+$	0	$-$
Product	$-$	∞	$+$	0	$-$	0	$+$	∞	$-$

Critical values

Notice that these zeros are on the bottom of the fraction.

The expression $=0$ when $x=-2$ or $x=3$.
The expression <0 when $x<-4$ or $-2<x<3$ or $x>5$.
The expression >0 when $-4<x<-2$ or $3<x<5$.
The expression is undefined when $x=-4$ or $x=5$.

6.5 Quadratic inequalities

6.5.1 Using the sign diagram

Example 6.9

Find the solution set of the inequality $(x+2)(1-x)>0$.

Solution:

Step 1: Find the critical values of x.

$x+2=0$ so $x=-2$

$1-x=0$ so $x=1$

Step 2: Draw up the table.
Put in the zeros.
Put in the + and − signs.
Put in the products.

		-2		1	
$x+2$	$-$	0	$+$	$+$	$+$
$1-x$	$+$	$+$	$+$	0	$-$
Product	$-$	0	$+$	0	$-$

Critical values

o indicates that the value at the end of the line is not included.

The expression $(x+2)(1-x)>0$ when $-2<x<1$.

Example 6.10

Find the solution set of the inequality $(x+3)(3-x) \leqslant 0$.

Solution:

Step 1: Find the critical values of x.

$x + 3 = 0$ so $x = -3$

$3 - x = 0$ so $x = 3$

Step 2: Draw up the table.
Put in the zeros.
Put in the + and − signs.
Put in the products.

		-3		3	
$x+3$	$-$	0	$+$	$+$	$+$
$3-x$	$+$	$+$	$+$	0	$-$
Product	$-$	0	$+$	0	$-$

Critical values

The expression $(x+3)(3-x) \leqslant 0$ when $x \leqslant -3$ or $x \geqslant 3$.

● indicates that the value at the end of the line is included.

6.5.2 Using the solutions of the equation

Example 6.11

Find the solution set of the inequality $x^2 - 5x + 6 > 0$.

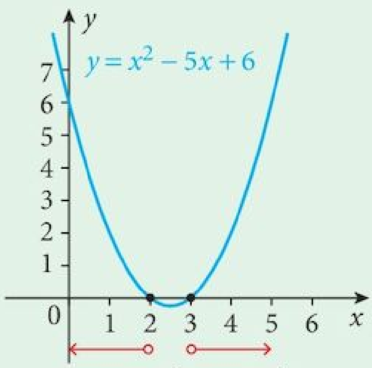
Solution:

Step 1: Find the solutions of the equation.

$x^2 - 5x + 6 = 0$

$x = 2$ or $x = 3$

Step 2: Sketch the graph of $y = x^2 - 5x + 6$.



Step 3: Choose the solutions.

$x < 2$ or $x > 3$

This technique can be used for simple functions such as linear or quadratic ones by checking when the graph is above or below the x-axis.

This technique can be used if you cannot factorise the expression.

Exercise 6.2

Draw sign diagrams for the following functions.

1 $f(x) = x - 3$

2 $f(x) = x + 2$

3 $f(x) = 4 - x$

4 $f(x) = (x+2)(x-2)$

5 $f(x) = (x+3)(x-4)$

6 $f(x) = (x+1)(4-x)$

7 $f(x) = (x-2)(x-5)$ **8** $f(x) = (x+3)(x+4)$ **9** $f(x) = (3-x)(6-x)$

10 $f(x) = \dfrac{(x+2)(x-3)}{x+1}$ **11** $f(x) = \dfrac{(x+1)}{(x+3)(x-5)}$ **12** $f(x) = \dfrac{(3-x)(4-x)}{(x+3)(x+5)}$

Solve these inequalities using a sign diagram.

13 $(x-2)(x-5) > 0$ **14** $(x+5)(x-1) < 0$ **15** $(x+6)(x+1) > 0$

16 $(x+1)(4-x) \leqslant 0$ **17** $(x+2)(x-3) \geqslant 0$ **18** $(2-x)(x+4) \geqslant 0$

19 $6 + x \leqslant x^2$ **20** $8 + x^2 > 6x$ **21** $x^2 + 4x \geqslant 5$

22 $0 < 6x^2 + x - 2$ **23** $4x^2 + 5x \geqslant 6$ **24** $5x^2 > 7x + 6$

25 $6x + 4 \leqslant x^2$ **26** $x^2 + 5x < 4$ **27** $x^2 + 1 \geqslant 6x$

28 $4x + 3 < 2x^2$ **29** $3x^2 \leqslant 9x + 4$ **30** $4x^2 + 3 < 10x$

31 A gardener is laying out a rectangular lawn. His specifications are that the area must be greater than $40\,\text{m}^2$ but the perimeter must be less than 40 m.

If the width of the lawn has to be less than its length, find the range of possible values for the width of the lawn.

Summary

Definition

Simultaneous means "at the same time". Thus, this means to find values of the variables that are solutions to both equations at the same time.

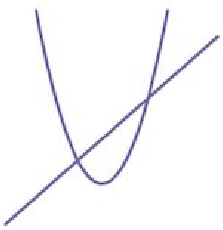
Solving equations

Given a linear equation $y = f(x)$ and a quadratic equation $y = g(x)$, eliminate the y to get $f(x) = g(x)$. This will be a quadratic equation to solve.

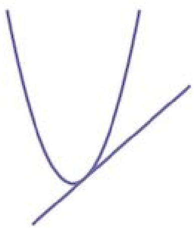
For a quadratic equation, test the discriminant to find the number of roots.

The discriminant test

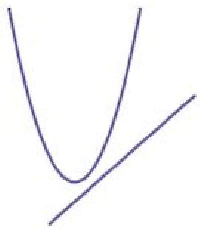
When trying to solve a problem involving a line and a curve, there are three possibilities:



$(b^2 - 4ac) > 0$
2 solutions;
line cuts the curve



$(b^2 - 4ac) = 0$
2 equal solutions;
line is a tangent



$(b^2 - 4ac) < 0$
no solutions;
line misses the curve

Inequalities

The rule for solving inequalities is the same as for equations. "Do the same thing to both sides - except, do not multiply or divide by zero or a quantity whose value you do not know (it might be zero) and, if you multiply or divide by a negative quantity, reverse the direction of the inequality (or equation)."

The sign diagram

When an expression has been fully factorised, use a sign diagram to test whether the function value is positive or negative by using its factors.

e.g. $(x+2)(1-x) > 0$

		-2		1	
$x+2$	$-$	0	$+$	$+$	$+$
$1-x$	$+$	$+$	$+$	0	$-$
Product	$-$	0	$+$	0	$-$

The value at the end of the line:
- ● is included
- ○ is not included

Quadratic inequalities

You can use your knowledge of the shape of the function to determine when it is positive and when it is negative.

Chapter 6 Summative Exercise

1 Determine whether or not the line intersects the curve, touches the curve, or does not intersect the curve.

a $y = 3x + 2$; $y = x^2 - 8x + 5$
b $y = x - 5$; $y = x^2 + 12x + 4$
c $4x + y = 18$; $y = 8 - 8x - 3x^2$
d $y = 6x + 8$; $y = 3x^2 + 12x + 7$
e $x + y = 5$; $y = 4 - 3x - x^2$
f $x + y = 1$; $y = 5x - 3 - 2x^2$
g $x + y = 4$; $xy = 8$
h $4x + y = 16$; $xy = 16$

2 Find the coordinates of the points of intersection of the lines and curves.

a $y = 3x + 9$; $y = 3x^2 + 6x + 3$
b $x + y = 6$; $y = 5x^2 + 4x - 4$
c $y = 4x - 13$; $y = 4x^2 - 24x + 11$
d $2x = y + 6$; $y = 4x - 2 - 2x^2$
e $y = x + 15$; $y = 3x^2 - 2x - 21$
f $y = 5x - 16$; $y = 2x^2 + 7x + 2$
g $y = x + 3$; $xy = 10$
h $y = 2x - 4$; $xy = 16$

3 Find the coordinates of the points where the lines touch the curves.

a $5x = y + 7$; $y = 2x^2 + x - 5$
b $4x + y = 9$; $y = 8x - 3 - 3x^2$
c $y = 14x - 81$; $y = 3x^2 - 16x - 6$
d $y = 5x - 10$; $3y = x^2 + 5x - 5$
e $6x + y = -4$; $2y = x^2 - 8x - 4$
f $2x + y = 9$; $y = 8 - 4x - x^2$
g $3x + 4y = 24$; $xy = 12$
h $5x + 4y = -40$; $xy = 20$

4 Factorise these expressions and use a sign diagram to solve the given inequality.

a $f(x) = x^2 - x - 20$; $f(x) > 0$
b $f(x) = x^2 - 3x - 18$; $f(x) < 0$
c $f(x) = 6x^2 - x - 2$; $f(x) \leqslant 0$
d $f(x) = 3x^2 - 13x - 10$; $f(x) \geqslant 0$
e $f(x) = 12 + 5x - 2x^2$; $f(x) \leqslant 0$
f $f(x) = 15 + x - 2x^2$; $f(x) \geqslant 0$
g $f(x) = 7x - 5 - 2x^2$; $f(x) > 0$
h $f(x) = 16x - 16 - 3x^2$; $f(x) < 0$

5 Draw a sign diagram for the following functions.

a $f(x) = \dfrac{x-3}{x+2}$ **b** $f(x) = \dfrac{6-x}{x+1}$

c $f(x) = \dfrac{(x+1)(x-3)}{x+2}$ **d** $f(x) = \dfrac{(x+3)(2-x)}{x-1}$

e $f(x) = \dfrac{x-2}{(x+3)(x+1)}$ **f** $f(x) = \dfrac{x+1}{(x+4)(x-2)}$

g $f(x) = \dfrac{(x-3)(x+2)}{(x+1)(x-4)}$ **h** $f(x) = \dfrac{(x+3)(x-2)}{(x+4)(x-4)}$

6 Find the range of values of the constant c such that the line $2y = 7x + c$ meets the curve $2y = x^2 - 5x + 6$ at two distinct points.

7 Find the coordinates of the points where the line $4y = 7x + 1$ intersects the curve $x^2 - 2x + y^2 - 4y = 60$.

8 Find the coordinates of the point where the line $5x - 6y = 0$ meets the curve $x^2 - y^2 = 11$.

9 Find the range of values of p such that the line $y = 2px - p$ intersects the curve $y = x^2$.

10 Aishya and Suhasini are sisters. Aishya is the eldest. The sum of their ages is 15 and the product of their ages is 54. Form two simultaneous equations relating their ages and solve them.

11 The perimeter of a rectangle is 38 cm and its area is 84 cm^2. Find the dimensions of the sides of the rectangle.

12 The sum of the roots of a quadratic equation is 2 and the product of the roots is −35.

a Write down two simultaneous equations connecting the roots, a and b, and solve them.

b What is the quadratic equation that has these two numbers as roots?

13 In order to produce a new ice cream flavour, the manufacturer mixes x kg of mango pulp with y kg of kiwi fruit pulp in such a way as to satisfy the following equations.

$x + y = 13$ and $x - y = \dfrac{24}{x}$

a Solve these equations to find corresponding values of x and y.

b Which solution is not practical?

14 A rectangular tile has dimensions x cm and $(8 - x)$ cm. Given that the rectangle's area must be at least 12 cm^2, find the set of possible values of x.

15 A gardener is laying out a new rectangular lawn. The specifications given to the gardener are as follows:

- the area must at least 40 m^2
- the perimeter must be 44 m

What are the minimum and maximum lengths of:

a the shorter edge of the lawn

b the longer edge of the lawn?

Chapter 6 Test

1 hour

1 Find the coordinates of the points of intersection of the curve $4y^2 - 8y = x^2 - 4x + 7$ and the straight line $3y = 2x - 1$. [5]

2 Find the coordinates of the points where the line $3x + 4y = 15$ intersects the curve $2xy = 9$. [5]

3 Solve the simultaneous equations.
$$\frac{3}{x} + \frac{1}{y} = 1$$
$$x - 2y = 6$$
[5]

4 Find the set of values of m for which the line $y = mx$ intersects the curve $x^2 + y^2 - 20x - 10y + 100 = 0$. [5]

5 Find the set of values of m for which the line $y = mx$ does not intersect the curve $y = 3x^2 - 12x + 3$. [5]

6 The line $3y = 2x - 4$ meets the curve $x^2 - 4x - 2y^2 + 2y - xy + 9 = 0$ at the points A and B. Find the coordinates of A and of B. [5]

7 The line $3y = 2x - 19$ intersects the curve $4x^2 + 9y^2 - 16x + 54y + 61 = 0$ at the points A and B. Find the coordinates of A and of B. [5]

8 The line $2x + 3y + 6 = 0$ meets the curve $xy + 12 = 0$ at the points A and B. Find the coordinates of A and of B. [5]

Examination Questions

1 Find the values of m for which the line $y = mx - 9$ is a tangent to the curve $x^2 = 4y$. [4]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2002, P1, Qu 2]

2 Find the values of k for which the line $y = kx - 2$ meets the curve $y^2 = 4x - x^2$. [4]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2003, P1, Qu 1]

3 Find the solution set of the quadratic inequality

(i) $x^2 - 8x + 12 > 0$ [3]

(ii) $x^2 - 8x < 0$. [2]

Hence find the solution set of the inequality $|x^2 - 8x + 6| < 6$. [2]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2003, P2, Qu 6]

4 Find the values of k for which the line $x + 3y = k$ and the curve $y^2 = 2x + 3$ do not intersect. [4]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2003, P1, Qu 1]

5 Find the values of k for which the line $y = x + 2$ meets the curve $y^2 + (x + k)^2 = 2$. [5]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2004, P2, Qu 4]

6 Find the set of values of x for which $(x - 6)^2 > x$. [3]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2005, P1, Qu 1]

7 Find the values of the constant c for which the line $2y = x + c$ is a tangent to the curve $y = 2x + \frac{6}{x}$. [4]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2005, P1, Qu 3]

8 **a** Find the value of m for which the line $y = mx - 3$ is a tangent to the curve $y = x + \frac{1}{x}$ and find the x-coordinate of the point at which this tangent touches the curve. [5]

b Find the value of c and of d for which $\{x : -5 < x < 3\}$ is the solution set of $x^2 + cx < d$. [2]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2006, P2, Qu 7]

9 A triangle has a base of length $(13 - 2x)$ m and a perpendicular height of x m.
Calculate the range of values of x for which the area of the triangle is greater than $3\,\text{m}^2$. [3]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2007, P2, Qu 1]

10 The line $y = 3x + k$ is a tangent to the curve $x^2 + xy + 16 = 0$.

(i) Find the possible values of k. [3]

(ii) For each of these values of k find the coordinates of the point of contact of the tangent with the curve. [2]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2008, P1, Qu 3]

11 Find the coordinates of the points where the straight line $y = 2x - 3$ intersects the curve $x^2 + y^2 + xy + x = 30$. [5]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2008, P2, Qu 3]

12 Find the set of values of x for which $(2x + 1)^2 > 8x + 9$. [4]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2008, P1, Qu 2]

13 Find the set of values of m for which the line $y = mx + 2$ does not meet the curve $y = x^2 - 5x + 18$. [5]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2008, P2, Qu 3]

14 Find the values of k for which $x^2 - 2(2k + 1)x + (k + 2) = 0$ has two equal roots. [4]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2009, P1, Qu 3]

7 The binomial theorem

Syllabus statements

- use the binomial theorem for the expansion of $(a+b)^n$ for positive integral n
- use the general term $\binom{n}{r}a^{n-r}b^r$, $0 < r \leqslant n$

(knowledge of the greatest term and properties of the coefficients is not required)

7.1 Introduction

In this chapter, we investigate the expansion of $(a+b)^n$ for positive integer values of n. Later, if you study Mathematics at a higher level, you will meet the problems created by asking if the theory can be extended to cover negative or rational values of n. For now, we focus on the simpler cases.

The process leads to a series. Series of different varieties were very important in the days before electronic calculation because using them was the only way that many mathematical functions could be calculated. Indeed, your calculator uses them without you realising it.

7.2 Powers of $(a + b)$

We can express powers of $(a + b)$ by expanding the brackets as follows:

$(a + b)$ is called "binomial". It has two elements.

$$(a + b)^2 = (a + b)(a + b) = a^2 + 2ab + b^2$$

$$(a + b)^3 = (a + b)(a + b)^2 = a^3 + 3a^2b + 3ab^2 + b^3$$

	$1a$	$1b$
a	$1a^2$	$1ab$
b	$1ab$	$1b^2$

	$1a^2$	$2ab$	$1b^2$
a	$1a^3$	$2a^2b$	$1ab^2$
b	$1a^2b$	$2ab^2$	$1b^3$

and so on.

Note how the coefficients in the exapansion are formed. $2 = 1 + 1$; $3 = 2 + 1$ and $3 = 1 + 2$.

These are just the sum of two coefficients in the previous expansion.

7.3 Pascal's triangle

We can extend the process to create this triangle of coefficients. Each row's inner elements are formed by adding pairs of numbers in the previous row. Here, $6 + 15 = 21$. Each row's outer elements are always 1.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

Exercise 7.1

Expand the following by mutiplying out the brackets and compare the coefficients with the values in Pascal's triangle.

1 $(a + b)^4$ **2** $(a + b)^5$ **3** $(a + b)^6$ **4** $(a + b)^7$

Pascal's triangle can be used to expand powers of binomial expressions, but it is really useful only for small powers. You would not choose to expand $(a + b)^{20}$ this way. It would require too much work extending the triangle.

7.4 Combinations

When we are calculating $(a + b)^3 = (a + b)(a + b)(a + b)$, we have to choose one element from each bracket and multiply them together.

We get the following:

Term	Brackets to choose a from	Brackets to choose b from	Number of ways to choose
a^3	3	0	${}^3C_3$ or ${}^3C_0$ $(= 1)$
a^2b	2	1	${}^3C_2$ or ${}^3C_1$ $(= 3)$
ab^2	1	2	${}^3C_1$ or ${}^3C_2$ $(= 3)$
b^3	0	3	${}^3C_0$ or ${}^3C_3$ $(= 1)$

It does not matter whether we choose a's or b's. Usually we count the number of ways to choose the b's.

So $(a + b)^3 = {}^3C_0\, a^3 + {}^3C_1\, a^2b + {}^3C_2\, ab^2 + {}^3C_3\, b^3$

$= a^3 + 3a^2b + 3ab^2 + b^3$

Remember that ${}^nC_r$ can also be written as ${}_nC_r$ or $\binom{n}{r}$.

Exercise 7.2

Use the ${}_nC_r$ button on your calculator to check that the results are, in fact, the same as those given by Pascal's triangle.

7.5 Expanding binomial expressions

Example 7.1

Expand $(a + b)^6$.

Solution:

Step 1: Write down the terms. Powers of a reduce from 6 to 0, powers of b increase from 0 to 6.

$(a + b)^6 = \square\, a^6 + \square\, a^5b + \square\, a^4b^2 + \square\, a^3b^3 + \square\, a^2b^4 + \square\, ab^5 + \square\, b^6$

Step 2: Write in the numbers from the correct row of Pascal's triangle or the binomial coefficients ${}^6C_r$.

$(a + b)^6 = \mathbf{1}a^6 + \mathbf{6}a^5b + \mathbf{15}a^4b^2 + \mathbf{20}a^3b^3 + \mathbf{15}a^2b^4 + \mathbf{6}ab^5 + \mathbf{1}b^6$

or

$(a + b)^6 = {}^6C_0\, a^6 + {}^6C_1\, a^5b + {}^6C_2\, a^4b^2 + {}^6C_3\, a^3b^3 + {}^6C_4\, a^2b^4 + {}^6C_5\, ab^5 + {}^6C_6\, b^6$

or

$(a+b)^6 = \binom{6}{0}a^6 + \binom{6}{1}a^5b + \binom{6}{2}a^4b^2 + \binom{6}{3}a^3b^3 + \binom{6}{4}a^2b^4 + \binom{6}{5}ab^5 + \binom{6}{6}b^6$

Step 3: If you have used binomial coefficients, work them all out.

$(a + b)^6 = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$

Exercise 7.3

Find the binomial expansion of the following expressions.

1 $(a+b)^8$ **2** $(1+x)^4$ **3** $(3+x)^5$ **4** $(2+x)^6$

7.6 The binomial theorem for a positive integer index

$$(a+b)^n = \binom{n}{0}a^n + \binom{n}{1}a^{n-1}b + \ldots + \binom{n}{r}a^{n-r}b^r + \ldots + \binom{n}{n}b^n$$

This is just a formal way of writing down the results we have just derived.

Where
$$\binom{n}{r} = \frac{n!}{r!(n-r)!}$$
$$= \frac{n(n-1)\ldots(n-r+1)}{r!}$$

Note that we have not proved that the results hold in all cases.

Exercise 7.4

Using the formula for ${}^nC_r$, confirm the following results.

1 $\binom{6}{1}=6$ **2** $\binom{6}{2}=\frac{6\times5}{2!}$ **3** $\binom{6}{3}=\frac{6\times5\times4}{3!}$

4 $\binom{12}{1}=12$ **5** $\binom{12}{2}=\frac{12\times11}{2!}$ **6** $\binom{12}{3}=\frac{12\times11\times10}{3!}$

7 $\binom{20}{1}=20$ **8** $\binom{20}{2}=\frac{20\times19}{2!}$ **9** $\binom{20}{3}=\frac{20\times19\times18}{3!}$

10 $\binom{n}{1}=n$ **11** $\binom{n}{2}=\frac{n(n-1)}{2!}$ **12** $\binom{n}{3}=\frac{n(n-1)(n-2)}{3!}$

7.7 More complicated expressions

You should take care when expanding more complicated expressions. A good rule is "Do not do too much all at once. Take your time!"

Example 7.2

Expand $(2-3x)^4$.

Solution:

Step 1: Write down the terms. Powers of 2 reduce from 4 to 0, powers of $(-3x)$ increase from 0 to 4.

At this stage, do not work out any coefficients, just get the terms down.

$$(2-3x)^4 = \square\, 2^4 + \square\, 2^3(-3x) + \square\, 2^2(-3x)^2 + \square\, 2(-3x)^3 + \square\, (-3x)^4$$

Step 2: Write in the numbers from the correct row of Pascal's triangle – or the binomial coefficients ${}^4C_r$.

$$(2-3x)^4 = \mathbf{1}\cdot 2^4 + \mathbf{4}\cdot 2^3(-3x) + \mathbf{6}\cdot 2^2(-3x)^2 + \mathbf{4}\cdot 2(-3x)^3 + \mathbf{1}\cdot(-3x)^4$$

Step 3: Calculate the coefficients and the signs.

$$(2-3x)^4 = 16 - 96x + 216x^2 - 216x^3 + 81x^4$$

Example 7.3

Expand $(1-2x)^{16}$ in ascending powers of x as far as the term in x^3.

Solution:

Step 1: Write down the terms. Powers of 1 reduce from 16 to 13, powers of $(-2x)$ increase from 0 to 3.

At this stage, do not work out any coefficients, just get the terms down.

$$(1-2x)^{16} = \square\, 1^{16} + \square\, 1^{15}(-2x) + \square\, 1^{14}(-2x)^2 + \square\, 1^{13}(-2x)^3 + \ldots$$

Step 2: Write in the binomial coefficients ${}^{16}C_r$.

$$(1-2x)^{16} = 1^{16} + \mathbf{16}\cdot 1^{15}(-2x) + \frac{\mathbf{16\times 15}}{\mathbf{2!}}\cdot 1^{14}(-2x)^2 + \frac{\mathbf{16\times 15\times 14}}{\mathbf{3!}}\cdot 1^{13}(-2x)^3 + \ldots$$

Step 3: Calculate the coefficients and the signs.

$$(1-2x)^{16} = 1 - 32x + 480x^2 - 4480x^3 + \ldots$$

Example 7.4

Using the result from Example 7.3, find 0.998^{16} correct to 6 d.p. without using a calculator.

Solution:

$(1 - 2x) = 0.998$ when $x = 0.001$

From Example 7.3,

$(1 - 2x)^{16} = 1 - 32x + 480x^2 - 4480x^3 + \ldots$

Substituting $x = 0.001$ gives

$(0.998)^{16} = 1 - 0.032 + 0.00048 - 0.00000448 + \ldots$

$= 0.968476$ (correct to 6 decimal places)

The next term is 0.00000002912. This will not affect the sixth decimal place.

Example 7.5

Find the term in x^3 in the expansion of $(3 - 4x)^7$.

Solution:

The term in x^3 must be $\quad {}^7C_3(3)^4(-4x)^3$

Notes 1: the power of the x term is 3

2: the powers sum to 7 so the other power is 4

3: the binomial coefficient is ${}^7C_3$, 7 from the question, 3 from the power of x.

The term in x^3 is $-181\,440x$.

Exercise 7.5

1 Use the binomial theorem to expand each expression.

a $(1 - x)^4$ **b** $(1 - 2x)^5$ **c** $(2 - x)^3$ **d** $(3 - 2x)^6$

e $\left(1+\frac{1}{2}x\right)^4$ **f** $\left(2-\frac{1}{4}x\right)^5$ **g** $\left(2+\frac{3}{4}x\right)^3$ **h** $\left(1-\frac{1}{5}x\right)^3$

i $(3 + x)^4$ **j** $(2 - 3x)^5$ **k** $\left(1+\frac{2}{3}x\right)^3$ **l** $(4 - x)^6$

m $\left(\frac{1}{2}-2x\right)^4$ **n** $\left(\frac{1}{3}+3x\right)^5$ **o** $\left(\frac{1}{4}-2x\right)^3$ **p** $\left(2+\frac{1}{2}x\right)^6$

2 Use the binomial theorem to expand each expression.

a $\left(1-\frac{1}{x}\right)^4$ **b** $(x^2 + 2)^5$ **c** $\left(x^2-\frac{1}{x}\right)^3$ **d** $\left(x-\frac{1}{x}\right)^6$

3 Find the coefficient of the term indicated in the square brackets in the expansion of each expression.

a $(3-2x)^9$ $[x^5]$ b $(1-2x)^{10}$ $[x^3]$ c $(2-x)^{12}$ $[x^6]$ d $(1-x)^{15}$ $[x^{12}]$

e $(1+2x)^{20}$ $[x^4]$ f $(2-x)^{30}$ $[x^{26}]$ g $(2-3x)^{10}$ $[x^8]$ h $(3-2x)^{15}$ $[x^{13}]$

4 Expand these series in ascending powers of x as far as the term shown in the square brackets:

a $(1-2x)^{15}$ $[x^3]$ b $(2+x)^{20}$ $[x^3]$ c $(1-3x)^{12}$ $[x^2]$ d $(1+2x)^{18}$ $[x^4]$

e $(1+2x)^{20}$ $[x^4]$ f $\left(\frac{1}{2}-x\right)^{12}$ $[x^2]$ g $\left(2-\frac{1}{2}x\right)^{16}$ $[x^3]$ h $\left(1-\frac{1}{3}x\right)^{15}$ $[x^3]$

5 By choosing a suitable binomial expansion of the form $(1 \pm kx)^n$ and a suitable value of x, find each of the following correct to 3 s.f.

a $(1.01)^{20}$ b $(0.98)^{12}$ c $(1.03)^{16}$ d $(0.998)^{15}$

Example 7.6

a Expand $(1+u)^3$.

b By substituting $u = x + x^2$, find the expansion of $(1 + x + x^2)^3$.

Solution:

a $(1+u)^3 = 1 + 3u + 3u^2 + u^3$

b $$(1 + x + x^2)^3 = 1 + 3(x + x^2) + 3(x + x^2)^2 + (x + x^2)^3$$
$$= 1 + 3(x + x^2) + 3(x^2 + 2x^3 + x^4) + (x^3 + 3x^4 + 3x^5 + x^6)$$
$$= 1 + 3x + 6x^2 + 7x^3 + 6x^4 + 3x^5 + x^6$$

Example 7.7

a Find the first four terms, in ascending powers of x, of the expansion of $(3+x)^6$.

b Use the result to find the coefficient of the term in x^3 in the expansion of $(1-2x)(3+x)^6$.

Solution:

a $$(3+x)^6 = 3^6 + 6.3^5x + 15.3^4x^2 + 20.3^3x^3 + \ldots$$
$$= 729 + 1458x + 1215x^2 + 540x^3 + \ldots$$

b $(1-2x)(3+x)^6$

$\times$	729	$1458x$	$1215x^2$	$540x^3$
1				$540x^3$
$-2x$			$-2430x^3$	

When multiplying out the brackets, the only elements you need are the two that produce x^3.

So, the coefficient of x^3 is -1890.

Example 7.8

Find the term independent of x in the binomial expansion of $\left(x+\frac{2}{x^2}\right)^{12}$.

The term "independent of x" is the term in x^0.

Solution:

The general term of the expansion is $^{12}C_r x^{12-r} \times \left(\frac{2}{x^2}\right)^r$

$= {}^{12}C_r 2^r x^{12-3r}$

The term independent of x is the term in x^0.

So $12-3r=0$

$r=4$

The term independent of x is $^{12}C_4\, 2^4 = 7920$

Exercise 7.6

1 **a** Find the expansion, in ascending powers of u, of $(1+u)^4$.

b By substituting $u = x + x^2$, find the expansion of $(1 + x + x^2)^4$.

2 Find the term independent of x in the expansion of $\left(x+\frac{1}{x}\right)^{10}$.

3 **a** Find the first four terms in the expansion, in ascending powers of x, of $(1-x)^6$.

b Find the coefficient of x^3 in the expansion of $(2-3x)(1-x)^6$.

4 Calculate the term independent of x in the binomial expansion of $\left(x^2+\frac{1}{2x^4}\right)^{12}$.

5 Find the coefficient of x^6 in the expansion of $(2x-1)^6(3x+1)$.

6 **a** Find the expansion, in ascending powers of u, of $(1-u)^5$.

b By substituting $u = x - x^2$, find the expansion of $(1 - x + x^2)^5$.

7 **a** Expand $\left(x-\frac{1}{x}\right)^8$.

b Find the value of k that creates a term of 140 in the expansion of $(k-2x)\left(x-\frac{1}{x}\right)^8$.

8 The binomial expansion of $(1+ax)^n$, where $n>0$, in ascending powers of x, is

$1 + 20x + 45a^2x^2 + bx^3 + \ldots$

Find the value of n, of a and of b.

Summary

Definition

A binomial expression is one that is of the form $(a + bx)$.

It is another name for a linear expression. It has two terms, hence "binomial".

The binomial theorem is concerned with finding $(a + bx)^n$ for different values of n.

Expanding $(a + b)^n$

The terms in the expansion will be:

$a^n \quad a^{n-1}b \quad a^{n-2}b^2 \quad a^{n-3}b^3 \ldots a^3b^{n-3} \quad a^2b^{n-2} \quad ab^{n-1} \quad b^n$

The sum of the indices for each term is n.

Binomial coefficients

There are two ways to find these: using Pascal's triangle, or the formula for combinations.

Pascal's triangle

Suitable for small values of n.

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
\cdots & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots & & \cdots
\end{array}
$$

Combinations

The values in Pascal's triangle are exactly the same as values of ${}^nC_r$ or $\binom{n}{r}$.

$${}^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$$

Small values of r

$$\binom{n}{1} = n \qquad \binom{n}{2} = \frac{n(n-1)}{2!} \qquad \binom{n}{3} = \frac{n(n-1)(n-2)}{3!}$$

Binomial expansion

$$(a+b)^n = a^n + \binom{n}{1}a^{n-1}b + \binom{n}{2}a^{n-2}b^2 + \binom{n}{3}a^{n-3}b^3 + \cdots$$

Answering questions

Remember to write down the expressions first.

Do not work out any coefficients at this stage.

Then, in the next line, calculate all the coefficients.

There are marks in the exam for writing down the expressions.

Do not lose them by trying to do too much at the same time.

Approximations

Choose a suitable value of x to give you the answer you need.

Chapter 7 Summative Exercise

1 The 9th row of Pascal's triangle is

1 9 36 84 126 126 84 36 9 1

What is the 10th row?

2 Expand the following using coefficients from Pascal's triangle.

a $(a+b)^3$ **b** $(a+b)^9$ **c** $(a+b)^{10}$

3 Expand the following using coefficients from Pascal's triangle.

a $(1+x)^4$ **b** $(1-x)^4$ **c** $(2+x)^4$ **d** $(x-2)^4$

4 Write out the following using the formula. Write the answers in the form $\dfrac{n(n-1)(n-2)}{r\,!}$

a $\dbinom{7}{4}$ **b** $\dbinom{9}{3}$ **c** $\dbinom{10}{3}$ **d** $\dbinom{14}{2}$

5 Use the binomial theorem to expand these.

a $(1-x)^3$ **b** $(2x-1)^4$ **c** $(3-x)^3$ **d** $(2-3x)^4$

6 Use the binomial theorem to expand these.

a $\left(2+\dfrac{1}{x}\right)^5$ **b** $\left(\dfrac{2}{x}-1\right)^4$ **c** $\left(x^2+\dfrac{1}{x^2}\right)^6$ **d** $\left(x+\dfrac{1}{x}\right)^8$

7 Find the coefficient of the term indicated in square brackets in the binomial expansion of these.

a $(2+3x)^{12}$ $[x^4]$ **b** $(1-2x)^{14}$ $[x^8]$ **c** $(3+2x)^{10}$ $[x^5]$

d $(2x-1)^9$ $[x^3]$ **e** $\left(x-\dfrac{1}{x}\right)^{12}$ $[x^4]$ **f** $\left(x^2-\dfrac{1}{x}\right)^{16}$ $[x^{23}]$

8 Expand $(1+2x)^8$ as far as the term in x^4.

By letting $x = 0.01$, find the value of 1.02^8 correct to 6 d.p.

9 Expand $\left(2-\dfrac{1}{x}\right)^{12}$ as far as the term in x^{-5}.

By letting $x = 100$, find the value of 1.99^{12} correct to 5 d.p.

10 a Expand $(2-3x)^6$ as far as the term in x^3.

b Use the result to find the coefficient of x^2 in the expansion of $(1+2x)(2-3x)^6$.

11 a Find the expansion, in ascending powers of x, as far as the term in x^3, of $(1-2x)^8$.

b Use the result to find the coefficient of x^3 in the expansion of $(2-3x)(1-2x)^8$.

12 **a** Use the binomial theorem to find $(1+\sqrt{2})^6$ in the form $a + b\sqrt{2}$.

b Use the binomial theorem to find $(1-\sqrt{2})^6$ in the form $a + b\sqrt{2}$.

c Multiply your results together and hence show that $(1+\sqrt{2})^6$ and $(1-\sqrt{2})^6$ are reciprocals of each other.

13 The binomial expansion of $(a-x)^n$, where $n > 0$, in ascending powers of x, is

$$531\,441 - 2\,125\,764x + 3\,897\,234x^2 + \ldots$$

Find the value of a and the value of n.

14 Find the coefficient of x^2 in the expansion of $(2-x)^4(1+x)^3$.

15 Find the term independent of x in the expansion of $\left(x^2 - \frac{2}{x}\right)^9$.

16 Find the term independent of x in the expansion of $\left(x + \frac{3}{x^2}\right)^{12}$.

17 In the expansion of $(a+x)^n$, the coefficients of x and x^2 are equal. Express n in terms of a.

18 The first three terms, in ascending powers of x, of the expansion of $(a+bx)(1-2x)^7$ are $3 - 40x + cx^2$. Find the values of a, b and c.

19 The first three terms of the expansion of $(a+b)^n$ are p, q and r.

a Find p, q and r in terms of a, b and n.

b Find n if $p = 10$, $q = 15$ and $r = 9$.

20 The first three terms in the expansion of $(1+ax)^n$ are $1 - 14x + 84x^2$. Find the value of a and the value of n.

Chapter 7 Test

1 hour

1 **a** In ascending powers of x, find the first three terms in the expansion of $(1+2x)^5$. [3]

b Hence find the coefficient of x^2 in the expansion of $(1+2x)^5(1+2x-3x^2)$. [3]

2 Find the term independent of x in the expansion of $\left(x^4 - \frac{1}{x^2}\right)^{12}$. [3]

3 **a** Expand $(2+x)^4 + (2-x)^4$. [3]

b Using the substitution $u = x^2$, solve the equation $(2+x)^4 + (2-x)^4 = 626$. [3]

4 Find, in its simplest form, the coefficient of x^4 in the expansion of:

a $(1+2x)^8$ [2]

b $\left(x + \frac{5}{x^2}\right)^{10}$ [2]

5 **a** In ascending powers of x, find the first three terms in the expansion of $(3 + 2x)^5$. [2]

b In the expansion of $(3 + 2x)^5 (a + bx)^3$, the constant term is 1944 and the coefficient of x^8 is -256. Find the value of a and the value of b. [4]

6 **a** Find the coefficient of x^4 in the expansion of $(3 - 2x)^7$. [2]

b Find the coefficient of x^4 in the expansion of $(2 + x)(3 - 2x)^7$. [3]

7 **a** Given that n is a positive integer, write down the first 3 terms, in ascending powers of x, in the expansion of $\left(1 - \frac{1}{3}x\right)^n$. [3]

b The coefficient of x^2 in the expansion of $(1 + x)\left(1 - \frac{1}{3}x\right)^n$ is $\frac{20}{3}$.
Find the value of n. [4]

8 Find the term independent of x in the expansion of $\left(x^3 - \frac{2}{x^2}\right)^{10}$. [3]

Examination Questions

1 Find the first 3 terms in the expansion, in ascending powers of x, of $(2 + x)^6$ and hence obtain the coefficient of x^2 in the expansion of $(2 + x - x^2)^6$. [4]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2002, P2, Qu 2]

2 **(i)** Find the first 3 terms in the expansion, in ascending powers of x, of $(2 - x)^5$. [3]

(ii) Hence find the value of the constant k for which the coefficient of x in the expansion of $(k + x)(2 - x)^5$ is -8. [2]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2003, P1, Qu 3]

3 Obtain

(i) the first 3 terms in the expansion, in descending powers of x, of $(3x - 1)^5$ [3]

(ii) the coefficient of x^4 in the expansion of $(3x - 1)^5(2x + 1)$. [2]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2003, P2, Qu 5]

4 **a** Calculate the term independent of x, in the binomial expansion of $\left(x - \frac{1}{2x^5}\right)^{18}$. [3]

b In the binomial expansion of $(1 + kx)^n$, where $n \geqslant 3$ and k is a constant, the coefficients of x^2 and x^3 are equal. Express k in terms of n. [4]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2004, P1, Qu 9]

5 Given that the expansion of $(a + x)(1 - 2x)^n$ in ascending powers of x, is $3 - 41x + bx^2 + \ldots$, find the values of the constants a, n and b. [6]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2004, P1, Qu 5]

6 **a** **(i)** Expand $(2 + x)^5$. [3]

(ii) Use your answer to part **(i)** to find the integers a and b for which $(2 + \sqrt{3})^5$ can be expressed in the form $a + b\sqrt{3}$. [3]

b Find the coefficient of x in the expansion of $\left(x - \frac{4}{x}\right)^7$. [3]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2005, P2, Qu 11]

7 The binomial expansion of $(1 + px)^n$, where $n > 0$, in ascending powers of x, is

$$1 - 12x + 28p^2x^2 + qx^3 + \ldots$$

Find the value of n, of p and of q. [6]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2005, P2, Qu 5]

8 **(i)** In the binomial expansion of $\left(x + \frac{k}{x^3}\right)^8$, where k is a positive constant, the term independent of x is 252.

Evaluate k. [4]

(ii) Using your value of k, find the coefficient of x^4 in the expansion of $\left(1 - \frac{x^4}{4}\right)\left(x + \frac{k}{x^3}\right)^8$. [3]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2006, P2, Qu 8]

9 Given that the coefficient of x^2 in the expansion of $(k + x)\left(2 - \frac{x}{2}\right)^6$ is 84, find the value of the constant k. [6]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2006, P1, Qu 6]

10 **(i)** Find the first 3 terms, in ascending powers of u, in the expansion of $(2 + u)^5$. [2]

(ii) By replacing u with $2x - 5x^2$, find the coefficient of x^2 in the expansion of $(2 + 2x - 5x^2)^5$. [4]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2007, P1, Qu 4]

11 **(i)** Expand $(1 + x)^5$. [1]

(ii) Hence express $(1 + \sqrt{2})^5$ in the form $a + b\sqrt{2}$, where a and b are integers. [3]

(iii) Obtain the corresponding result for $(1 - \sqrt{2})^5$ and hence evaluate $(1 + \sqrt{2})^5 + (1 - \sqrt{2})^5$. [2]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2007, P2, Qu 5]

12 Find the coefficient of x^3 in the expansion of

(i) $(1 + 3x)^8$ [2]

(ii) $(1 - 4x)(1 + 3x)^8$ [3]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2008, P2, Qu 5]

13 **(i)** Find, in ascending powers of x, the first 3 terms of the expansion of $(2 - 3x)^5$. [3]

The first 3 terms in the expansion of $(a + bx)(2 - 3x)^5$ in ascending powers of x are $64 - 192x + cx^2$.

(ii) Find the value of a, and of b and of c. [5]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2008, P1, Qu 9]

14 Find the coefficient of x^4 in the expansion of

(i) $(1 + 2x)^6$ [2]

(ii) $\left(1-\frac{x}{4}\right)(1+2x)^6$ [3]

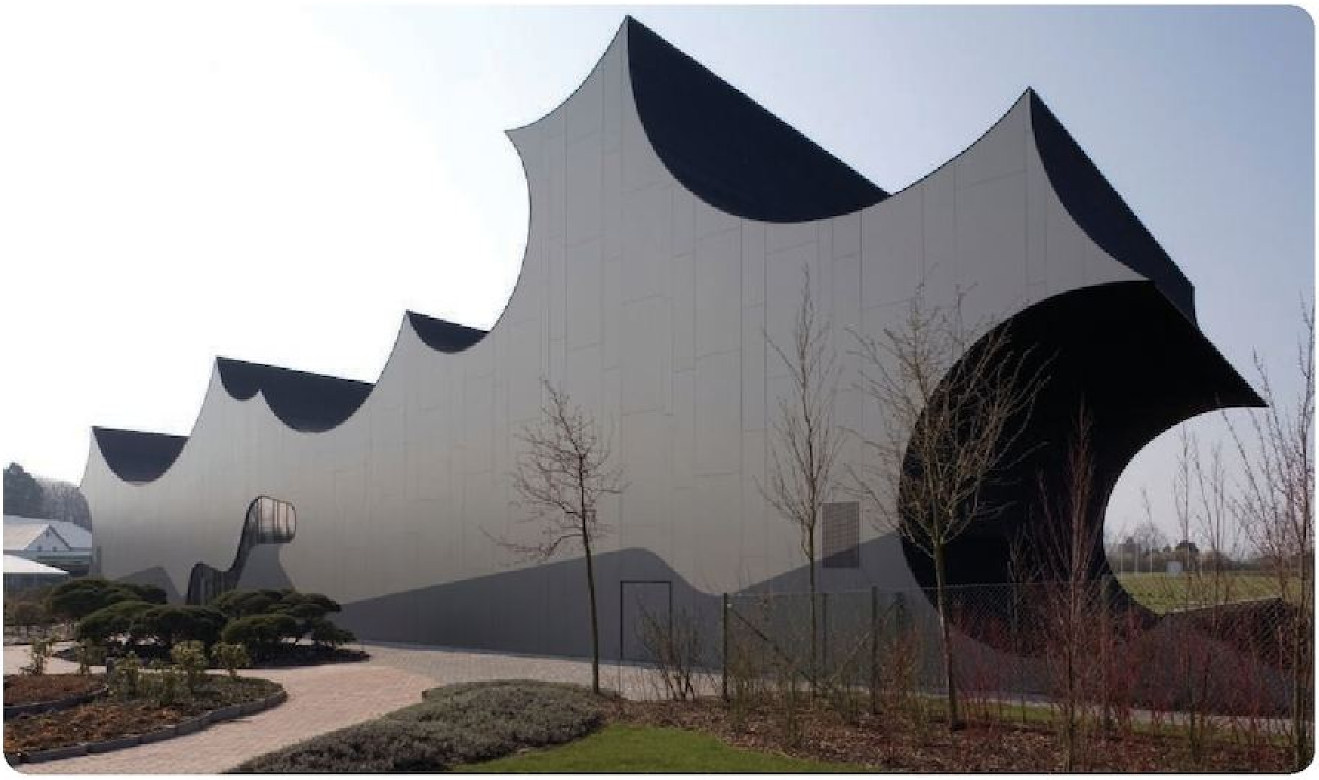
[Cambridge IGCSE Additional Mathematics 0606, Jun 2009, P2, Qu 4]

15 **(i)** Find the first four terms, in ascending powers of x, in the expansion of $\left(2-\frac{x}{2}\right)^6$. [4]

(ii) Find the coefficient of x^3 in the expansion of $(1+x)^2\left(2-\frac{x}{2}\right)^6$. [2]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2009, P2, Qu 2]

8 Polynomial factorisation



Syllabus statements

- know and use the remainder and factor theorems
- find factors of polynomials
- solve cubic equations

8.1 Introduction

In the last chapter, you looked at the Binomial theorem.

It is called "binomial" because the expression $(a + bx)$ has 2 terms.

Similarly, the expression $(a + bx + cx^2)$ is called a **trinomial**, although, for other reasons, we call it a quadratic.

Extending this idea, we get the term **polynomial**. "Poly-" comes from the Greek for "many", without specifying exactly how many.

While the ideas presented here can be extended to polynomials with a large number of terms, the most that you are likely to meet are those extending to x^3 or possibly x^4.

8.2 Polynomial division

What do you get when you divide $(x^4 + 3x^3 + 6x^2 + 5x + 5)$ by $(x^2 + x + 1)$?

If $x = 10$, this would become $13\,655 \div 111$ which you could calculate by long division.

Compare these two long division problems.

$$
\begin{array}{r}
1\,2\,3\phantom{\text{ Remainder}} \\
1\,1\,1\overline{)\,1\,3\,6\,5\,5}\phantom{\text{ Remainder}} \\
\underline{1\,1\,1}\downarrow\downarrow\phantom{\text{ Remainder}} \\
2\,5\,5\downarrow\phantom{\text{ Remainder}} \\
\underline{2\,2\,2}\downarrow\phantom{\text{ Remainder}} \\
3\,3\,5\phantom{\text{ Remainder}} \\
\underline{3\,3\,3}\phantom{\text{ Remainder}} \\
2\text{ Remainder}
\end{array}
\qquad\qquad
\begin{array}{r}
x^2 + 2x + 3\phantom{\text{ Remainder}} \\
x^2 + x + 1\overline{)\,x^4 + 3x^3 + 6x^2 + 5x + 5}\phantom{\text{ Remainder}} \\
\underline{x^4 + x^3 + x^2}\;\downarrow\;\;\downarrow\phantom{\text{ Remainder}} \\
2x^3 + 5x^2 + 5x\;\;\downarrow\phantom{\text{ Remainder}} \\
\underline{2x^3 + 2x^2 + 2x}\;\;\downarrow\phantom{\text{ Remainder}} \\
3x^2 + 3x + 5\phantom{\text{ Remainder}} \\
\underline{3x^2 + 3x + 3}\phantom{\text{ Remainder}} \\
2\text{ Remainder}
\end{array}
$$

In fact, they are identical when $x = 10$.

However, there is an important difference:

When dividing one integer by another, we say, for example, "How many times does 111 divide into 136?", giving the answer 1.

When doing the equivalent polynomial division, we say "How many times does x^2 divide into x^4?", giving the answer x^2.

It does not matter about the $(+x + 1)$ nor about what follows x^4. Only the first term is important.

Sometimes when you do polynomial division, there are negative terms and the subtractions give you negative results. Don't worry; after a bit of practice, dealing with these will become simple.

Example 8.1

Divide $3x^3 - 2x^2 + x - 3$ by $x + 2$.

Solution:

$$
\begin{array}{lll}
 & & 3x^2 - 8x + 17 \\
 & & x + 2\overline{)\,3x^3 - 2x^2 + x - 3} \\
\mathbf{1} & 3x^2 \times (x + 2): & 3x^3 + 6x^2 \;\downarrow\;\;\downarrow \\
 & \text{Subtract} & -8x^2 + x \;\;\downarrow \\
\mathbf{2} & -8x \times (x + 2): & -8x^2 - 16x \;\;\downarrow \\
 & \text{Subtract} & 17x - 3 \\
\mathbf{3} & 17 \times (x + 2): & 17x + 34 \\
 & \text{Subtract} & -37
\end{array}
$$

1 How many times does x divide into $3x^3$?

2 How many times does x divide into $-8x^2$?

3 How many times does x divide into $17x$?

So the quotient is $3x^2 - 8x + 17$ and the remainder is -37.

Check: $(x + 2)(3x^2 - 8x + 17) - 37 = (3x^3 - 8x^2 + 17x) + (6x^2 - 16x + 34) - 37$

$= 3x^3 - 2x^2 + x - 3$

Exercise 8.1

Find the quotient and the remainder when the polynomial is divided by the expression in square brackets.

1 $2x^3 + 3x^2 + 4x + 2 \quad [x + 3]$

2 $3x^3 + 2x^2 + 4x - 3 \quad [x + 2]$

3 $2x^3 + 3x^2 - 4x - 2 \quad [x + 1]$

4 $x^3 - 3x^2 - 2x - 1 \quad [x + 4]$

5 $x^3 + 2x^2 - x + 2 \quad [x + 1]$

6 $2x^3 - 3x^2 + 4x + 5 \quad [x + 2]$

7 $2x^3 - 3x^2 + 4x - 2 \quad [x + 3]$

8 $3x^3 - 2x^2 - 4x + 3 \quad [x + 4]$

9 $2x^3 + 3x^2 + 4x + 2 \quad [x - 1]$

10 $3x^3 + 2x^2 + 4x - 3 \quad [x - 4]$

11 $2x^3 + 3x^2 - 4x - 2 \quad [x - 3]$

12 $x^3 - 3x^2 - 2x - 1 \quad [x - 2]$

13 $x^3 + 2x^2 - x + 2 \quad [x - 3]$

14 $2x^3 - 3x^2 + 4x + 5 \quad [x - 4]$

15 $2x^3 - 3x^2 + 4x - 2 \quad [x - 1]$

16 $3x^3 - 2x^2 - 4x + 3 \quad [x - 2]$

17 $2x^3 - 3x^2 + 4x - 2 \quad [x^2 + x - 1]$

18 $3x^3 - 2x^2 - 4x + 3 \quad [x^2 - x - 1]$

19 $x^3 + 2x^2 - 3x + 4 \quad [x^2 - x + 1]$

20 $2x^3 - 3x^2 + x + 2 \quad [x^2 + x + 1]$

21 $2x^3 + 3x^2 + 4x + 2 \quad [x^2 + 2x - 2]$

22 $3x^3 - 2x^2 - 4x + 3 \quad [x^2 - 3x - 1]$

23 $x^3 + 2x^2 - 3x + 4 \quad [x^2 - 2x + 1]$

24 $2x^3 - 3x^2 - x - 2 \quad [x^2 + x + 2]$

8.3 The division algorithm

When you divide 35 by 4, you get 8, remainder 3.

Value	is called the
35	Dividend (N)
4	Divisor (D)
8	Quotient (Q)
3	Remainder (R)

So $\frac{35}{4} = 8$, Remainder $= 3$ $\qquad \frac{N}{D} = Q$, Remainder $= R$

We can also write this as

$$35 = 4 \times 8 + 3 \qquad N = D \times Q + R$$

Important! The remainder must be less than the divisor: $R < D$

When applied to polynomial expressions, you get exactly the same sort of results and the formula still applies, but every element is a function of x.

$f(x) = D(x) \times Q(x) + R(x)$

However, when we say **the remainder must be less than the divisor**, what does it mean?

We cannot say (from Example 8.1) that $-37 < (x + 2)$ because it might not be true.

This statement must be changed to:

The degree of the remainder must be less than the degree of the divisor deg(R) < deg(D)

This is very important!

Thus, if the divisor is linear (degree 1), the remainder must have degree 0. It must be a number.

Check: Look at your results from Exercise 8.1.

In questions 1 – 16, the remainders were numbers.

In questions 17 – 24, the remainders were linear or a number.

8.4 The remainder theorem

$f(x) = (x - a) \times Q(x) + R$

Furthermore, if we substitute the value $x = a$, we get

$f(a) = (a - a) \times Q(a) + R$

$= R$ since $(a - a) = 0$

In other words, we have a way of finding the remainder without doing the division.

This result is important and is called the **Remainder Theorem**.

> If $f(x) = (x - a) \times Q(x) + R$
> Then $f(a) = R$

Exercise 8.2

In each of these questions, taken from Exercise 8.1, identify the value of a from $[x - a]$ and calculate $f(a)$. Check that your answers agree with your results from Exercise 8.1. Note: some values of a are negative.

1 $f(x) = 2x^3 + 3x^2 + 4x + 2$ $[x + 3]$

2 $f(x) = 3x^3 + 2x^2 + 4x - 3$ $[x + 2]$

3 $f(x) = 2x^3 + 3x^2 - 4x - 2$ $[x + 1]$

4 $f(x) = x^3 - 3x^2 - 2x - 1$ $[x + 4]$

5 $f(x) = x^3 + 2x^2 - x + 2$ $[x + 1]$

6 $f(x) = 2x^3 - 3x^2 + 4x + 5$ $[x + 2]$

7 $f(x) = 2x^3 - 3x^2 + 4x - 2$ $[x + 3]$

8 $f(x) = 3x^3 - 2x^2 - 4x + 3$ $[x + 4]$

9 $f(x) = 2x^3 + 3x^2 + 4x + 2$ $[x - 1]$

10 $f(x) = 3x^3 + 2x^2 + 4x - 3$ $[x - 4]$

11 $f(x) = 2x^3 + 3x^2 - 4x - 2$ $[x - 3]$

12 $f(x) = x^3 - 3x^2 - 2x - 1$ $[x - 2]$

13 $f(x) = x^3 + 2x^2 - x + 2$ $[x - 3]$

14 $f(x) = 2x^3 - 3x^2 + 4x + 5$ $[x - 4]$

15 $f(x) = 2x^3 - 3x^2 + 4x - 2$ $[x - 1]$

16 $f(x) = 3x^3 - 2x^2 - 4x + 3$ $[x - 2]$

Find the remainder when the function f(x) is divided by the expression in brackets:

17 $f(x) = 2x^3 - 3x^2 + 4x - 2 \quad [2x - 1]$

18 $f(x) = 3x^3 - 2x^2 - 4x + 3 \quad [3x - 2]$

19 $f(x) = x^3 + 2x^2 - x + 2 \quad [2x + 3]$

20 $f(x) = 2x^3 - 3x^2 + 4x + 5 \quad [3x + 1]$

Example 8.2

When the cubic polynomial $f(x) = x^3 + ax^2 + bx + 3$ is divided by $(x - 2)$, the remainder is 7 and when it is divided by $(x - 3)$ the remainder is 15. Find the values of a and b.

Solution:

Using the remainder theorem: $f(2) = 8 + 4a + 2b + 3 = 7$ [1]

and $f(3) = 27 + 9a + 3b + 3 = 15$ [2]

[1] gives $4a + 2b = -4$

[2] gives $9a + 3b = -15$

Solving the simultaneous equations: $a = -3$ and $b = 4$

Exercise 8.3

1 When the polynomial $f(x) = 2x^3 + 3x^2 + ax + 2$ is divided by $(x - 1)$, the remainder is 8. Find the value of a.

2 When the polynomial $f(x) = 3x^3 + ax^2 + 4x - 3$ is divided by $(x + 1)$, the remainder is -5. Find the value of a.

3 When the polynomial $f(x) = x^3 - 3x^2 - 2x + a$ is divided by $(x - 2)$, the remainder is 1. Find the value of a.

4 When the polynomial $f(x) = ax^3 + bx^2 - 4x + 6$ is divided by $(x - 2)$, the remainder is -6. When it is divided by $(x + 1)$, the remainder is -3. Find the value of a and the value of b.

5 When the polynomial $f(x) = x^3 + ax^2 + bx - 2$ is divided by $(x + 1)$, the remainder is -4. When it is divided by $(x - 3)$, the remainder is -8. Find the value of a and the value of b.

6 When the polynomial $f(x) = ax^3 - 3x^2 + bx + 2$ is divided by $(x - 1)$, the remainder is 5. When it is divided by $(x + 2)$, the remainder is -46. Find the remainder when it is divided by $(x - 2)$.

7 The remainder when the polynomial $f(x) = 2x^3 + ax^2 - x - 3$ is divided by $(x + 2)$ is 3 times the remainder when it is divided by $(x + 1)$. Find the remainder when it is divided by $(x - 2)$.

8 The cubic polynomial $2x^3 - 3x^2 + ax - 4$ is divisible by $(x - 2)$. Find the value of a.

9 The quartic polynomial $x^4 + ax^2 + b$ is divisible by $(x + 1)(x - 2)$. Find the value of a and the value of b.

10 The quartic polynomial $x^4 - 2x^3 + ax^2 + bx - 3$ is divisible by $(x^2 - 1)$. Find the value of a and the value of b.

8.5 The factor theorem

You saw in questions 8, 9 and 10 of Exercise 8.3 that it is possible to get a remainder of zero.

When this happens, $f(x) = (x - a)Q(x) + R$

becomes $\qquad f(x) = (x - a)Q(x)$

and we have factorised $f(x)$.

This leads to **The Factor Theorem**.

> For any polynomial $f(x)$, if $f(a) = 0$ then **$(x - a)$ is a factor of $f(x)$** and we can use polynomial division to find the second factor.

You will already know that if the roots of the quadratic equation $x^2 + px + q = 0$ are a and b, then the equation can be written as $(x - a)(x - b) = 0$.

In the same way, if the roots of the cubic equation $x^3 + px^2 + qx + r = 0$ are a, b and c, the equation can be written as $(x - a)(x - b)(x - c) = 0$.

This result is a consequence of the factor theorem.

Example 8.3

a Show that $(x - 1)$ is a factor of $f(x) = x^3 - 3x^2 - 13x + 15$.

b Find the quotient when $f(x)$ is divided by $(x - 1)$.

c Factorise $f(x)$.

Solution:

a $f(1) = 1 - 3 - 13 + 15 = 0$

Hence, by the factor theorem, $(x - 1)$ is a factor of $f(x)$.

b $f(x) = (x - 1)Q(x)$

$$
\begin{array}{rrrrr}
 & & x^2 & -\ 2x & -\ 15 \\ \cline{2-5}
(x-1) & \big)\, x^3 & -\ 3x^2 & -\ 13x & +\ 15 \\
 & x^3 & -\ x^2 & \downarrow & \downarrow \\
 & & -\ 2x^2 & -\ 13x & \downarrow \\
 & & -\ 2x^2 & +\ 2x & \downarrow \\
 & & & -\ 15x & +\ 15 \\
 & & & -\ 15x & +\ 15 \\
 & & & & 0
\end{array}
$$

c $f(x) = (x - 1)(x^2 - 2x - 15)$

$(x^2 - 2x - 15)$ factorises to $(x - 5)(x + 3)$ so:

$f(x) = (x - 1)(x - 5)(x + 3)$

Example 8.4

Find the cubic equation whose roots are −2, 2 and 3.

Solution:

If the roots are as given, the equation will be of the form:

$(x+2)(x-2)(x-3)=0$

Expanding these brackets gives

$x^3-3x^2-4x+12=0$

8.6 Factorising polynomials

If we find a linear factor, as in Example 8.3, the quotient $Q(x)$ will always be one degree less than $f(x)$. We can then repeat the process to find a linear factor of $Q(x)$ (our new $f(x)$), and so on until we have found all the linear factors, or, more likely, until we have reduced it to a quadratic factor, from which we can find the remaining factors using the quadratic formula.

However, remember that not many polynomials can be factorised. Also, not all cubic polynomials have three real roots. They must have one real root, but the others may not be real or may not be "nice" real numbers (they might be irrational).

Example 8.5

a Factorise the polynomial $f(x)=x^3-5x^2+2x+8$.

b Hence solve the equation $f(x)=0$.

Solution:

a **Step 1: Integer search**

$f(1)=1-5+2+8=6$: no good.

$f(2)=8-20+4+8=0$: found one!

Try the integers 1, 2, 3, −1, −2, −3. Only then go on to 4, 5, −4, −5 if necessary.

$f(2)=0$, hence, by the factor theorem, $(x-2)$ is a factor of $f(x)$.

Step 2: Factorise to find $Q(x)$

$f(x)=(x-2)(x^2+3x-4)$

Note that if one of the roots is not a small integer, you could waste a lot of time searching. You need to find a better technique. These are part of the A-level course.

Step 3: Factorise $Q(x)$

$$f(x)=(x-2)(x^2+3x-4)$$
$$=(x-2)(x+1)(x-4)$$

We also know that any suitable values must be a factor of 8 in this case.

b $f(x)=(x-2)(x+1)(x-4)=0$

$x\in\{-1,2,4\}$

Exercise 8.4

1 **a** Show that $(x - 1)$ is a factor of $f(x) = x^3 - x^2 - 4x + 4$.

b Factorise $f(x)$ completely.

c Solve the equation $f(x) = 0$.

2 **a** Show that $(x + 2)$ is a factor of $f(x) = 2x^3 + 11x^2 - 16x - 60$.

b Factorise $f(x)$ completely.

c Solve the equation $f(x) = 0$.

3 **(i)** Factorise each of these expressions completely.

(ii) Solve the equation $f(x) = 0$.

a	$f(x) = x^3 - 2x^2 - 5x + 6$	**b**	$f(x) = x^3 + 4x^2 - x - 4$	**c**	$f(x) = x^3 + 6x^2 + 3x - 10$
d	$f(x) = x^3 - x^2 - 8x + 12$	**e**	$f(x) = x^3 - 2x^2 - 11x + 12$	**f**	$f(x) = x^3 + 2x^2 - 4x - 8$
g	$f(x) = x^3 - 5x^2 - 4x + 20$	**h**	$f(x) = x^3 + 8x^2 + 17x + 10$	**i**	$f(x) = x^3 - 3x^2 - 10x + 24$
j	$f(x) = x^3 - 2x^2 - 9x + 18$	**k**	$f(x) = x^3 - 4x^2 - 3x + 18$	**l**	$f(x) = x^3 - 3x^2 - 10x + 24$

4 **(i)** Factorise each of these expressions completely.

(ii) Solve the equation $f(x) = 0$.

a	$f(x) = 2x^3 + 3x^2 - 3x - 2$	**b**	$f(x) = 3x^3 + 10x^2 - 9x - 4$	**c**	$f(x) = 2x^3 + 3x^2 - 5x - 6$
d	$f(x) = 2x^3 + 9x^2 + 10x + 3$	**e**	$f(x) = 3x^3 + 7x^2 - 7x - 3$	**f**	$f(x) = 2x^3 - 9x^2 + 13x - 6$
g	$f(x) = 2x^3 + 7x^2 - 5x - 4$	**h**	$f(x) = 3x^3 + 4x^2 - 5x - 2$	**i**	$f(x) = 2x^3 + x^2 - 12x + 9$
j	$f(x) = 2x^3 + x^2 - 8x - 4$	**k**	$f(x) = 3x^3 - 8x^2 + 3x + 2$	**l**	$f(x) = 2x^3 - 3x^2 - 5x + 6$

Summary

Polynomial division To divide a polynomial by a smaller one, for example, to divide $4x^3 + 3x^2 - 5x + 4$ by $x - 3$, follow the same rules as with long division with numbers:

		$4x^2 - 15x - 44$	
		$x - 3\,\overline{)\,4x^3 + 3x^2 - 5x + 4}$	
1	$4x^2 \times (x - 3)$:	$\underline{4x^3 - 12x^2}$ ↓ ↓	**1** How many times does x divide into $4x^3$?
	Subtract	$-15x^2 + x$ ↓	
2	$-15x \times (x - 3)$:	$\underline{-15x^2 + 45x}$	**2** How many times does x divide into $-15x^2$?
	Subtract	$-44x + 4$	
3	$-44 \times (x - 3)$:	$\underline{-44x + 132}$	**3** How many times does x divide into $-44x$?
	Subtract	-128	

So the quotient is $4x^2 - 15x - 44$ and the remainder is (-128).

The division algorithm $f(x) = D(x) \times Q(x) + R(x)$ The degree of the remainder must be less than the degree of the divisor.

The remainder theorem	If	$f(x) = (x - a) \times Q(x) + R,$
	then	$f(a) = R$

The remainder when $f(x)$ is divided by $(x - a)$ is $f(a)$.

The factor theorem	For any polynomial, $f(x)$,	
	If	$f(a) = 0$
	then	$(x - a)$ is a factor of $f(x)$
	and	$f(x) = (x - a) \times Q(x)$

Factorising polynomials

1 Search for a factor, using the Factor Theorem.
Try easy numbers, like 1, 2, 3, −1, −2, −3, then larger ones if necessary. Remember, there may not be any simple factors.

2 Divide out the factor to create a polynomial of smaller degree.

3 Repeat the process, or, if possible, factorise the new polynomial.

Chapter 8 Summative Exercise

1 Find the quotient and remainder when the given polynomial is divided by the expression in square brackets.

a	$x^3 - 4x^2 + 5x - 6$	$[x + 3]$	**b**	$2x^3 + x^2 - 7x + 9$	$[x - 4]$
c	$3x^3 - 5x^2 + 2x - 4$	$[x + 5]$	**d**	$4x^3 + 2x^2 - 5x + 3$	$[x - 2]$
e	$x^4 - 5x^3 + 3x^2 - 2x - 7$	$[x + 1]$	**f**	$3x^4 + 2x^3 - 4x^2 - x + 3$	$[x - 3]$
g	$3x^3 + x^2 - 5x + 4$	$[x^2 - 2x + 3]$	**h**	$4x^4 + 5x^3 - 6x^2 - 7x + 4$	$[x^2 + 3x - 2]$

2 Find the remainder when the polynomial $f(x)$ is divided by the expression in square brackets.

a	$f(x) = x^3 - 4x^2 + 5x - 6$	$[x - 2]$	**b**	$f(x) = 2x^3 + x^2 - 7x + 9$	$[2x - 1]$
c	$f(x) = 3x^3 - 5x^2 + 2x - 4$	$[x + 3]$	**d**	$f(x) = 4x^3 + 2x^2 - 5x + 3$	$[2x - 3]$
e	$f(x) = x^4 - 5x^3 + 3x^2 - 2x - 7$	$[x + 2]$	**f**	$f(x) = 3x^4 + 2x^3 - 4x^2 - x + 3$	$[x - 4]$
g	$f(x) = 3x^3 + x^2 - 5x + 4$	$[x - 1]$	**h**	$f(x) = 4x^4 + 5x^3 - 6x^2 - 7x + 4$	$[x + 1]$

3 When the cubic polynomial $f(x) = 4x^3 + ax^2 + x - 5$ is divided by $(x - 3)$, the remainder is 88. Find the value of a.

4 When the cubic polynomial $f(x) = 3x^3 + 5x^2 + ax + 1$ is divided by $(x + 2)$, the remainder is 9. Find the value of a.

5 When the cubic polynomial $f(x) = ax^3 - 6x^2 + 8x - 3$ is divided by $(x - 1)$, the remainder is 1. Find the value of a.

6 The remainder, when the cubic polynomial $f(x) = 4x^3 + ax^2 - 7x + 5$ is divided by $(x - 2)$ is 7 times the remainder when it is divided by $(x - 1)$.
Find the value of a and the remainder when it is divided by $(x + 2)$.

7 **a** Show that $(x-2)$ is a factor of the cubic polynomial $f(x) = x^3 + 3x^2 - 4x - 12$.

b Factorise $f(x)$ completely.

c Solve the equation $f(x) = 0$.

8 **a** Show that $(x+2)$ is a factor of the cubic polynomial $f(x) = 2x^3 - 3x^2 - 11x + 6$.

b Factorise $f(x)$ completely.

c Solve the equation $f(x) = 0$.

9 **a** Show that $(x-3)$ is a factor of the cubic polynomial $f(x) = 4x^3 - 12x^2 - 9x + 27$.

b Factorise $f(x)$ completely.

c Solve the equation $f(x) = 0$.

10 The cubic polynomial $f(x) = 2x^3 + x^2 - (a + a^2)x + a^2$ has a factor $(x-1)$.

a Find the value of a.

b Factorise $f(x)$ completely.

c Solve the equation $f(x) = 0$.

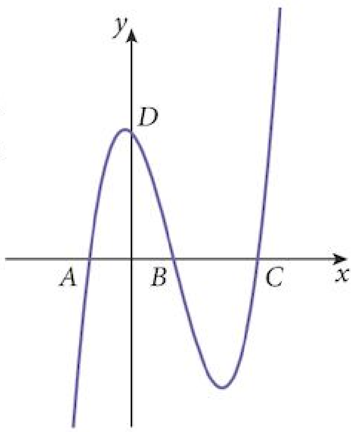
11 $f(x)$ is a cubic polynomial. One of its factors is x.
$f(x) - f(x-3) = 3x(3x-5)$ **[1]**

a Show that the remainder when $f(x)$ is divided by $(x-3)$ is 36.

b By letting $x = 0$, show that $(x+3)$ is a factor of $f(x)$.

c By writing $f(x) = x(x+3)(ax+b)$ and using **[1]**, find the value of a and the value of b.

12 The diagram shows the graph of the function
$y = x^3 + ax^2 + bx + c$, where a, b and c are integers.
The coordinates of A, B and C are $(-1, 0)$, $(1, 0)$ and $(3, 0)$ respectively.
Find the values of a, b and c and the coordinates of the point D, where the graph crosses the y-axis.



13 Find the coordinates of the point where the line $y = 3x - 5$ meets the curve $y = x^3 - 9x^2 + 26x - 20$.

14 **a** Show that $x = 2$ is a root of the equation $x^3 - 4x^2 + 3x + 2 = 0$.

b Hence solve the equation completely.

c What is the sum of the roots of the equation?

15 A cubic equation $f(x) = 0$ has roots α, β and γ.

  a Write the cubic expression $f(x)$ as a product of three linear factors.

  b Expand your expression.

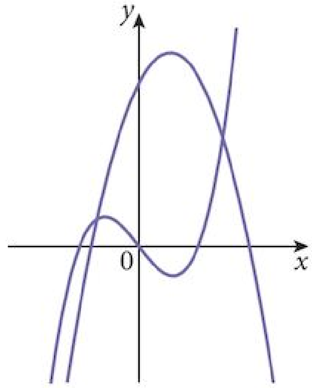
  c Where, in your expansion, does the sum of the roots $(\alpha + \beta + \gamma)$ appear?

  d Where, in your expansion, does the product of the roots $(\alpha\beta\gamma)$ appear?

  e What is the composition of the other coefficient in the expression for $f(x)$?

16 The diagram shows the graphs of the functions $y = f(x) = x^3 - 2x$ and $y = g(x) = 6 + 3x - 2x^2$.

By factorising the expression $f(x) - g(x)$ and using a sign diagram, solve the inequality $f(x) > g(x)$.



Chapter 8 Test

1 hour

1 **a** Show that $(3x - 1)$ is a factor of $f(x) = 3x^3 - x^2 + 3x - 1$. [1]

  b Hence show that $f(x)$ has only one real root. [3]

2 The cubic polynomial $f(x) = 6x^3 - 7x^2 - 11x + d$ has a factor $(x - 1)$.

  a Find the value of d. [2]

  b Find the remainder when the polynomial is divided by $(x + 2)$. [1]

3 The cubic polynomial $3x^3 + ax^2 + bx - 4$ has a remainder of -13 when divided by $(x + 1)$ and a remainder of 20 when divided by $(x - 2)$. Find the value of a and the value of b. [5]

4 Solve the equation $x^2(5 + x) = 2(2x^2 + 13x - 12)$. [6]

5 The cubic polynomial $x^3 - 3x^2 + ax + b$ has a factor $(x - 2)$.

  a Find a linear relationship between a and b. [1]

  b Find the remainder when the polynomial is divided by:

    (i) $(x - 1)$ [1]

    (ii) $(x - 3)$ [1]

  c Show that the sum of these remainders is 6. [1]

6 The cubic polynomial $f(x) = x^3 + ax^2 - 8x + b$ has a factor $(x - 2)$ and also a factor $(x + 3)$.

  a Find the value of a and the value of b. [5]

  b Find the remainder when $f(x)$ is divided by $(x + 1)$. [1]

  c Hence solve the equation $f(x) = 0$. [2]

7 The cubic polynomial $f(x) = x^3 + ax^2 + bx + 40$ has a factor $(x - 2)$ and leaves a remainder of -8 when divided by $(x - 3)$.

a Find the value of a and the value of b. [5]

b Factorise the polynomial completely. [3]

c Hence solve the equation $f(x) = 0$. [2]

Examination Questions

1 The cubic polynomial $f(x)$ is such that the coefficient of x^3 is -1 and the roots of the equation $f(x) = 0$ are 1, 2 and k. Given that $f(x)$ has a remainder of 8 when divided by $x - 3$, find

(i) the value of k,

(ii) the remainder when $f(x)$ is divided by $x + 3$. [6]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2002, P2, Qu 6]

2 Given that $4x^4 - 12x^3 - b^2x^2 - 7bx - 2$ is exactly divisible by $2x + b$,

(i) show that $3b^3 + 7b^2 - 4 = 0$, [2]

(ii) find the possible values of b. [5]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2003, P2, Qu 1]

3 The expression $x^3 + ax^2 + bx - 3$, where a and b are constants, has a factor of $x - 3$ and leaves a remainder of 15 when divided by $x + 2$. Find the value of a and of b. [5]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2003, P1, Qu 3]

4 Given that $6x^3 + 5ax - 12a$ leaves a remainder of -4 when divided by $x - a$, find the possible values of a. [7]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2004, P2, Qu 7]

5 The remainder when $2x^3 + 2x^2 - 13x + 12$ is divided by $x + a$ is three times the remainder when it is divided by $x - a$.

(i) Show that $2a^3 + a^2 - 13a + 6 = 0$. [3]

(ii) Solve this equation completely. [5]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2005, P1, Qu 10]

6 The function $f(x) = x^3 - 6x^2 + ax + b$, where a and b are constants, is exactly divisible by $x - 3$ and leaves a remainder of -55 when divided by $x + 2$.

(i) Find the value of a and of b. [4]

(ii) Solve the equation $f(x) = 0$. [4]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2005, P2, Qu 9]

7 The cubic polynomial $f(x)$ is such that the coefficient of x^3 is 1 and the roots of the equation $f(x) = 0$ are -2, $1 + \sqrt{3}$, and $1 - \sqrt{3}$.

(i) Express $f(x)$ as a cubic polynomial in x with integer coefficients. [3]

(ii) Find the remainder when $f(x)$ is divided by $x - 3$. [2]

(iii) Solve the equation $f(-x) = 0$. [2]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2006, P1, Qu 6]

8 The cubic polynomial $f(x)$ is such that the coefficient of x^3 is 1 and the roots of the equation $f(x) = 0$ are 1, k and k^2. It is given that $f(x)$ has a remainder of 7 when divided by $x - 2$.

(i) Show that $k^3 - 2k^2 - 2k - 3 = 0$. [3]

(ii) Hence find a value for k and show that there are no other real values of k which satisfy this equation. [5]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2006, P1, Qu 10]

9 **a** The remainder when the expression $x^3 - 11x^2 + kx - 30$ is divided by $x - 1$ is 4 times the remainder when this expression is divided by $x - 2$. Find the value of the constant k. [4]

b Solve the equation $x^3 - 4x^2 - 8x + 8 = 0$, expressing non-integer solutions in the form $a \pm \sqrt{b}$, where a and b are integers. [5]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2007, P1, Qu 8]

10 Solve the equation $x^2(2x + 3) = 17x - 12$. [6]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2008, P1, Qu 6]

11 A function f is such that $f(x) = ax^3 + bx^2 + 3x + 4$. When $f(x)$ is divided by $x - 1$, the remainder is 3. When $f(x)$ is divided by $2x + 1$, the remainder is 6. Find the value of a and of b. [5]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2008, P1, Qu 4]

12 Solve the equation $2x^3 + 3x^2 - 32x + 15 = 0$. [6]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2009, P1, Qu 6]

Term test 2A (Chapters 5–8)

1 hour

1 **a** Write the function $f(x) = 3x^2 - 12x + 16$ in completed square form $a(x - b)^2 + c$, stating the values of a, b and c. [3]

b Hence show that the equation $f(x) = 0$ has no real roots. [1]

2 A function f is defined as $f : x \mapsto ax^2 + bx$.

Given that $f(2) = 10$ and $f(3) = 27$, find the values of a and of b. [4]

3 **a** In ascending powers of x, write down the first four terms of the expansion of $(1 + 2x)^6$. [3]

b Find the coefficient of x^3 in the expansion of $(4 - 3x)(1 + 2x)^6$. [3]

4 The cubic polynomial $f(x) = 6x^3 + ax^2 + bx + 4$ has a factor $(x + 4)$ and leaves a remainder of -10 when divided by $(x - 1)$.

a Find the value of a and the value of b. [5]

b Factorise the polynomial completely. [3]

c Hence solve the equation $f(x) = 0$. [2]

5 Use the substitution $x = 2^u$ to find the values of u such that $2^{2u} + 32 = 3 \times 2^{2+u}$ [5]

6 Find the coordinates of the points A and B where the line $y = x + 2$ meets the curve $x^2 + y^2 = 20$. [5]

7 **a** The coefficient of x^3 in the expansion of $(3 + ax)^5$ is -3 times the coefficient of x^4.

Find the value of a. [3]

b The coefficient of x^2 in the expansion of $(3 + ax)^5$ is 405 times the coefficient of x^2 in the expansion of $\left(1 - \frac{2}{3}x\right)^n$, where a is the same value as in part **a**, and n is a positive integer.

Find the value of n. [3]

9 Straight lines

Syllabus statements

- interpret the equation of a straight line graph in the form $y = mx + c$
- solve questions involving mid-point and length of a line
- know and use the condition for two lines to be parallel or perpendicular

9.1 Introduction

In this chapter, we extend your knowledge of straight lines, finding their equations and applying simple coordinate geometry to solve problems. Much of this you will already have met.

9.2 Lines and line segments

Object	Length	End points	Sketching/graphing
Line	∞	none	
Half line	∞	1	
Line segment	finite	2	

These are all straight. You should not talk about a curved line.

Note that it is very common to use the word "line" when it is actually a "line segment" that is being referred to.

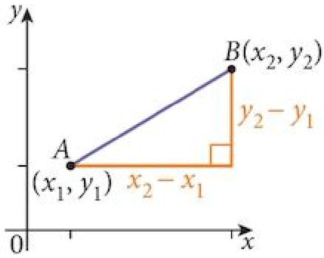
The phrase "the line *AB*" should, in most cases, be "the line segment *AB*".

9.3 Basic coordinate geometry

You should already be familiar with the following results.

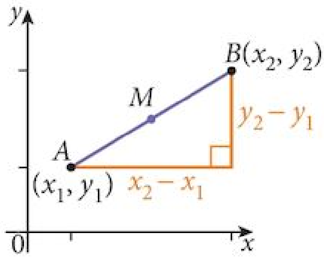
9.3.1 The distance between two points

$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$



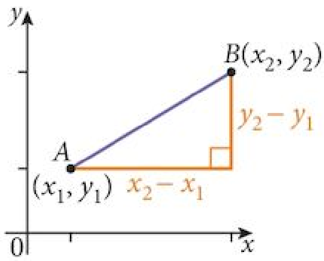
9.3.2 The mid-point of a line segment

The mid-point of AB is $M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$



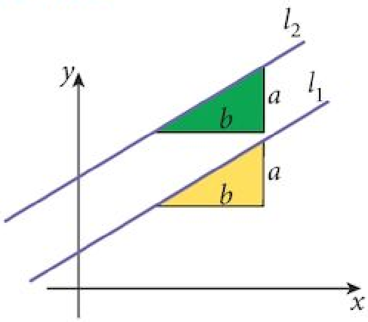
9.3.3 The gradient of a line segment

The gradient of AB is $m = \frac{y_2 - y_1}{x_2 - x_1}$



9.3.4 Parallel and perpendicular lines

If two lines are parallel,
they have the same gradient.

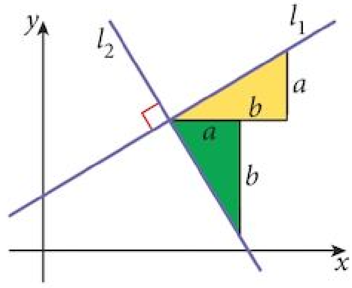


If two lines are perpendicular:

l_1 has gradient $\quad m_1 = \frac{a}{b}$

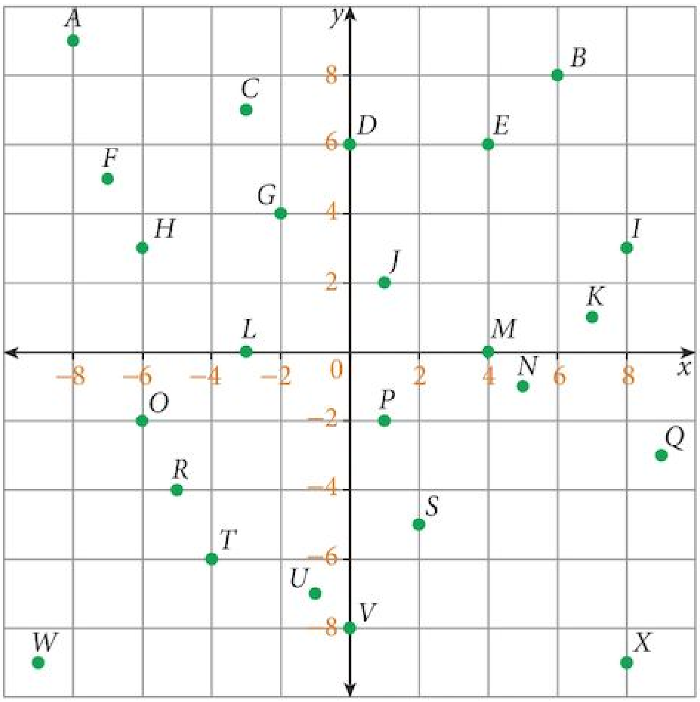
l_2 has gradient $\quad m_2 = -\frac{b}{a}$

$$m_1 \times m_2 = -1$$



Exercise 9.1

You are given the following collection of points. Questions **1** to **13** all relate to this diagram.



1 Find the length of each of these line segments.

a	*JE*	**b**	*CS*	**c**	*TM*	**d**	*DX*	**e**	*RK*
f	*OB*	**g**	*GM*	**h**	*LS*	**i**	*WR*	**j**	*FQ*

2 Find the mid-point of each of these line segments.

a	*AI*	**b**	*FC*	**c**	*HX*	**d**	*RK*	**e**	*OV*
f	*WQ*	**g**	*CK*	**h**	*TE*	**i**	*LU*	**j**	*TM*

3 Find the gradient of each of these line segments.

a	*MQ*	**b**	*OC*	**c**	*GV*	**d**	*FU*	**e**	*RJ*
f	*DR*	**g**	*MB*	**h**	*AH*	**i**	*NI*	**j**	*QT*
k	*SK*	**l**	*PS*	**m**	*HQ*	**n**	*HP*	**o**	*OG*

4 Identify the line segment from those in question **3** that is parallel to:

a IQ	**b** GN	**c** LR	**d** VS	**e** BJ
f CG	**g** NS	**h** HW	**i** GP	**j** CK
k KN	**l** AC	**m** KO		

5 Identify the line segment from those in question **3** that is perpendicular to:

a AE	**b** GK	**c** TV	**d** UP	**e** EI
f MU	**g** MN	**h** AG	**i** PR	**j** ST
k DW	**l** JL	**m** SX	**n** AR	

6 Show that $EINJ$ is a square and find its area.

7 Show that $CFLJ$ is a parallelogram and find the coordinates of the point of intersection of its diagonals.

Altitude: **noun.**

Geometry: the perpendicular distance from the vertex of a figure to the side opposite the vertex.

8 **a** Show that triangle FNU is isosceles.

b Find the coordinates of the mid-point of its shorter side.

c Find the altitude of the triangle and hence find its area.

9 **a** Show that triangle RVW is isosceles.

b Find the coordinates of the mid-point of the side that is not one of the equal pair.

c Find the altitude of the triangle and hence find its area.

10 **a** Show that triangle BCN is isosceles.

b Find the coordinates of the mid-point of the side that is not one of the equal pair.

c Find the altitude of the triangle and hence find its area.

11 **a** Find the coordinates of the point Y such that the quadrilateral $EQVY$ is a parallelogram.

b Show that the diagonals of $EQVY$ bisect each other.

c Show that the parallelogram is a square and find its area.

12 Find the coordinates of the point Y such that $YUSV$ is a rhombus.

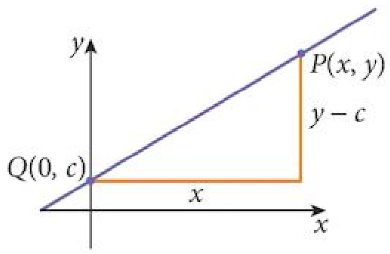
13 Find the coordinates of the points Y and Z such that $YNZU$ is a rhombus with sides of length $3\sqrt{10}$.

9.4 The equation of a line

We create an equation of a line by writing down two expressions for the gradient and then rearranging the equation. There are three basic forms of the equation.

9.4.1 The gradient-intercept form

Given a line that intercepts the y-axis at the point $Q(0, c)$, with gradient m and a general point, $P(x, y)$ lying on the line:



The gradient of the line, $m = \dfrac{y-c}{x}$

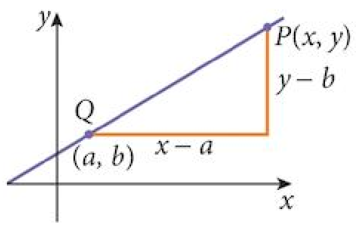
Rearranging gives $\boxed{y = mx + c}$

Remember, m is the gradient c is the y-intercept.

This is the most common form of the equation used in this course.

9.4.2 The gradient-point form

Given a line passing through the point $Q(a, b)$, with gradient m and a general point, $P(x, y)$ lying on the line:



The gradient of the line, $m = \dfrac{y-b}{x-a}$

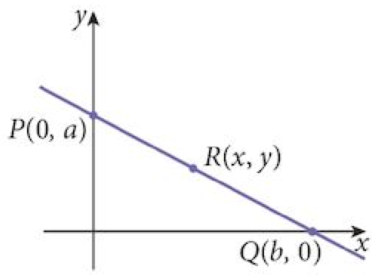
Rearranging gives $\boxed{(y-b) = m(x-a)}$

Remember, m is the gradient (a, b) is a point on the line.

This is the most useful form of the equation used in later courses.

9.4.3 The double intercept form

Given a line passing through the intercepts $P(0, a)$ and $Q(b, 0)$:



The gradient of the line, $\dfrac{y-a}{x} = -\dfrac{a}{b}$

Rearranging gives $\boxed{ax + by = ab}$

Each of these different forms can be converted to one of the others by rearranging. They are really different ways of writing the same thing.

Example 9.1

Find the gradient and the y-intercept of these lines:

a $3x + 4y = 12$

b $y - 3 = 2(x + 4)$

c $4x - 2y + 9 = 0$

Solution:

a $3x + 4y = 12$

Rearrange: $y = -\frac{3}{4}x + 3$

So: gradient $= -\frac{3}{4}$ y-intercept is $(0, 3)$

b $y - 3 = 2(x + 4)$

Rearrange: $y = 2x + 11$

So: gradient $= 2$ y-intercept is $(0, 11)$

c $4x - 2y + 9 = 0$

Rearrange: $y = 2x + 4.5$

So: gradient $= 2$ y-intercept is $(0, 4.5)$

Example 9.2

Find the equation of the perpendicular bisector of the line segment PQ where P is $(4, 2)$ and Q is $(8, -6)$.

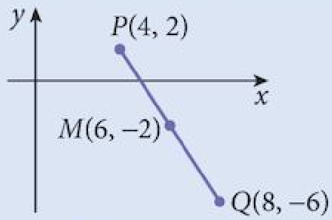
Solution:

The gradient of PQ is $\frac{-6-2}{8-4} = -2$

The gradient of the perpendicular is $-\frac{1}{-2} = \frac{1}{2}$

The mid-point is $M(6, -2)$.

The equation of the perpendicular bisector is $y + 2 = \frac{1}{2}(x - 6)$.



There is no need to simplify the equation if you are not asked to do so.

Example 9.3

a Find the points of intersection, P and Q, of the line $y = 2x - 3$ and the curve $y = x^2 - 2x - 8$.

b Find an equation of the perpendicular bisector of the line segment PQ in the form $ax + by + c = 0$.

c Find the points of intersection, R and S, of this perpendicular bisector with the curve.

Solution:

a Solve the equations $y = 2x - 3$

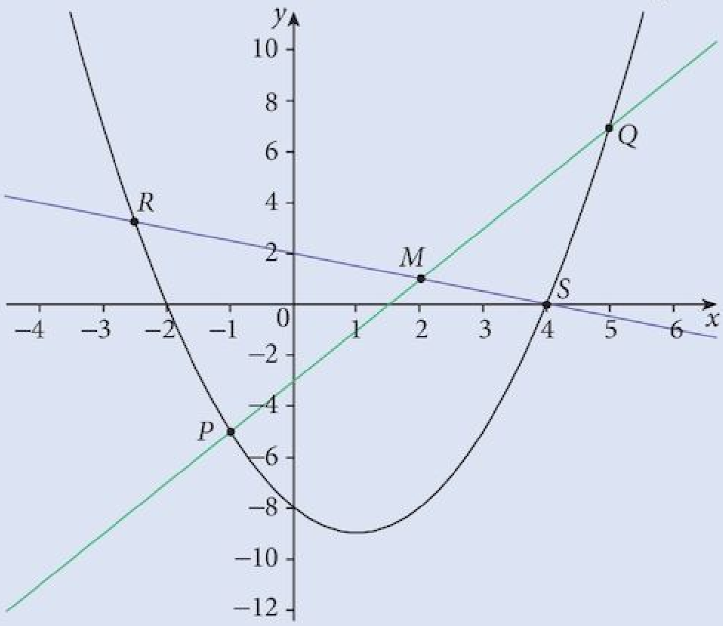
$$y = x^2 - 2x - 8$$

Gives $P(-1, -5)$ and $Q(5, 7)$

It does not matter which is P and which is Q.

b The gradient of PQ is $\frac{7+5}{5+1} = 2$

The gradient of the perpendicular is therefore $= -\frac{1}{2}$



Notice how the diagram is distorted by making the axis scales different.

The two lines are perpendicular but they do not look as if they are here. Only when plotted on axes with a 1 : 1 scale would they appear perpendicular.

When you draw diagrams, be aware of this.

The mid-point of PQ is $M(2, 1)$.

The equation of the perpendicular bisector is $y - 1 = -\frac{1}{2}(x - 2)$

You are asked for a specific format of the equation.

Rearranging: $x + 2y - 4 = 0$

c Solve the equations $x + 2y - 4 = 0$

$$y = x^2 - 2x - 8$$

Gives $R(-2.5, 3.25)$ and $S(4, 0)$

Exercise 9.2

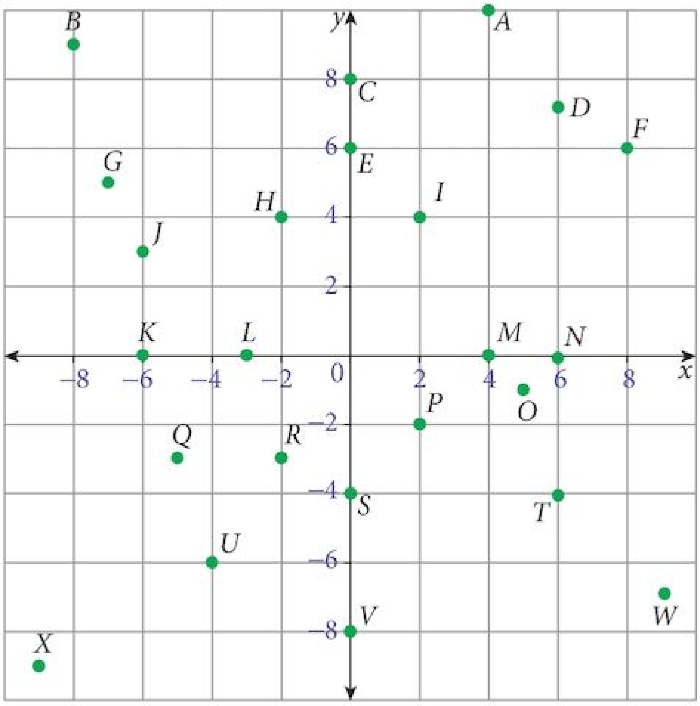
For this exercise, use the collection of points on the diagram below.

1 Find an equation of the line passing through the given points in the format $y = mx + c$.

a SM	**b** LH	**c** XW	**d** TX	**e** QI
f BI	**g** JR	**h** OQ	**i** CF	**j** AI

2 Find an equation of the line passing through the given points in the format $ax + by = c$.

a	*CN*	**b**	*LE*	**c**	*KV*	**d**	*SN*	**e**	*VM*
f	*KE*	**g**	*LV*	**h**	*CM*	**i**	*EN*	**j**	*KC*



3 Find an equation of the line passing through the given points in the format $y - y_1 = m(x - x_1)$ where (x_1, y_1) is the first named point.

a	*PV*	**b**	*UH*	**c**	*JQ*	**d**	*HS*	**e**	*RP*
f	*BG*	**g**	*IA*	**h**	*PD*	**i**	*OQ*	**j**	*JM*

4 Find an equation of the perpendicular bisector of the line segment given in the format $y - y_1 = m(x - x_1)$, where (x_1, y_1) is the mid-point of the line segment.

a	*KI*	**b**	*BD*	**c**	*LS*	**d**	*QO*	**e**	*UT*
f	*HM*	**g**	*EM*	**h**	*QW*	**i**	*GW*	**j**	*RD*

Summary

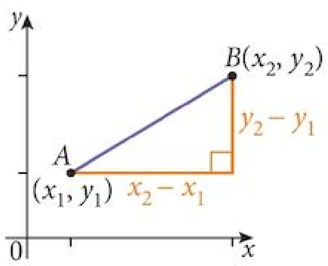
Coordinate geometry results — $A(x_1, y_1)$ and $B(x_2, y_2)$ are two points.

The distance between two points

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

The mid-point M of the line segment joining two points

$$M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

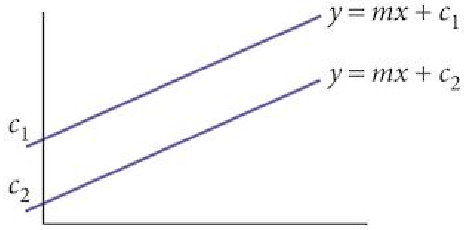


The gradient m of the line joining two points

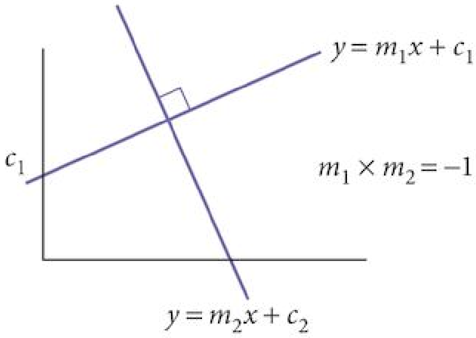
$$m = \left(\frac{y_2 - y_1}{x_1 - x_2}\right)$$

Parallel and perpendicular lines

Parallel lines have the same gradient.



Perpendicular lines:
If the gradients are m_1 and m_2, then $m_1 \times m_2 = -1$



The equations of a line:

Gradient–intercept form	$y = mx + c$	m is the gradient, c is the y-intercept.
Gradient–point form	$y - y_1 = m(x - x_1)$	m is the gradient, (x_1, y_1) is a point on the line.
Double intercept form	$ax + by = ab$	has intercepts $(b, 0)$ and $(0, a)$.

Chapter 9 Summative Exercise

Exercise 9.3

1 Find the distance between each pair of points.

a $A(-3, 9)$ and $B(3, 1)$ **b** $C(4, 4)$ and $D(-1, -8)$ **c** $E(-3, -4)$ and $F(5, 11)$

2 Find the mid-point of each line segment.

a $A(-7, 5)$ and $B(3, -7)$ **b** $C(-3, -10)$ and $D(5, 12)$ **c** $E(5, 9)$ and $F(-11, -13)$

3 Find the gradient of each line segment.

a $A(4, 6)$ and $B(-3, -8)$ **b** $C(8, -5)$ and $D(4, 7)$ **c** $E(-2, 10)$ and $F(10, 6)$

For questions **4** to **7**, use the points $A(-8, 8)$, $B(1, 8)$, $C(6, 7)$, $D(8, -6)$, $E(-6, -6)$, $F(0, -3)$ and $G(-6, 4)$.

4 a Find a pair of parallel line segments.

b Find a pair of perpendicular line segments.

c Find the point H such that $ACHG$ is a parallelogram.

5 Find the equations of these lines in the format $y = mx + c$.

a FC **b** BE **c** AG **d** BD

6 Find the equations of these lines in the format $y - y_1 = m(x - x_1)$.

a GD b AG c CE d BC

7 Find the equations of these lines in the format $ax + by = c$.

a FE b GB c AC d FD

8 The point A is $(-2, -5)$ and B is $(2, -1)$.

Find the equation of the perpendicular bisector of the line segment AB.

9 The point C is $(-3, 3)$ and D is $(5, 1)$.
Find the equation of the perpendicular bisector of the line segment CD.

10 Find the equation of the perpendicular bisector of the line segment joining the points where the line $2x + y = 5$ meets the curve $2y = x^2 - 4x + 6$.

11 Find the equation of the perpendicular bisector of the line segment joining the points where the line $y = 2x + 6$ meets the curve $xy = 8$.

12 $ABCD$ is a rhombus, where A is $(2, 5)$, B is $(12, p)$, C is $(4, -1)$ and D is (q, r).

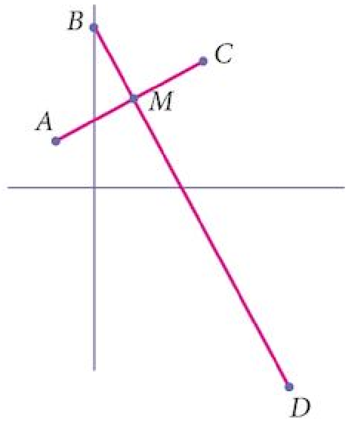
a Find the equation of the diagonal AC.

b Find the equation of the second diagonal BD.

c Find the values of the constants, p, q and r.

d Find the area of the rhombus $ABCD$.

13 The points, $A(-1, 2)$, $B(0, p)$, $C(3, 4)$ and $D(q, r)$ form a kite in which the diagonals meet at M.



a Find the coordinates of M.

b Find the equation of the diagonal BD.

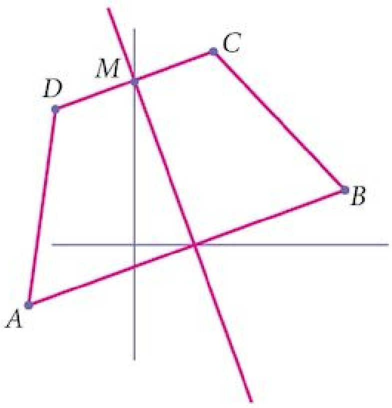
c Find the value of p.

The distance $MD = 3BM$.

d Find the value of q and the value of r.

e Find the area of the kite.

14 $ABCD$ is an isosceles trapezium in which A is $(-1, -3)$ and B is $(7, 3)$. Also, $2CD = AB$.



a Find the equation of the line of symmetry of the trapezium.

The point M is the mid-point of CD and lies on the y-axis.

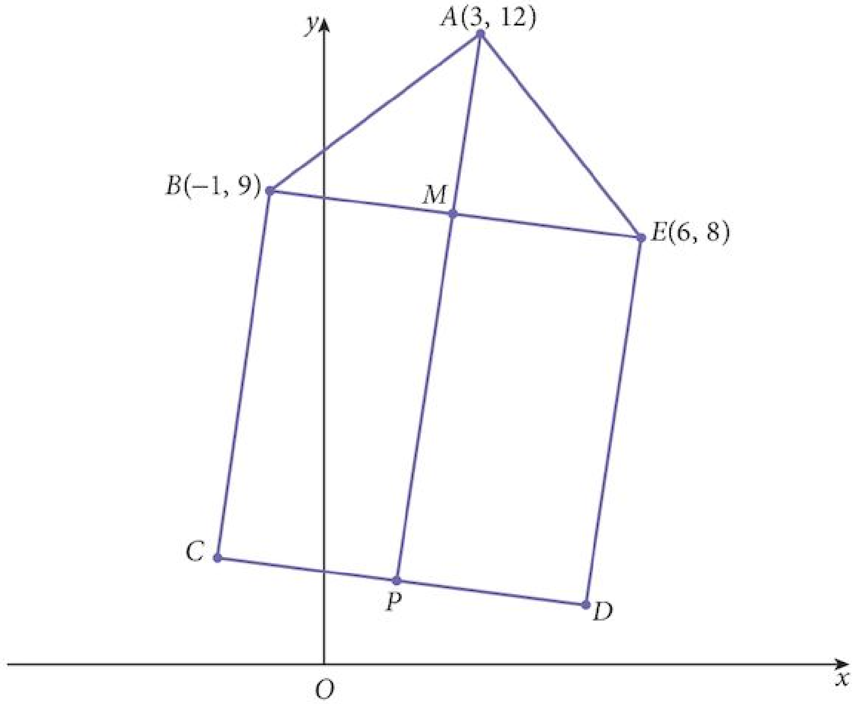
b Find the coordinates of M, C and D.

c Find the area of the trapezium.

Chapter 9 Test

1 hour

1 **Do not solve this question by creating an accurate diagram.**



In the diagram, the points $A(3, 12)$, $B(-1, 9)$, C, D and $E(6, 8)$ are the vertices of a pentagon. The line through AP is the perpendicular bisector of BE. M is the mid-point of BE and $MP = 2AM$. $BCDE$ is a rectangle.

a Find the equation of the line AP. [4]

b Find the coordinates of the point C and of the point D. [2]

c Find the area of the pentagon $ABCDE$. [4]

2 The line $y = x - 7$ intersects the curve $y = x^2 - 4x - 7$ at the points A and B.

a Find the distance AB. [6]

b Find the equation of the perpendicular bisector of AB. [4]

3 The points $A(0, 8)$, $B(8, 5)$ and $C(4, -1)$ form a triangle. M is the mid-point of BC, N is the mid-point of AC and P is the mid-point of AB.

a Find the coordinates of the points M, N and P. [3]

b Find the equation of the lines AM and BN. [4]

c Find the coordinates of the points of intersection of the lines AM and BN. [2]

d Show that this point also lies on the line CP. [1]

4 The line $2y = 3x + 6$ intersects the curve $xy = 12$ at points A and B. Find:

a the coordinates of A and of B [4]

b the distance AB, expressed in the form $a\sqrt{b}$, where a and b are integers [2]

c the equation of the perpendicular bisector of AB. [4]

Examination Questions

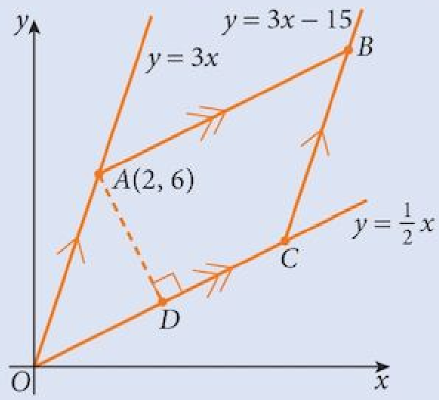
1 The line $2y = 3x - 6$ intersects the curve $xy = 12$ at the points P and Q.
Find the equation of the perpendicular bisector of PQ. [8]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2002, P1, Qu 9]

2 Find the distance between the points of intersection of the curve $y = 3 + \frac{4}{x}$ and the line $y = 4x + 9$. [6]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2003, P1, Qu 5]

3 **Solutions to this question by accurate drawing will not be accepted.**



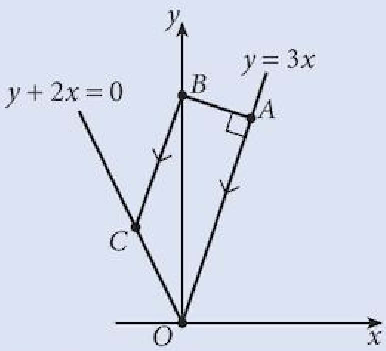
The diagram, which is not drawn to scale, shows a parallelogram $OABC$ where O is the origin and A is the point (2, 6). The equations of OA, OC and CB are $y = 3x$, $y = \frac{1}{2}x$ and $y = 3x - 15$ respectively. The perpendicular from A to OC meets OC at the point D. Find

(i) the coordinates of C, B and D. [8]

(ii) the perimeter of the parallelogram $OABC$, correct to 1 decimal place. [3]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2003, P1, Qu 11]

4



The diagram shows a trapezium $OABC$, where O is the origin. The equation of OA is $y = 3x$ and the equation of OC is $y + 2x = 0$. The line through A perpendicular to OA meets the y-axis at B, and BC is parallel to AO. Given that the length of OA is $\sqrt{250}$ units, calculate the coordinates of A, of B and of C. [10]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2004, P2, Qu 11]

5 The equation of a curve C is $2y = x^2 + 4$. The equation of the line L is $y = 3x - k$, where k is an integer.

(i) Find the largest value of the integer k for which L intersects the curve. [4]

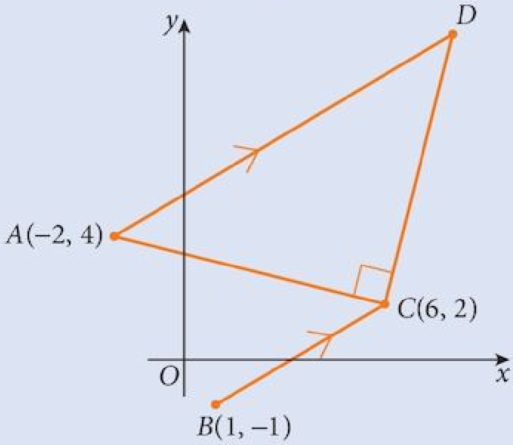
(ii) In the case where $k = -2$, show that the line joining the points of intersection of L and C is bisected by the line $y = 2x + 5$. [5]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2007, P2, Qu 8]

6 The straight line $5y + 2x = 1$ meets the curve $xy + 24 = 0$ at the points A and B. Find the length of AB, correct to one decimal place. [6]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2004, P1, Qu 5]

7 **Solutions to this question by accurate drawing will not be accepted.**



In the diagram the points A, B and C have coordinates $(-2, 4)$, $(1, -1)$ and $(6, 2)$ respectively. The line AD is parallel to BC and angle $ACD = 90°$.

(i) Find the equations of AD and CD. [6]

(ii) Find the coordinates of D. [2]

(iii) Show that the triangle ACD is isosceles. [2]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2004, P2, Qu 10]

8 The line $4y = 3x + 1$ intersects the curve $xy = 28x - 27y$ at the point $P(1, 1)$ and at the point Q. The perpendicular bisector of PQ intersects the line $y = 4x$ at the point R. Calculate the area of the triangle PQR. [9]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2004, P1, Qu 11]

9 The line $x + y = 10$ meets the curve $y^2 = 2x + 4$ at the points A and B. Find the coordinates of the mid-point of AB. [5]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2005, P2, Qu 2]

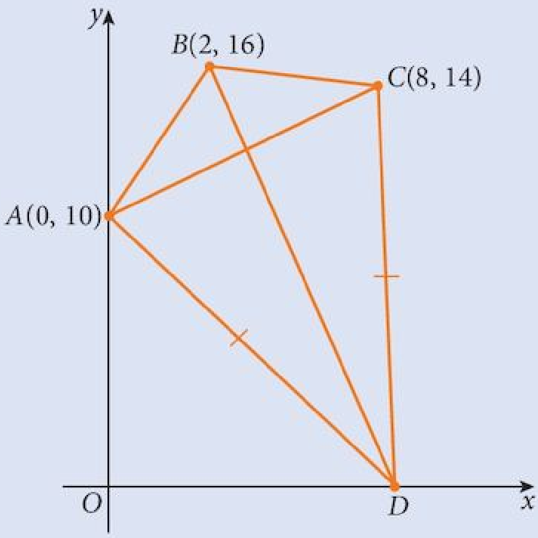
10 The line $y + 4x = 23$ intersects the curve $xy + x = 20$ at two points, A and B. Find the equation of the perpendicular bisector of the line AB. [6]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2006, P1, Qu 2]

11 The straight line $2x + y = 14$ intersects the curve $2x^2 - y^2 = 2xy - 6$ at the points A and B. Show that the length of AB is $24\sqrt{5}$ units. [7]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2006, P2, Qu 5]

12 **Solutions to this question by accurate drawing will not be accepted.**



The diagram, which is not drawn to scale, shows a quadrilateral $ABCD$ in which A is (0, 10), B is (2, 16) and C is (8, 14).

(i) Show that the triangle ABC is isosceles. [2]

The point D lies on the x-axis and is such that $AD = CD$. Find

(ii) the coordinates of D [2]

(iii) the ratio of the area of triangle ABC to the area of triangle ACD. [3]

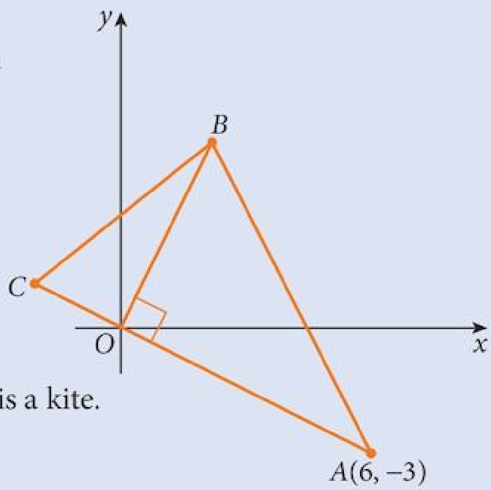
[Cambridge IGCSE Additional Mathematics 0606, Nov 2005, P1, Qu 10]

13 The straight line $3x = 2y + 18$ intersects the curve $2x^2 - 23x + 2y + 50 = 0$ at the points A and B. Given that A lies below the x-axis and that the point P lies on AB such that $AP : PB = 1 : 2$, find the coordinates of P. [6]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2007, P1, Qu 3]

14 Solutions to this question by accurate drawing will not be accepted.

The diagram shows a triangle ABC in which A is the point $(6, -3)$. The line AC passes through the origin O. The line OB is perpendicular to AC.



(i) Find the equation of OB. [2]

The area of triangle AOB is 15 units^2.

(ii) Find the coordinates of B. [3]

The length of AO is 3 times the length of OC.

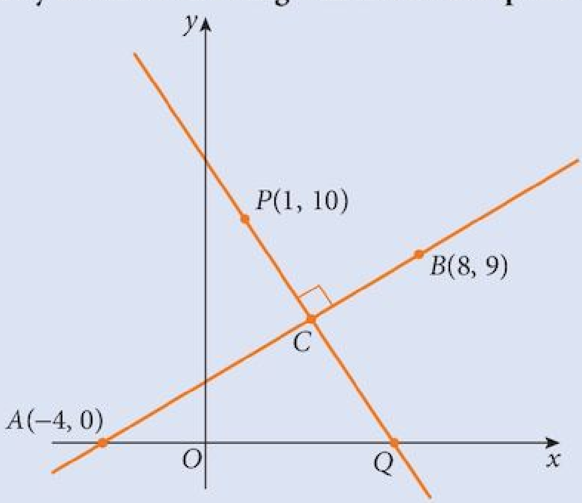
(iii) Find the coordinates of C. [2]

The point D is such that the quadrilateral $ABCD$ is a kite.

(iv) Find the area of $ABCD$. [2]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2007, P2, Qu 11]

15 Solutions to this question by accurate drawing will not be accepted.



The diagram shows a line AB passing through the points $A(-4, 0)$ and $B(8, 9)$. The line through the point $P(1, 10)$, perpendicular to AB, meets AB at C and the x-axis at Q. Find

(i) the coordinates of C and of Q [7]

(ii) the area of triangle ACQ. [2]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2009, P1, Qu 10]

10 The derived function

15 Differentiation and Integration

- understand the idea of a derived function
- use the notations $f'(x)$, $f''(x)$, $\frac{dy}{dx}$, $\frac{d^2y}{dx^2}$
- use the derivatives of the standard function x^n (for any rational n) together with constant multiples, sums and differences of these
- apply differentiation to gradients, tangents and normals

10.1 Introduction

The problem that you are going to solve, essentially, is “What do we get when we divide 0 by 0?” We usually say that this is undefined, but we can obtain sensible results by looking at the ratio $\frac{a}{b}$ where the two numbers a and b gradually get smaller and smaller and so create a sequence of values which, we hope, will lead somewhere.

We begin by looking at function diagrams. Similar results can be obtained, as you will see, by looking at the graphs of functions.

10.2 Function diagrams

Example 10.1

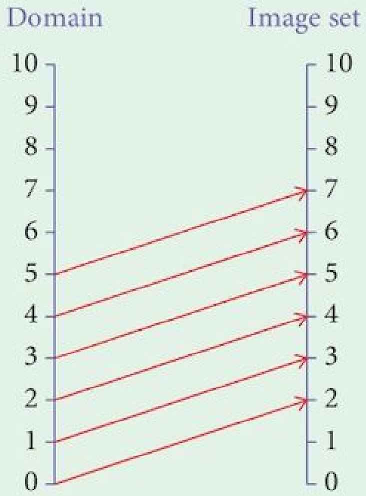
Draw the function diagram for the function $f(x) = x + 2, x \in \mathbb{R}$.

Solution:

Step 1: Create a table of values:

x	0	1	2	3	4	5
$f(x)$	2	3	4	5	6	7

Step 2: Draw the function diagram, connecting the values in the domain with their images.



Note that we could have extended the domain to include negative values. Also, we have drawn arrows only for integer values. The domain is the set of all real numbers but we cannot draw them all !

We will be interested in the effect that the function has on sections of the domain.

To do this, we take a section of the domain and the equivalent section of the image set.

We then find the ratio: $\dfrac{\text{image size}}{\text{object size}}$

The questions we can ask are: is this ratio constant?

if not, how does it vary with the size of the object set?

or, how does it vary with the position of the object set?

Example 10.2

You are given the function $f(x) = x + 2$.

a With a lower bound of $x = 2$, find the value of the ratio for an object set of size:

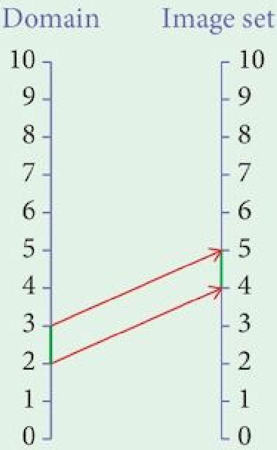
(i) 1 **(ii)** 2 **(iii)** 3

b With an object set of size 2, find the value of the ratio for an object set with a lower bound of:

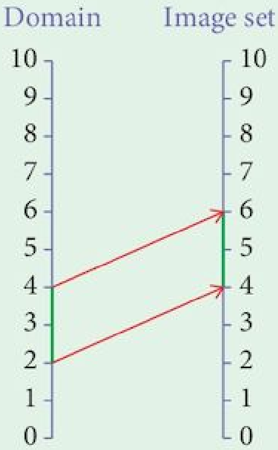
(i) 1 **(ii)** 2 **(iii)** 3

Solution:

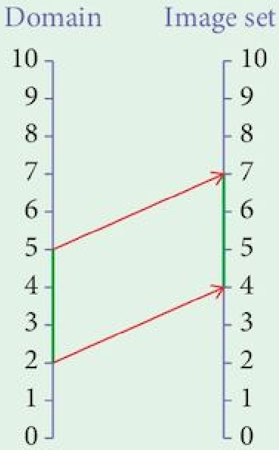
a **(i)**



(ii)

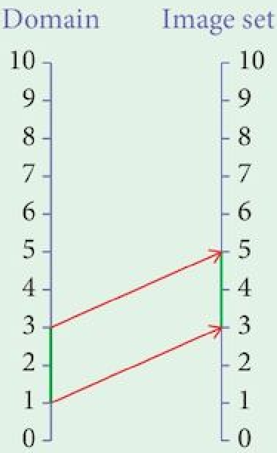


(iii)

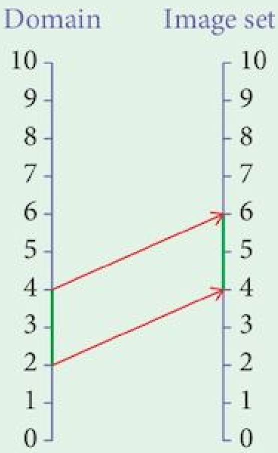


Ratio = $\dfrac{5-4}{3-2}=1$ $\quad\dfrac{6-4}{4-2}=1$ $\quad\dfrac{7-4}{5-2}=1$

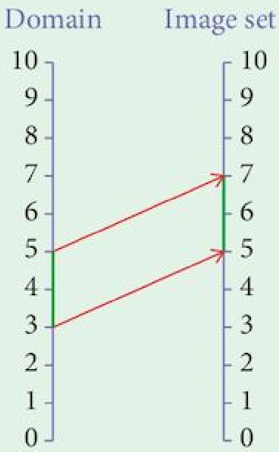
b **(i)**



(ii)



(iii)



Ratio = $\dfrac{5-3}{3-1}=1$ $\quad\dfrac{6-4}{4-2}=1$ $\quad\dfrac{7-5}{5-3}=1$

In each case, the ratio $\dfrac{\text{image size}}{\text{object size}}$ was equal to 1.

This was not very exciting but this function was a very simple one. We will meet more interesting functions later but first, try this exercise to make sure that you understand the process involved here. Most of these functions are fairly simple.

Exercise 10.1

For each of these functions, draw a function diagram and calculate the ratio $\dfrac{\text{image size}}{\text{object size}}$ for different combinations of (i) lower bound; (ii) size of object set.

1 $f(x) = x - 3$

2 $f(x) = 2x$

3 $f(x) = 3x + 1$

4 $f(x) = 4x - 1$

5 $f(x) = x^2$

6 Make a conjecture about the result you will get from each of the functions in questions **1**–**4**.

You will investigate the function in question **5** in the next section. The ultimate objective, following the work of Newton and Leibniz, is to see what happens when we make the size of the object set infinitesimally small.

10.3 The derived function

You are going to investigate the **derived function**.

This has the formula:

$$f'(x) = \underset{h \to 0}{\text{Lt}} \left[\frac{f(x+h) - f(x)}{h} \right]$$

Derived or **developed** from another function.

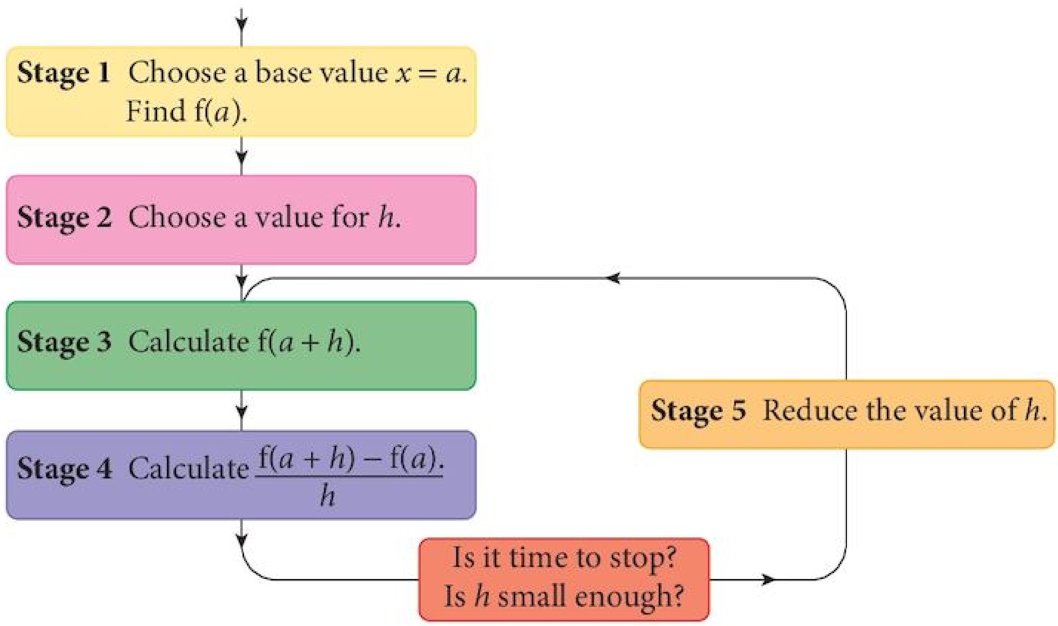
Although this may look intimidating, it is, in fact, another way of writing the ratio that you found in section 10.2.

We will use a well known problem-solving technique to investigate this formula.

Specialisation	Investigate specific values of x.
Generalisation	Try to find a pattern.
Conjecture	Guess a formula.
Convince	Prove the formula you guessed.

$$f'(a) = \underset{h \to 0}{\text{Lt}} \left[\frac{f(a+h) - f(a)}{h} \right]$$

We will tackle the formula in stages.



Problem 10.1

Using your calculator, find the smallest value of h that you can add to 2 that displays in full. Make a note of this value.

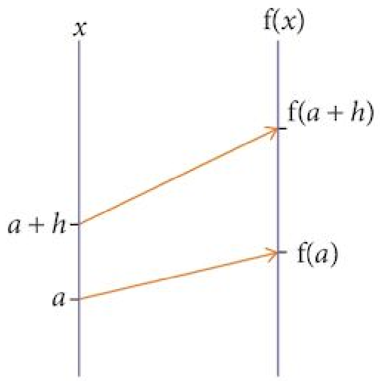
For my calculator, the smallest value of h that I can add is 0.000000001 (10^{-9}).

So, $2 + 10^{-9} = 2.000000001$

but $2 + 10^{-10} = 2$

This gives us a stopping point.

Sometimes, we will not even be able to make h this small.



The expression

$$\frac{\mathrm{f}(a+h)-\mathrm{f}(a)}{h}$$

is the ratio that you found in section 10.2.

10.4 The function $\mathrm{f}(x) = x^2$

Problem 10.2

Copy and complete this table for the function $\mathrm{f}(x) = x^2$.

x	x^2
6	
5	
4	
3	
2	
1	

You will already have made a table of some of these values in Exercise 10.1, question 5.

You are now ready to begin the investigation.

Specialise

Stage 1: Choose $a = 1$

Start simply and be systematic.

Problem 10.3

Copy and complete this table.

	$a = 1$	$\mathrm{f}(a) =$			
	Stage 2		**Stage 3**		**Stage 4**
	h	$a + h$	$\mathrm{f}(a + h)$	$\mathrm{f}(a + h) - \mathrm{f}(a)$	$\dfrac{\mathrm{f}(a + h) - \mathrm{f}(a)}{h}$
Stage 5	5				
	4				
	3				
	2				
	1				

You could set this up on a spreadsheet and save a lot of time.

Question: What do you think will happen if we let $h = 0$?

This is not possible, but we can extend the table with smaller values of h.

Stage 5	0.1				
	0.01				
	0.001				
	0.0001				
	0.00001				
	10^{-6}				
	10^{-7}				
	10^{-8}				
	10^{-9}				

Question: If we could make h smaller still, what do you think the result would be?

Problem 10.4

Repeat the exercise with different values of a.

Specialise Stage 1: Choose $a = 2$

Specialise Stage 1: Choose $a = 3$

Specialise Stage 1: Choose $a = 4$

Record your results in a copy of this table:

a	1	2	3	4
$f'(a)$				

Generalise Can you see a pattern?

Problem 10.5

Conjecture Can you guess a rule?

If $f(x) = x^2$, $f'(x) = \boxed{???}$

Convince Copy and complete these statements.

If $f(x) = x^2$

$$f(x+h) = \boxed{???}$$

$$f(x+h) - f(x) = \boxed{???}$$

$$\frac{f(x+h) - f(x)}{h} = \boxed{???}$$

Finally, what happens when $h \to 0$?

$$f'(x) = \boxed{???}$$

10.5 Higher indices

Problem 10.6

Repeat problems 10.2 to 10.5, but this time use the function $f(x) = x^3$.

Problem 10.7

Repeat problems 10.2 to 10.5, but this time use the function $f(x) = x^4$.

Problem 10.8

Copy and complete this table showing your results so far:

$f(x)$	$f'(x)$
x^2	
x^3	
x^4	

Generalise Can you see a pattern?

Problem 10.9

Conjecture Can you guess a rule?

If $f(x) = x^n$; $f'(x) = \boxed{???}$

Convince Copy and complete these statements:

If $f(x) = x^n$

$$f(x+h) = \boxed{???}$$

Expand this using the binomial theorem as far as the term in x^3.

$$f(x+h) - f(x) = \boxed{???}$$

$$\frac{f(x+h) - f(x)}{h} = \boxed{???}$$

Finally, what happens when $h \to 0$?

What happens to the terms in the series that you have not written down?

$$f'(x) = \boxed{???}$$

The project you have just completed, using powers of x, follows the pattern that we have to use for any function. You will meet more of them later.

You have found the **derived function**, also called the **derivative**, and the process of finding this for more complicated functions is called **differentiation**. You will meet these words a lot from now on.

Exercise 10.2

Write down the derived function in each of the following cases:

1 $f(x) = x^4$ **2** $f(x) = x^5$ **3** $f(x) = x^6$ **4** $f(x) = x^{10}$ **5** $f(x) = x^{100}$

10.6 More complicated functions

Question: What can we do to make functions more complicated?

There are various things that we can do but, for now, we will be interested in two ways:

1 Multiplying a function by a constant

2 Adding (or subtracting) one function to (from) another.

10.6.1 Multiplying by a constant: $g(x) = k \times f(x)$

From the definition of the derivative:

$$g'(x) = \underset{h \to 0}{\text{Lt}} \left[\frac{g(x+h) - g(x)}{h} \right]$$

But $\quad g(x) = k \times f(x)$

So $\quad g(x+h) = k \times f(x+h)$

and $\quad g(x+h) - g(x) = k \times f(x+h) - k \times f(x)$

$$= k\,[f(x+h) - f(x)]$$

$$g'(x) = \operatorname*{Lt}_{h \to 0}\left[\frac{k[f(x+h) - f(x)]}{h}\right]$$

$$g'(x) = k\operatorname*{Lt}_{h \to 0}\left[\frac{f(x+h) - f(x)}{h}\right]$$

$$\boxed{\mathbf{g'(x) = k \times f'(x)}}$$

k is independent of *h* so we can take it right outside.

The expression in blue is just the definition of f′(*x*).

Example 10.3

Find the derivative of the following functions:

a $f(x) = 3x^2$ **b** $g(x) = 7x^3$ **c** $h(x) = 8x^5$

Solution:

a $f(x) = 3x^2 = 3 \times x^2$

so $f'(x) = 3 \times (2x) = 6x$

b $g(x) = 7x^3 = 7 \times x^3$

so $g'(x) = 7 \times (3x^2) = 21x^2$

c $h(x) = 8x^5 = 8 \times x^5$

so $h'(x) = 8 \times (5x^4) = 40x^4$

10.6.2 Adding one function to another: m(*x*) = f(*x*) + g(*x*)

From the definition of the derivative:

$$m'(x) = \operatorname*{Lt}_{h \to 0}\left[\frac{m(x+h) - m(x)}{h}\right]$$

But $\quad m(x) = f(x) + g(x)$

So $\quad m(x+h) = f(x+h) + g(x+h)$

and $\quad m(x+h) - m(x) = [f(x+h) + g(x+h)] - [f(x) + g(x)]$

$$= [f(x+h) - f(x)] + [g(x+h) - g(x)]$$

$$m'(x) = \operatorname*{Lt}_{h \to 0}\left[\frac{[f(x+h) - f(x)] + [g(x+h) - g(x)]}{h}\right]$$

We can split the limit.

$$m'(x) = \operatorname*{Lt}_{h \to 0}\left[\frac{f(x+h)-f(x)}{h}\right] + \operatorname*{Lt}_{h \to 0}\left[\frac{g(x+h)-g(x)}{h}\right]$$

The expression in blue is just the definition of $f'(x)$.

The expression in magenta is just the definition of $g'(x)$.

This split works for limits but we haven't proved that it is okay.

$$\boxed{m'(x) = f'(x) + g'(x)}$$

Treat each function independently.

Similarly, if $\quad m(x) = f(x) - g(x)$

then $\quad m'(x) = f'(x) - g'(x)$

Example 10.4

Find the derivative of the following functions:

a $f(x) = x^2 + x^3$ **b** $g(x) = x^5 - x^7$ **c** $h(x) = 2x^3 + 4x^2$

Solution:

a $f(x) = x^2 + x^3$

so $f'(x) = 2x + 3x^2$

b $g(x) = x^5 - x^7$

so $g'(x) = 5x^4 - 7x^6$

c $h(x) = 2x^3 + 4x^2$

so $h'(x) = 2 \times (3x^2) + 4 \times (2x)$

$h'(x) = 6x^2 + 8x$

Using both the sum rule and the constant multiplier rule.

10.7 Other powers of *x*

We proved our results for positive integer powers of x using the binomial theorem. In fact, the binomial theorem also applies to other powers of x such as negative or rational ones.

The binomial theorem for these cases is beyond the scope of this syllabus but the results hold once we make adjustments.

As a consequence, we can find the derivative of any function made up of powers of x using the sum/difference rule and the constant multiplier rule.

However, ${}^{-3}C_2$ or ${}^{\frac{1}{2}}C_2$ do not make sense.

What would they mean?

Example 10.5

Find the derivative of the following functions:

a $f(x) = x^2 + \dfrac{1}{x^2}$ **b** $g(x) = 2x^3 + 4\sqrt{x}$ **c** $h(x) = \sqrt[3]{x} + \sqrt{x^3}$

Solution:

a

$$f(x) = x^2 + \frac{1}{x^2}$$
$$= x^2 + x^{-2}$$

So $f'(x) = 2x + (-2)\,x^{-3}$

$$f'(x) = 2x - \frac{2}{x^3}$$

b

$$g(x) = 2x^3 + 4\sqrt{x}$$

Write as powers: $g(x) = 2x^3 + 4x^{\frac{1}{2}}$

So $g'(x) = 2 \times (3x^2) + 4 \times \left(\frac{1}{2}x^{-\frac{1}{2}}\right)$

$$g'(x) = 6x^2 + \frac{2}{\sqrt{x}}$$

c

$$h(x) = \sqrt[3]{x} + \sqrt{x^3}$$

Write as powers: $h(x) = x^{\frac{1}{3}} + x^{\frac{3}{2}}$

So $h'(x) = \frac{1}{3}x^{-\frac{2}{3}} + \frac{3}{2}x^{\frac{1}{2}}$

$$h'(x) = \frac{1}{3\sqrt[3]{x^2}} + \frac{3\sqrt{x}}{2}$$

Exercise 10.3

1 Differentiate:

a 1 (Hint: write 1 as $1x^0$) **b** 4 (Hint: write 4 as $4x^0$)

2 Differentiate each of the following functions:

a $e(x) = x^4$ **b** $f(x) = x^6$ **c** $g(x) = x^9$ **d** $h(x) = x^{12}$

e $e(x) = \dfrac{1}{x}$ **f** $f(x) = \dfrac{1}{x^2}$ **g** $g(x) = \dfrac{1}{x^3}$ **h** $h(x) = \dfrac{1}{x^9}$

i $e(x) = \sqrt{x}$ **j** $f(x) = \sqrt[3]{x}$ **k** $g(x) = \sqrt[4]{x}$ **l** $h(x) = \sqrt[5]{x}$

m $e(x) = \sqrt{x^3}$ **n** $f(x) = \sqrt[3]{x^2}$ **o** $g(x) = \sqrt[4]{x^3}$ **p** $h(x) = \sqrt{x^5}$

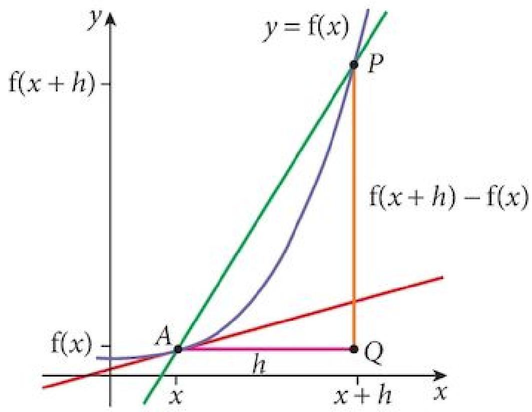
q $e(x) = \dfrac{1}{\sqrt{x}}$ **r** $f(x) = \dfrac{1}{\sqrt[3]{x}}$ **s** $g(x) = \dfrac{1}{\sqrt[4]{x^3}}$ **t** $h(x) = \dfrac{1}{\sqrt[5]{x^3}}$

3 Differentiate each of the following functions:

a $f(x) = x^2 + x^4$	**b** $g(x) = x^3 - \sqrt{x}$	**c** $h(x) = \sqrt{x} + \sqrt[3]{x}$
d $f(x) = 4x^2 - 3x^4$	**e** $g(x) = 3\sqrt{x} + 4x$	**f** $h(x) = 5x - 3\sqrt{x^3}$
g $f(x) = \frac{1}{x} + \frac{1}{x^2}$	**h** $g(x) = \frac{3}{x^2} - \frac{4}{x^3}$	**i** $h(x) = \frac{5}{x^7} + \frac{4}{x^5}$
j $f(x) = \frac{4}{\sqrt{x}} - \frac{3}{\sqrt[3]{x}}$	**k** $g(x) = \frac{3}{\sqrt[4]{x^3}} + \frac{2}{\sqrt[3]{x}}$	**l** $h(x) = \frac{10}{\sqrt[5]{x^2}} - \frac{8}{\sqrt[4]{x^3}}$

10.8 The gradient function

We can give another interpretation of the definition of the derivative which uses graphs.



Given is the graph of the function $y = f(x)$ and a point $A\ (x, f(x))$.

We choose a value for h and find the point $P\ (x + h, f(x + h))$.

Draw the line through A and P.

(This called a **secant** line).

From the triangle AQP, the gradient of AP is given by

$$\frac{f(x+h)-f(x)}{h}$$

This is exactly the same formula that we used for the derived function.

Notice that as A is fixed (as before) and we reduce the value of h, the point P will move down the curve and eventually end up at the point A.

The secant line AP will eventually become the tangent line, shown in **red**.

This means that the derived function will give us a formula for the **gradient function**.

However, when we do this we usually use a different notation.

Derived function	Gradient Function	Interpret as
h	δx	a small change in x
$f(x+h) - f(x)$	δy	a small change in y
$f'(x)$	$\frac{dy}{dx}$	differentiate y with respect to x

You have to get used to using the different types of notation.

Notice that dx does not mean $d \times x$.

Note that as $h \mapsto 0$, both δx and δy also $\mapsto 0$.

We say that $\underset{h \mapsto 0}{\mathrm{Lt}} \left[\frac{\delta y}{\delta x}\right] = \frac{dy}{dx}$

Example 10.6

Find the derivative of the following:

a $f(x) = x^2 + x^3$ **b** $y = x^2 + x^3$ **c** $x^2 + x^3$

Solution:

a
$$f(x) = x^2 + x^3$$
$$f'(x) = 2x + 3x^2$$

b
$$y = x^2 + x^3$$
$$\frac{dy}{dx} = 2x + 3x^2$$

c
$$x^2 + x^3$$
$$(x^2 + x^3)' = 2x + 3x^2$$

Notice the different styles of notation used here.

The mathematics is exactly the same but the notation is not. You must match your notation to that of the question.

In particular, notice that you cannot get $\frac{dy}{dx}$ if there is no "y" in the question.

Example 10.7

Find the gradient of the curve $y = x^2 - 3x + 2$ at the point where $x = 5$.

Solution:

The gradient of the curve is given by $\frac{dy}{dx}$.

Hence we differentiate the function.

$$y = x^2 - 3x + 2$$
$$\frac{dy}{dx} = 2x - 3$$

Substitute the value $x = 5$: $\frac{dy}{dx} = 2 \times 5 - 3 = 7$

Also, when $x = 5$, $y = 25 - 15 + 2 = 12$

So, at the point (5, 12), the curve has a gradient of 7.

Using function notation we would say that $f'(5) = 7$

Example 10.8

Find the coordinates of the points on the curve $y = x^3 - 3x^2 - 12x + 7$ where the gradient is 12.

Solution:

The gradient of the curve is given by $\dfrac{dy}{dx}$.

Hence we differentiate the function.

$$y = x^3 - 3x^2 - 12x + 7$$

$$\frac{dy}{dx} = 3x^2 - 6x - 12$$

Solve the equation: $3x^2 - 6x - 12 = 12$

giving $3x^2 - 6x - 24 = 0$

or $x^2 - 2x - 8 = 0$

whose solutions are $x = -2$ or $x = 4$

The corresponding y-coordinates are $y = 11$ and $y = -25$

Hence the points are $(-2, 11)$ and $(4, -25)$.

Example 10.9

At the point where $x = 2$, the function $y = 2x^3 + kx^2 - 8x + 4$ has a gradient of 28.

Find **a** the value of the constant k

b the coordinates of the point.

Solution:

a

The gradient of the curve is given by $\dfrac{dy}{dx}$.

Hence we differentiate the function.

$$y = 2x^3 + kx^2 - 8x + 4$$

$$\frac{dy}{dx} = 6x^2 + 2kx - 8$$

Substitue the value $x = 2$: $\dfrac{dy}{dx} = 24 + 4k - 8$

Solve the equation: $16 + 4k = 28$

giving $k = 3$

b

The y-coordinate is $y = 16 + 12 - 16 + 4$

$= 16$

So, the coordinates of the point are $(2, 16)$.

Example 10.10

A curve has equation $y = 2x^3 + ax^2 + bx - 6$, where a and b are constants. The gradient of the curve at the point (2, 6) is 16. Find

a the value of a and of b.

b the coordinates of the other point on the curve where the gradient is 16.

Solution:

a

We have two unknown constants and so we need to solve 2 simultaneous equations.

The first we can obtain from the point on the curve

When $x = 2$, $y = 6$. $\quad 6 = 2 \times (2)^3 + a \times (2)^2 + b \times (2) - 6$

giving $\quad 4a + 2b = -4$

or $\quad 2a + b = -2 \quad$ [1]

We can obtain the second equation from the gradient.

The gradient of the curve is given by $\frac{dy}{dx}$.

Hence we differentiate the function.

$$y = 2x^3 + ax^2 + bx - 6$$

$$\frac{dy}{dx} = 6x^2 + 2ax + b$$

Substitue the value $x = 2$ $\quad \frac{dy}{dx} = 24 + 4a + b = 16$

giving $\quad 4a + b = -8 \quad$ [2]

Finally, we solve the simultaneous equations [1] and [2].

to give $\quad a = -3$ and $b = 4$.

b

Substituting the values for a and b gives

$$\frac{dy}{dx} = 6x^2 - 6x + 4$$

Hence we need to solve the equation

$$16 = 6x^2 - 6x + 4$$

or $\quad x^2 - x - 2 = 0$

from $\quad$ which $x = -1$ or $x = 2$

When $x = -1$, $\quad y = -15$

So the other point is $\quad (-1, -15)$

Remember: we already know one solution.

Exercise 10.4

1 Differentiate the following functions.

a $x^3 - x^2$

b $f(x) = 3x^4 + 4x^3$

c $y = 2x^5 + 4x^4 - 6x^3$

d $3x^2 + \frac{4}{x}$

e $g(x) = \frac{2}{x^2} - \frac{3}{x^3}$

f $y = 4x^2 - \frac{4}{\sqrt{x}}$

g $2x^3 - \frac{6}{x^2}$

h $h(x) = \frac{3}{x} - \frac{6}{x^2}$

i $y = 6\sqrt{x} + \frac{8}{\sqrt{x}}$

j $9\sqrt[3]{x} - 4\sqrt{x}$

k $k(x) = 12\sqrt[4]{x^3} + 4\sqrt{x^3}$

l $y = 3x^2 + 4 - \frac{6}{x^3}$

2 The function $f(x) = ax^2 + 3$. It is known that $f'(4) = 16$. Find the value of a.

3 If $y = 3x^2 + 4x$, find the value of x for which $\frac{dy}{dx} = 22$.

4 Find the gradient of the curve $y = x^3 - 2x^2 + 3x - 2$ at the point where $x = 2$.

5 Find the points on the curve $y = x^3$ where the gradient is 12.

6 If $y = 2x^3 - \frac{k}{x^2}$, and $\frac{dy}{dx} = 14$ when $x = 1$, find the value of k.

7 The gradient of the curve $y = x^3 - ax^2 - 3x + 2$ at the point, A, where $x = 2$ is -7. Find the value of a and the y coordinate of the point A.

8 Find the coordinates of the two points on the curve $y = 2x^3 - 3x^2 - 32x + 6$ where the gradient is 4.

9 A curve has equation $y = ax^3 + bx^2 - 24x + 6$, where a and b are constants. The gradient of the curve at the point $(6, -30)$ is 48. Find

a the value of a and of b.

b the coordinates of the other point on the curve where the gradient is 48.

10 A curve has equation $y = ax^3 + \frac{b}{x}$, where a and b are constants. The curve passes through the point $(1, 1)$. The gradient when $x = -1$ is 15.

Find

a the value of a and of b.

b the coordinates of the other points on the curve where the gradient is 15.

10.9 A third interpretation of the derivative

We have been finding the instantaneous ratio of the change in the function value relative to a change in the input value. If the input values were measuring time, we could describe this as a "rate of change".

In fact, we use the same phrase when the input values are not time.

Thus $\frac{dy}{dx}$ represents a **rate of change** of y with respect to x.

If the rate of change $\left(\frac{dy}{dx}\right)$ is positive then the value of the function is increasing at that point.

If the rate of change $\left(\frac{dy}{dx}\right)$ is negative then the value of the function is decreasing at that point.

Later in the course, you will meet an application of this. The rate at which your position is changing is usually called your velocity. Also, the rate at which your velocity is changing is usually called your acceleration.

10.10 The second derivative

One application is given above in which we can find a rate of change and then find how fast that rate of change is changing.

Mathematically, the process is simple. We just keep differentiating.

Differentiating a function produces another function – the derivative.

When we differentiate this new function we get the **second derivative**.

This has several uses and the notation we use is:

Function	1st Derivative	2nd Derivative
$f(x)$	$f'(x)$	$f''(x)$
y	$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$

For higher derivatives we use $f^n(x)$ and $\frac{d^ny}{dx^n}$

The expression $\frac{dy}{dx}$ is sometimes thought of as $\frac{d}{dx}(y)$ with the operator $\frac{d}{dx}$ meaning "differentiate ( ) with respect to" x.

Then, $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ means "differentiate $\frac{dy}{dx}$ with respect to x". That gives us $\frac{d^2y}{dx^2}$.

Example 10.11

If $f(x) = 4x^3 - 3x^2 + 2x - 5$, find **a** $f'(x)$, **b** $f''(x)$.

Solution:

a $f(x) = 4x^3 - 3x^2 + 2x - 5$

$f'(x) = 12x^2 - 6x + 2$

b $f''(x) = 24x - 6$

Example 10.12

a If $f(x) = 4x^3 - 6x^2 + 8x - 9$, find $f''(x)$.

b If $y = \frac{2}{x} + 12\sqrt{x}$, find $\frac{d^2y}{dx^2}$.

Solution:

a $f(x) = 4x^3 - 6x^2 + 8x - 9$

Differentiate once: $f'(x) = 12x^2 - 12x + 8$

Differentiate again: $f''(x) = 24x - 12$

b $y = \frac{2}{x} + 12\sqrt{x}$

Differentiate once: $\frac{dy}{dx} = \frac{-2}{x^2} + \frac{6}{\sqrt{x}}$

Differentiate again: $\frac{d^2y}{dx^2} = \frac{4}{x^3} - \frac{3}{\sqrt{x^3}}$

Exercise 10.5

1 Find the second derivative of each of the following functions.

a $x^4 + x^3$

b $f(x) = 2x^5 + 3x^4 - 4x^2$

c $y = 3x^7 - 2x^6 + 5x^5$

d $2x^2 - \frac{1}{x}$

e $g(x) = \frac{3}{x^2} + \frac{2}{x^3}$

f $y = 3x^2 - \frac{8}{\sqrt{x}}$

g $4x^3 + \frac{2}{x^2}$

h $h(x) = \frac{2}{x} + \frac{1}{x^2}$

i $y = 12\sqrt{x} + \frac{8}{\sqrt{x}}$

j $18\sqrt[3]{x} - 8\sqrt{x}$

k $k(x) = 16\sqrt[4]{x^3} - 8\sqrt{x^3}$

l $y = 2x^2 + 5 - \frac{1}{x^3}$

10.11 Tangents and normals

We have seen that the derivative can be used to find the gradient of a curve. We can use this information to find the equation of the tangent and of the normal to the curve at that point.

Remember that tangents and normals are perpendicular to each other, so once you know the gradient of one of them, it is easy to find the gradient of the other using $m_1 m_2 = -1$.

Example 10.13

a Find the gradient of the curve $y = 2x^2 - 3x - 5$ at the point where $x = 3$.

b Hence find the equation of the tangent to the curve at this point.

Solution:

a $$y = 2x^2 - 3x - 5$$

So $$\frac{dy}{dx} = 4x - 3$$

When $x = 3$, $$\frac{dy}{dx} = 9$$

b Also, when $x = 3$, $y = 4$.

The equation of the tangent is

$$y - 4 = 9(x - 3)$$

You can write this in any required format.

Example 10.14

a Find the equation of the normal to the curve $y = x^2$ at the point where $x = 1$.

b Find the coordinates of the point where this normal cuts the curve again.

Solution:

a $$y = x^2$$

So $$\frac{dy}{dx} = 2x$$

When $x = 1$, $$\frac{dy}{dx} = 2$$

Also, when $x = 1$, $$y = 1$$

The gradient of the normal is $$-\frac{1}{2}$$

The equation of the normal is

$$y - 1 = -\frac{1}{2}(x - 1)$$

or $$2y = 3 - x$$

b The normal intersects the curve when

$$3 - x = 2x^2$$

or $$2x^2 + x - 3 = 0$$

$$(x - 1)(2x + 3) = 0$$

$$x = 1 \text{ or } x = -\frac{3}{2}$$

We know that $x = 1$ is a root, so the other root is $x = -\frac{3}{2}$.

So, the normal cuts the curve again at $\left(-\frac{3}{2}, \frac{9}{4}\right)$

Example 10.15

a Find the equation of the tangent to the curve $y = x^3 - 2x^2 - 5x + 6$ at the point where $x = -1$.

b Find the coordinates of the point where this tangent cuts the curve again.

Solution:

a
$$y = x^3 - 2x^2 - 5x + 6$$

So
$$\frac{dy}{dx} = 3x^2 - 4x - 5$$

When $x = -1$,
$$\frac{dy}{dx} = 2$$

Also, when $x = -1$,
$$y = 8$$

The equation of the tangent is
$$y - 8 = 2(x + 1)$$

or
$$y = 2x + 10$$

b The tangent intersects the curve when
$$2x + 10 = x^3 - 2x^2 - 5x + 6$$

or
$$x^3 - 2x^2 - 7x - 4 = 0$$

We need to solve this cubic equation.

However, we already know two of the roots.

There are two equal roots at the point where a tangent meets a curve.

So
$$x^3 - 2x^2 - 7x - 4 = (x + 1)(x + 1)(x - a)$$

By inspection,
$$a = 4$$

Check!
$$\begin{aligned}&(x + 1)(x + 1)(x - 4)\\ &= (x^2 + 2x + 1)(x - 4)\\ &= x^3 - 2x^2 - 7x - 4\end{aligned}$$

When $x = 4$,
$$y = 18$$

So, the tangent cuts the curve again at (4, 18).

Exercise 10.6

1 Find the equation of the tangent to the curve $y = x^2 - 3x + 4$ at the point where

  a $x = 2$

  b $x = -1$

2 Find the equation of the tangent to the curve $y = 8 - 2x - x^2$ at the point where

  a $x = 1$

  b $x = -2$

3 You are given the curve $y = 6 + 4x + x^2 - x^3$.

(i) Find the equation of the tangent to the curve.

(ii) Find the coordinates of the point where the tangent cuts the curve again for each of the points where

a $x = 1$

b $x = -1$

4 You are given the curve $xy = 12$.

a Find the equation of the tangent to the curve at the point where $x = 2$.

b Find the equation of the normal to the curve at this point.

c Find the coordinates of the point where this normal cuts the curve again.

5 A curve has the equation $y = x^2 + \frac{1}{x}$.

a Find the equation of the tangent to the curve at the point where $x = 1$.

b Find the coordinates of the point where this tangent cuts the curve again.

6 A curve has the equation $y = 4 - 3x + 4x^2 - x^3$.

a Find the equation of the tangent to the curve at the point where $x = 3$.

b Find the coordinates of the point where this tangent cuts the curve again.

7 You are given the curve $x^2y = 12$.

a Find the equation of the tangent to the curve at the point where $x = 4$.

b Find the coordinates of the point where this tangent cuts the curve again.

c Find the equation of the normal to the curve at the point where $x = 4$.

8 A curve has the equation $y = 2 + x - \frac{8}{x}$.

a Find the equation of the tangent to the curve at the point where $x = 2$.

b Find the equation of the normal to the curve at the same point.

c Find the coordinates of the point where this normal cuts the curve again.

9 A curve has the equation $y = 2x - \frac{4}{x}$.

a Find the equation of the normal to the curve at the point where $x = 2$.

b Find the coordinates of the point where this normal cuts the curve again.

Summary

The **derived function** (or **derivative**) is what we get when we **differentiate** a function.

$\frac{dy}{dx}$ means "differentiate y with respect to x".

The definition of the derivative	$f'(x) = \underset{h \mapsto 0}{\mathrm{Lt}}\left[\frac{f(x+h)-f(x)}{h}\right]$
Differentiating powers of x	If $f(x) = x^n$ Then $f'(x) = nx^{n-1}$
More complicated functions	Constant multiplier rule: If $f(x) = k \times g(x)$ Then $f'(x) = k \times g'(x)$ Sum (difference) rule: If $f(x) = g(x) \pm h(x)$ Then $f'(x) = g'(x) \pm h'(x)$
The gradient function	Algebraically, this is identical to the derived function. Thus we can use derivatives to investigate gradients
Higher derivatives	We can continue to differentiate as much as we like. 2nd derivative $\frac{d^2y}{dx^2}$ 3rd derivative $\frac{d^3y}{dx^3}$ and so on. Some functions eventually become zero if you differentiate far enough. Others get more complicated.
Tangents and normals	To find the equation of the tangent at the point where $x = a$. **a** Differentiate the function and find $f'(a)$. The gradient of the curve is the same as the gradient of the tangent. **b** Find $b = f(a)$. **c** The equation of the tangent is $(y - b) = f'(a) \times (x - a)$. **d** The equation of the normal is $(y - b) = \frac{-1}{f'(a)} \times (x - a)$.

Chapter 10 Summative Exercise

1 Differentiate each of the following functions:

a $f(x) = x^5$ **b** $f(x) = x^8$ **c** $f(x) = x^{11}$ **d** $f(x) = x^{15}$

e $y = \dfrac{1}{x^4}$ **f** $y = \dfrac{1}{x^5}$ **g** $y = \dfrac{1}{x^6}$ **h** $y = \dfrac{1}{x^{10}}$

2 Differentiate each of the following functions:

a $g(x) = x^6 - x^3$ **b** $g(x) = x^7 - \sqrt{x^3}$ **c** $g(x) = \sqrt{x} + \sqrt[3]{x}$

d $y = \dfrac{3}{x} + \dfrac{2}{x^5}$ **e** $y = \dfrac{7}{x^4} - \dfrac{6}{\sqrt[3]{x}}$ **f** $y = \dfrac{8}{\sqrt[4]{x}} - \dfrac{10}{\sqrt[5]{x}}$

3 Find the gradient of the curve $y = 2x^3 + 3x^2 - 8x - 3$ at the point where $x = 3$.

4 Find the gradient of the curve $y = x^2 + \dfrac{1}{x}$ at the point where $x = 2$.

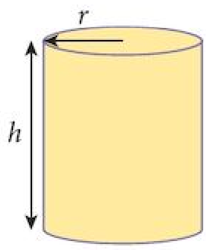
5 Find the x-coordinates of the points on the curve $y = x^3 - 3x^2 - 18x + 2$ where the gradient is 3.

6 Find the points on the curve $y = 2 - \dfrac{8}{x}$ where the gradient is 2.

7 An aluminium food container is constructed as a cylinder so that its height, h, is 3 times its radius, r.

a Find the surface area, S, of the can in terms of r, the radius of the cylinder.

b Find $\dfrac{dS}{dr}$.

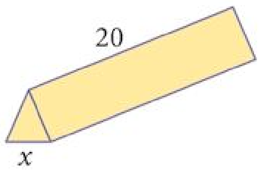


8 The volume of a sphere is given by the formula $V = \dfrac{4}{3}\pi r^3$.

Find $\dfrac{dV}{dr}$.

9 A triangular prism, whose cross section is an equilateral triangle of side x cm, and whose length is 20 cm, has a volume given by $V = 5\sqrt{3}x^2$.

Find $\dfrac{dV}{dx}$.



10 A curve has equation $y = 4x^2 - 6x - 8$.

a Find the equation of the tangent of the curve at the point where $x = 2$.

b Find the equation of the normal of the curve at the same point.

11 A curve, C_1, has equation $8y = x^2 + 16$.

A second curve, C_2, has equation $y = \frac{16}{x}$.

a Find the point of intersection of the two curves.

b Find the equation of the tangent of C_1 and of C_2 at their point of intersection.

c Show that, at the point of intersection, the two curves are perpendicular to each other.

A technical name for this is "orthogonal".

12 A curve has equation $y = 6x^3 + ax^2 + bx - 12$.

It passes through the point $(1, -4)$.

At the point where it crosses the y-axis, its gradient is -8.

a Find the value of a and the value of b.

b Find the coordinates of the other point where its gradient is -8.

13 The tangents to the curve $y = 15 - 2x - x^2$ at the points where $x = -3$ and $x = 2$ meet at the point P.

a Find the coordinates of the point P.

The normals to the curve at these same points meet at the point Q.

b Find the coordinates of the point Q.

14 A curve has equation $y = 2x^3 - 6x^2 - 4x + 10$.

a Find the equation of the tangent to the curve at the point where $x = 2$.

b Find the coordinates of the point where this tangent cuts the curve again.

c Find the equation of the normal to the curve at the point where $x = 2$.

d Find the area bounded by the normal, the x-axis and the y-axis.

Chapter 10 Test

1 hour

1 If $y = \sqrt{x} + \frac{1}{\sqrt{x}}$,

find the value of $\frac{\mathrm{d}y}{\mathrm{d}x}$ when $x = 4$. [3]

2 a Show that $\frac{2x^3 - 3}{x^2}$ can be written as $2x - \frac{3}{x^2}$. [1]

b If $y = \frac{2x^3 - 3}{x^2}$ find:

(i) $\frac{\mathrm{d}y}{\mathrm{d}x}$ [2]

(ii) $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ [2]

3 The equation of a curve is $y = x^3 - 7x + 6$.

a Find the gradient of the curve where it cuts the y-axis. [2]

b Find the coordinates of the points where it cuts the x-axis. [3]

c Find the gradient of the curve at the points where it cuts the x-axis. [1]

4 a Find the equation of the tangent to the curve $y = x^3 + 2x^2 - 4x + 1$ at the point, A, where $x = -1$. [4]

b Find the coordinates of the point, B, where the tangent intersects the curve again. [4]

c Find the equation of the perpendicular bisector of AB. [4]

5 A function, f, is such that $f(x) = x^3 + ax^2 + bx - 9$.

It is given that $(x + 3)$ is a factor of both $f(x)$ and $f'(x)$.

a Find the value of a and the value of b. [5]

b Find the remainder when $f(x)$ is divided by $(x - 3)$. [2]

6 a Sketch the curve $y = (3x + 2)(3 - 4x)$ for $-2 \leqslant x \leqslant 2$, stating the coordinates of the points where the graph cuts the axes. [3]

b By expanding the brackets, find the equation of the tangent to the curve at the point where x is the positive root of $(3x + 2)(3 - 4x) = 0$. [4]

11 Trigonometric functions

Syllabus statements

- know the trigonometric functions of angles of any magnitude
- solve simple trigonometric equations

11.1 Introduction

In this chapter you extend your knowledge and use of trigonometric functions. In order to do so, we have to re-define them so that we can talk about angles greater than 90°. In fact, we can talk about angles of any size, positive of negative.

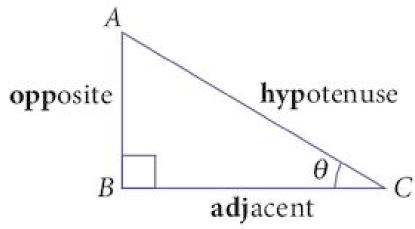
11.2 Trigonometric ratios

You will already be familiar with the following ratios:

$$\sin\theta = \frac{\text{opp}}{\text{hyp}}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}}$$



Pronounce them as:
sin: *sine*
cos: *coz*
tan: *tan*

These, along with Pythagoras' theorem, allow us to solve problems involving right-angled triangles.

One difficulty is that these definitions apply only to angles less than 90° ($\theta < 90°$).

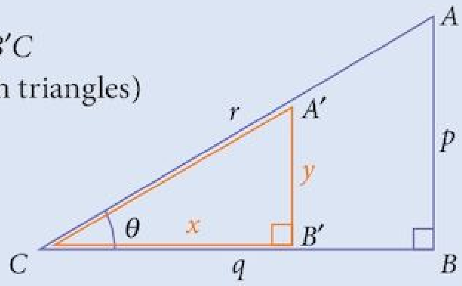
In order to move forward and consider angles of any size, we must change the definition while still allowing the ratios above to work.

Example 11.1

Given a right-angled triangle ABC where angle B is 90° and the lengths of AB, BC, and AC are p, q and r respectively, and angle $C = \theta$, what are the lengths of the similar triangle formed by dilating the triangle using a scale factor of $\frac{1}{r}$?

Solution:

Let the dilated triangle be $A'B'C$
(C is a common vertex in both triangles)



Let $CB' = x$

and $A'B' = y$

If the scale factor is $\frac{1}{r}$

then $A'C = 1$

and $x = \frac{1}{r} \times q = \frac{q}{r} = \cos\theta$

$y = \frac{1}{r} \times p = \frac{p}{r} = \sin\theta$

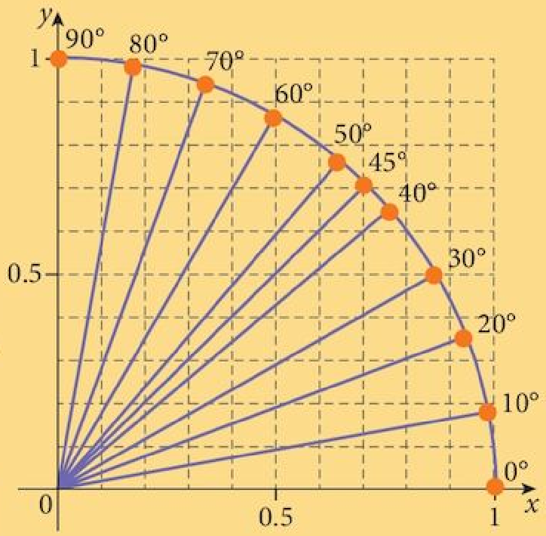
Dilation is the process of changing the size of objects using a scale factor.
You have probably used word "**enlargement**" but that is not really suitable if you are making them smaller!
When your pupils dilate, they can get larger or smaller.

So, if the hypotenuse of a triangle is 1, the other lengths will be $\cos\theta$ and $\sin\theta$.

This provides us with an alternative definition of sine and cosine.

11.3 Defining sine and cosine

Problem 11.1



1 On a large sheet of graph paper, draw a quadrant of a circle as shown opposite.

The centre of the circle should be at the origin.

Use a grid scale that will allow as much paper to be used as possible, while the scale goes from 0 to 1.

Draw radii on your graph every 10° and also at 45°.

2 Copy and complete the table of x- and y-coordinates of the points where the radii touch the circle.

Measure these correct to 2 decimal places.

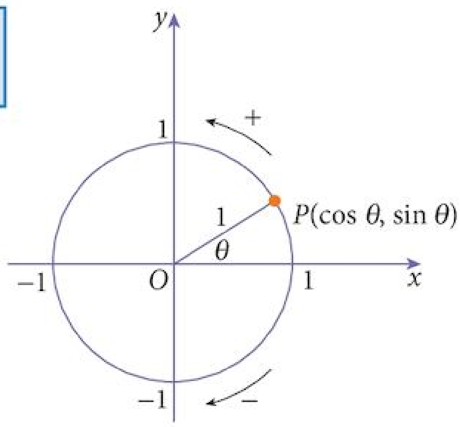
Angle	x	y
0°	1.00	0.00
10°		
20°		
…		
90°	0.00	1.00

3 When your table is complete, use your calculator to find the sine and cosine of the angles you measured in your table.

How accurate were your measurements?

We can now define sine and cosine as follows:

> The coordinates of a point P as it moves anticlockwise around a unit circle are $(\cos\theta, \sin\theta)$ where θ is the angle xPO.



Angle xOP is formed from the line segment OP and the positive x axis.

Notice that with this definition, we do not have to stop at 90°. We could go on forever.

We could even go in the opposite direction to investigate negative angles.

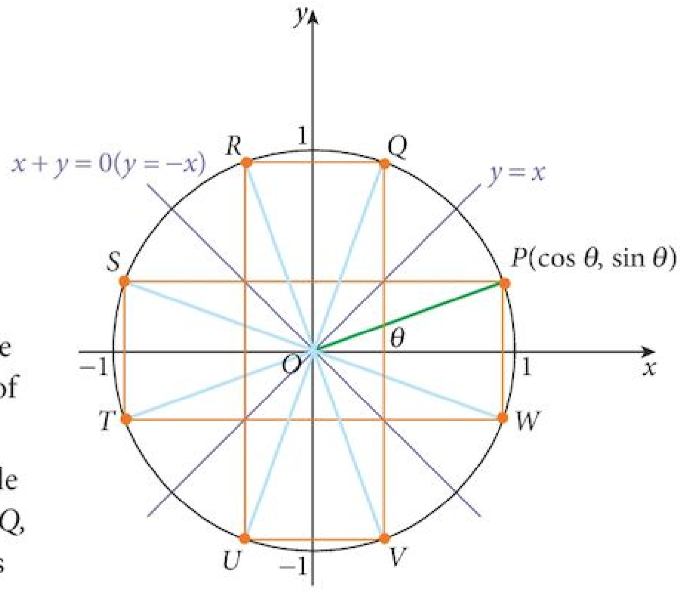
Remember: x-coordinate ≡ cosine; y-coordinate ≡ sine

11.4 Symmetries of the unit circle

Any diameter of a circle is a line of symmetry.

However, for our purposes, we will be interested in

1. the x-axis ($y = 0$)
2. the y-axis ($x = 0$)
3. the line $y = x$
4. the line $x + y = 0$ ($y = -x$)



The circle also has rotational symmetry of infinite order but we will be interested only in rotations of $\pm 90°$ and $180°$.

From the single point, P ($\cos\theta$, $\sin\theta$) where angle $xPO = \theta$, we can generate the seven other points Q, R, S, T, U, V, and W by reflection in the four lines mentioned, or by rotation.

By symmetry, the coordinates of these points will all consist of the same values, but not necessarily in the same order, and some will be negative values.

Problem 11.2

Copy and complete the table for each of the seven additional points.

In each case, find

a the angle made with the positive x-axis(ϕ)

b the cosine and sine of the angle in terms of $\cos\theta$ and/or $\sin\theta$.

Point	Angle made with the positive x-axis (ϕ)	$\cos\phi$	$\sin\phi$
P	θ	$\cos\theta$	$\sin\theta$
Q	$90° - \theta$		
R			
S			
T			
U			
V			
W	$-\theta$ or $360° - \theta$		

Your table contains many useful formulae such as

$\sin(-\theta) = \sin\theta$ $\qquad$ $\sin(90° - \theta) = \cos\theta$

$\cos(-\theta) = -\cos\theta$ $\qquad$ $\cos(90° - \theta) = \sin\theta$

How many more can you find?

For deeper understanding, it is often more useful to remember *how* to get the formulae than to simply memorise them.

Example 11.2

State two positive angles and two negative angles which will give identical radii in the unit circle as an angle of:

a $125°$ $\qquad$ **b** $255°$

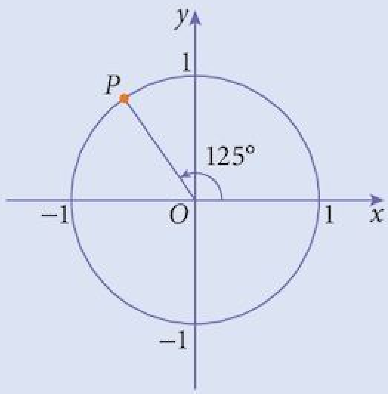
Solution:

a The point P, where angle $xOP = 125°$ is as shown.

Other angles which give an identical radius OP are found by adding or subtracting multiples of $360°$ to/from this.

Giving: $\quad 485°, 845°, \ldots$ or $-235°, -595°, \ldots$

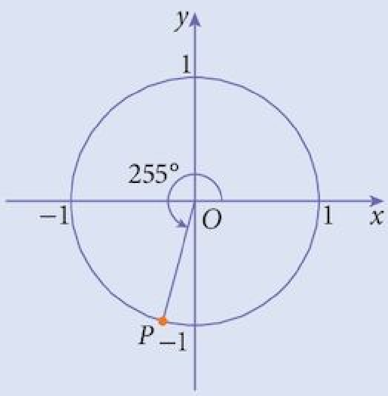
Angle xOP is formed from the line segment OP and the positive x axis.



b The point P, where angle $xOP = 255°$ is as shown.

Other angles which give an identical radius OP are found by adding or subtracting multiples of $360°$ to/from this.

Giving: $\quad 615°, 975°, \ldots$ or $-105°, -465°, \ldots$



Example 11.3

Express each of the following as the same function of an acute angle:

a cos 115° **b** sin 190° **c** cos 290°

Solution:

a Angle $xOP = 115°$

$\theta = 180° - 115°$

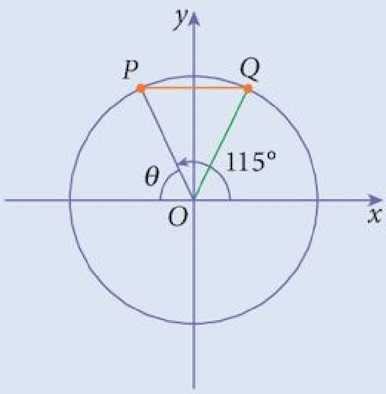
$= 65°$

Reflecting OP in the y-axis gives OQ.

By symmetry,

angle $xOQ = 65°$

Hence $\cos 115° = -\cos 65°$



b Angle $xOP = 190°$

$\theta = 190° - 180°$

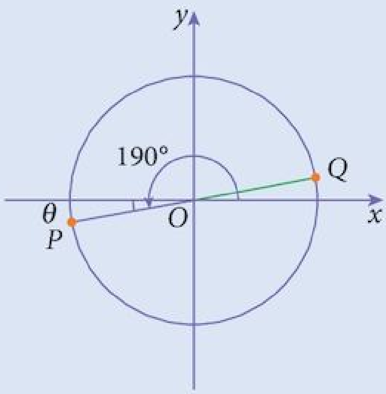
$= 10°$

Rotating OP through 180° about the origin gives OQ.

By symmetry,

angle $xOQ = 10°$

Hence $\sin 190° = -\sin 10°$



c Angle $xOP = 290°$ (the reflex angle)

$\theta = 360° - 290°$

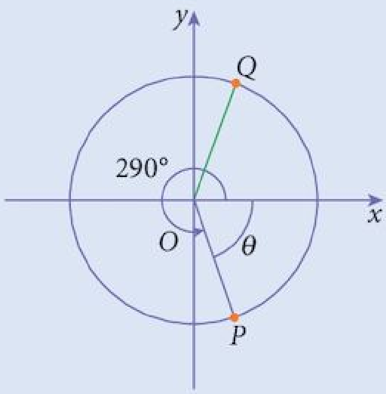
$= 70°$

Reflecting OP in the x-axis gives OQ.

By symmetry,

angle $xOQ = 70°$

Hence $\cos 290° = \cos 70°$



11.5 Graphs of the trigonometric functions

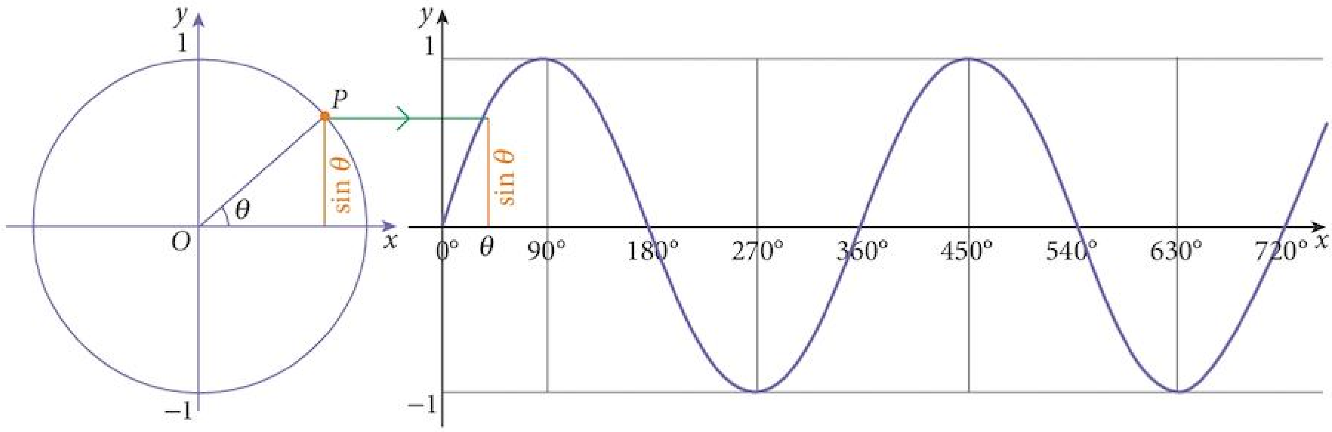
We use the unit circle to show how we get the graphs of sin θ and cos θ.

11.5.1 The graph of sin θ

Below is a unit circle next to the coordinate axes.

As the point P moves positively (anticlockwise) around the circle, we can transfer the y-coordinate which is the measurement for sin θ across to the graph. This produces the following result.

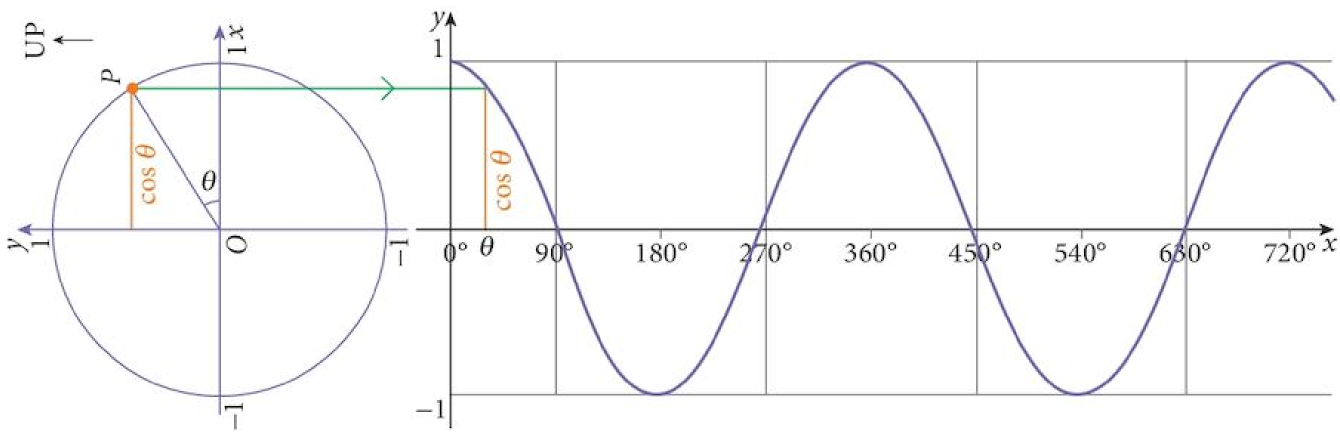
We start with sin θ because it is easier to deal with.



11.5.2 The graph of cos θ

The measurement of cosine is the x-coordinate. In order to graph this in the same way, we would need to go down the page instead of across, like we did with sin θ. To avoid this, we will turn the unit circle around.

As the point P moves positively around the circle, we can transfer the x-coordinate which is the measurement for cos θ down to the graph. This produces the following result.

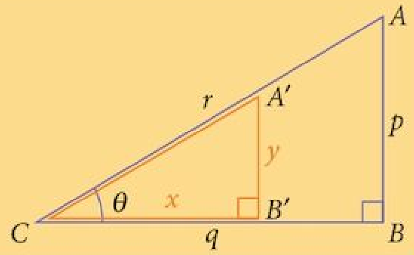


11.6 The tangent function and its graph

The third trigonometric function you have used is the tangent function.

Problem 11.3

Given a right-angled triangle ABC where angle B is 90° and the lengths of AB, BC, and AC are p, q and r respectively and angle $C = \theta$, what are the lengths of the similar triangle formed by dilating the triangle using a scale factor of $\frac{1}{q}$?



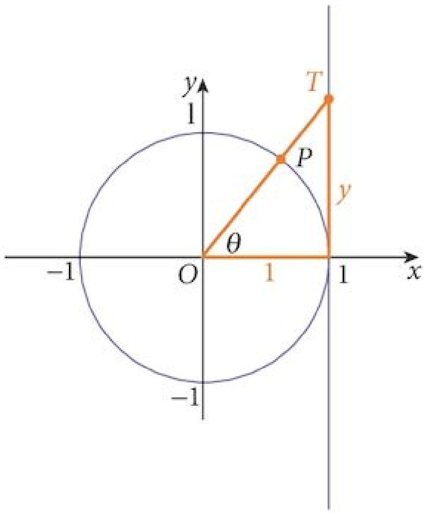
The results of Problem 11.3 are:

$$x = 1$$
$$y = \tan\theta$$
$$c = \frac{r}{q}$$

The length c is used later.

We can accommodate this in our unit circle diagram as follows:

The tangent measurement is made from the x-axis along a line that is a tangent to the unit circle at (1, 0). We measure to the point where the radius OP (extended) crosses the line.



The coordinates of the point T are $(1, \tan\,\theta)$.

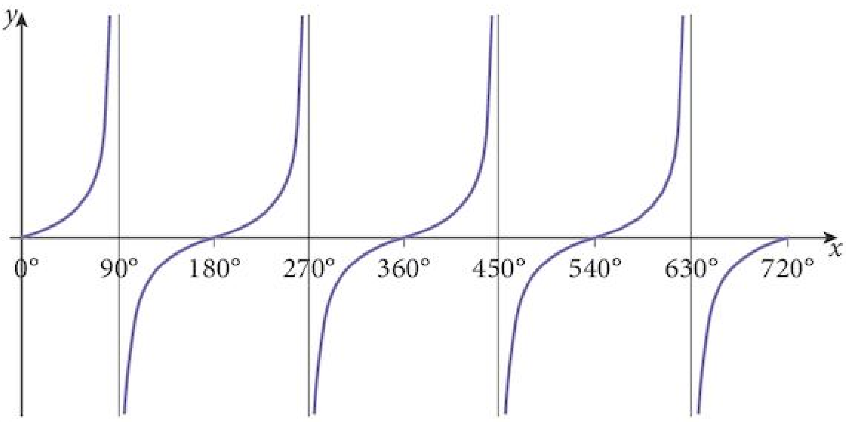
In fact, this is why we call it a tangent.

Note that, for an obtuse angle, the radius OP has to be extended beyond O, not P, in order to cross the tangent line.

Such a line would cross at a negative value.

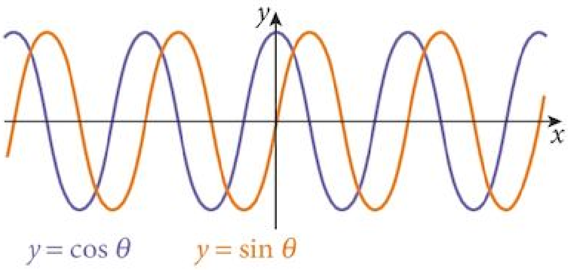
As the point P moves around the circle, the point T, giving the tangent value, will move up the line to infinity when $\theta = 90°$. When $\theta = 91°$, $\tan\,\theta$ is very negative and, as θ approaches 180°, the value of $\tan\,\theta$ will gradually rise to zero before the process is repeated as θ increases from 180° to 360°.

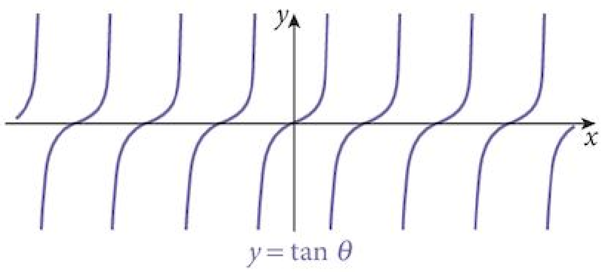
Thus, we get the following graph of $\tan\,\theta$.



11.7 Properties of the trigonometric functions

Extending the domains of the functions, we get the following graphs, extending in both directions:





Property	$y = \cos\theta$	$y = \sin\theta$	$y = \tan\theta$
Function	not 1 : 1	not 1 : 1	not 1 : 1
Inverse	none	none	none
Periodic, repeats after	360°	360°	180°
Amplitude	1	1	∞

There will be no inverse function unless we restrict the domain. We will deal with this later.

The sine and cosine graphs are the same shape but not in the same place.
(They are offset from one another.)

Exercise 11.1

1 For each angle, state two other positive angles and two negative angles which will give the same x and y coordinates in the unit circle.

a 30°	**b** 80°	**c** 120°	**d** 150°	**e** 180°
f 210°	**g** 255°	**h** 270°	**i** 310°	**j** 340°
k 420°	**l** 530°	**m** 630°	**n** 710°	**o** 1050°

2 Express each of the following as the same function of an acute angle.

a cos 120°	**b** sin 110°	**c** tan 135°	**d** cos 160°	**e** sin 150°
f tan 210°	**g** cos 225°	**h** sin 240°	**i** tan 255°	**j** cos 260°
k sin 280°	**l** tan 300°	**m** cos 310°	**n** sin 325°	**o** tan 330°

3 Express each of the following as the same function of an acute angle.

a cos (−15°)	**b** sin (−40°)	**c** tan (−65°)	**d** cos (−70°)	**e** sin (−85°)
f tan (−115°)	**g** cos (−130°)	**h** sin (−160°)	**i** tan (−165)	**j** cos (−170°)
k sin (−210°)	**l** tan (−240°)	**m** cos (−300°)	**n** sin (−320°)	**o** tan (−350°)

4 Express each of the following as the sine of an acute angle.

a cos 120°	**b** cos 135°	**c** cos 150°	**d** cos 160°	**e** cos 210°
f cos 225°	**g** cos 240°	**h** cos 255°	**i** cos 260°	**j** cos 280°
k cos 300°	**l** cos 315°	**m** cos 350°	**n** cos 430°	**o** cos 650°

5 As stated, the functions $y = \cos x$, $y = \sin x$ and $y = \tan x$ are not 1 : 1.

a In each case suggest a restricted domain, as large as possible, that is 1 : 1.

b Is it possible to have the same domain restriction in all cases?

c There are many possible answers to part **a**. Which do you think are most useful and why?

11.8 Inverse trigonometric functions

From question **5** of Exercise 11.1, you will realise that we have some difficulty with the inverse trigonometric functions. If we want to solve an equation such as $\sin \theta = 0.3$, we will need an inverse function. This is often written as $\sin^{-1} x$.

Careful! This is often confused with the reciprocal function $(\sin x)^{-1} = \dfrac{1}{\sin x}$

Sometimes **arcsin** x is used instead of $\sin^{-1} x$.

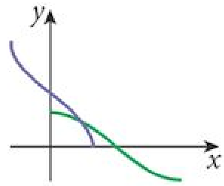
The domain of the function $y = \sin x$ is a set of angles.

The range is a set of numbers.

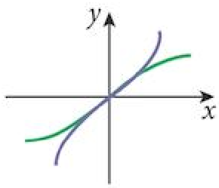
The domain of the inverse is the range of the function.
The range of the inverse is the domain of the function.

Function	$y = \cos \theta$	$y = \sin \theta$	$y = \tan \theta$
Restricted domain	$0° \leqslant \theta \leqslant 180°$ $[0°, 180°]$	$-90° \leqslant \theta \leqslant 90°$ $[-90°, 90°]$	$-90° < \theta < 90°$ $(-90°, 90°)$

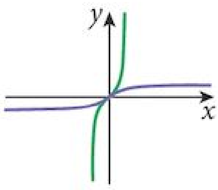
The graphs of the inverse functions are:



$y = \cos x$
$y = \cos^{-1} x$



$y = \sin x$
$y = \sin^{-1} x$



$y = \tan x$
$y = \tan^{-1} x$

Note that the range of the inverse functions are often called the **principal values**.

11.9 The periodic properties of trigonometric functions

You can see from the graphs of trigonometric functions that they are periodic.
The graphs are waves and so different angles give identical cosines, sines and tangents.
The basic graphs are repeated every 360°.

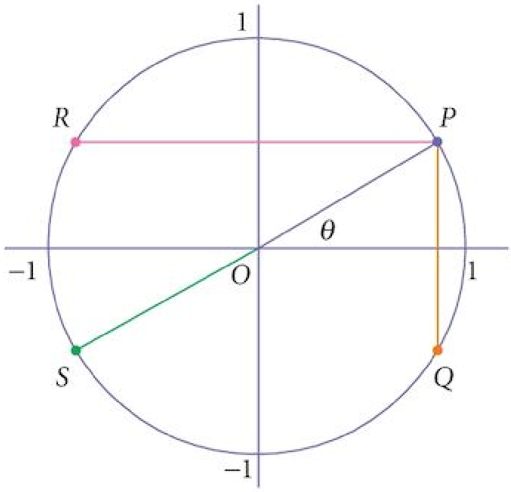
The reason for this is demonstrated by the unit circle.

The point P, whose radius from O makes an angle θ with the x-axis has:

the same cosine as point Q,

the same sine as point R,

the same tangent as point S.



Furthermore, the angle θ is not the only angle that would leave the point P in that position.

If we increase or decrease θ by 360°, or any multiple of 360°, the point P will travel completely around the circle a number of times and end up in exactly the same place, so it will have exactly the same cosine, sine and tangent as it has now.

The same applies also to the points Q, R and S.

We can use this property when solving trigonometric equations.

Example 11.4

Find two positive and two negative angles that have

a the same cosine as 70°

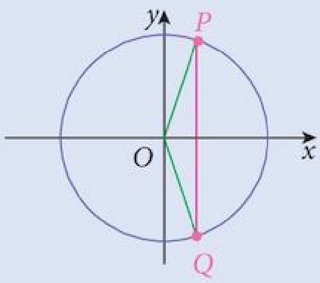
b the same tangent as −50°.

Solution:

a If the radius OP makes an angle of 70° with the x-axis, then other angles that will place the point in the same position are 430°, 790°, 1150°, . . . and −290°, −650°, −1010°, . . .

Point P has the same cosine as any angle that places the point in the same position as point Q.

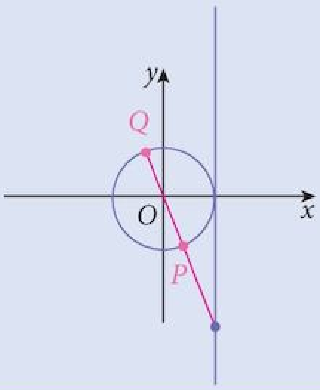
These angles are −70°, −430°, −790°, . . . and 290°, 650°, 1010°, . . .



b If the radius OP makes an angle of −50° with the x-axis, then other angles that will place the point in the same position are 310°, 670°, 1030°, . . . and −410°, −770°, −1130°, . . .

Point P has the same tangent as any angle that places the point in the same position as point Q.

These angles are 130°, 490°, 850°, . . . and −230°, −590°, −950°, . . .



11.10 Solving trigonometric equations

Example 11.5

Solve each of the following equations for $0° < \theta \leq 360°$.

a $\cos\theta = 0.3$ **b** $\sin\theta = -0.7$ **c** $\tan\theta = -2.5$

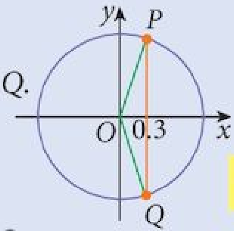
Solution:

a $\cos\theta = 0.3$

is equivalent to $\theta = \cos^{-1} 0.3$

Step 1: Draw the line $x = 0.3$

This intersects the unit circle at two points P and Q.



Cosines are measured by the x-coordinate.

Step 2: Find the principal value from your calculator.

$$\theta = 72.54°$$

This is the first value of P.

Step 3: Find the alternative values for θ at both P and Q.

P: $\{\ldots, -1007.46°, -647.46°, -287.46°, 72.54°, 432.54°, 792.54°, 1152.54°, \ldots\}$
Q: $\{\ldots, -1152.54°, -792.54°, -432.54°, -72.54°, 287.46°, 647.46°, 1007.46°, \ldots\}$

Step 4: Select the ones needed to satisfy the question.

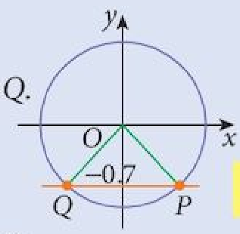
$\theta \in \{72.54°, 287.46°\}$

b $\sin\theta = -0.7$

is equivalent to $\theta = \sin^{-1}(-0.7)$

Step 1: Draw the line $y = -0.7$

This intersects the unit circle at two points P and Q.



Sines are measured by the y-coordinate.

Step 2: Find the principal value from your calculator.

$$\theta = -44.43°$$

This is the first value of P.

Step 3: Find the alternative values for θ at both P and Q.

P: $\{\ldots, -1124.43°, -764.43°, -404.43°, -44.43°, 315.57°, 675.57°, 1035.57°, \ldots\}$
Q: $\{\ldots, -855.57°, -495.57°, -135.57°, 224.43°, 584.43°, 944.43°, 1304.43°, \ldots\}$

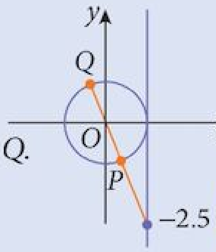
Step 4: Select the ones needed to satisfy the question.

c $\tan\theta = -2.5$

is equivalent to $\theta = \tan^{-1}(-2.5)$

Step 1: Draw the line from the -2.5 measurement through the origin.

This intersects the unit circle at two points P and Q.



Tangents are measured along the tangent line.

Step 2: Find the principal value from your calculator.

$$\theta = -68.20°$$

This is the first value of P.

Step 3: Find the alternative values for θ at both P and Q.

P: $\{\ldots, -1148.20°, -788.20°, -428.20°, -68.20°, 291.80°, 651.80°, 1011.80°, \ldots\}$
Q: $\{\ldots, -968.20°, -608.20°, -248.20°, 111.80°, 471.80°, 831.80°, 1191.80°, \ldots\}$

Step 4: Select the ones needed to satisfy the question.

$\theta \in \{111.80°, 291.80°\}$

Example 11.6

Solve the equation $\cos(3\theta - 60°) = 0.3$ for $-180° < \theta \leqslant 180°$ giving solutions correct to 1 d.p.

Solution:

We follow exactly the same steps 1–3 as in Example **11.5a**.

This gives us the following values for $(3\theta - 60°)$:

$$(3\theta - 60°) = \{\ldots, -1007.46°, -647.46°, -287.46°, 72.54°, 432.54°, 792.54°, 1152.54°, \ldots)$$

or $$(3\theta - 60°) = \{\ldots, -1152.54°, -792.54°, -432.54°, -72.54°, 287.46°, 647.46°, 1007.46°, \ldots\}$$

adding 60° to both sides:

$$3\theta = \{\ldots, -947.46°, -587.46°, -227.46°, 132.54°, 492.54°, 852.54°, 1212.54°, \ldots\}$$

or $$3\theta = \{\ldots, -1092.54°, -732.54°, -372.54°, -12.54°, 347.46°, 707.46°, 1067.46°, \ldots\}$$

dividing both sides by 3:

$$\theta = \{\ldots, -315.82°, -195.82°, -75.82°, 44.18°, 164.18°, 284.18°, 404.18°, \ldots\}$$

or $$\theta = \{\ldots, -364.18°, -244.18°, -124.18°, -4.18°, 115.82°, 235.82°, 355.82°, \ldots\}$$

Step 4: Select the solutions needed to satisfy the question.

$\theta \in \{-124.18°, -75.82°, -4.18°, 44.18°, 115.82°, 164.18°\}$

Note that while this might seem a cumbersome technique, it is possible to reduce the amount of work, but if you do not do it properly, you will lose some solutions. Be careful!

Example 11.7

Solve the equation $4 \sin x \cos x = \sin x$, for $0° \leqslant x \leqslant 360°$.

Solution:

$$4 \sin x \cos x = \sin x$$
$$4 \sin x \cos x - \sin x = 0$$
$$\sin x\,(4 \cos x - 1) = 0$$

giving $\sin x = 0$ or $\cos x = 0.25$

$x = 0°, 180°, 360°$ $\qquad$ $x = 75.5°, 284.5°$

so $x = 0°, 75.5°, 180°, 284.5°, 360°$

Note that you cannot divide both sides by $\sin x$.

Exercise 11.2

Answers to questions in this exercise should either be exact or, if approximate, given to 1 decimal place.

Make sure your calculator is in Degree mode.

1 Solve each equation for $-180° \leqslant \theta \leqslant 180°$.

a $\cos\theta = 0.25$ **b** $\sin\theta = 0.4$ **c** $\tan\theta = 0.7$ **d** $\cos\theta = 0.9$

e $\sin\theta = 0.85$ **f** $\tan\theta = 0.2$ **g** $\cos\theta = -0.3$ **h** $\sin\theta = -0.5$

i $\tan\theta = -2.0$ **j** $\cos\theta = -0.75$ **k** $\sin\theta = -0.9$ **l** $\tan\theta = -4.5$

m $\cos\theta = 0$ **n** $\sin\theta = -1$ **o** $\tan\theta = 6$

2 Solve each equation for $0° \leqslant \theta \leqslant 360°$.

a $\cos 2\theta = 0.4$ **b** $\sin 3\theta = 0.7$ **c** $\tan 4\theta = 3.5$ **d** $\cos 3\theta = 0.25$

e $\sin 4\theta = 0.45$ **f** $\tan 2\theta = 0.2$ **g** $\cos 4\theta = -0.6$ **h** $\sin 2\theta = -0.8$

i $\tan 3\theta = -3.0$ **j** $\cos\frac{\theta}{2} = 0.75$ **k** $\sin\frac{\theta}{3} = 0.7$ **l** $\tan\frac{\theta}{2} = -0.5$

m $\cos(-\theta) = 0.2$ **n** $\sin(-\theta) = -0.2$ **o** $\tan(-\theta) = 3$

3 Solve each equation for $0° \leqslant \theta \leqslant 360°$.

a $\cos(2\theta + 30°) = 0.25$ **b** $\sin(3\theta - 40°) = 0.2$ **c** $\tan(4\theta - 50°) = 1.5$

d $\cos(3\theta - 60°) = -0.5$ **e** $\sin(4\theta + 100°) = -0.4$ **f** $\tan(2\theta + 20°) = -1.6$

g $\cos(20° - 2\theta) = 0.3$ **h** $\sin(45° - 2\theta) = -0.5$ **i** $\tan\left(60° - \frac{\theta}{2}\right) = -0.7$

4 Solve each equation for $-180° \leqslant \theta \leqslant 180°$.

a $2\cos^2\theta + \cos\theta = 0$ **b** $3\sin^2\theta = \sin\theta$ **c** $\tan^2\theta - 2\tan\theta = 0$

d $6\cos^2\theta - \cos\theta - 2 = 0$ **e** $2\sin^2\theta + \sin\theta - 1 = 0$ **f** $6\tan^2\theta - 13\tan\theta + 6 = 0$

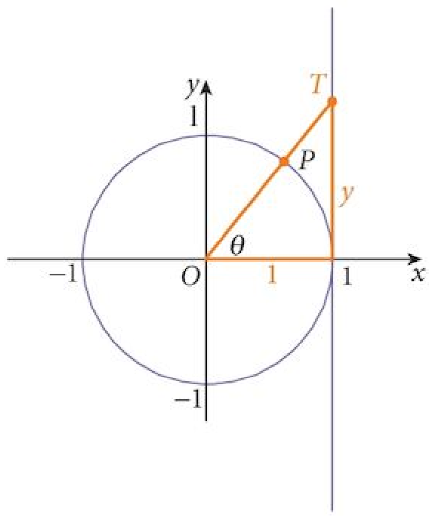
Summary

Definitions

As a point moves around a unit circle in an anticlockwise direction starting from (1, 0), the coordinates of the point P, for an angle of rotation θ, are $(\cos\theta, \sin\theta)$.

The coordinates of the point T, where the radius, extended, crosses the tangent at (1, 0) are $(1, \tan\theta)$.

It does not matter how large the angle θ is. It could even be negative.



Inverse trigonometric functions

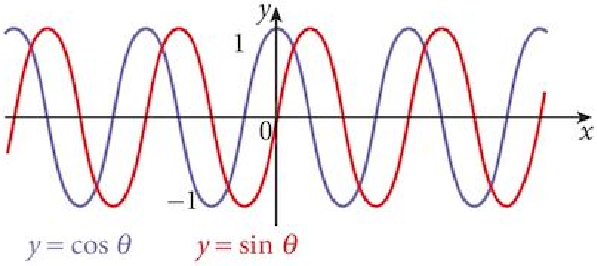
Trigonometric functions are not 1 : 1, so in order to find angles, we have to restrict the domain.

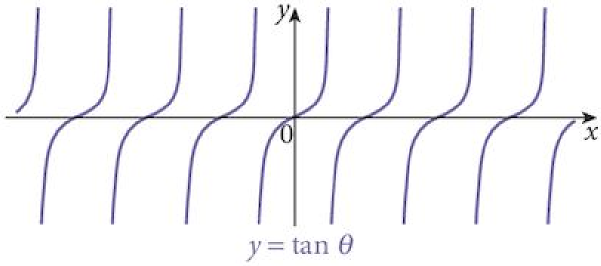
$y = \sin x \quad -180° \leqslant x \leqslant 180°$

$y = \cos x \quad 0° \leqslant x \leqslant 180°$

$y = \tan x \quad -180° < x < 180°$

Graphs of trigonometric functions





Amplitude

The distance between the axis of the graph and the maximum points.

For the standard sine and cosine functions, the amplitude is 1.

Note that, athough periodic, the amplitude of the tangent function is infinite.

Period

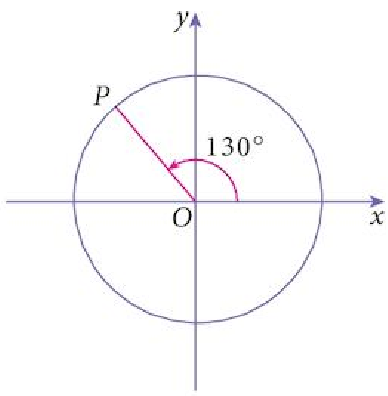
The angle between consecutive identical points on the graph.

For the standard functions, the period is 360°.

Solving trigonometric equations

You can use the unit circle to find equivalent angles with the same sine, cosine or tangent values. Make sure that you find all the solutions to an equation.

Chapter 11 Summative Exercise



1 In the diagram, the angle $xOP = 130°$.

a Write down two positive angles such that $\sin x = \sin 130°$.

b Write down two negative angles such that $\sin x = \sin 130°$.

c Write down two positive angles such that $\cos x = \cos 130°$.

d Write down two negative angles such that $\cos x = \cos 130°$.

e Write down two positive angles such that $\tan x = \tan 130°$.

f Write down two negative angles such that $\tan x = \tan 130°$.

2 Express each of these as the same function of an acute angle.

a cos 160°	**b** sin 110°	**c** tan 140°
d cos 250°	**e** sin 220°	**f** tan 260°
g cos 320°	**h** sin 310°	**i** tan 340°

3 Solve each equation for $0° \leqslant x \leqslant 360°$.

a $\cos x = 0.3$ b $\sin x = 0.4$ c $\tan x = 1.5$

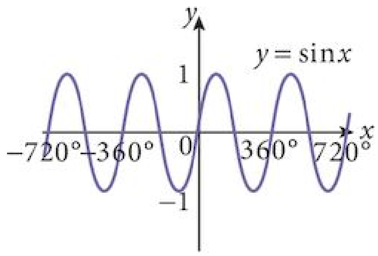
d $\cos x = -0.2$ e $\sin x = -0.3$ f $\tan x = -0.9$

4 Solve each equation for $-180° < x \leqslant 180°$.

a $\cos 2x = 0.6$ b $\sin 3x = 0.8$ c $\tan 2x = 0.9$

d $\cos 3x = -0.8$ e $\sin 2x = -0.9$ f $\tan 3x = -2.5$

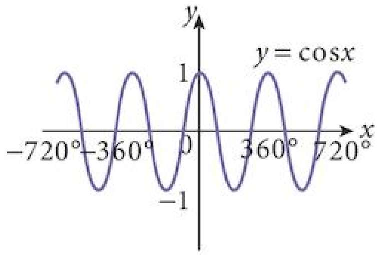
5 The diagram at right shows the graph of the function $f(x) = \sin x$ and the one below it shows the graph of the function $f(x) = \cos x$, both for the domain $-720° \leqslant x \leqslant 720°$.



a The standard restriction of the domain (called the principal values) to create a 1 : 1 function is $-90° \leqslant x \leqslant 90°$.

Choose another domain for which the function is 1 : 1 and which contains the value $x = -180°$.

b For the cosine function, state the standard restricted domain of the function and choose another domain for which the function is 1 : 1 and which contains the value $x = -450°$.



6 Solve each equation for $0° \leqslant x \leqslant 360°$.

a $\sin 2x = \cos 36°$ b $1 + 2\sin 2x = 0$

c $\sin(2x - 60°) = -\frac{1}{2}$ d $5\tan(x + 40°) = 6$

7 Solve each equation for $-180° < x \leqslant 180°$.

a $3(\sin x - \cos x) = 2(\sin x + \cos x)$ b $2(\sin x + \cos x) = 3\cos x$

c $3(\sin x + \cos x) = 2\cos x$ d $4(\cos x - 2\sin x) = 3(\cos x - 3\sin x)$

8 Solve each equation for $0° \leqslant x \leqslant 360°$.

a $\sin^2 x + 2\sin x\cos x - 3\cos^2 x = 0$ b $12\sin^2 x + \sin x - 6 = 0$

c $20\cos^2 x - 7\cos x - 3 = 0$ d $2\tan^2 x - 7\tan x - 4 = 0$

9 Solve each equation for $-180° < x \leqslant 180°$.

a $2\sin x\cos x = \cos x$ b $\sin x + 3\sin x\cos x = 0$

c $4\tan x\cos x - \tan x = 0$ d $3\sin x\tan x + 3\sin x - \tan x - 1 = 0$

e $2\cos x\tan x - 4\cos x + \tan x - 2 = 0$

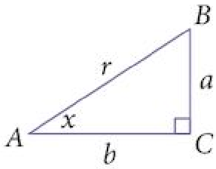
10 Angles A and B both lie in the range $0° \leqslant x \leqslant 360°$.

Both $\sin A$ and $\cos B$ are negative. Find the possible range of values of:

a $A + B$ b $A - B$

Answer questions **11** to **13** without using a calculator.

Instead, use the diagram on the right and Pythagoras' theorem.



11 In the triangle, $\cos x = \dfrac{4}{5}$.

a The angle y is obtuse and $\cos y = -\cos x$. Find $\sin y$.

b The angle z is a reflex angle and $\cos z = -\cos x$. Find $\tan z$.

12 a The angle x is obtuse and $12 \tan x = -5$. Find $\sin x$.

b The angle y lies between 270° and 360° and $17 \cos y = 8$. Find $\sin y$.

13 a The angle x is reflex and $25 \cos x = -7$. Find $\sin x$.

b The angle y lies between 270° and 360° and $41 \sin y = -9$. Find $\cos x$.

Chapter 11 Test

1 hour

1 Solve these equations for $0° \leqslant x \leqslant 360°$.

a $\sin 2x = -0.2$ [2]

b $\sin x = 3 \cos x$ [2]

c $1 + 2 \cos 2x = 0$ [2]

2 a On the same diagram, sketch the curves $y = \sin x$ and $y = \sin 2x - 1$ for $0° \leqslant x \leqslant 360°$. [4]

b Hence state the number of solutions of the equation $\sin 2x - \sin x = 1$ in this range. [2]

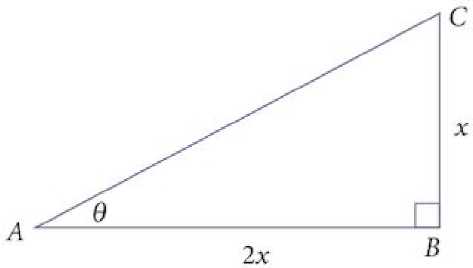
3 Find all angles in the range $0° \leqslant x \leqslant 360°$ which satisfy these equations:

a $2 \sin(x + 20°) = 1$ [3]

b $2(\sin x + \cos x) = 3 \sin x$ [3]

c $6 \cos^2 x - \cos x - 1 = 0$ [4]

4



ABC is a right-angled triangle, with $AB = 2x$ and $BC = x$.

a Find an expression for $\sin \theta$. [3]

b Find an expression for $\sec \theta$. [2]

c Find the value of θ correct to 1 d.p. [2]

5 **a** The cubic polynomial $f(x) = 6x^3 + x^2 - 11x - 6$ has a factor $(x + 1)$.
Solve the equation $f(x) = 0$. [5]

b Hence find solutions to the equation $6(\sin^3\theta - 1) = \sin\theta\,(11 - \sin\theta)$ for values of θ in the range $0° \leqslant \theta \leqslant 360°$. [6]

Term test 3A (Chapters 9–11)

1 hour

1 Find the distance between the points where the line $y = 5x - 11$ cuts the curve $y = 2x^2 - 5x - 3$. [6]

2 **a** Show that $\dfrac{(2x^3 - 6)}{x}$ can be written as $2x^2 - \dfrac{6}{x}$. [1]

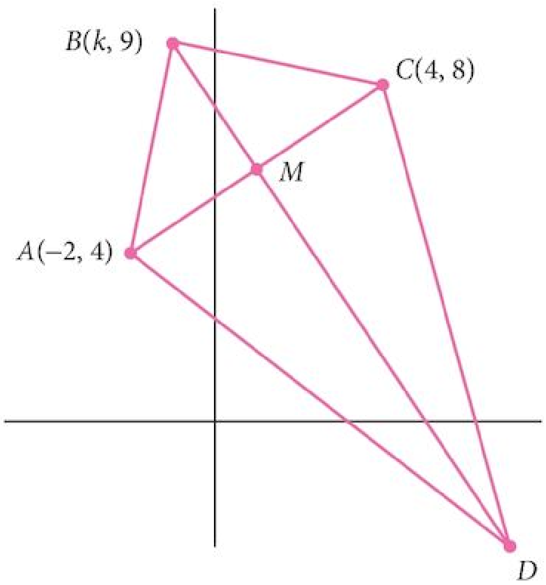
b Hence, if $xy = 2x^3 - 6$, find $\dfrac{dy}{dx}$. [2]

3 Solve these equations.

a $3\sin x + 4\cos x = 0$, for $0° \leqslant x \leqslant 360°$ [3]

b $2\cos(x - 10°) = -1$, for $0° \leqslant x \leqslant 360°$ [3]

4



The points $A(-2, 4)$ and $C(4, 8)$ are vertices of the kite $ABCD$.

M is the mid-point of AC. The vertex B of the kite has coordinates $(k, 9)$. The length $MD = 4BM$.

a Find the equation of the perpendicular bisector of AC. [3]

b Find the value of k. [1]

c Find the coordinates of D. [1]

d Find the area of the kite. [3]

5 Differentiate with respect to x:

a $(2 - x^2)^3$ [2]

b $xy = 2 - x$ [2]

c $f(x) = \dfrac{2}{\sqrt{x}} + \sqrt{x}$ [2]

6 Find all angles x, $0° \leqslant x \leqslant 360°$, which satisfy these equations.

a $4 \sin 2x - 3 \cos 2x = 0$ [4]

b $3 (\cos x - \sin x) = 5 (\cos x + \sin x)$ [4]

c $3 \sin(x - 40°) = 2$ [3]

12 Circular measure

Syllabus statements

- solve problems involving the arc length and sector area of a circle, including knowledge and use of radian measure

12.1 Measuring an angle

The simplest measure of rotation is the **revolution**.

However, we need to measure angles smaller than a one complete revolution.

The oldest of these systems is the use of **degrees**.

There are 360° in one revolution.

However, it turns out that degrees are not very useful when doing advanced mathematics, and so a different system is usually used.

We measure angles using **radians**.

Try this problem and you will see why.

Problem 12.1

Step 1: Check your calculator and make sure that you know how to change the angle measurement system from degrees (usually Deg) to radians (usually Rad).

Step 2: We are going to investigate the limit $\underset{h\to 0}{\mathrm{Lt}}\left[\frac{\sin h}{h}\right]$

You will see why in a later chapter.

Copy and complete the following table:

h	$\frac{\sin h}{h}$ degrees	$\frac{\sin h}{h}$ radians
0.1	0.01745	0.99833
0.01		
0.001		
0.0001		
…		

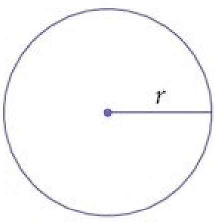
Keep going until your results have converged.

Step 3: What were your final results?

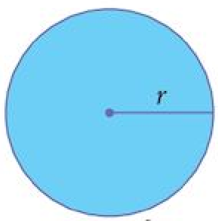
Which of these results would you rather use later?

12.2 Mensuration of the circle

You should know how to calculate the circumference C and area A of a circle.



$C = 2\pi r$

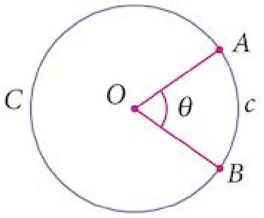


$A = \pi r^2$

From these, we can calculate the length of an arc or the area of a sector.

12.2.1 Length of an arc (using degrees)

We use ratios to find the length of the arc AB.



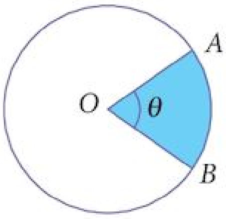
$$\frac{c}{C} = \frac{\theta}{360^\circ}$$

$$c = \frac{\theta}{360^\circ} \times 2\pi r$$

$C(2\pi r)$ is the circumference of the circle; c is the circumference of the arc AB.

12.2.2 Area of a sector (using degrees)

Similarly, we use ratios to find the area of the sector OAB.



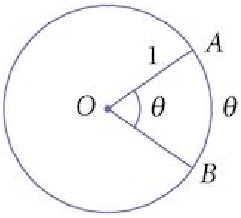
$$\frac{a}{A} = \frac{\theta}{360^\circ}$$

$$a = \frac{\theta}{360^\circ} \times \pi r^2$$

$A(\pi r^2)$ is the area of the circle; a is the area of the sector OAB.

12.3 The radian

Radians are defined so that the angle made at the centre of a circle of radius 1 unit is the same size as the length of the arc.

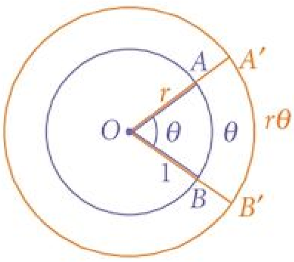


The arc length $AB = \theta$ when the radius $= 1$.

So the angle representing 1 revolution $=$ circumference

$= 2\pi$

If we dilate the circle by scale factor r (the new radius of the circle), the arc length becomes $r\theta$.

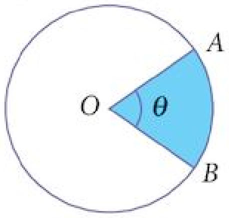


$$A'B' = r\theta$$

$$\boxed{\text{arc length} = r\theta}$$

Also, the circumference becomes $2\pi r$.

We can find the area of the sector, a, in the same way as we did using degree measurement. All that changes is the angle representing 1 revolution:



$$\frac{a}{A} = \frac{\theta}{2\pi}$$

$$a = \frac{\theta}{2\pi} \times \pi r^2$$

$$\boxed{\text{sector area} = \frac{1}{2}r^2\theta}$$

12.4 Equivalent angle measurements

Revolutions	Degrees	Radians	Radians (decimals)
1	360°	2π	6.283185
$\frac{1}{2}$	180°	π	3.142159
$\frac{1}{4}$	90°	$\frac{\pi}{2}$	1.570796
$\frac{1}{6}$	60°	$\frac{\pi}{3}$	1.047198
$\frac{1}{8}$	45°	$\frac{\pi}{4}$	0.785400
$\frac{1}{12}$	30°	$\frac{\pi}{6}$	0.523599

Calculators may give radians in decimal format but we prefer to write them as rational multiples of π.

Notice that 1 radian is just a little smaller than 60°. Actually, to one decimal place, it is 57.3°.

To convert, multiply by $\frac{180}{\pi}$ or $\frac{\pi}{180}$. Which gives the larger result?

It has become traditional to use the ° symbol to represent degrees. In the same way, some people use ^{c} (for circular) to indicate that the angle is measured in radians. However, it may be safer to use "rad" or no symbol at all.

In reality, unlike the measurements of mass, length and time which have dimension, angles are dimensionless scalars (just numbers).

Exercise 12.1

1 Convert these angles from degree measure to radian measure, expressing them as a rational multiple of π:

a 10°	**b** 20°	**c** 22.5°	**d** 30°	**e** 75°	**f** 105°	**g** 120°
h 135°	**i** 150°	**j** 210°	**k** 225°	**l** 240°	**m** 270°	**n** 300°
o 315°	**p** 330°	**q** 450°	**r** 540°	**s** 630°	**t** 720°	**u** 1080°

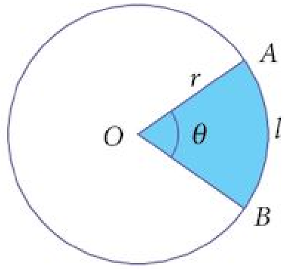
2 Convert these angles from radian measure to degree measure, giving the answer to one decimal place:

a	0.01745	**b**	0.08727	**c**	0.2618	**d**	0.4363	**e**	0.8727	**f**	1.2217	**g**	1.7453
h	1.9199	**i**	2.8798	**j**	3.4034	**k**	4.4506	**l**	4.9742	**m**	5.5851	**n**	6.0214
o	6.5	**p**	7.0	**q**	7.5	**r**	8.0	**s**	8.5	**t**	9.0	**u**	10.0

3 Find **(i)** the length, l, of the arc

and **(ii)** the area of the sector

in the following cases (all lengths in cm).



a $r = 5; \quad \theta = \frac{\pi}{6}$ **b** $r = 8; \quad \theta = \frac{\pi}{3}$

c $r = 10; \quad \theta = \frac{\pi}{12}$ **d** $r = 20; \quad \theta = \frac{\pi}{2}$

e $r = 30; \quad \theta = \frac{\pi}{3}$ **f** $r = 50; \quad \theta = \frac{\pi}{6}$

4 Find **(i)** the angle subtended at the centre

and **(ii)** the area of the sector in the following cases (l = arc length, and all lengths in cm):

a $r = 5; \quad l = 10$ **b** $r = 8; \quad l = 15$

c $r = 4; \quad l = 20$ **d** $r = 20; \quad l = 45$

e $r = 30; \quad l = 30$ **f** $r = 50; \quad l = 120$

5 Find **(i)** the angle subtended at the centre (correct to 2 d.p.)

and **(ii)** the length of the arc (correct to 2 d.p.) in the following cases (a = area, and all lengths in cm):

a $r = 5; \quad a = 10$ **b** $r = 8; \quad a = 15$

c $r = 4; \quad a = 20$ **d** $r = 20; \quad a = 45$

e $r = 30; \quad a = 30$ **f** $r = 50; \quad a = 120$

6 Find **(i)** the radius of the circle (correct to 3 s.f.)

and **(ii)** the area of the sector (correct to 3 s.f.) in the following cases (all lengths in cm):

a $l = 5; \quad \theta = \frac{\pi}{6}$ **b** $l = 12; \quad \theta = \frac{\pi}{3}$

c $l = 20; \quad \theta = \frac{5\pi}{12}$ **d** $l = 30; \quad \theta = \frac{3\pi}{2}$

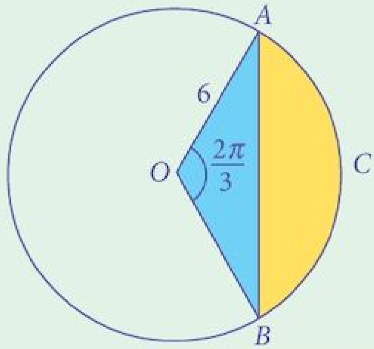
e $l = 24; \quad \theta = \frac{2\pi}{3}$ **f** $l = 50; \quad \theta = \frac{11\pi}{6}$

12.5 Circle geometry

Example 12.1

Find the area of the orange shaded segment (ABC) of the circle.

Solution:



Area of segment ABC = Area of sector $OACB$ − Area of triangle OAB

Area of sector $OACB$ $=\frac{1}{2}\times 6^2\times\frac{2\pi}{3}$

Area of triangle OAB $=\frac{1}{2}\times 6^2\times\sin\left(\frac{2\pi}{3}\right)$

Area of segment $=\frac{1}{2}\times 6^2\times\left[\frac{2\pi}{3}-\sin\left(\frac{2\pi}{3}\right)\right]$

$=18\left[\frac{2\pi}{3}-\frac{\sqrt{3}}{2}\right]$

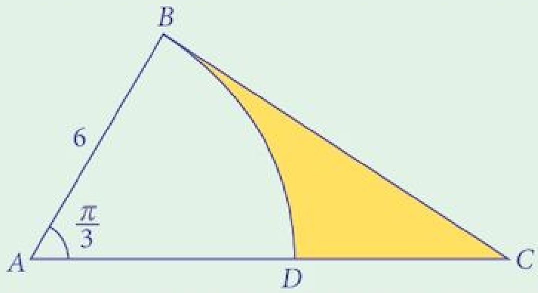
$=12\pi-9\sqrt{3}$

Area of triangle $=\frac{1}{2}ab\sin O$

Example 12.2

In the triangle ABC, $AB=6$ and angle $BAC=\frac{\pi}{3}$.
BD is the arc of a circle, centre A, and BC is a tangent to the circle.

Find the area of the shaded region BCD.



Solution:

Angle ABC is a right angle.

$$BC=6\tan\left(\frac{\pi}{3}\right)=6\sqrt{3}$$

$$\text{Area } ABC=\frac{1}{2}\times 6\times 6\sqrt{3}=18\sqrt{3}$$

$$\text{Area sector } ABD=\frac{1}{2}\times 6^2\times\frac{\pi}{3}=6\pi$$

$$\text{Shaded area}=6\left(3\sqrt{3}-\pi\right)$$

Any tangent to a circle is always perpendicular to the radius.

12.6 Solving equations

As you have seen, radians are just an alternative angle measuring system which turns out to be more useful from a mathematical point of view, even if, in real life, degrees are used by more people.
As in Chapter 11, we can solve trigonometric equations using either system. However, there are some things you need to note:

1. **Which system to use?**

   This is usually indicated in questions by a phrase such as: for $0° \leqslant x \leqslant 360°$ or for $0 \leqslant x \leqslant 2\pi$.

   Keep an eye out for these indicators. Using the wrong system will cost you marks.

2. **How to change from one to the other.**

   Your calculator is equipped to use both systems, usually selected by a function mode key.

   You must find out how to change the mode of your calculator. They are not all the same.

3. **Special angles**

   You must be aware of the special angles (usually rational multiples of π).

   Most calculators cannot provide answers in this format but they are preferable to decimal answers since they will be irrational numbers.

Your calculator also has facilities for a third angle measuring system called grads which we do not use. Grads were introduced as an attempt to decimalise angles. There are 100 grads in a right angle. This system has only limited use and you will not need it, but at least you know what it is.

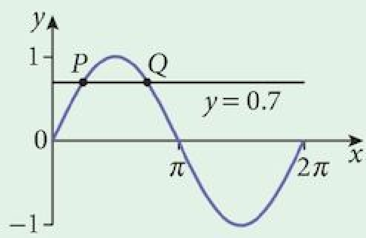
Example 12.3

Solve the following equations for $0 \leqslant x \leqslant 2\pi$.

a $\sin x = 0.7$ **b** $\cos x = 0.5$ **c** $\tan x = -\frac{1}{\sqrt{3}}$

Solution:

a $\sin x = 0.7$



The diagram shows two positions, P and Q, where $\sin x = 0.7$.

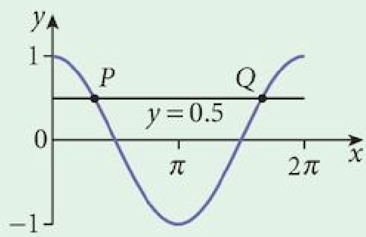
Your calculator will give the solution for P as $x = 0.775$.

To find the value for Q: $x = \pi - 0.775$

$x = 2.366$

So, the solution set is: $x \in \{0.775, 2.366\}$

b $\cos x = 0.5$



The diagram shows two positions, P and Q, where $\cos x = 0.5$.

Your calculator will give the solution for P as $x = 1.075$.

However, this is a special value that you should recognise.

$$x = \frac{\pi}{3}$$

To find the value for Q: $x = 2\pi - \frac{\pi}{3}$

$$x = \frac{5\pi}{3}$$

So, the solution set is: $x \in \left\{\frac{\pi}{3}, \frac{5\pi}{3}\right\}$

c $\tan x = -\frac{1}{\sqrt{3}}$

This is another value that you should recognise.

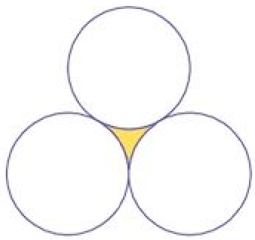
The solution set is $x \in \left\{\frac{5\pi}{6}, \frac{11\pi}{6}\right\}$

Exercise 12.2

In this exercise, all lengths and areas should be found accurately. Surds and π should be left in the answers.

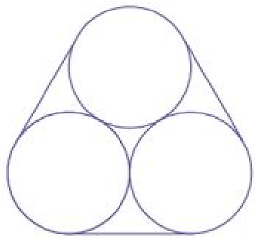
1 The diagram shows three identical circles, each of radius 10 cm. Each circle touches the other two.

Find the area of the shaded region in the centre of the diagram.



2 The three circles from question **1** represent water pipes (end view). They are fastened together using a rubber band.

Assuming the rubber band is taut and there is no overlap of material, find the length of the rubber band.

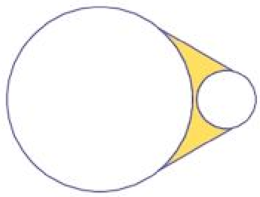


3 The diagram on the right shows a large circle of radius 30 cm touching a small circle of radius 10 cm.

The two circles are held together by a band passing around their edges.

a Find the length of the band.

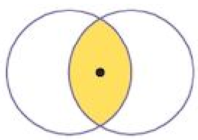
b Find the area shaded in the diagram.



4 The diagram on the right shows two circles, each of radius 10 cm. Each circle passes through the centre of the other one.

a Find the perimeter of the shaded region.

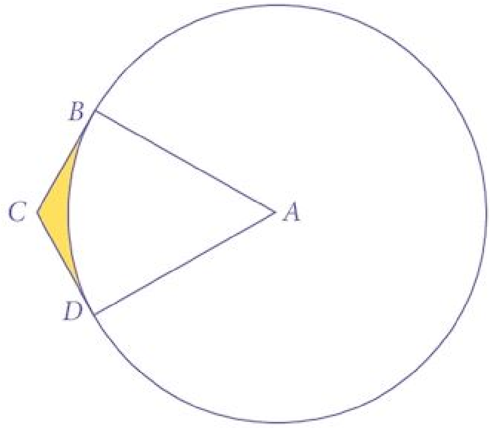
b Find the area of the shaded region.



5 The diagram on the right shows a kite, $ABCD$.
The arc BD is part of a circle, centre A.
Angle $BAD = \frac{\pi}{3}$ rad; angle $BCA = \frac{2\pi}{3}$ rad.
$AB = 20$ cm.

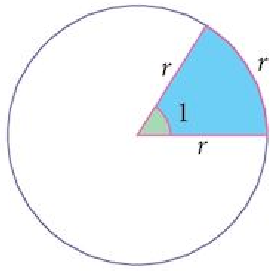
a Find the perimeter of the shaded region.

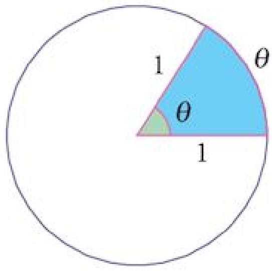
b Find the area of the shaded region.



Summary

Definitions One radian is the angle swept through that generates an arc at the circumference equal to the radius.





Alternatively, if the radius is 1 unit, the angle, in radians is equal to the arc length.

Equivalences If the radius is 1 unit,

the circumference $= 2\pi$

so 1 revolution $(360°) = 2\pi$ radians

$180° = \pi$ radians

$90° = \frac{\pi}{2}$ radians

and so on.

Fractions of π are preferable to decimals if they are "nice" fractions.

Formulae The length of an arc $= r\theta$

The area of a sector $= \frac{1}{2}r^2\theta$

Solving trigonometric equations Make sure your calculator is set to radian mode before you solve an equation in radians. Try to get used to working in radians rather than finding answers in degrees and then converting them.

Remember Later, when we are doing calculus, we must always work in radians.

Chapter 12 Summative Exercise

1 Convert these angles from degree measure to radian measure, leaving your answers as rational multiples of π.

a $15°$ **b** $30°$ **c** $40°$ **d** $75°$ **e** $90°$

f $120°$ **g** $135°$ **h** $210°$ **i** $225°$ **j** $300°$

2 Convert these angles from radian measure to degree measure.

a $\frac{\pi}{12}$ **b** $\frac{\pi}{6}$ **c** $\frac{\pi}{4}$ **d** $\frac{\pi}{3}$ **e** $\frac{5\pi}{12}$

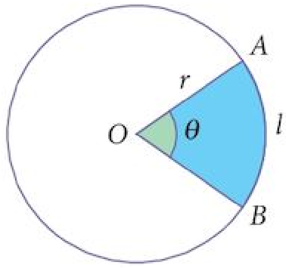
f $\frac{4\pi}{3}$ **g** $\frac{17\pi}{12}$ **h** $\frac{3\pi}{10}$ **i** $\frac{13\pi}{18}$ **j** $\frac{15\pi}{9}$

3 In the following cases, find:

(i) the length of AB

(ii) the area of the sector AOB.

Express your answers as a rational multiple of π.



a $r = 6\,;\, \theta = \frac{\pi}{4}$ **b** $r = 10\,;\, \theta = \frac{5\pi}{12}$ **c** $r = 18\,;\, \theta = \frac{5\pi}{6}$

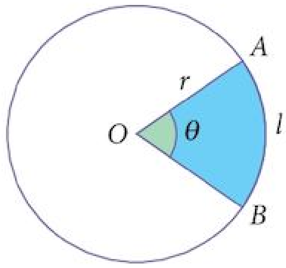
d $r = 24\,;\, \theta = \frac{3\pi}{4}$ **e** $r = 32\,;\, \theta = \frac{3\pi}{8}$ **f** $r = 15\,;\, \theta = \frac{8\pi}{5}$

4 In the following cases, find:

(i) the angle subtended at the centre

(ii) the area of the sector AOB.

Express your answers as a rational multiple of π.



a $r = 12\,;\, l = \frac{9\pi}{2}$ **b** $r = 4\,;\, l = \frac{8\pi}{5}$ **c** $r = 9\,;\, l = 3\pi$

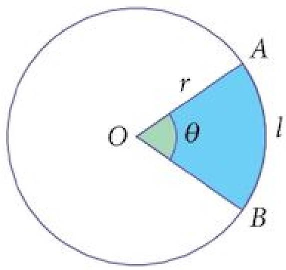
d $r = 18\,;\, l = 15\pi$ **e** $r = 16\,;\, l = 12\pi$ **f** $r = 6\,;\, l = 8\pi$

5 In the following cases, where a is the area of the sector, find:

(i) the radius of the circle

(ii) the length of the arc AB.

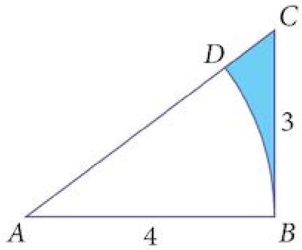
Express your answers as a rational multiple of π.



a $a = 3\pi\,;\, \theta = \frac{2\pi}{3}$ **b** $a = \frac{75\pi}{8}\,;\, \theta = \frac{3\pi}{4}$ **c** $a = \frac{45\pi}{2}\,;\, \theta = \frac{5\pi}{9}$

d $a = 30\pi\,;\, \theta = \frac{3\pi}{5}$ **e** $a = \frac{40\pi}{3}\,;\, \theta = \frac{5\pi}{12}$ **f** $a = 15\pi\,;\, \theta = \frac{5\pi}{6}$

6 In the diagram, angle ABC is $\dfrac{\pi}{2}$. Find:

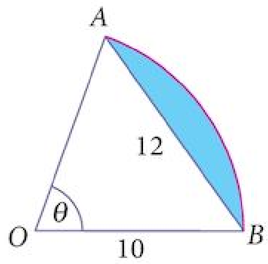


a the angle BAC in radians, correct to 1 d.p.

b the perimeter of the shaded part

c the area of the shaded part.

7 The arc AB is part of a circle, centre O, radius 10 cm. The chord AB has length 12 cm. Find:

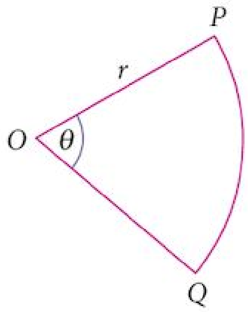


a the angle θ, in radians, correct to 1 d.p.

b the length of the arc AB

c the area of the shaded region.

8 A piece of wire of length 20 cm is bent to form a sector of a circle, centre O as shown. Given that the radius of the circle is r cm, express:

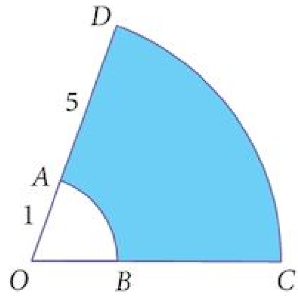


a θ in terms of r

b the area A of the sector, in terms of r.

9 AB and CD are arcs of concentric circles, centre O. $OA = 1$ m and $OD = 6$ m.

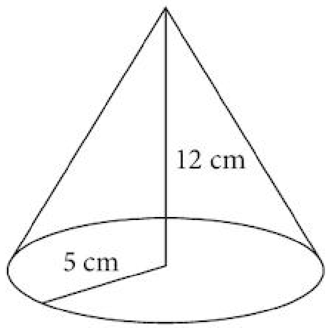
If the perimeter of the shaded area is 24 m, find:



a the angle θ in radians

b the area of the shaded region.

10 A cone has a base radius of 5 cm and a height of 12 cm. The curved surface of the cone is cut and flattened to form a sector of a circle. Find:



a the angle of the sector, in radians

b the area of the curved surface of the cone.

Chapter 12 Test

1 hour

1 Find all angles x, $0 \leqslant x \leqslant 2\pi$, which satisfy the equation:

a $\cos(3 - x) = 0.6$ [4]

b $3\tan(2x + 1) = -2$ [4]

c $6\sin^2 x - \sin x - 1 = 0$. [4]

2 In the figure, OAB is the sector of a circle, centre O.

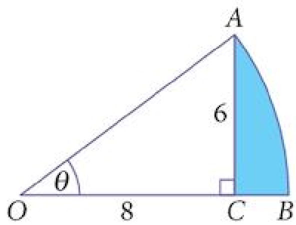
OCA is a right-angled triangle, with $OC = 8$ cm and $CA = 6$ cm.

Find:

a the angle θ in radians, correct to 3 s.f. [2]

b the perimeter of the shaded region [4]

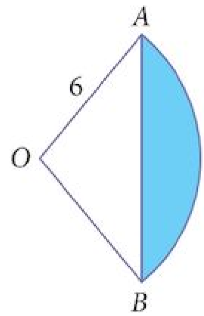
c the area of the shaded region. [3]



3 The triangle OAB is equilateral and OAB is a sector of the circle, centre O, radius 6 cm.

a Find the perimeter of the shaded segment of the circle. [2]

b Find the area of the shaded segment of the circle. [5]



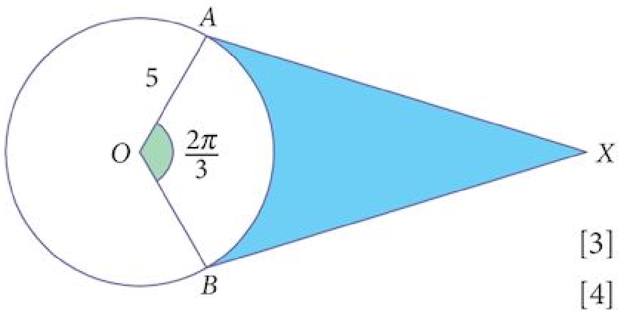
4 The figure shows a circle, radius 5 units.

Lines XA and XB are tangents to the circle at the points A and B.

Angle $AOB = \dfrac{2\pi}{3}$ radians.

a Find the perimeter of the shaded region. [3]

b Find the area of the shaded region. [4]



5 **a** Sketch, on the same diagram, the graphs of $y = |2\cos x|$ and $y = \dfrac{4x}{3\pi}$ for $0 \leqslant x \leqslant 2\pi$. [4]

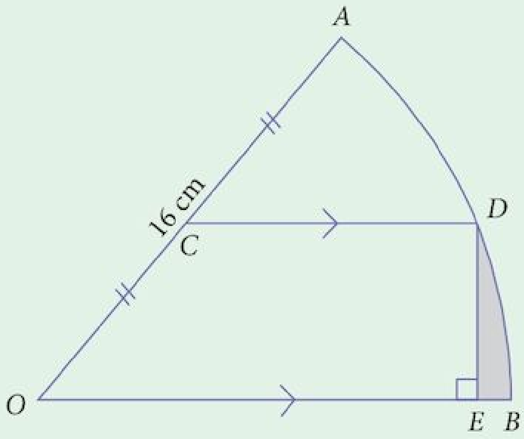
b State, for the range $0 \leqslant x \leqslant 2\pi$, the number of solutions of:

(i) $|3\pi\cos x| = 2x$ [3]

(ii) $3\pi\cos x = 2x$ [3]

Examination Questions

1



In the diagram, OAB is a sector of a circle, centre O and radius 16 cm, and the length of the arc AB is 19.2 cm. The mid-point of OA is C and the line through C parallel to OB meets the arc AB at D. The perpendicular from D to OB meets OB at E.

(i) Find the angle OAB in radians. [2]

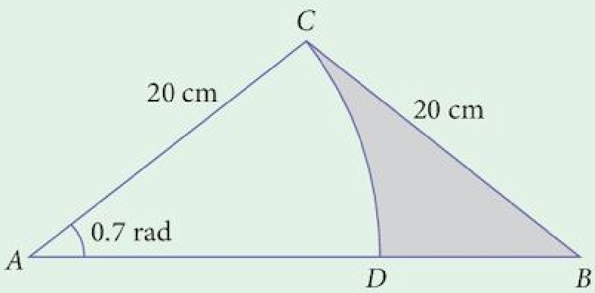
(ii) Find the length of DE. [2]

(iii) Show that the angle DOE is approximately 0.485 radians. [2]

(iv) Find the area of the shaded region. [4]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2003, P2, Qu 10]

2



The diagram shows an isosceles triangle ABC in which $BC = AC = 20$ cm, and angle $BAC = 0.7$ radians. DC is an arc of a circle, centre A.

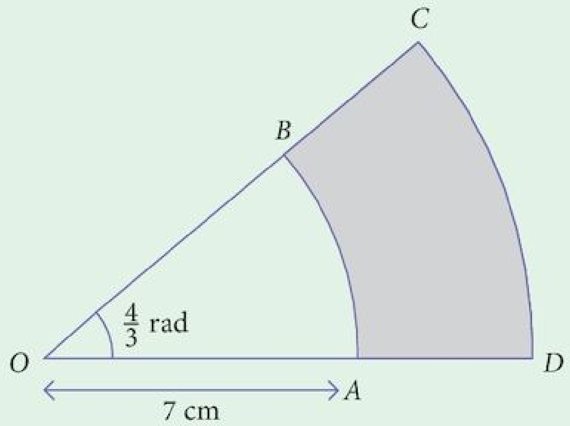
Find, correct to 1 decimal place:

(i) the area of the shaded region, [4]

(ii) the perimeter of the shaded region. [4]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2004, P1, Qu 10]

3

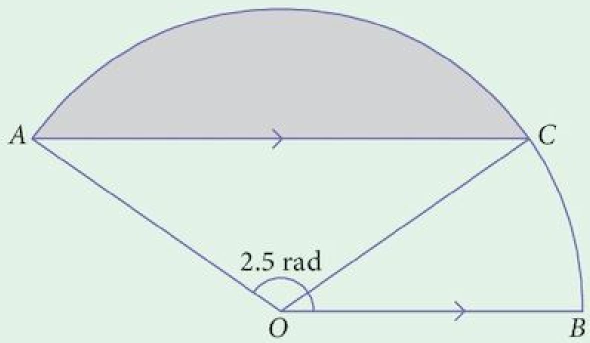


The diagram shows a sector *COD* of a circle, centre *O*, in which angle $COD = \dfrac{4}{3}$ radians.

The points *A* and *B* lie on *OD* and *OC* respectively, and *AB* is an arc of a circle, centre *O*, of radius 7 cm. Given that the area of the shaded region *ABCD* is 48 cm^2, find the perimeter of this shaded region. [6]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2004, P1, Qu 4]

4



The diagram shows a sector *OACB* of a circle, centre *O*, in which angle $AOB = 2.5$ radians. The line *AC* is parallel to *OB*.

(i) Show that the angle $AOC = (5 - \pi)$ radians. [3]

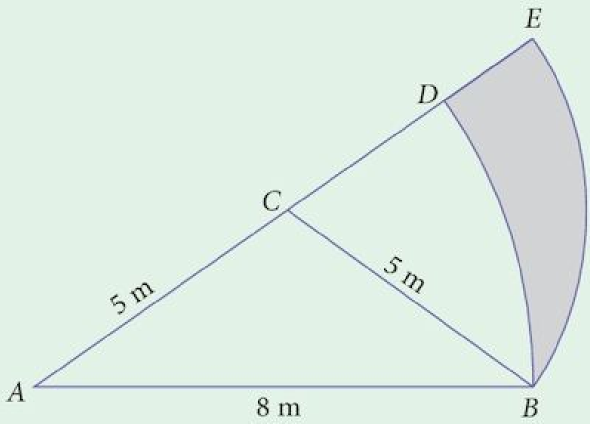
Given that the radius of the circle is 12 cm, find

(ii) the area of the shaded region, [3]

(iii) the perimeter of the shaded region. [3]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2006, P2, Qu 11]

5



The diagram shows an isosceles triangle ACB in which $AB = 8$ m, $BC = CA = 5$ m. $ABDA$ is a sector of the circle, centre A and radius 8 m. $CBEC$ is a sector of the circle, centre C and radius 5 m.

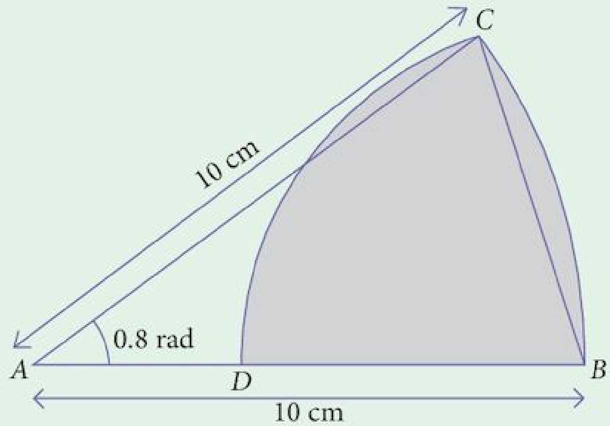
(i) Show that the angle $BCE = 1.287$ radians correct to 3 decimal places. [2]

(ii) Find the perimeter of the shaded region. [4]

(iii) Find the area of the shaded region. [4]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2007, P1, Qu 10]

6



The diagram shows a sector ABC of a circle, centre A and radius 10 cm, in which the angle $BAC = 0.8$ radians. The arc CD of a circle has centre B and the point D lies on AB.

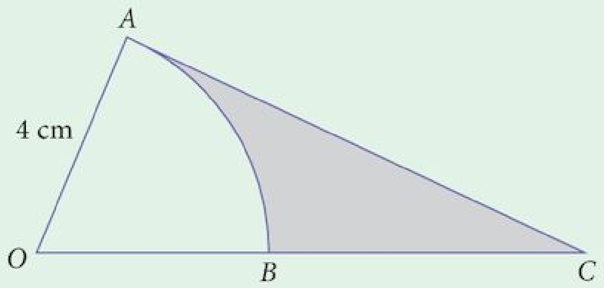
(i) Show that the length of the straight line BC is 7.79 cm, correct to 2 decimal places. [2]

(ii) Find the perimeter of the shaded region. [4]

(iii) Find the area of the shaded region. [4]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2007, P1, Qu 10]

7



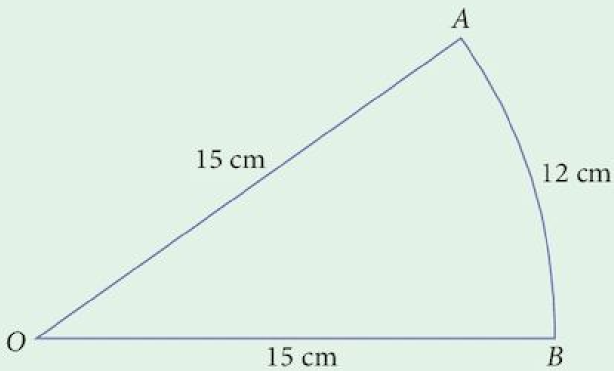
The diagram shows a sector OAB of a circle, centre O and radius 4 cm. The tangent to the circle at A meets the line OB extended at C. Given that the area of the sector OAB is 10 cm^2, calculate

(i) the angle AOB in radians, [2]

(ii) the perimeter of the shaded region. [4]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2008, P1, Qu 7]

8



The diagram shows a sector AOB of a circle, centre O, radius 15 cm. The length of the arc AB is 12 cm.

(i) Find, in radians, the angle AOB. [2]

(ii) Find the area of the sector AOB. [2]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2009, P1, Qu 1]

9 Solve the equation $3\sin\left(\frac{x}{2}-1\right)=1$ for $0 < x < 6\pi$ radians. [5]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2007, P2, Qu 2]

13 Applications of the derivative

Syllabus statements

- apply differentiation to stationary points, connected rates of change, small increments and approximations and practical maxima and minima problems
- use the first and second derivative tests to discriminate between maxima and minima

13.1 Introduction

In Chapter 10 you met the derived function and the first simple application of it.

In this chapter, we look at further applications of the derived function and introduce an extension to the theory so that we can deal with functions that are a little more complicated than series made up of powers of x.

13.2 Stationary points

When sketching a graph, it is often helpful to know where the significant points on the graph are.

Some significant points are:

- points where the graph cuts the axes
- points where the tangent is horizontal
- points to which the graph is heading as it goes towards its extremes.

Of these, we find the first by solving the simultaneous equations

$$y = f(x), \text{ and one of } x = 0 \text{ or } y = 0.$$

The third we find by investigating what happens as x or y approaches infinity.

The second is what we are interested in here.

We solve the problem by finding $\frac{dy}{dx}$ and then solving the equation $\frac{dy}{dx} = 0$.

These points are called **stationary points** because at these points the value of y is instantaneously at rest.

Stationary points are also called turning points.

Example 13.1

Find the stationary points of the following graphs:

a $y = x^2 - 4x + 7$

b $y = x^3 - 3x^2 - 9x + 4$

Solution:

a

$$y = x^2 - 4x + 7$$

$$\frac{dy}{dx} = 2x - 4$$

We need $\frac{dy}{dx} = 0$: $\quad 0 = 2x - 4$

We could have solved this problem by completing the square.

$$x = 2$$

Solving for y when $x = 2$ gives $\quad y = 3$

So there is one stationary point at $(2, 3)$.

b

$$y = x^3 - 3x^2 - 9x + 4$$

$$\frac{dy}{dx} = 3x^2 - 6x - 9$$

We need $\frac{dy}{dx} = 0$: $\quad 0 = 3x^2 - 6x - 9$

$$0 = 3(x^2 - 2x - 3)$$

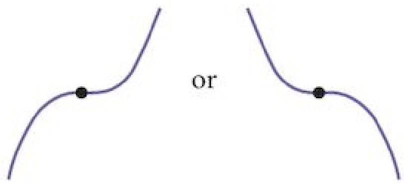
Solve the quadratic: $\quad x = -1, 3$

Solving for y when $x = -1$ or 3 gives: $y = 9, -23$

So there are two stationary points at $(-1, 9)$ and $(3, -23)$.

13.3 The nature of stationary points

There are three types of stationary point:

		
Maximum	Minimum	Point of inflexion

Imagine that you are walking along the curve from left to right.

For the maximum, when the gradient is zero, you are turning **right**.

For a minimum, when the gradient is zero, you are turning **left**.

For a point of inflexion, as you pass the stationary point, you **change the direction** that you are turning from right to left, or from left to right.

This is also a point of inflexion but it is not stationary because the gradient is not zero when you change the direction of turning.

Notice that "**maximum**" really means "**local maximum**".

Some curves have more than one maximum; for other curves, parts of the curve rise above a local maximum.

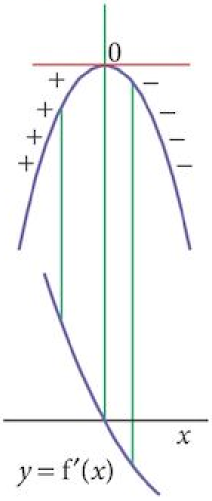
The curve on the right has two minima and two maxima, although the extremes of the curve shoot off to infinity.

Even though one maximum is larger than the other, we call them both "maximum" (plural maxima), leaving out the "local".

13.4 Identifying the nature of stationary points

There are two techniques that you can use, both based on the gradient of the curve on either side of the stationary point.

13.4.1 The maximum

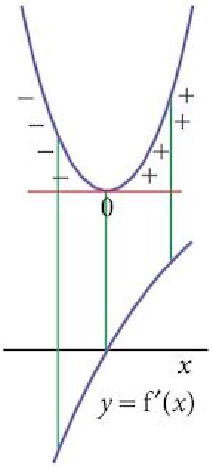


At the maximum, the gradient is zero. (That's how we find them.)
To the left of the maximum, the gradient is positive.
To the right of the maximum, the gradient is negative.

If we draw the graph of the gradient of the curve $[y = f'(x)]$, it must cross the x-axis at the stationary value, being positive to the left of that value, and negative to the right.

The gradient of the gradient graph $[(f'(x))' = f''(x)]$ must be negative.

13.4.2 The minimum

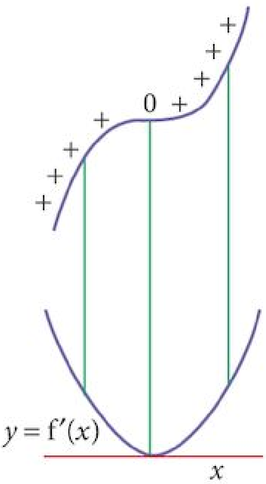


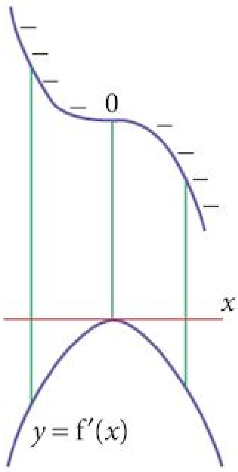
At the minimum, the gradient is zero.
To the left of the minimum, the gradient is negative.
To the right of the minimum, the gradient is positive.

If we draw the graph of the gradient of the curve $[y = f'(x)]$, it must cross the x-axis at the stationary value, being negative to the left of that value, and positive to the right.

The gradient of the gradient graph $[(f'(x))' = f''(x)\ ]$ must be positive.

13.4.3 A stationary point of inflexion





At a stationary point of inflexion the gradient is zero, but the gradient is either positive on both sides or negative on both sides of the stationary point.

If we draw the graph of the gradient of the curve $[y = f'(x)]$, it must touch the x-axis at the stationary value.

The graph will have either a maximum or a minimum at this point.

The gradient of the gradient graph $[(f'(x))' = f''(x)]$ must be zero.

13.4.4 Identification techniques

Technique 1: We could find the value of the gradient to the left and right of the stationary point.

Example 13.2

Find the nature of the stationary points of the following graph:

$$y = x^3 - 3x^2 - 9x + 4$$

Solution:

Step 1: Find the stationary points.

These are $(-1, 9)$ and $(3, -23)$

From Example 13.1

The gradient function is:

$$\frac{dy}{dx} = 3x^2 - 6x - 9$$

Step 2: Test the gradient for values of x close to the stationary points:

x	$\frac{dy}{dx}$	Nature
−1.1 −0.9	1.23 −1.17	+ − Maximum
2.9 3.1	−1.17 1.23	− + Minimum

Actual gradient values do not matter. We just want to know whether they are positive or negative.

The point $(-1, 9)$ is a maximum and the point $(3, -23)$ is a minimum.

Technique 2: We find the value of the second derivative at the stationary point.

Example 13.3

Find the nature of the stationary points of the following graph:

$$y = x^3 - 3x^2 - 9x + 4$$

Solution:

Step 1: Find the stationary points.

These are $(-1, 9)$ and $(3, -23)$

From Example 13.1

The gradient function is

$$\frac{dy}{dx} = 3x^2 - 6x - 9$$

Step 2: Find the second derivative.

$$\frac{d^2y}{dx^2} = 6x - 6$$

When $x = -1$: $\quad \frac{d^2y}{dx^2} = -12$ (negative $\rightarrow$ Maximum)

When $x = 3$: $\quad \frac{d^2y}{dx^2} = 12$ (positive $\rightarrow$ Minimum)

So $(-1, 9)$ is a maximum and $(3, -23)$ is a minimum.

Again, the actual values are not important.

To summarise:

$\frac{dy}{dx}$	$\frac{d^2y}{dx^2}$	Nature
0	+ ve	Maximum
0	− ve	Minimum
0	0	Stationary point of inflexion

Points of inflexion are not in the syllabus so you will not be asked about them.

Exercise 13.1

1 Find the coordinates of the points on each of the following curves at which the gradient is zero.

a $y = x^2 - 5x + 4$ **b** $y = x^2 + 4x + 3$ **c** $y = 4 - x^2$

d $y = 2 - 3x + x^2$ **e** $y = 5 - 4x - x^2$ **f** $y = 20 - 12x - 3x^2$

g $y = 3x - x^3$ **h** $y = x^3 - 6x^2$ **i** $y = 2x^3 - 6x^2 - 18x + 2$

j $y = 8x^2 - x^4$ **k** $y = 32x - x^4$ **l** $y = 3x^4 - 8x^3 - 30x^2 + 72x$

2 Find the coordinates of the stationary points on each of the following curves and determine their nature.

a $y = 8 - 2x - x^2$ **b** $y = x^2 - 6x + 9$ **c** $y = 7 + 4x - x^2$

d $y = x^2 + 10x + 5$ **e** $y = 6x^2 - x - 1$ **f** $y = 10x^2 + 3x - 1$

g $y = 4x - 3x^3$ **h** $y = x^3 - 3x^2$ **i** $y = x^3 - 12x + 2$

j $y = 8x^2 - x^4$ **k** $y = 27x - 2x^4$ **l** $y = 3x^4 - 4x^3 - 12x^2 + 5$

3 Find the coordinates of the stationary points on each of the following curves and determine their nature.

a $y = 4x + \frac{1}{x}$ **b** $y = x^2 - \frac{16}{x}$ **c** $y = 15\sqrt{x^5} - 15\sqrt{x^3} + 6$

d $y = x + \frac{4}{x^2}$ **e** $y = 27\sqrt{x} + \frac{4}{x}$ **f** $y = \frac{54 - x^3}{x}$

13.5 Small increments

From the gradient function, $\frac{dy}{dx} \approx \frac{\delta y}{\delta x}$

When h (or δx) is almost zero.

We can transform this to become $\boxed{\delta y \approx \frac{dy}{dx} \times \delta x}$

Remember that δx is "a small change in x" and δy is "a small change in y".

We can use this to approximately calculate values of functions that are close to ones we do know.

Example 13.4

Given that $2^3 = 8$, calculate an approximate value for 2.001^3.

Of course, we could calculate this accurately with a calculator.

Solution:

We start with the function $y = x^3$

Differentiate it: $\frac{dy}{dx} = 3x^2$

Put $x = 2$

and $\delta x = 0.001$

Then $\frac{dy}{dx} = 3(2)^2 = 12$

$$\delta y \approx \frac{dy}{dx} \times \delta x$$

$$= 12 \times 0.001$$

$$= 0.012$$

so $2.001^3 \approx 2^3 + 0.012$

$$= 8.012$$

Check with a calculator to see how accurate this answer is.

Example 13.5

You are given the function $f(x) = x^4 - 3x^3 + 6x^2 - 8x + 5$.

The value of the function when $x = 1$ is 1.

Find an approximate value when $x = 1.01$.

Solution:

We start with the function $\quad f(x) = x^4 - 3x^3 + 6x^2 - 8x + 5$

Differentiate it: $\quad f'(x) = 4x^3 - 9x^2 + 12x - 8$

Put $\quad x = 1$

and $\quad \delta x = 0.01$

Then

$$f'(1) = 4(1)^3 - 9(1)^2 + 12(1) - 8$$
$$= -1$$
$$\delta y \approx \frac{dy}{dx} \times \delta x$$
$$= -1 \times 0.01$$
$$= -0.01$$

so $\quad f(1.01) \approx 1 - 0.01$

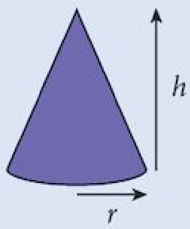
$$= 0.99$$

Again, check with a calculator to see how accurate this answer is.

Example 13.6

The volume of a cone is given by $V = \frac{1}{3}\pi r^2 h$.

Approximately how much does the volume change if we increase the radius by an amount p, if the shape of the cone remains the same?



Solution:

If the shape remains the same: $\quad \frac{h}{r} = \text{constant} = c$

So $\quad h = rc$

and $\quad V = \frac{1}{3}\pi c r^3$

We cannot work with 2 variables so we eliminate one of them.

Differentiate this with respect to r:

$$\frac{dV}{dr} = \pi c r^2$$
$$\delta V \approx \frac{dV}{dr} \times \delta r$$
$$= \pi c r^2 \times p$$
$$= \pi c p r^2$$

$$\text{Increase in volume} \approx \pi p r h$$

We do not always have to use y and x.

Exercise 13.2

1 If $y = \sqrt{x}$, estimate the value of

a $\sqrt{101}$ **b** $\sqrt{82}$ **c** $\sqrt{626}$

2 If $y = \sqrt[3]{x}$, estimate the value of

a $\sqrt[3]{1001}$ **b** $\sqrt[3]{126}$ **c** $\sqrt[3]{3370}$

3 If $f(x) = 3x - x^3$, estimate the value of

a $f(2.01)$ **b** $f(3.001)$ **c** $f(-3.9)$

4 Estimate the value of $f(2.1)$ if

a $f(x) = x^3 - 6x^2$ **b** $f(x) = x^4 - 3x^2 + 2x$ **c** $f(x) = 6x + \dfrac{1}{x}$

5 The volume of a cube is 1000 cm^3. Estimate the reduction in surface area if the length of each side is reduced by 0.1 cm.

13.6 The chain rule

You have seen that $\dfrac{dy}{dx} \approx \dfrac{\delta y}{\delta x}$

Since δy and δx are measurable quantities (even if they are minute),

we can write $\dfrac{\delta y}{\delta x} = \dfrac{\delta y}{\delta u} \times \dfrac{\delta u}{\delta x}$

or, with a longer chain, $\dfrac{\delta y}{\delta x} = \dfrac{\delta y}{\delta u} \times \dfrac{\delta u}{\delta v} \times \dfrac{\delta v}{\delta x}$

We can make the chain as long as we like. Usually, we only need one intermediate variable.

Then, as each of these quantities shrinks to zero,

we get $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

or, with a longer chain, $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dy} \times \dfrac{dv}{dx}$

13.7 Connected rates of change

A **rate of change** is a measure of how much one thing is changing when a related measurement changes. Often the related measurement is **time** but it does not have to be.

$\dfrac{dx}{dt}$ measures the rate at which x is changing with time.

and $\dfrac{dy}{dx}$ measures the rate at which y is changing with x.

We can use the **chain rule** to find one rate of change from another.

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dv} \times \frac{dv}{dx}$$

We will also need one more result which can be derived from the chain rule.

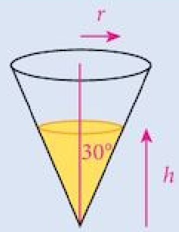
The reciprocal rule:

$$\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$$

This reciprocal rule works only for 1st derivatives.

Example 13.7

A conical container with its vertex at the bottom and semi-vertical angle 30° is being filled with water at a rate of 3 litres min^{-1}.
How fast is the water level rising when it is 10 cm deep?



Solution:

If the depth of the water is h cm and the radius of the water surface is r cm,

$$r = h \tan 30$$

so $$h = r\sqrt{3} \quad [1]$$

The surface area is $$A = \pi r^2$$

and the volume of water is $$V = \frac{1}{3}\pi r^2 h$$

$$= \frac{\sqrt{3}}{3}\pi r^3 \quad [2]$$

The rate at the water level is rising is $\frac{dh}{dt}$.

This is what we are required to find.

The rate at which the volume is increasing is

$$\frac{dV}{dt} = \frac{3000}{60}\ \text{cm}^3\text{s}^{-1}$$

$$= 50\ \text{cm}^3\text{s}^{-1} \quad [3]$$

Create a suitable chain $$\frac{dh}{dt} = \frac{dh}{dr} \times \frac{dr}{dV} \times \frac{dV}{dt}$$

Find the derivatives: [1]: $$\frac{dh}{dr} = \sqrt{3}$$

[2]: $$\frac{dV}{dr} = \sqrt{3}\pi r^2$$

Complete the chain
$$\frac{dh}{dt} = \frac{dh}{dr} \times \frac{dr}{dV} \times \frac{dV}{dt}$$
$$= \sqrt{3} \times \frac{1}{\sqrt{3}\pi r^2} \times 50$$
$$= \frac{50}{\pi r^2}$$

When $h = 10$
$$r = \frac{10}{\sqrt{3}}$$

and
$$\frac{dh}{dt} = \frac{50 \times 3}{100\pi}$$
$$= 0.48\,\text{cm}\,\text{s}^{-1}$$

Exercise 13.3

1 Use the chain rule to prove that $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$

2 The side of a square is increasing at a rate of 5 cm s^{-1}. Find the rate of increase of the area of the square when the side is 50 cm.

3 The area of a square is increasing at a rate of 20 cm^2 s^{-1}. Find the rate of increase of the side of the square when the area is 100 cm^2.

4 The radius of a circle is increasing at a rate of 2 cm s^{-1}. Find the rate of increase of the area of the circle when the radius is 10 cm.

5 The area of a circle is increasing at a rate of 5 cm^2 s^{-1}. Find the rate of increase of the radius of the circle when the radius is 10 cm.

6 The volume of a sphere is increasing at a rate of 40π cm^3 s^{-1}. Find the rate of increase of the radius of the sphere when it is 10 cm.

7 The radius of a sphere is increasing at a rate of 5 cm s^{-1}. Find the rate of increase of the surface area of the sphere when the radius is 15 cm.

8 The volume of a sphere is increasing at a rate of 20 cm^3 s^{-1}. Find the rate of increase of the surface area of the sphere when the radius is 40 cm.

9 Oil is dripping from the engine of a car to form a circular puddle under the car. The area of the puddle is increasing at a rate of 2 cm^2 s^{-1}. Find the rate of increase of the radius of the puddle when the area is 50 cm^2.

10 The surface area of a cube is increasing at a rate of 60 cm^2 s^{-1}. Find the rate of increase of the length of an edge of the cube when it is 10 cm.

11 A spherical balloon is being blown up by a mechanical pump which pumps 7200π cm^3 s^{-1} of air into the balloon. Find the rate of increase of the radius of the balloon when its volume is 36000π cm^3.

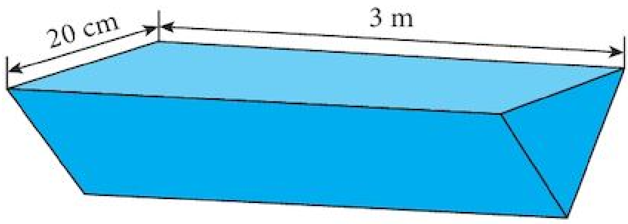
12 A container of water in the shape of an inverted cone with a semi-vertical angle of 30° is initially full of water. The water drips out of the cone through a hole in the vertex at a rate of 2 cm^3 s^{-1}.

a Find the rate at which the depth of water is changing when the radius of the water surface is 5 cm.

b Find the rate at which the radius of the water surface is changing when the depth of the water is 10 cm.

13 The surface area of a cube is increasing at a rate of 6 cm^2 s^{-1}. Find the rate of increase of the length of an edge of the cube when it is 10 cm.

14 A drinking trough is in the shape of a triangular prism. The length of the trough is 3 m and the top is rectangular with a width of 20 cm. The ends of the trough are equilateral triangles. Water is flowing into it at a rate of 15 cm^3 s^{-1}. Find the rate of increase of the depth of the water when the surface area is 2000 cm^2.



13.8 Composite functions

Forming **composite functions** is the third way that we can create more complicated functions out of simple ones.

In order to differentiate these functions, we need the **chain rule**:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Example 13.8

a If $y = (4x^2 - 3x - 5)^3$, find $\dfrac{dy}{dx}$

b If $y = \dfrac{1}{x^2 - 4x}$, find $\dfrac{dy}{dx}$

Solution:

a $y = (4x^2 - 3x - 5)^3$

Separate the functions:

$y = u^3$ where $u = 4x^2 - 3x - 5$

We could have used the binomial theorem to expand this function but that would have meant more work.

Differentiate each one

$$\frac{dy}{du} = 3u^2 \qquad \text{and} \qquad \frac{du}{dx} = 8x - 3$$

Using the chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= 3u^2 \times (8x - 3)$$

Substitute for u

$$= 3(4x^2 - 3x - 5)^2 (8x - 3)$$

b

$$y = \frac{1}{x^2 - 4x}$$

$$= (x^2 - 4x)^{-1}$$

Separate the functions:

$$y = u^{-1} \qquad \text{where} \quad u = x^2 - 4x$$

Differentiate each one

$$\frac{dy}{du} = -u^{-2} \qquad \text{and} \qquad \frac{du}{dx} = 2x - 4$$

Using the chain rule:

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

$$= -u^{-2} \times (2x - 4)$$

Substitute for u

$$= \frac{4 - 2x}{(x^2 - 4x)^2}$$

An alternative expression for the derivative of a composite function comes from using the function notation.

If $\quad y = f(g(x))$

We can write $\quad y = f(u) \quad$ where $\quad u = g(x)$

Differentiating as before $\quad \dfrac{dy}{du} = f'(u) \quad$ and $\quad \dfrac{du}{dx} = g'(x)$

Then, using the chain rule $\quad \dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

which gives $\quad \dfrac{dy}{dx} = f'(u) \times g'(x)$

Eliminating u $\quad \dfrac{dy}{dx} = f'(g(x)) \times g'(x)$

Alternatively, $\quad [f(g(x))]' = f'(g(x)) \times g'(x)$

The process is rather like peeling an onion.

Step 1: Differentiate the outside function first, leaving the inner function alone.

Step 2: Differentiate the inner function and multiply the two together.

Note that for more complicated composites, you have a longer chain but the process is just the same.

Exercise 13.4

1 Differentiate each of the following functions:

a $y=(x+3)^4$ **b** $y=(x^2-2x)^3$ **c** $y=(x^2-2x+3)^4$

d $y=(1-5x)^{12}$ **e** $y=(1-2x+3x^2)^4$ **f** $y=((4x-3)^2-2x)^3$

g $y=\dfrac{1}{4x+1}$ **h** $y=\dfrac{3}{(2x+1)^2}$ **i** $y=\dfrac{2}{(x^2+3x)^3}$

j $y=\sqrt{2+3x}$ **k** $y=\sqrt[3]{1-2x}$ **l** $y=\sqrt[4]{1+2x-3x^2}$

m $y=\dfrac{1}{\sqrt{4x+1}}$ **n** $y=\dfrac{3}{\sqrt[3]{2x+1}}$ **o** $y=\dfrac{2}{\sqrt[4]{x^2+3x}}$

2 Find the second derivative of each of the following functions:

a $(4x+1)^3$ **b** $\sqrt{3x-2}$ **c** $\dfrac{1}{\sqrt{2x+4}}$

d $\dfrac{1}{\sqrt{4x+1}}$ **e** $\dfrac{9}{\sqrt[3]{2x+1}}$ **f** $\dfrac{1}{\sqrt{2x}}+\dfrac{1}{\sqrt{2x+1}}$

3 **a** Given that $y=(x^2+2x)^3$, find $\dfrac{dy}{dx}$

b Expand $(x^2+2x)^3$ using the binomial theorem.

c Differentiate your answer from part **b** and show that it is equivalent to your answer from **a**.

4 Find the gradient of the curve $y=\dfrac{1}{(x-3)^2}$ at the point where $x=4$.

5 Find the point on the curve $y=\dfrac{1}{(x-2)^2}$ where the gradient is 2.

6 A curve has the equation $y=2+\dfrac{8}{(x-2)^2}$.

a Find the equation of the tangent to the curve at the point where it crosses the y-axis.

b Find the equation of the normal to the curve at the same point.

c Find the coordinates of the point where this normal cuts the curve again.

13.9 Maximising and minimising

With the exception of quadratic functions (where we can complete the square) and trigonometric functions that can be simplified, the standard technique to find a maximum or a minimum is to find a formula for the quantity that you are interested in and then differentiate it with respect to the variable.

So far, we have differentiated polynomial expressions only. However, all continuous functions can be differentiated (although it is possible to create functions that cause trouble). Later, we will look at trigonometric and exponential functions.

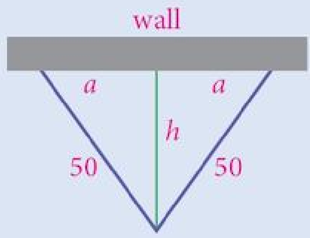
Example 13.9

A farmer wants to create a temporary pen for his sheep.

He has 100 m of fencing and a stone wall available.

He decides that the shape of his pen will be an isosceles triangle.

How far from the wall should the vertex of the triangle be in order to create the maximum area inside the pen?



Solution:

Let the distance of the vertex from the wall be h and let the fence join the wall at two points, distance $2a$ apart.

Area of pen $\quad A = ha$

But $\quad h^2 + a^2 = 50^2$

So $\quad A = h\sqrt{50^2 - h^2}$

We can write this as $\quad A = \sqrt{50^2 h^2 - h^4}$

Differentiate both sides with respect to h. $\quad \dfrac{dA}{dh} = \dfrac{1}{2} \times \dfrac{(5000h - 4h^3)}{\sqrt{50^2 h^2 - h^4}}$

We need $\dfrac{dA}{dh} = 0$ $\quad 0 = \dfrac{1}{2} \times \dfrac{(5000h - 4h^3)}{\sqrt{50^2 h^2 - h^4}}$

A fraction = 0 when the numerator = 0.

$0 = 5000h - 4h^3$

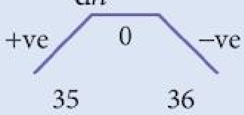
$h = 0$ will obviously not give maximum area!

So $\quad h = 0 \quad$ or $\quad h^2 = 1250$

$h = 35.36$

When $h = 35$, $\quad \dfrac{dA}{dh} = 1.4$

When $h = 36$, $\frac{dA}{dh} = -2.65$

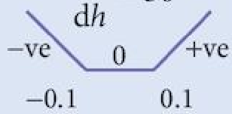


This is a maximum.

The preferred method is to find the 2nd derivative but, at this stage, we cannot do that for this function.

When $h = -0.1$, $\frac{dA}{dh} = -50$

When $h = 0.1$, $\frac{dA}{dh} = 50$

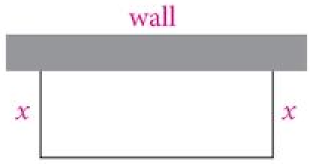


This is a minimum.

Thus, the maximum area is when $h = 35.36$ m and the area is 1250 m^2.

Exercise 13.5

1 A farmer wants to create a temporary pen for his sheep. He has 100 m of fencing and a stone wall available. He decides that the shape of his pen will be a rectangle. Find the dimensions of the pen so that the area inside is a maximum.



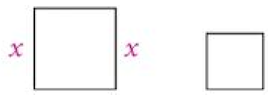
2 A sheep pen is constructed in the open field. It is rectangular in shape with one pair of edges of length x m. The perimeter of the pen is 100 m. Show that, in order to make the area a maximum, the pen must be square.

3 The farmer needs to pen his ram seperately from his ewes. In order to do this, he makes two square pens from his 100 m of fencing.

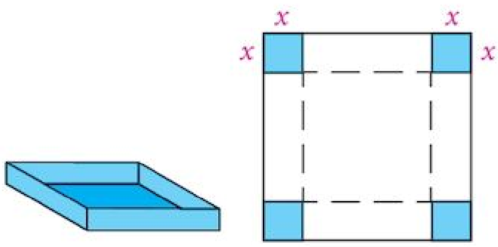
If the larger pen has sides of x m,

a find the dimensions of the smaller pen

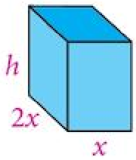
b find the value of x in order to create the minimum total area.



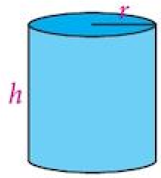
4 An open box is made from a square of board of side 30 cm. The corners of the square are removed and the edges are folded up to create the box. If each corner square cut out has side x cm, find the dimensions of the box that will create a maximum volume.



5 A closed box is constructed to hold muesli. It must have a volume of 3000 cm^3. The base of the box is a rectangle with one edge twice as long as another. In order to reduce costs, the surface area must be as small as possible. Find the dimensions of the box in order to do this.



6 A metal can to hold baked beans is in the shape of a cylinder and is constructed to have a volume of 200 cm^3. The base radius is r cm and the height is h cm. For environmental reasons, the amount of metal used in making the can must be minimised.



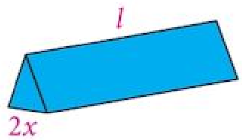
a Find the dimensions of the can in order to do this.

b What is the relationship between h and r?

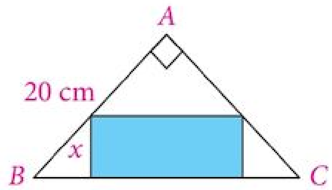
c If you wanted to make a can with twice the volume, what should the dimensions be?

d In practice, is this what happens? (You should visit your local store to investigate this.)

7 A container is made in the shape of a triangular prism with a length of l cm.
The cross section is an equilateral triangle of edge $2x$ cm.
The volume of the container is to be 500 cm^2.
Find the dimensions of the container in order to minimise its surface area.



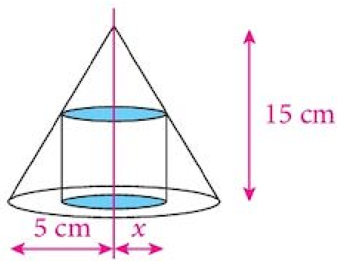
8 ABC is an isosceles triangle whose equal sides, $AB = AC = 20$ cm. A rectangle is drawn inside the triangle as shown so that one edge lies along the edge BC of the triangle. The height of the rectangle is x cm. Find the value of x in order to maximise the area of the rectangle.



9 A right circular cone of base radius 5 cm and height 15 cm has, within it, a cylinder of base radius x cm so that the axis of the cylinder is also the axis of the cone.

a Show that the volume of the cylinder is given by $V = 15\pi x^2 - 3\pi x^3$.

b Find the maximum value of V and the corresponding value of x.



Summary

Stationary points

Stationary means that the value of the function is not changing, even for a very short period of time.

The gradient at a stationary point is zero.

So **1** find $\frac{dy}{dx}$

2 solve $\frac{dy}{dx} = 0$

Types of stationary point

There are three types of stationary point:

		 or 
Maximum	Minimum	Stationary point of inflexion

"Maximum" means "local maximum". "Minimum" means "local minimum".

Identifying stationary points

1 Investigate the value of the gradient either side (but close to) the stationary point.

2 In most cases the better option is to find the value of the second derivative at the stationary point.

–ve gives a maximum; +ve gives a minimum.

Small Increments

$$\delta y = \frac{dy}{dx}\delta y$$

The chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$$

Connected rates of change

The chain rule $\frac{dV}{dt} = \frac{dV}{da} \times \frac{da}{db} \times \frac{db}{dt}$

You can make the chain as long as you like provided you start and finish in the correct places.

Composite functions

If $y = f(g(x))$

Where $y = f(u)$ and $u = g(x)$

Then, using the chain rule $\dfrac{dy}{dx} = \dfrac{dy}{du} \times \dfrac{du}{dx}$

Alternatively, $[f(g(x))]' = f'(g(x)) \times g'(x)$

Maximising and minimising

1 Find a formula for the variable you are trying to maximise (minimise).

2 Differentiate and put the derivative = 0.

3 Check the second derivative to confirm either maximum or minimum.

Chapter 13 Summative Exercise

1 Find the stationary values of the following functions and determine their nature.

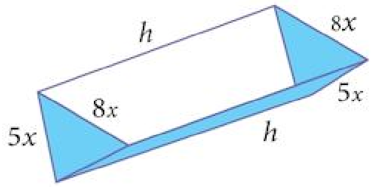
a $y = 2x^3 - 3x^2 - 12x + 5$

b $y = 6x^4 - 12x^3 - 30x^2 + 72x - 5$

c $y = x + \dfrac{4}{x} + 2$

d $y = 16\sqrt{x} + \dfrac{27}{x}, x \geq 0$

2 A water trough is in the shape of a triangular prism.
The dimensions are shown in the diagram.
The volume of the trough is 900 units³.



a Show that the total surface area of the trough (no lid),

$A = 24x^2 + 10xh$.

b Write the surface area A in terms of x alone.

c Find $\dfrac{dA}{dx}$ and find the value of x that produces a stationary value of A.

d Find this stationary value of A and determine its nature.

3 Given $10^3 = 1000$, use the function $y = x^3$ to find an approximate value of 10.001^3.

4 Given $f(x) = (1 + 2x)^7$ and $f(0) = 1$, find an approximate value of $f(0.001)$.

5 The circumference of a circle is increasing at a rate of 4 cm s^{-1}. Find the rate of increase of:

a the radius of the circle

b the area of the circle when the radius is 10 cm.

6 Find the approximate value of $f(3.01)$ if:

a $f(x) = x^3 - 4x^2 + 3x$

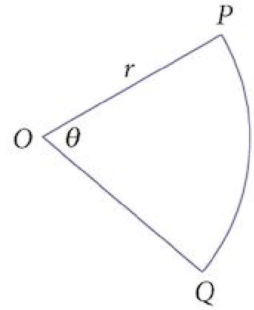
b $f(x) = 4x^2 - \dfrac{18}{x^2}$

7 A solid cylinder has a radius of r cm and a height of $4r$ cm.

a Write down expressions for **(i)** the volume V and **(ii)** the total surface area A of the cylinder.

b Find an approximate increase in **(i)** the volume V and **(ii)** the total surface area A of the cylinder if r increases by a small amount p, from 4 to $(4 + p)$.

8 A wire, of length 16 cm is bent to form a sector of a circle, centre O and radius r.

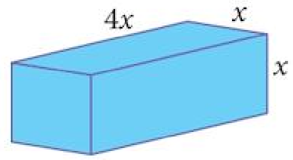


a Express θ in terms of r.

b Find the rate of change of the angle θ if the radius is increasing at a rate of 0.5 cm min^{-1} when $r = 2$ cm.

c Find the rate of increase of the area of the sector if the radius is increasing at a rate of 0.5 cm min^{-1} when $r = 2$ cm.

9 A block of ice in the shape of a cuboid with a square cross-section of side x, and length $4x$, is left outside and is gradually melting.



a Find, in terms of x:

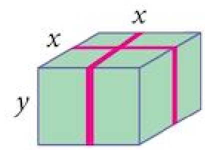
(i) the volume of the block of ice

(ii) the total surface area of the block of ice.

The ice is melting in such a way that the surface area is shrinking at a rate of 6 m^2 min^{-1}.

b Find the rate of change of the volume of the block when $x = 3$ m.

10 Sarah has received a birthday present contained in a box tied up with red ribbon.



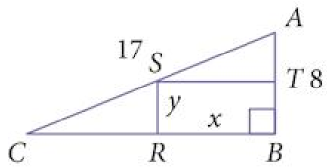
The ribbon passes all around the box in the manner shown and has a total length of 240 cm. The box has a square base of side x cm, and a height of y cm.

a Find an expression for the total length of ribbon.

b Find the value of x for which the volume has a stationary value and determine the nature of this stationary value.

11 In the right-angled triangle ABC, $AB = 8$ cm and $AC = 17$ cm. The rectangle $BRST$ is drawn so that S lies on AC, T lies on AB and R lies on BC.

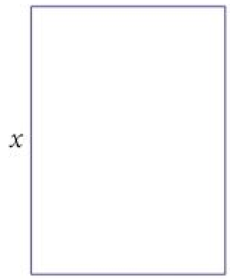
Given that $BR = x$ cm and $BT = y$ cm, find a relationship between x and y. Find the area of the rectangle in terms of x and hence calculate the maximum value of the area as x varies.



Chapter 13 Test

1 hour

1 A piece of wire of length 150 cm is formed into a rectangle as shown. The length of one side is x cm.



a Show that the area, A cm^2, of the rectangle can be written as $A = 75x - x^2$. [2]

b Given that x can vary, find the stationary value of A and determine the nature of this stationary value. [4]

2 A curve has equation $y = 4x + \dfrac{1}{x-4}$.

a Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$. [4]

b Find the coordinates of the stationary points on the curve. [4]

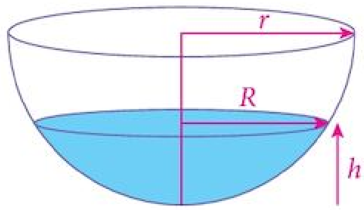
c Determine the nature of each of the stationary points. [2]

3 **a** Given $y = \sqrt{(2x+1)^3}$, find $\dfrac{dy}{dx}$. [2]

b The value of x is increased by a small amount, p, from 4 to $4 + p$.

Find the approximate corresponding increase in the value of y. [2]

4 When a hemispherical bowl, of radius r cm, contains water to a depth of h cm, the volume of water in the bowl is given by $V = \frac{\pi}{3}(3rh^2 - h^3)$.



a Show that the radius of the water surface, R, is given by $R = \sqrt{2rh - h^2}$ [2]

b Water is being poured into the bowl at a constant rate of 400 cm^3 min^{-1}.

Find, in terms of r, the rate of increase of the water surface area when $2h = r$. [4]

5 The surface area, S, and volume, V, of a sphere are given by $S = 4\pi r^2$ and $V = \dfrac{4}{3}\pi r^3$.

If r is increasing at a rate of 0.2 cm s^{-1} when $r = 3$, find the corresponding rate of increase of

a the surface area, S [4]

b the volume, V. [4]

6 **a** Find the coordinates of the stationary points on the curve $y = x^2 + \dfrac{54}{x}$. [4]

b Determine the nature of the stationary points. [2]

Examination Questions

1 A curve has the equation $y = \dfrac{8}{2x-1}$.

(i) Find an expression for $\dfrac{dy}{dx}$. [3]

(ii) Given that y is increasing at a rate of 0.2 units per second when $x = -0.5$, find the corresponding rate of change of x. [2]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2005, P2, Qu 1]

2 A cuboid has a total surface area of 120 cm^2. Its base measures x cm by $2x$ cm and its height is h cm.

(i) Obtain an expression for h in terms of x. [2]

Given that the volume of the cuboid is V cm^3,

(ii) show that $V = 40x - \dfrac{4x^3}{3}$. [1]

Given that x can vary,

(iii) show that V has a stationary value when $h = \dfrac{4x}{3}$. [4]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2006, P2, Qu 9]

3 A curve has the equation $y = \sqrt{x} + \dfrac{9}{\sqrt{x}}$.

(i) Find expressions for $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$. [4]

(ii) Show that the curve has a stationary value when $x = 9$. [1]

(iii) Find the nature of this stationary value. [2]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2007, P1, Qu 5]

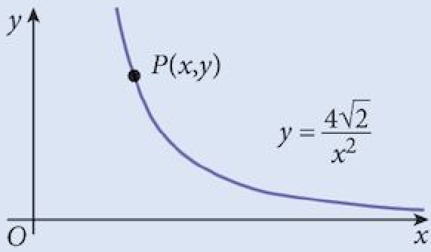
4 Two variables, x and y, are related by the equation $y = 6x^2 + \dfrac{32}{x^3}$.

(i) Obtain an expression for $\dfrac{dy}{dx}$. [2]

(ii) Use your expression to find the approximate change in the value of y when x increases from 2 to 2.04. [3]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2009, P2, Qu 5]

5



The diagram shows part of the curve $y = \frac{4\sqrt{2}}{x^2}$. The point $P(x, y)$ lies on this curve.

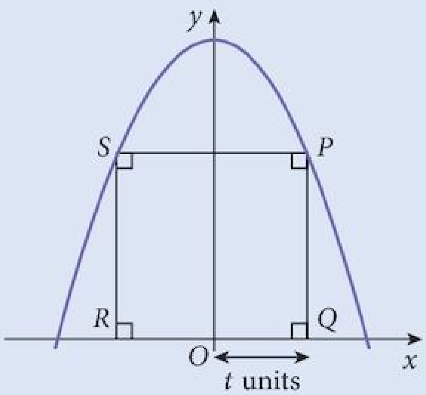
(i) Write down an expression, in terms of x, for $(OP)^2$. [1]

(ii) Denoting $(OP)^2$ by S, find an expression for $\frac{dS}{dx}$. [2]

(iii) Find the value of x for which S has a stationary value and the corresponding value of OP. [3]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2008, P2, Qu 7]

6



The diagram shows part of the curve $y = 27 - x^2$. The points P and S lie on this curve. The points Q and R lie on the x-axis and $PQRS$ is a rectangle. The length of OQ is t units.

(i) Find the length of PQ in terms of t and hence show that the area, A square units, of $PQRS$ is given by $A = 54t - 2t^3$. [2]

(ii) Given that t can vary, find the value of t for which A has a stationary value. [3]

(iii) Find this stationary value of A and determine its nature. [3]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2009, P2, Qu 8]

7 The equation of a curve is $y = \frac{8}{(3x-4)^2}$.

(i) Find the gradient of the curve where $x = 2$. [3]

(ii) Find the approximate change in y when x increases from 2 to $2 + p$, where p is small. [2]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2006, P1, Qu 3]

14 Matrices

Syllabus statements

- display information in the form of a matrix of any order and interpret the data in a given matrix
- state the order of a given matrix
- solve problems involving the calculation of the sum and product (where appropriate) of two matrices and interpret the results
- calculate the product of a scalar quantity and a matrix
- use the algebra of 2×2 matrices (including the zero, $\mathbf{O}$, and identity, $\mathbf{I}$, matrix)
- calculate the determinant and inverse, $\mathbf{A}^{-1}$, of a non-singular 2×2 matrix and solve simultaneous linear equations.

14.1 Introduction

This is another topic that you have met before. In this chapter, we extend your knowledge of the use of matrices in solving problems.

We begin with a reminder of the rules for combining matrices and finding the inverse. We then move on to the use of the inverse in solving problems such as finding the solution of simultaneous equations.

Although you can already do this for simple equations, the ultimate aim is to find a way of solving large systems of simultaneous equations. This is usually done by computers using matrix techniques.

Example 14.1

A survey of the price of fruit produced the following results:

Item	Supermarket A	Supermarket B
Bananas	\$6 per kg	\$5 per kg
Dragon Fruit	\$4 each	\$5 each
Mangoes	\$10 per kg	\$9 per kg
Oranges	\$3 each	\$4 each
Papayas	\$5 per kg	\$4 per kg
Pineapples	\$7 per kg	\$6 per kg
Watermelon	\$4 per kg	\$5 per kg

Provided that we can remember what each row represents and what each column represents, all we really need are the values from the table.

This is particularly true if the data is going to be fed into a computer database. The computer will not forget what the rows mean.

We can create this matrix from the data:

$$\begin{pmatrix} 6 & 5 \\ 4 & 5 \\ 10 & 9 \\ 3 & 4 \\ 5 & 4 \\ 7 & 6 \\ 4 & 5 \end{pmatrix}$$

14.2 Notation

We describe matrices by their shape. The matrix in Example 14.1 is a (7 by 2) matrix.

That is, 7 rows and 2 columns. We call this the **order** of the matrix.

Matrices are written within rounded brackets (...)

Sometimes the notation (7×2) is used. However, the "$\times$" does not mean multiply.

Sometimes square brackets are used. [...]

We use bold, upper case letters as names of matrices: **A**, **B**, **C**, ...

The numbers in a matrix are called the **elements** or **entries**.

A matrix with only one row is called a **row matrix**:

$(2 \quad 4 \quad 7 \quad 3)$ is a (1 by 4) matrix.

Similarly, we have a **column matrix**: $\begin{pmatrix} 5 \\ -8 \\ 3 \end{pmatrix}$ is a (3 by 1) matrix.

We do not use separators (commas) between the elements.

You cannot easily write in bold characters, so you should indicate a matrix in some other way.

$\underline{A}$, or $\vec{A}$, or $\underset{\sim}{A}$ are all methods for writing the names of matrices.

You must write them clearly.

14.3 Matrix algebra

14.3.1 Equality

Two matrices are equal if:

1 they have the same order

2 each pair of corresponding elements are equal.

14.3.2 Addition and subtraction

Two matrices can be added or subtracted if:

1 they have the same order

2 corresponding elements represent the same sort of thing.

To perform the addition, corresponding elements are added together.

Note: 2 is not a problem if all elements are numbers but you cannot add the cost of mangoes to the height of a wall.

Example 14.2

If $\mathbf{A} = \begin{pmatrix} 2 & 4 & 3 \\ -1 & 5 & 1 \\ 6 & -2 & -3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & 0 & 3 \\ 2 & 1 & 3 \\ -1 & 4 & 1 \end{pmatrix}$

find **a** $\mathbf{A} + \mathbf{B}$ **b** $\mathbf{A} - \mathbf{B}$

Solution:

a

$$\mathbf{A} + \mathbf{B} = \begin{pmatrix} 2 & 4 & 3 \\ -1 & 5 & 1 \\ 6 & -2 & -3 \end{pmatrix} + \begin{pmatrix} 2 & 0 & 3 \\ 2 & 1 & 3 \\ -1 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 4 & 6 \\ 1 & 6 & 4 \\ 5 & 2 & -2 \end{pmatrix}$$

b

$$\mathbf{A} - \mathbf{B} = \begin{pmatrix} 2 & 4 & 3 \\ -1 & 5 & 1 \\ 6 & -2 & -3 \end{pmatrix} - \begin{pmatrix} 2 & 0 & 3 \\ 2 & 1 & 3 \\ -1 & 4 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 4 & 0 \\ -3 & 4 & -2 \\ 7 & -6 & -4 \end{pmatrix}$$

14.3.3 Scalar multiplication

From the previous example,

$$\mathbf{A} + \mathbf{A} = \begin{pmatrix} 2 & 4 & 3 \\ -1 & 5 & 1 \\ 6 & -2 & -3 \end{pmatrix} + \begin{pmatrix} 2 & 4 & 3 \\ -1 & 5 & 1 \\ 6 & -2 & -3 \end{pmatrix} = \begin{pmatrix} 4 & 8 & 6 \\ -2 & 10 & 2 \\ 12 & -4 & -6 \end{pmatrix}$$

In ordinary algebra, we would usually write $\quad \mathrm{A} + \mathrm{A} = 2\mathrm{A}$.

We do the same here: $\quad \mathbf{A} + \mathbf{A} = 2\mathbf{A}$

Notice, however, that while **A** is bold, 2 is not.

2 is a number (scalar) and not a matrix.

Thus we have found a sensible interpretation of "multiplying a matrix by a scalar".

Every element in the matrix has been multiplied by 2.

> To calculate $n\mathbf{M}$, where **M** is a matrix and n is a scalar, we multiply every element of **M** by n.

14.3.4 The zero matrix

Technically, zero is the identity for addition.

In other words, when you add zero to something, it does not change.

In the same way, we can have a zero matrix.

$$0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

There is a zero matrix of every order.

While this idea may seem trivial, the zero matrix is just as important as the number zero.

Example 14.3

Find the values of the variables a, b, c and d in the matrix equation:

$$\begin{pmatrix} a & 4 \\ 1 & 5 \end{pmatrix} + \begin{pmatrix} 3 & b \\ c & 2 \end{pmatrix} = \begin{pmatrix} 5 & 10 \\ 4 & d \end{pmatrix}$$

Solution:

Adding the matrices gives us four equations: $a + 3 = 5$

$4 + b = 10$

$1 + c = 4$

$5 + 2 = d$

from which $\quad a = 2,\ b = 6,\ c = 3$ and $d = 7$

Exercise 14.1

Use matrices **A** to **F** for questions 1 to 3.

$$\mathbf{A} = \begin{pmatrix} -1 & 3 \\ 2 & -1 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 4 & -2 & -3 \end{pmatrix} \qquad \mathbf{C} = \begin{pmatrix} 1 & -4 & 2 & 3 \\ -2 & 3 & 1 & 2 \\ -1 & 2 & 5 & 0 \end{pmatrix}$$

$$\mathbf{D} = \begin{pmatrix} 2 & 4 \\ -1 & -3 \end{pmatrix} \qquad \mathbf{E} = \begin{pmatrix} 3 & 1 & 2 \\ -2 & -1 & 0 \\ -1 & 2 & 3 \end{pmatrix} \qquad \mathbf{F} = \begin{pmatrix} -2 & 5 & -3 & -1 \\ 0 & -2 & 4 & 3 \\ -1 & 0 & 0 & 4 \end{pmatrix}$$

1 What are the orders of the matrices **A** to **F**?

2 Can you find **B** + **E**? Explain.

3 Calculate the following:

a 2**A** b 4**B** c 3**C** d **A** + **D**

e **C** – **F** f 2**D** – **A** g 2**F** + 3**C** h **A** – **A**

4 Find the values of the variables a, b, c and d in this matrix equation.

$$\begin{pmatrix} a & 4 & 3 \\ -1 & 5 & 1 \\ c & -2 & -3 \end{pmatrix} = \begin{pmatrix} 2 & 4 & a+b+d \\ -1 & b & 1 \\ 6 & -2 & -3 \end{pmatrix}$$

5 Find the values of the variables a, b, c and d in this matrix equation.

$$\begin{pmatrix} a & 3 \\ -2 & 4 \end{pmatrix} + \begin{pmatrix} -2 & 5 \\ b & -1 \end{pmatrix} = \begin{pmatrix} 2 & b-c \\ 5 & a+d \end{pmatrix}$$

14.3.5 Matrix multiplication

What does it mean to multiply matrices?

We must find a sensible use of matrices that looks like multiplication.

Example 14.4

Aisyah wanted to buy some fruit for the family.

Her shopping list was:

Bananas	2 kg
Dragon Fruit	3
Mangoes	4 kg
Papayas	6 kg

Use the same supermarket data from Example 14.1.

Which supermarket should she go to in order to get the lowest total cost?

Solution:

We can create a matrix for her order:

$$\begin{pmatrix} 2 & 3 & 4 & 6 \end{pmatrix}$$

Note that this is a row matrix.

The cost of her shopping at Supermarket A would be

$2 \times 6 + 3 \times 4 + 4 \times 10 + 6 \times 5 = \$94$

We choose to write this calculation using matrices in this format:

$$\begin{pmatrix} 2 & 3 & 4 & 6 \end{pmatrix} \begin{pmatrix} 6 \\ 4 \\ 10 \\ 5 \end{pmatrix} = 94$$

Transposing both matrices would give exactly the same result:

$$\begin{pmatrix} 6 & 4 & 10 & 5 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 4 \\ 6 \end{pmatrix} = 94$$

Notice how the first is a row matrix and the second is a column matrix.

Numbers are multiplied together in pairs, in order, and the results added to get the total.

Similarly, the cost of her shopping at Supermarket B would be

$2 \times 5 + 3 \times 5 + 4 \times 9 + 6 \times 4 = \$85$

We choose to write this calculation using matrices in this format:

$$\begin{pmatrix} 2 & 3 & 4 & 6 \end{pmatrix} \begin{pmatrix} 5 \\ 5 \\ 9 \\ 4 \end{pmatrix} = 85$$

Again, transposing both matrices would give exactly the same result.

Notice how the calculation is done:

1. Each quantity is multiplied by the price of that fruit and the total cost is found by addition.
2. The first matrix is written as a row matrix and the second is written as a column matrix.

We could combine these calculations into one like this:

$$\begin{pmatrix} 2 & 3 & 4 & 6 \end{pmatrix} \begin{pmatrix} 6 & 5 \\ 4 & 5 \\ 10 & 9 \\ 5 & 4 \end{pmatrix} = \begin{pmatrix} 94 & 85 \end{pmatrix}$$

This transposed alternative would work just as well.

$$\begin{pmatrix} 6 & 4 & 10 & 5 \\ 5 & 5 & 9 & 4 \end{pmatrix} \begin{pmatrix} 2 \\ 3 \\ 4 \\ 6 \end{pmatrix} = \begin{pmatrix} 94 \\ 85 \end{pmatrix}$$

Now, each calculation is a row multiplied by a column.

In order for this to work, the number of elements in the row must be the same as the number in the column for them to be compatible.

We call this process **matrix multiplication**.

We can extend the idea to include more fruit order rows, or more Supermarket price columns.

Example 14.5

Multiply the following two matrices.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 3 & 4 & 1 & 3 & 2 \\ 3 & 4 & 2 & 5 & 6 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 45 & 39 & 42 \\ 31 & 33 & 26 \\ 36 & 46 & 44 \end{pmatrix}$$

Each row is multiplied by 3 columns to produce 9 elements in the result matrix.

The number 33 in the centre is the magenta row multiplied by the orange column.

When we multiply:

$(m \text{ by } n) \quad \times \quad (n \text{ by } q) \quad = \quad (m \text{ by } q)$

same

You should always check that the matrices you are multiplying together are compatible and that each calculation is sensible.

Example 14.6

The table below shows the current state of the inter-house netball league.

	Played	Won	Drew	Lost
Amethyst	7	2	2	3
Diamond	8	6	2	0
Saphire	8	3	0	5
Topaz	7	1	2	4

A win scores 3 points, a draw scores 1 point and a loss scores 0 points.

a Write down 2 matrices from which to calculate the total score of each team.

b Multiply your matrices together to find the current total scores for each team.

Solution:

a The matrix for games results is:

$$\begin{pmatrix} 2 & 2 & 3 \\ 6 & 2 & 0 \\ 3 & 0 & 5 \\ 1 & 2 & 4 \end{pmatrix}$$

The matrix for points per game is:

$$\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix}$$

The number of games played is not important. If we included it, we would need another row in the points matrix awarding zero points for playing a game.

b Check for compatibility and work out the order of the result:

(4 by 3) × (3 by 1) = (4 by 1)

Calculate the result:

$$\begin{pmatrix} 2 & 2 & 3 \\ 6 & 2 & 0 \\ 3 & 0 & 5 \\ 1 & 2 & 4 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 8 \\ 20 \\ 9 \\ 5 \end{pmatrix}$$

Notice that the rows represent the teams.

The current totals for each team are: Amethyst 8, Diamond 20, Saphire 9, Topaz 5.

Exercise 14.2

1 A manufacturer sells chocolate bars for the following prices (prices in $):

Crunch	Milk	Dark	Bubbly	Nutty
3	4	6	2	5

At festival time it packages them in Festival boxes to make them more attractive. Each box contains the following number of bars of chocolate:

Crunch	Milk	Dark	Bubbly	Nutty
5	4	1	3	2

The bars are not all the same size. The mass of each, in grams, is as follows:

Crunch	Milk	Dark	Bubbly	Nutty
100	150	400	80	200

a Write down two matrices whose product will give the total cost of the Festival Box.

b Use your matrices to calculate the total cost of a Festival Box.

c Write down two matrices whose product will give the total mass of the Festival Box.

d Use your matrices to calculate the total mass of a Festival Box.

2 A DIY store sells collections of screws as follows:

Screw size	15 mm	25 mm	40 mm	50 mm	75 mm
Economy	100	50	30	10	5
Enthusiast	200	150	100	100	50
Professional	500	400	400	200	100
Price ($ per 100)	4	5	6	8	10

a Write down two matrices whose product will give the total number of screws in each type of pack.

b Use your matrices to calculate the total number of screws in each type of pack.

c Write down two matrices whose product will give the total cost of each type of pack.

d Use your matrices to calculate the total cost of each type of pack.

3 A fruit stall has orders for the following:

Customer \ Fruit	Apples	Oranges	Watermelon	Mangoes	Papayas
A	10	5	1	6	3
B	8	10	2	12	1
C	5	6	1	4	2
D	4	5	3	8	1
Price ($ each)	3	4	8	2	6

a Write down two matrices whose product will give the total cost for each customer.

b Use your matrices to calculate the total cost for each customer.

c Write down two matrices whose product will give the total orders for each type of fruit.

d Use your matrices to calculate the total orders for each type of fruit.

e Write down a product of three matrices that will give the total value of all the orders.

f Use your matrices to calculate this total value.

4 A manufacturer assembles machinery from different components as follows:

Component \ Item	P	Q	R	S	T	Price ($) each	Tax ($) each
Resistors	20	30	10	15	25	2	0
Capacitors	10	40	15	20	30	5	0
Microchips	5	4	1	3	4	20	5
Circuit board	1	1	1	1	1	30	10

The government charges tax on microchips and circuit boards at the rate shown. It is necessary to show the tax charged seperately from the price. The final price will be the manufacturing price plus the tax.

a Write down two matrices whose product will give the manufacturing price and, seperately, the tax charged for each item as a matrix.

b Calculate the product of your matrices.

c Write down a matrix which, when multiplied by your answer to **b** will give the total price of each item.

d Calculate the total price for each item.

e Write down a product of three matrices that will produce the same result.

f Use your matrices to calculate this total value.

5 A bus company has three types of bus: Bus, Coach and Express. They are all the same model but have different numbers of seats allowing different levels of comfort:

Bus 60 Coach 50 Express 36

Over a busy holiday period, on its major route, it sends 4 Buses, 8 Coaches and 3 Expresses. It charges \$20 for a Bus seat, \$30 for a Coach seat and \$50 for a seat on an Express.

If all the buses are full:

a Explain what the following matrix product will produce:

$$\underset{\mathbf{A}}{\begin{pmatrix} 4 & 8 & 3 \end{pmatrix}} \quad \underset{\mathbf{B}}{\begin{pmatrix} 60 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 36 \end{pmatrix}} \quad \underset{\mathbf{C}}{\begin{pmatrix} 20 \\ 30 \\ 50 \end{pmatrix}}$$

b Calculate the product.

c The matrix product could be written as (**AB**)**C**, or as **A**(**BC**).

What difference does it make to the final answer if you change the order of the calculations?

d What information would the product **(i) AB, (ii) BC** give you?

e What information would the product **AC** give you?

6 A wholesaler sells rice in four grades, all in 5 kg bags. Economy costs \$4, Brown costs \$8, Jasmine \$11 and Basmati costs \$20. They supply four customers whose orders (number of 5 kg bags) are shown in the table.

Grade / Customer	Economy	Brown	Jasmine	Basmati
A	20	15	10	5
B	12	10	8	12
C	4	8	12	20
D	10	20	15	15

a Use matrices to calculate the total cost to each customer.

b Use matrices to calculate the total mass of each type of rice sold.

14.4 More matrix algebra

14.4.1 The associative law

Matrix multiplication is **associative**.

$$(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$$

From Exercise 14.2 Question 5.

You can move the brackets to make the calculation easier.

14.4.2 The commutative law

Matrix multiplication is **NOT commutative**.

$$\mathbf{AB} \neq \mathbf{BA}$$

Most of the time, they will be incompatible if you change the order. Even if they are still compatible the result may be different.

You cannot change the order of the matrices in a multiplication.

14.4.3 The distributive law

Matrix multiplication is **distributive**.

$$(\mathbf{A} + \mathbf{B})\mathbf{C} = \mathbf{AC} + \mathbf{BC}$$

and

$$\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$$

You can expand the brackets or factorise expressions.

14.5 The algebra of (2 by 2) matrices

14.5.1 Compatibility

If we multiply two (2 by 2) matrices, the result will also be a (2 by 2) matrix.

$(2 \text{ by } 2) \times (2 \text{ by } 2) = (2 \text{ by } 2)$

This is true for any order of square matrices. (2 by 2) matrices can be used to transform the x-y plane. Higher orders are used to transform 3- or 4-dimensional spaces.

14.5.2 The identity matrix (I)

An **identity** matrix is one which has no effect when we multiply another matrix by it.

The word **identity** has another meaning later.

$$\underbrace{(n \text{ by } n)}_{\text{Must be square}} \times (n \text{ by } q) = (n \text{ by } q)$$

same

If the matrix is to have no effect, it must be a **square** matrix.

The identity element for multiplication of numbers is 1.

The identity element for addition of numbers is 0.

The (2 by 2) identity matrix is $\mathbf{I}_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

We often leave out the "2".

For any (2 by 2) matrix **A**: $\mathbf{IA} = \mathbf{AI} = \mathbf{A}$

Choose a matrix and try it.

This does not conflict with the commutative Law. It is a special case for Identity matrices.

14.5.3 Inverse matrices

An **inverse** matrix is one which reverses the effect of a matrix.

There is no such thing as "division" of matrices. We have to go back to see what "division" means.

When we use a matrix to transform an object to produce an image, the **inverse** matrix will transform the image back to the original object. Thus, it reverses the effect of the matrix.

So, we need to find a matrix $\mathbf{A}^{-1}$ such that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

In fact, $\mathbf{A}^{-1}\mathbf{A} = \mathbf{AA}^{-1}$

Example 14.7

If $\mathbf{A} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$, find a matrix $\mathbf{A}^{-1}$ so that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

Solution:

If $\mathbf{A}^{-1} = \begin{pmatrix} p & r \\ q & s \end{pmatrix}$, then $\begin{pmatrix} p & r \\ q & s \end{pmatrix}\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

we have 4 simultaneous equations:

$pa + rb = 1$ [1] $\quad pc + rd = 0$ [2]

$qa + sb = 0$ [3] $\quad qc + sd = 1$ [4]

[1] $\times c$ $\quad pac + rbc = c$

[2] $\times a$ $\quad pca + rda = 0$

from which $\quad r = \dfrac{c}{bc - ad}$ or $r = \dfrac{-c}{ad - bc}$

similarly $\quad p = \dfrac{d}{ad - bc}$

From [3] and [4], we get

$$q = \frac{-b}{ad - bc}$$

Similarly $$s = \frac{a}{ad - bc}$$

So, $$\mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

14.5.4 The determinant of a matrix

The quantity $ad - bc$ is called the **determinant** of the matrix.

We use the notation $\det(\mathbf{A})$ or just $\det \mathbf{A}$.

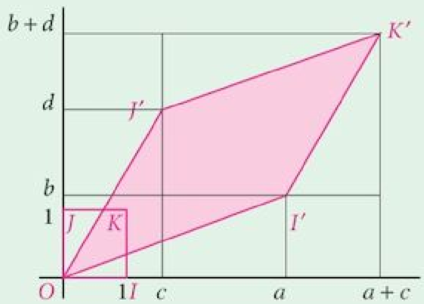
Only square matrices have a determinant.

Example 14.8

If $\mathbf{A} = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$, find the area of the image of the unit square $OIKJ$ where O is $(0, 0)$, I is $(1, 0)$, K is $(1, 1)$ and J is $(0, 1)$.

Solution:

$$\begin{pmatrix} a & c \\ b & d \end{pmatrix}\begin{array}{c} \begin{array}{cccc} O & I & K & J \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{cccc} O' & I' & K' & J' \end{array} \\ \begin{pmatrix} 0 & a & a+c & c \\ 0 & b & b+d & d \end{pmatrix} \end{array}$$



The image after the transformation will be a parallelogram.

The area is $(a + c)(b + d) - ab - cd - 2cb = ad - bc$

The area of the object (unit) square = 1.

Thus $ad - bc$ is the area scale factor for the transformation.

14.5.5 Forming the inverse of a matrix

Step 1	Find the determinant of the matrix.	$ad - bc$
Step 2	Swap the elements in the leading diagonal	a and d
Step 3	Multiply the elements in the other diagonal by -1.	c and b
Step 4	Divide each element by the determinant.	

Example 14.9

Find the inverse of the matrix $\mathbf{A} = \begin{pmatrix} 4 & 2 \\ 5 & 3 \end{pmatrix}$.

Solution:

Step 1: Find the determinant $\det(\mathbf{A}) = 4 \times 3 - 2 \times 5 = 2$

Step 2: Swap the elements in the leading diagonal $\begin{pmatrix} 3 & \\ & 4 \end{pmatrix}$

Step 3: Mulitply the other two elements by -1 $\begin{pmatrix} 3 & -2 \\ -5 & 4 \end{pmatrix}$

Step 4: Divide each element by the determinant $\begin{pmatrix} \frac{3}{2} & -1 \\ -\frac{5}{2} & 2 \end{pmatrix}$

$$\mathbf{A}^{-1} = \begin{pmatrix} \frac{3}{2} & -1 \\ -\frac{5}{2} & 2 \end{pmatrix}$$

14.5.6 Singular matrices

A matrix whose determinant is zero is called **singular**.

If $\det(\mathbf{A}) \neq 0$, it is called **non-singular**.

The consequences of this are:

1 We cannot find its inverse; the matrix does not have an inverse.

2 The transformed unit square has area = 0.

In fact, the parallelogram becomes a line and we lose a dimension.

Example 14.10

Find the determinant of the matrix $\mathbf{A} = \begin{pmatrix} 4 & 2 \\ 6 & 3 \end{pmatrix}$

and show that all points on the image of the unit square lie on a line.

Solution:

Step 1: Find the determinant: $\det(\mathbf{A}) = 4 \times 3 - 2 \times 6 = 0$

The images of the unit square are

$$\begin{pmatrix} 4 & 2 \\ 6 & 3 \end{pmatrix}\begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 4 & 6 & 2 \\ 0 & 6 & 9 & 3 \end{pmatrix}$$

The points (0, 0), (4, 6), (6, 9) and (2, 3) all lie on the line $2y = 3x$.

In fact, the image of any point in the plane will be on this line.

14.6 Solving matrix equations

To solve an equation, **you must do the same thing to both sides**.

This is a universal rule.

We can manipulate the equation $\mathbf{AB} = \mathbf{C}$

by multiplying both sides by a matrix $\mathbf{D}$.

Following the rule above, we could get $\mathbf{DAB} = \mathbf{DC}$

Multiplying both sides **in front**. This is called **pre-multiplying**.

or, we could get $\mathbf{ABD} = \mathbf{CD}$

Multiplying both sides on **the right**. This is called **post-multiplying**.

WARNING

But, we cannot get $\mathbf{DAB} = \mathbf{CD}$

nor $\mathbf{ABD} = \mathbf{DC}$

nor $\mathbf{ADB} = \mathbf{DC}$

We have not followed the rule. We have done different things to each side.

To solve the matrix equation $\mathbf{AX} = \mathbf{B}$

we pre-multiply by $\mathbf{A}^{-1}$ $\quad \mathbf{A}^{-1}(\mathbf{AX}) = \mathbf{A}^{-1}\mathbf{B}$

But multiplication is associative: $\quad (\mathbf{A}^{-1}\mathbf{A})\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$

$$\mathbf{IX} = \mathbf{A}^{-1}\mathbf{B}$$

So $\quad \mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$

$\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$
$\mathbf{IX} = \mathbf{X}$

When $\mathbf{X}$ is a column matrix (usually written $\mathbf{x}$), then so is $\mathbf{B}$ (usually written $\mathbf{b}$).

To solve the matrix equation $\mathbf{Ax} = \mathbf{b}$

we pre-multiply by $\mathbf{A}^{-1}$ $\quad \mathbf{A}^{-1}(\mathbf{Ax}) = \mathbf{A}^{-1}\mathbf{b}$

But multiplcation is associative: $\quad (\mathbf{A}^{-1}\mathbf{A})\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$

$$\mathbf{Ix} = \mathbf{A}^{-1}\mathbf{b}$$

So $\quad \mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$

Example 14.11

If $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ -4 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 2 \\ 5 & 4 \end{pmatrix}$, solve the equation $\mathbf{AX} = \mathbf{B}$.

Solution:

We need to find the inverse of **A**.

$$\det(A) = (2)\times(1) - (-4)\times(-1)$$
$$= -2$$

So $\mathbf{A}^{-1} = -\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 4 & 2 \end{pmatrix}$

$$\mathbf{AX} = \mathbf{B}$$

gives $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$

so $\mathbf{X} = -\frac{1}{2}\begin{pmatrix} 1 & 1 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 5 & 4 \end{pmatrix}$

It is easier to leave the fraction till the end.

$$= -\frac{1}{2}\begin{pmatrix} 6 & 6 \\ 14 & 16 \end{pmatrix}$$

$$= \begin{pmatrix} -3 & -3 \\ -7 & -8 \end{pmatrix}$$

14.7 Solving simultaneous equations

You already know how to solve simultaneous equations with two unknowns. This is an alternative method.

Problem 14.1

a Show that the pair of simultaneous equations

$$2x + 3y = 16$$
$$3x - 4y = 7$$

can be written as $\begin{pmatrix} 2 & 3 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 16 \\ 7 \end{pmatrix}$

This equation could also come from another problem: "What object point is transformed onto (16, 7) by the matrix?"

b Writing $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ 3 & -4 \end{pmatrix}$

Find $\mathbf{A}^{-1}$.

c Pre-multiply both sides of the matrix equation by $\mathbf{A}^{-1}$ to solve the equations.

Note: it is often easier to write $\mathbf{A}^{-1}$ as $\dfrac{1}{ad-bc}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$

rather than include the determinant within the matrix. The determinant may not be a nice value and you can divide by it at the end.

Example 14.12

Solve the pair of simultaneous equations:

$4x - 5y = 22$

$2x + 7y = -8$

Solution:

Write the equations in matrix form

$$\begin{pmatrix} 4 & -5 \\ 2 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 22 \\ -8 \end{pmatrix}$$

Find the inverse of the coefficient matrix

$$\frac{1}{38}\begin{pmatrix} 7 & 5 \\ -2 & 4 \end{pmatrix}$$

Pre-multiply both sides by the inverse matrix

$$\frac{1}{38}\begin{pmatrix} 7 & 5 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 4 & -5 \\ 2 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{38}\begin{pmatrix} 7 & 5 \\ -2 & 4 \end{pmatrix}\begin{pmatrix} 22 \\ -8 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{38}\begin{pmatrix} 114 \\ -76 \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$$

Exercise 14.3

1 The matrices **A** and **B** are given by $\mathbf{A} = \begin{pmatrix} 1 & -2 \\ 3 & 2 \end{pmatrix}$, and $\mathbf{B} = \begin{pmatrix} 2 & -2 \\ 1 & 4 \end{pmatrix}$.
Find matrices **P** and **Q** such that

a $\mathbf{P} = \mathbf{A}^2 - 2\mathbf{B}$

b $\mathbf{Q} = \mathbf{A}(\mathbf{B}^{-1})$

2 The matrices $\mathbf{A}$ and $\mathbf{B}$ are given by $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ -1 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 4 & 2 \\ 6 & 5 \end{pmatrix}$.

Find the values of p and q such that

a $\mathbf{A}^2 + 7\mathbf{I} = p\mathbf{A}$

b $\mathbf{B}^2 + 8\mathbf{I} = q\mathbf{B}$

3 The matrices $\mathbf{A}$ and $\mathbf{B}$ are given by $\mathbf{A} = \begin{pmatrix} 3 & -1 \\ -1 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & 1 \\ 6 & 4 \end{pmatrix}$.

Show that

a $5\mathbf{A}^{-1} = 5\mathbf{I} - \mathbf{A}$

b $2\mathbf{B}^{-1} = 6\mathbf{I} - \mathbf{B}$

4 Given that $\mathbf{A} = \begin{pmatrix} 4 & 2 \\ m & 1 \end{pmatrix}$ and that $\mathbf{A} + \mathbf{A}^{-1} = k\mathbf{I}$, where m and k are constants and $\mathbf{I}$ is the identity matrix, evaluate m and k.

5 Given that $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 3 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} -1 & 1 \\ 0 & 2 \end{pmatrix}$, find the matrices $\mathbf{X}$, $\mathbf{Y}$ and $\mathbf{Z}$ such that

a $\mathbf{X} = \mathbf{B}^2 + 2\mathbf{A}$

b $\mathbf{AY} = \mathbf{B}$

c $\mathbf{ZA} = \mathbf{B}$

6 Given that $\mathbf{B} = \begin{pmatrix} 1 & -1 \\ 1 & 2 \end{pmatrix}$, find $(\mathbf{B}^{-1})^2$.

7 Use matrices to solve the pairs of simultaneous equations:

a $2x - 5y = 24$, $4x - 3y = 20$

b $3x - y = 11$, $x - y = 7$

c $4x - 2y = 22$, $3x + 2y = 6$

d $5x - 2y = 13$, $2x - 6y = 26$

e $2x + y = 7$, $x + y = 3$

f $x - 3y = 4$, $4x - 3y = -2$

g $2x - y = 5$, $3x - 2y = 11$

h $2x - 3y = 9$, $x + y = 2$

i $3x - y = 11$, $9x - 2y = 28$

j $x + y = 5$, $4x + 3y = 19$

k $2x - 3y = 9$, $4x - y = 13$

l $4x - 7y = -5$, $3x + 2y = 18$

m $2x - 3y = 5$, $3x - 4y = 7$

n $3x - 2y = 7$, $2x + 3y = -4$

o $3x + 8y = 22$, $2x + 12y = 8$

Summary

Definitions A matrix is a data structure that can be used for many purposes. It looks a bit like a table in which the position of each element is important. You cannot move the element around without changing the matrix.

The order of a matrix The **order** of a matrix is given by the number of rows and columns of the matrix. Remember rows first, then columns. So (2 by 3) would have 2 rows and 3 columns.

Matrix algebra We can perform many operations on a matrix, usually using appropriate well-known algebraic operation symbols.

Addition/subtraction We add/subtract equivalent elements. This means that the matrices must have the same order. Also, corresponding elements must represent the same sort of thing.

Scalar multiplication In the matrix $k\mathbf{A}$, each element of $\mathbf{A}$ is multiplied by the scalar k.

The zero matrix Every element of the zero matrix is 0.

Matrix multiplication Matrices can be multiplied by each other only if they are **compatible.**

$$(m \text{ by } \underbrace{n) \quad \times \quad (n}_{\text{same}} \text{ by } q) \quad = \quad (m \text{ by } q)$$

Remember, you multiply a row by a column: $\begin{pmatrix} 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} 6 \\ 5 \\ 4 \end{pmatrix} = (28)$

$$\begin{array}{ccccc} p & \times & q & & (p, q) \\ \text{row} & & \text{column} & & \text{element} \end{array}$$

(2 by 2) Matrices Square matrices can be multiplied by each other if they are the same size, to produce another matrix of the same size.

Determinant The determinant det $(\mathbf{A}) = ad - bc$.
If det $(\mathbf{A}) = 0$, the matrix is **singular**.

Commutativity Matrix multiplication is **not** commutative: $\mathbf{AB} \neq \mathbf{BA}$.

Associativity Matrix multiplication is associative: $\mathbf{A(BC)} = \mathbf{(AB)C}$.

The identity matrix The matrix that has no effect when you multiply another matrix by it.

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Inverse Matrix $\mathbf{M^{-1}\,M = MM^{-1} = I}$

Forming $\mathbf{M^{-1}}$
1: Swap the elements in the leading diagonal.
2: Make the other two elements negative.

3: Divide by the determinant. Singular matrices do not have an inverse.

$$\mathbf{M} = \begin{pmatrix} a & c \\ b & d \end{pmatrix} \qquad \mathbf{M}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

Solving matrix equations

If $\mathbf{AX} = \mathbf{B}$ then $\mathbf{X} = \mathbf{A}^{-1}\mathbf{B}$

If $\mathbf{YA} = \mathbf{B}$ then $\mathbf{Y} = \mathbf{BA}^{-1}$

You must multiply on the correct side.

Solving simultaneous equations

Write them as $\mathbf{Ax} = \mathbf{b}$

then $\mathbf{x} = \mathbf{A}^{-1}\mathbf{b}$

Chapter 14 Summative Exercise

For questions **1** to **3**, use the following matrices:

$$\mathbf{A} = \begin{pmatrix} -1 & 1 & 2 \\ 3 & -2 & 1 \\ 4 & 1 & -3 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} 3 & -1 & 4 \\ 2 & 5 & -3 \end{pmatrix} \qquad \mathbf{C} = \begin{pmatrix} -2 & 3 \\ 5 & -8 \end{pmatrix}$$

$$\mathbf{D} = \begin{pmatrix} 4 & -5 \\ -2 & 3 \end{pmatrix} \qquad \mathbf{E} = \begin{pmatrix} -2 & 4 & -1 \\ 3 & -3 & 5 \end{pmatrix} \qquad \mathbf{F} = \begin{pmatrix} 1 & 0 & 2 \\ -1 & 3 & -2 \\ 2 & 1 & 0 \end{pmatrix}$$

1 Write down the order of matrices **A** to **F**.

2 Calculate:

a $2\mathbf{A} - 3\mathbf{F}$ **b** $3\mathbf{B} + 2\mathbf{E}$ **c** $4\mathbf{C} - \mathbf{D}$

3 Calculate:

a **BA** **b** **EF** **c** **CD**

d **CB** **e** **AF** **f** **DE**

4 The enrolment in a secondary school is shown in this table.

	Yr 9	Yr 10	Yr 11
Boys	50	35	35
Girls	40	60	55

Students are attached to houses in the following proportions:

	Boys	Girls
Red	20%	40%
Yellow	40%	40%
Blue	40%	20%

a Write down two compatible matrices that, when multiplied together, give the numbers of students from each year attached to each house, in the format shown at right:

$$\begin{array}{l} \\ \text{Red} \\ \text{Yellow} \\ \text{Blue} \end{array} \begin{array}{c} \begin{array}{ccc} \text{Yr 9} & \text{Yr 10} & \text{Yr 11} \end{array} \\ \begin{pmatrix} X & X & X \\ X & X & X \\ X & X & X \end{pmatrix} \end{array}$$

b Find a column matrix which, when post-multiplied by your answer to **a** will produce a list of the total number of students attached to each house.

5 A tourist attraction charges the following entry fees for different categories of visitor:

Child	Adult	Senior citizen
\$20	\$50	\$25

Over a particular weekend, they received the following numbers of visitors:

	Child	Adult	Senior citizen
Saturday	60	100	30
Sunday	80	150	40

a Write the entry fee numbers as a column matrix, **C**.

b Write the visitor numbers as a matrix called **V**.

c By calculating the product **VC**, find the total income on each day for the attraction.

d Multiply this product by a suitable matrix **T** to produce the total income over the weekend.

e If you were to multiply your matrix **T** by the matrix **V**, what information would the product provide?

6 **a** When a (2 by 2) matrix is used to transform shapes, what information does the determinant of the matrix provide?

b When does a (2 by 2) matrix **not** have an inverse?

For questions **7** to **11**, use the matrices at the start of this Summative Exercise.

7 Find the determinant of the matrix **a** **C** **b** **D**.

8 Find the inverse of the matrix **a** **C** **b** **D**.

9 **a** Find **(i)** $\mathbf{C}^{-1}\mathbf{D}$ **(ii)** $\mathbf{D}\mathbf{C}^{-1}$

b Which of your answers is the solution of the equation **(i)** $\mathbf{XC} = \mathbf{D}$ **(ii)** $\mathbf{CX} = \mathbf{D}$?

10 Solve the equations **(i)** $\mathbf{DY} = \mathbf{C}$ **(ii)** $\mathbf{YD} = \mathbf{C}$

11 Use the matrices **C** and **D** to solve the pairs of simultaneous equations:

a $2x - 3y = 12$
$5x - 8y = 31$

b $4x - 5y = -24$
$2x - 3y = -14$

12 Solve the pairs of simultaneous equations:

a $4x + 5y = 19$
$6x - 7y = -73$

b $3y - 2x = -25$
$5x + 2y = -4$

Chapter 14 Test

1 Given that $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 6 & 2 \\ 10 & 4 \end{pmatrix}$, find:

a $2\mathbf{B} - 3\mathbf{A}$ [2]

b $\mathbf{A}^{-1}$ [2]

c the matrix $\mathbf{X}$ such that $\mathbf{X}\mathbf{B}^{-1} = \mathbf{A}$ [4]

d the matrix $\mathbf{Y}$ such that $\mathbf{YA} = \mathbf{B}$. [4]

2 If $\mathbf{A} = \begin{pmatrix} 7 & -5 \\ 2 & 0 \end{pmatrix}$ and $\mathbf{A} = \mathbf{B} + 4\mathbf{I}$, where $\mathbf{I}$ is the identity matrix, find $\mathbf{B}^{-1}$. [3]

3 Given that $\mathbf{M} = \begin{pmatrix} 7 & -6 \\ -8 & 7 \end{pmatrix}$ and $\mathbf{N} = \begin{pmatrix} 5 & -3 \\ 2 & 0 \end{pmatrix}$, find:

a $\mathbf{M}^{-1}$ [2]

b the matrix $\mathbf{C}$ such that $\mathbf{CM} = \mathbf{N}$ [4]

c the matrix $\mathbf{D}$ such that $\mathbf{MD} = \mathbf{N}$. [4]

4 Given the matrix $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ -2 & -4 \end{pmatrix}$, find:

a $\mathbf{A}^2$ [2]

b $3\mathbf{A} - 2\mathbf{I}$, where $\mathbf{I}$ is the identity matrix [2]

c $\mathbf{A}^{-1}$ [2]

d Use your results to solve the simultaneous equations:

$3x + y = 9$

$x + 2y = -2$ [3]

5 Matrices **A**, **B** are such that

$\mathbf{A} = \begin{pmatrix} -2a & -3b \\ 3a & 5b \end{pmatrix}$ $\mathbf{B} = \begin{pmatrix} a & 3b \\ 4a & -2b \end{pmatrix}$

a Find $\mathbf{A}^{-1}$ [2]

b Find the matrix $\mathbf{X}$ such that $\mathbf{AX} = \mathbf{B}$. [4]

Examination Questions

1 A company produces 4 types of central heating radiator, known as types A, B, C and D. A builder buys radiators for all the houses on a new estate. There are 20 small houses, 30 medium-sized houses and 15 large houses.

A small house needs 3 radiators of type A, 2 of type B and 2 of type C.
A medium-sized house needs 2 radiators of type A, 3 of type C and 3 of type D.
A large house needs 1 radiator of type B, 6 of type C and 3 of type D.

The costs of the radiators are \$30 for type A, \$40 for type B, \$50 for type C and \$80 for type D.

Using matrix multiplication twice, find the total cost to the builder of all the radiators for the estate. [6]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2002, P1, Qu 5]

2 Given that $\mathbf{A} = \begin{pmatrix} 4 & 2 \\ 3 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & 1 \\ -2 & 3 \end{pmatrix}$, write down the inverse of **A** and of **B**. [3]

Hence find

(i) the matrix **C** such that $2\mathbf{A}^{-1} + \mathbf{C} = \mathbf{B}$, [2]

(ii) the matrix **D** such that $\mathbf{BD} = \mathbf{A}$. [2]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2003, P2, Qu 7]

3 A flower show is held over a three day period — Thursday, Friday and Saturday. The table below shows the entry price per day for an adult and for a child, and the number of adults and children attending each day.

	Thursday	Friday	Saturday
Price (\$) - Adult	12	10	10
Price (\$) - Child	5	4	4
Number of adults	300	180	400
Number of children	40	40	150

(i) Write down two matrices such that their product will give the amount of entry money paid on Thursday and hence calculate this product. [2]

(ii) Write down two matrices such that the elements of their product will give the amount of entry money paid for each of Friday and Saturday and hence calculate this product. [2]

(iii) Calculate the total amount of entry money paid over the three day period. [1]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2005, P2, Qu 2]

4 The matrices $\mathbf{A}$ and $\mathbf{B}$ are given by $\mathbf{A} = \begin{pmatrix} -2 & -1 \\ 6 & 2 \end{pmatrix}$, and $\mathbf{B} = \begin{pmatrix} 0 & -1 \\ 4 & 3 \end{pmatrix}$.

Find matrices $\mathbf{P}$ and $\mathbf{Q}$ such that

(i) $\mathbf{P} = \mathbf{B}^2 - 2\mathbf{A}$ [3]

(ii) $\mathbf{Q} = \mathbf{B}(\mathbf{A}^{-1})$ [4]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2006, P1, Qu 5]

5 The table shows the number of games played and the results of the five teams in a football league.

	Played	Won	Drew	Lost
Parrots	8	5	3	0
Quails	7	4	1	2
Robins	8	4	0	4
Swallows	7	2	1	4
Terns	8	1	1	6

A win earns 3 points, a draw 1 point and a loss 0 points. Write down two matrices which upon multiplication display in their product the total number of points earned by each team and hence calculate these totals. [4]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2006, P2, Qu 2]

6 It is given that $\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 5 & p \end{pmatrix}$ and that $\mathbf{A} + \mathbf{A}^{-1} = k\mathbf{I}$, where p and k are constants and $\mathbf{I}$ is the identity matrix. Evaluate p and k. [6]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2005, P2, Qu 6]

7 A large airline has a fleet of aircraft consisting of 5 aircraft of type A, 8 of type B, 4 of type C and 10 of type D. The aircraft have 3 classes of seat known as Economy, Business and First. The table below shows the number of these seats available in each of the four types of aircraft.

Type of aircraft \ Class of seat	Economy	Business	First
A	300	60	40
B	150	50	20
C	120	40	0
D	100	0	0

(i) Write down two matrices whose product shows the total number of seats in each class.

(ii) Evaluate the product of the matrices.

On a particular day, each aircraft made one flight. 5% of the Economy seats were empty, 10% of the Business seats were empty and 20% of the First seats were empty.

(iii) Write down a matrix whose product with the matrix found in part **(ii)** will give the total number of empty seats on that day.

(iv) Evaluate the total. [6]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2006, P1, Qu 5]

8 Given that $\mathbf{A} = \begin{pmatrix} 4 & -1 \\ -3 & 2 \end{pmatrix}$, use the inverse matrix of $\mathbf{A}$ to

(i) solve the simultaneous equations

$$y - 4x + 8 = 0$$
$$2y - 3x + 1 = 0$$

(ii) find the matrix $\mathbf{B}$ such that $\mathbf{BA} = \begin{pmatrix} -2 & 3 \\ 9 & -1 \end{pmatrix}$ [8]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2006, P2, Qu 8]

9 Given that $\mathbf{A} = \begin{pmatrix} 2 & 3 \\ -2 & -1 \end{pmatrix}$, and $\mathbf{B} = \begin{pmatrix} 8 & 10 \\ -4 & 2 \end{pmatrix}$, find the matrices $\mathbf{X}$ and $\mathbf{Y}$ such that

(i) $\mathbf{X} = \mathbf{A}^2 + 2\mathbf{B}$ [3]

(ii) $\mathbf{YA} = \mathbf{B}$ [4]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2007, P2, Qu 7]

10 Given that $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}$, find the values of each of the constants m and n for which

$$\mathbf{A}^2 + m\mathbf{A} = n\mathbf{I}$$

where $\mathbf{I}$ is the identity matrix. [4]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2007, P1, Qu 1]

11 A cycle shop sells three models of racing cycles, A, B and C. The table below shows the numbers of each model sold over a four week period and the cost of each model in \$.

Week \ Model	A	B	C
1	8	12	14
2	7	10	2
3	10	12	0
4	6	8	4
Cost (\$)	300	500	800

In the first two weeks the shop banked 30% of all money received, but in the last two weeks the shop only banked 20% of all money received.

(i) Write down three matrices such that matrix multiplication will give the total amount of money banked over the four-week period. [2]

(ii) Hence evaluate this amount. [4]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2007, P2, Qu 4]

12 Given that $\mathbf{A} = \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 3 & -5 \\ 0 & 2 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} 4 \\ 1 \end{pmatrix}$, calculate

(i) $\mathbf{AB}$, [2]

(ii) $\mathbf{BC}$, [2]

(iii) the matrix $\mathbf{X}$ such that $\mathbf{AX} = \mathbf{B}$. [4]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2008, P2, Qu 9]

13 Given that $\mathbf{A} = \begin{pmatrix} 13 & 6 \\ 7 & 4 \end{pmatrix}$, find the inverse matrix $\mathbf{A}^{-1}$ and hence solve the simultaneous equations

$$13x + 6y = 41$$
$$7x + 4y = 24$$ [4]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2008, P2, Qu 1]

14 Given that $\mathbf{A} = \begin{pmatrix} 7 & 6 \\ 3 & 4 \end{pmatrix}$, find $\mathbf{A}^{-1}$ and hence solve the simultaneous equations

$$7x + 6y = 17$$
$$3x + 4y = 3$$ [4]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2009, P2, Qu 2]

15

Team \ Place	1st	2nd	3rd	4th
Harriers	6	3	1	2
Strollers	3	2	4	3
Road Runners	2	5	5	0
Olympians	1	2	2	7

The table shows the results achieved by four teams in twelve events of an athletics match. In each event, 1st place scores 5 points, 2nd place scores 3 points, 3rd places scores 2 points and 4th place scores 1 point.

(i) Write down two matrices whose product shows the total number of points scored by each team. [2]

(ii) Evaluate this product of matrices. [2]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2009, P1, Qu 2]

16 It is given that $\mathbf{A} = \begin{pmatrix} 2 & -1 \\ 4 & 3 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & -3 \\ 2 & 0 \end{pmatrix}$, and $\mathbf{C} = \begin{pmatrix} 5 \\ -2 \end{pmatrix}$. Find

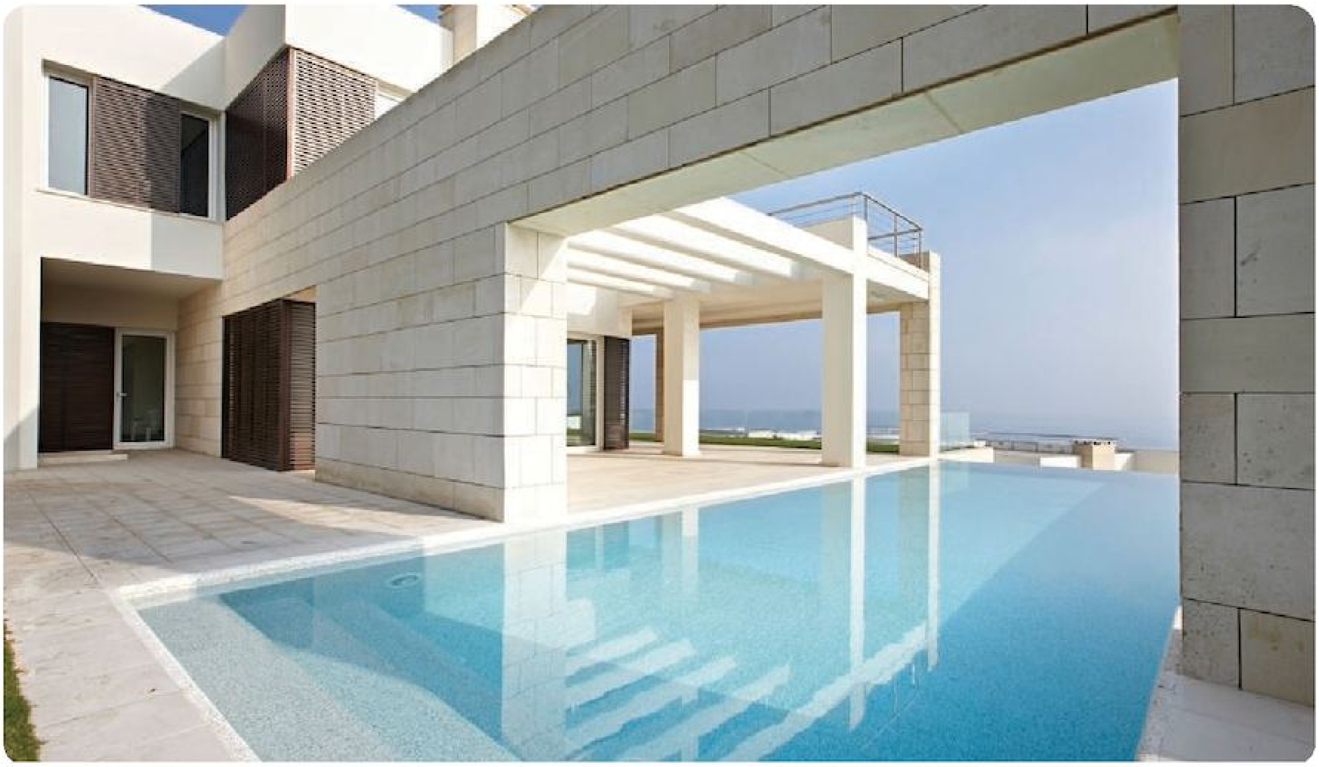
(i) $\mathbf{AB}$, [2]

(ii) $\mathbf{BC}$, [2]

(iii) $\mathbf{A}^{-1}$, and hence find the matrix $\mathbf{X}$ such that $\mathbf{AX} = \mathbf{B}$. [4]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2009, P2, Qu 8]

15 Integration



Syllabus statements

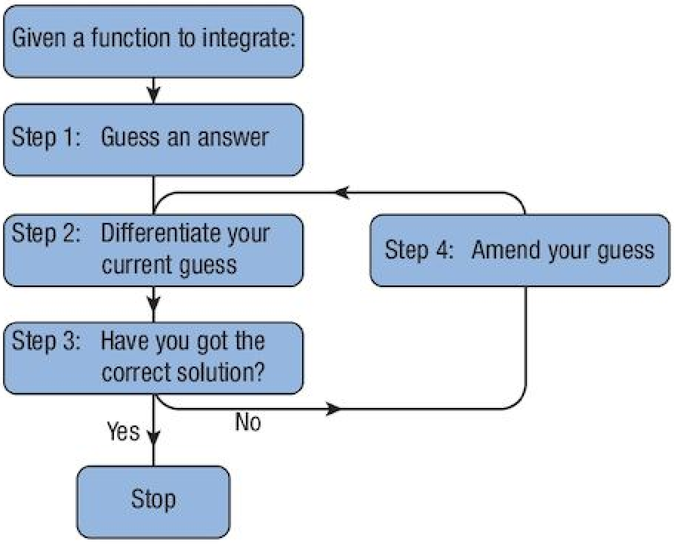
- understand integration as the reverse process of differentiation
- integrate sums of terms in powers of x, excluding $\frac{1}{x}$
- integrate functions of the form $(ax+b)^n$ (excluding $n=-1$)
- evaluate definite integrals and apply integration to the evaluation of plane areas

15.1 Introduction

Integration is the inverse process of differentiation. However, the process of creating the derived function is many-to-one. This means that we have to be careful when we try to reverse the process. Every continuous function can be differentiated. Most functions cannot be integrated. However, functions made up of powers of x can be integrated, so we start with those.

15.2 The integration process

The process is really an iterative one. You get closer to the answer as you work through it. When you become really experienced, you can jump to the correct answer at an earlier stage.



You may feel that this is a little unsatisfactory.

You do need to realise that you can only integrate a function that you can obtain by differentiating another function. The problem sometimes is to find the correct function, if there is one.

15.3 Integral notation

When we differentiate, we use the notation $\frac{\mathrm{d}}{\mathrm{d}x}(\cdots)$ to mean "**differentiate (...) with respsect to *x***".

In a similar way, we use the notation $\int(\cdots)\,\mathrm{d}x$ to mean "**integrate (...) with respect to *x***".

The reason for using this notation will be explained later.

15.4 Rules of integration

The simple rules follow from the equivalent rules for differentiating.

15.4.1 The constant multiplication rule

If a function is multiplied by a constant, set aside the constant and concentrate on the function.

so $\int k\,\mathrm{f}(x)\,\mathrm{d}x = k\int \mathrm{f}(x)\,\mathrm{d}x$

15.4.2 Addition/subtraction rule

If two simple functions are added or subtracted, treat each one individually.

so $\int \mathrm{f}(x)+\mathrm{g}(x)\,\mathrm{d}x = \int \mathrm{f}(x)\,\mathrm{d}x + \int \mathrm{g}(x)\,\mathrm{d}x$

or $\int \mathrm{f}(x)-\mathrm{g}(x)\,\mathrm{d}x = \int \mathrm{f}(x)\,\mathrm{d}x - \int \mathrm{g}(x)\,\mathrm{d}x$

15.5 Integrating polynomial functions

We know that when we differentiate a power of x, that power is reduced by 1.

f(x)	f′(x)
x^5	$5x^4$
x^4	$4x^3$
x^3	$3x^2$
x^2	$2x$
x	1
x^0	0
x^{-1}	$-x^{-2}$
x^{-2}	$-2x^{-3}$
x^n	nx^{n-1}

Which power of x is missing from the right hand column?

When we integrate one of the functions on the right, we get the one on the left.

Example 15.1

Find **a** $\int x^3\,dx$

b $\int 12x^3\,dx$

Solution:

a

Step 1: Try $f(x) = x^4$

Step 2: $f'(x) = 4x^3$

Step 3: Not quite right. Correct power, wrong constant.

Step 4: Try $f(x) = \frac{1}{4}x^4$

$f'(x) = x^3$ Correct!

Thus $\int x^3\,dx = \frac{1}{4}x^4$

b

$$\int 12x^3\,dx = 12\int x^3\,dx$$
$$= 12\left(\frac{1}{4}x^4\right)$$
$$= 3x^4$$

> *Problem 15.1*
>
> **a** Specialise, starting with $f'(x) = x$, to find f(x), then increase the power of x.
>
> **b** Find a general rule for $f'(x) = x^n$.

15.6 Problems that we often encounter

15.6.1 Zero!

When we differentiate a function such as $f(x) = x^3 - 2x^2 + 2x + 3$

the result is $f'(x) = 3x^2 - 4x + 2 + 0$

We do not usually write the zero but you must remember that it is there and **you must include it when you are integrating**, even if it not there in the question.

We can add zero to anything.

Example 15.2

a Given that $f'(x) = 4x^3 - 9x^2 + 2x - 3$, find f(x).

b You are given that when $x = 1$, $f(x) = 6$.

Find the value of the arbitrary constant in **a**.

Solution:

a Include the zero! $f'(x) = 4x^3 - 9x^2 + 2x - 3 + 0$

Ignore the constants and integrate each term individually:

$$f(x) = 4\left(\frac{1}{4}x^4\right) - 9\left(\frac{1}{3}x^3\right) + 2\left(\frac{1}{2}x^2\right) - 3x + c$$

Notice that we do not know where the zero came from so we include a general constant c.

Finally, tidy up the terms $f(x) = x^4 - 3x^3 + x^2 - 3x + c$

This is called the **general solution**.

b Putting $x = 1$ $\quad f(1) = 1 - 3 + 1 - 3 + c$

$= 6$

thus $\quad c = 10$

and $\quad f(x) = x^4 - 3x^3 + x^2 - 3x + 10$

This is called the **particular solution**.

Note that you should add the "c" only to **one side of the equation**. There is no point in adding it to both. As constants, they will combine to give a single value so we only put a single value in.

15.6.2 x^{-1}

You should have noticed that x^{-1} does not appear in the list of derivatives of powers of x. (See the table in section 15.5.)

It can be integrated, but the result is not a power of x and so it is excluded from this section of the syllabus.

15.7 Looking at the gradient function

The gradient function shows that the differential equation $\frac{dy}{dx} = f(x)$ represents the gradient of a curve. However, there are many curves with the same gradient function.

Example 15.3

a Identify the curves whose gradient function is $\frac{dy}{dx} = 2x$.

b Find the equation of the curve with the gradient function in **a** which has a y-intercept (0, 4).

Solution:

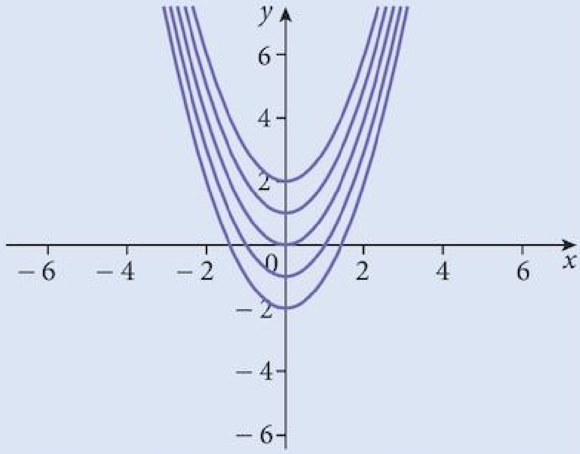
a Integrating, $y = x^2 + c$

This represents a family of parallel curves.

b The curve passes through (0, 4).

$4 = 0 + c$

Thus $y = x^2 + 4$



Note that the curves are parallel because for any particular pair of curves, the **vertical** distance between them is constant.

15.8 The arbitrary constant (constant of integration)

Every time we integrate something, we must include the zero and this will generate an arbitrary constant. The solution we get which includes the arbitrary constant is called the **general solution**.

Sometimes, as in Examples 15.2 and 15.3, we have more information that allows us to find a value for the arbitrary constant. The solution which includes this value is called a **particular solution**.

If the differential equation that we started with is of the form $\frac{d^2y}{dx^2} = f(x)$ we would need to integrate twice, creating two arbitrary constants.

In order to find both of these we would need extra information, which we could be given in various ways, for example

(i) two values of x and the corresponding values of $f(x)$

(ii) for one value of x, the value of both $f(x)$ and $f'(x)$

Example 15.4

Identify the curve whose second derivative is $\frac{d^2y}{dx^2} = 6x - 4$

and which passes through the points (1, 2) and (2, 7).

Solution:

If $\frac{d^2y}{dx^2} = 6x - 4$

then $\frac{dy}{dx} = 3x^2 - 4x + c$

and $y = x^3 - 2x^2 + cx + k$

The curve passes through (1, 2), so $2 = 1 - 2 + c + k$

or $3 = c + k$ [1]

It also passes through (2, 7), so $7 = 8 - 8 + 2c + k$

giving $7 = 2c + k$ [2]

Solving [1] and [2], gives $c = 4$ and $k = -1$

so the equation of the curve is $y = x^3 - 2x^2 + 4x - 1$

Example 15.5

Identify the curve whose second derivative is $\frac{d^2y}{dx^2} = 12x$

and which passes through the point (0, 4) with gradient 2.

Solution:

$$\frac{d^2y}{dx^2} = 12x$$

Integrate both sides with respect to x.

$$\frac{dy}{dx} = 6x^2 + c$$

The gradient at (0, 4) is 2, so $2 = 0 + c$

giving $c = 2$

and $\frac{dy}{dx} = 6x^2 + 2$

Integrate again $y = 2x^3 + 2x + k$

A different constant.

This passes through (0, 4), so $4 = 0 + 0 + k$

giving $k = 4$

The particular solution is $y = 2x^3 + 2x + 4$

Exercise 15.1

1 Find $\mathrm{f}(x)$ if:

a $\mathrm{f}'(x) = x^2$ **b** $\mathrm{f}'(x) = x^3$ **c** $\mathrm{f}'(x) = x^4$

d $\mathrm{f}'(x) = 6x^2$ **e** $\mathrm{f}'(x) = 8x^3$ **f** $\mathrm{f}'(x) = 15x^4$

g $\mathrm{f}'(x) = \frac{1}{x^2}$ **h** $\mathrm{f}'(x) = \frac{1}{x^3}$ **i** $\mathrm{f}'(x) = \frac{1}{x^4}$

j $\mathrm{f}'(x) = -\frac{5}{x^2}$ **k** $\mathrm{f}'(x) = -\frac{6}{x^3}$ **l** $\mathrm{f}'(x) = -\frac{9}{x^4}$

2 Find y if:

a $\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 - x^3$ **b** $\frac{\mathrm{d}y}{\mathrm{d}x} = 8x^3 - 2x$ **c** $\frac{\mathrm{d}y}{\mathrm{d}x} = 10x^4 - 3x^3$

d $\frac{\mathrm{d}y}{\mathrm{d}x} = 9x^2 + \frac{1}{x^2}$ **e** $\frac{\mathrm{d}y}{\mathrm{d}x} = x^3 - \frac{2}{x^3}$ **f** $\frac{\mathrm{d}y}{\mathrm{d}x} = x^4 + \frac{3}{x^4}$

g $\frac{\mathrm{d}y}{\mathrm{d}x} = 4\sqrt{x} + \frac{2}{\sqrt{x}}$ **h** $\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt{x^3} - \frac{2}{\sqrt[3]{x}}$ **i** $\frac{\mathrm{d}y}{\mathrm{d}x} = \sqrt[3]{x} + \frac{3}{\sqrt[3]{x^2}}$

3 Find y if the derivative is as shown and the curve passes through the point given:

a $\frac{\mathrm{d}y}{\mathrm{d}x} = 4x - 6x^2$ $(0, 3)$ **b** $\frac{\mathrm{d}y}{\mathrm{d}x} = 5x^4 - 8x^3$ $(0, -2)$

c $\frac{\mathrm{d}y}{\mathrm{d}x} = 10x^4 - 4x^3$ $(1, 5)$ **d** $\frac{\mathrm{d}y}{\mathrm{d}x} = 9x^2 + \frac{1}{x^2}$ $(1, -4)$

e $\frac{\mathrm{d}y}{\mathrm{d}x} = 8x^3 - \frac{3}{x^2}$ $(1, 6)$ **f** $\frac{\mathrm{d}y}{\mathrm{d}x} = 6x + \frac{4}{x^3}$ $(2, 15)$

g $\frac{\mathrm{d}y}{\mathrm{d}x} = 6\sqrt{x} + \frac{2}{\sqrt{x}}$ $(1, 10)$ **h** $\frac{\mathrm{d}y}{\mathrm{d}x} = 15\sqrt{x^3} - \frac{2}{\sqrt[3]{x}}$ $(1, 7)$

i $\frac{\mathrm{d}y}{\mathrm{d}x} = 12\sqrt[3]{x} + \frac{1}{\sqrt[3]{x^2}}$ $(8, 150)$ **j** $\frac{\mathrm{d}y}{\mathrm{d}x} = 6\sqrt{x} - 10\sqrt{x^3}$ $(4, -100)$

4 Find each of the following integrals:

a $\int 8x^3 + 6x^2\,\mathrm{d}x$ **b** $\int 8y^3 + 6y^2\,\mathrm{d}y$ **c** $\int 6\sqrt{t} + \frac{3}{\sqrt{t}}\,\mathrm{d}t$

d $\int 5v^4 - \frac{2}{v^2}\,\mathrm{d}v$ **e** $\int 2\sqrt[3]{y} + 6\sqrt{y}\,\mathrm{d}y$ **f** $\int 6\sqrt{t^3} + \frac{3}{\sqrt{t^3}}\,\mathrm{d}t$

5 The equation of a curve has a second derivative given by

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 6x + 6.$$

At the point $(-1, 0)$, through which it passes, it has a gradient of 1.
Find the equation of the curve.

6 The equation of a curve has a second derivative given by

$$\frac{d^2y}{dx^2} = \frac{-2}{x^3}.$$

At the point (1, 4), through which it passes, it has a gradient of 3.
Find the equation of the curve.

7 The equation of a curve has a second derivative given by

$$\frac{d^2y}{dx^2} = 6x + 6.$$

The curve passes through the points (1, −1) and (2, 4).
Find the equation of the curve.

8 The equation of a curve has a second derivative given by

$$\frac{d^2y}{dx^2} = 6x - \frac{2}{x^3} + \frac{12}{x^5}.$$

The curve has a stationary point at (1, 0).
Find the equation of the curve and the nature of the stationary point.

9 A cubic curve of the form $y = ax^3 + bx^2 + cx + d$ has two stationary points, one at (1, 5) and another at (3, 1).

Find the equation of the curve and the nature of the stationary points.

10 **a** Use the composite rule to differentiate the function $y=(2x+1)^3$.
Hence find y if $y = \int (2x+1)^2\, dx$. (use c as your constant of integration)

b Use the binomial theorem to expand $(2x+1)^2$.
Integrate the resulting expression. (use k as your constant of integration)

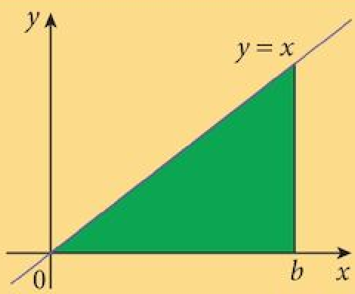
c Use the binomial theorem to expand your answer to part **a**.
Compare the expanded versions of your answers.
What is the relationship between the constants of integration?

15.9 The area function

Problem 15.2

a Find the area between the curve $y = x$ and the x-axis between the values $x = 0$ and $x = b$.

b What is the relationship between the area you found in **a** and the integral of the function $f'(x) = x$?



In general, to find the area between the curve $y = f(x)$ and the x-axis between the values $x = 0$ and $x = b$, we slice the area into thin sections which are roughly rectangular.

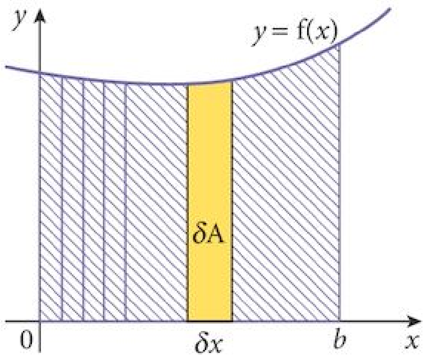
Each rectangle has width δx and height $f(x)$.

The area of each slice, $\delta A \approx f(x)\,\delta x$. **[1]**

The total area is found by adding them all together.
However, when we reduce the width δx, the addition becomes one of continuous values rather than discrete ones.

When we add together a set of discrete values we use the symbol Σ. This is a greek letter S (Sigma).

When we add together a set of continuous values, we use a different letter S: $\int$. This is the integral sign.



Returning to **[1]**, as $\delta x \mapsto 0$, $\dfrac{\delta A}{\delta x} \mapsto \dfrac{dA}{dx}$

[1] becomes $\dfrac{dA}{dx} = f(x)$

This is a differential equation that we solve by **integrating both sides with respect to x**.

We get $\displaystyle\int \frac{dA}{dx}\,dx = \int f(x)\,dx$ **[2]**

This reduces to $\displaystyle\int 1\,dA = \int f(x)\,dx$ **[3]**

which becomes $A = \displaystyle\int f(x)\,dx$ **[4]**

Finally, we put $x = b$ $A = F(b)$

Thus we can find areas by integration.

$\int \ldots dx$ means **integrate with respect to x**.

Note that you cannot have an integral sign by itself. It must be acccompanied by dx (or dy or some such).

Usually, when solving questions, we leave out lines [2] and [3] of this theory. They are included here so that you can see that we are still following the rule "**Do the same thing to both sides**".

Example 15.6

Find the area between the curve $y = 6x^2 + 4x$ and the x-axis between the values $x = 0$ and $x = 2$.

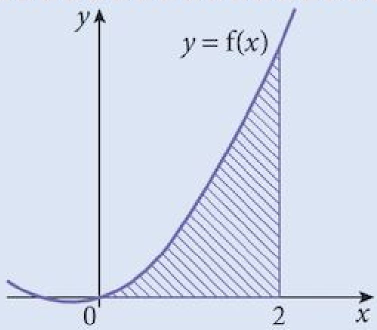
Solution:

$$A = \int 6x^2 + 4x\,dx$$

$$= 2x^3 + 2x^2$$

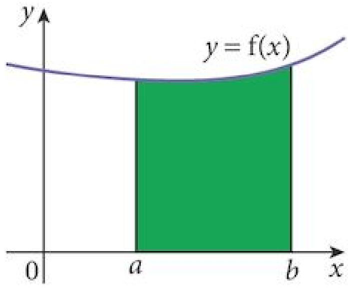
When $x = 2$ $A = 16 + 8$

$$= 24$$



Note we did not use "c"

15.10 The area of the region between $x = a$ and $x = b$



The process described can be used to find the area between a curve and the x-axis from $x = 0$ to $x = b$.

If $F'(x) = f(x)$,

the area from $x = 0$ to $x = b$ is $F(b)$

the area from $x = 0$ to $x = a$ is $F(a)$

$\Rightarrow$ the area from $x = a$ to $x = b$ is $F(b) - F(a)$.

We have a modified notation for this:

$$A = \int_{x=a}^{x=b} f(x)\,dx$$

or just

$$A = \int_a^b f(x)\,dx$$

if it clear that a and b are values of x.

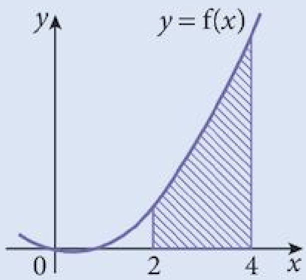
When we have limits to the integration like this, we call it a **definite integral**.

Without limits, it is an **indefinite integral**.

Example 15.7

Find the area between the curve $y = 6x^2 + 4x$ and the x-axis between the values $x = 2$ and $x = 4$.

Solution:



$$A = \int_2^4 6x^2 + 4x\,dx$$

$$= \left[2x^3 + 2x^2 + c\right]_2^4$$

$$A = \underbrace{\left[128 + 32 + c\right]}_{x=4} - \underbrace{\left[16 + 8 + c\right]}_{x=2}$$

$$= \left[160 + c\right] - \left[24 + c\right]$$

$$= 136$$

Notice that, in this calculation, the value of c does not matter. For that reason, we do not usually include it when calculating a definite integral. It will always be eliminated.

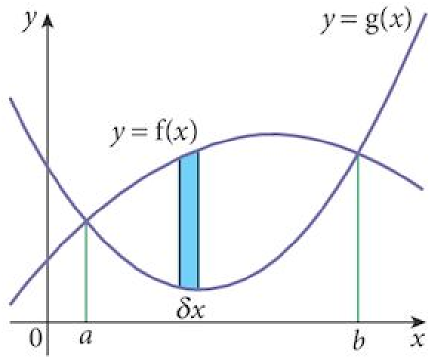
In the theory section, we used "the area from $x = 0$ to $x = b$ is $F(b)$".
In fact, this is $F(b) - F(0)$ and the "c" would have been eliminated.

15.11 The area of the region between two curves

We can extend these ideas to find the area between two curves.

We slice the area into thin rectangles of width δx.

The height of each rectangle is $f(x) - g(x)$.



Note that this height must always be positive in order to give a positive area. This is only true for $a \leqslant x \leqslant b$.

The area of each slice, $\mathscr{E}A \approx \left[f(x) - g(x)\right]\mathscr{E}x$

from which, $A = \int_a^b \left[f(x) - g(x)\right] dx$

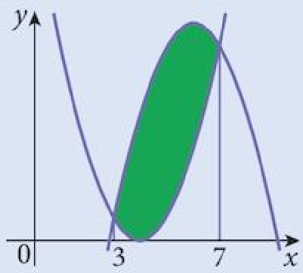
Note that this is exactly what we did when finding the area under a curve since, in that case $g(x) = 0$, the zero function.

Example 15.8

Find the area between the curves $y = -x^2 + 12x - 26$ and $y = x^2 - 8x + 16$.

Solution:

First we need to find where the curves intersect.



Solving the simultaneous equations $y = -x^2 + 12x - 26$

$y = x^2 - 8x + 16$

gives (3, 1) and (7, 9)

$$A = \int_3^7 \left[\left(-x^2 + 12x - 26\right) - \left(x^2 - 8x + 16\right)\right] dx$$

$$A = \int_3^7 -2x^2 + 20x - 42 \, dx$$

$$A = \left[-\frac{2}{3}x^3 + 10x^2 - 42x\right]_3^7$$

$$= \left[-32\frac{2}{3}\right] - [-54]$$

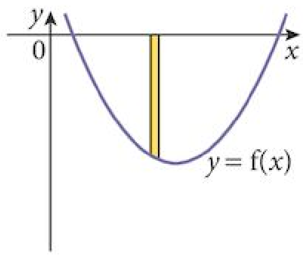
$$= 21\frac{1}{3}$$

15.12 Curves below the x-axis

When a curve lies below the x-axis the height of the rectangle is given by

top – bottom

$[0 - f(x)]$



and the area is therefore given by

$$A = \int -f(x)\,dx$$

The top function is $y = 0$. An area cannot be negative, but an integral can be.

This means that we have to know where the curve intersects the x-axis or with another function in order to determine which function is the top one for any value of x.

Sometimes, we have to calculate two areas and add them together.

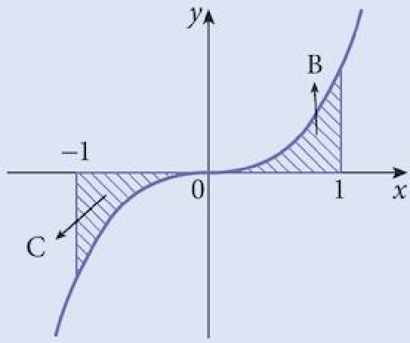
Example 15.9

a Find $A = \int_{-1}^{1} x^3\,dx$.

b Find the area between the curve $y = x^3$ and the x-axis between the values $x = -1$ and $x = 1$.

Solution:

a
$$A = \int_{-1}^{1} x^3\,dx$$
$$= \left[\frac{1}{4}x^4\right]_{-1}^{1}$$
$$= \left[\frac{1}{4} - \frac{1}{4}\right]$$
$$= 0$$



b The curve intersects the x-axis at $x = 0$.
The integral for area C will be negative because it is below the x-axis.
The integral for area B will be positive because it is above the x-axis.
By symmetry, these two areas will be equal.

We could calculate

$$A = \int_{-1}^{0} \left[-x^3\right] dx + \int_{0}^{1} x^3\,dx$$

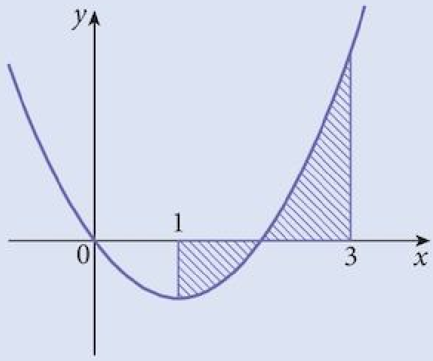
But it might be quicker to find

$$A = 2\int_{0}^{1} x^3\,dx = \frac{1}{2}$$

Example 15.10

Find the area between the curve $y = \dfrac{x^3 - 2x^2}{x}$ and the x-axis between the values $x = 1$ and $x = 3$.

Solution:



We have two problems with this question:

1: We need to write the function as a series of powers of x.

2: We can see from the graph that the function crosses the x-axis at $x = 2$.

1: $$\frac{x^3 - 2x^2}{x} = x^2 - 2x$$

2: To calculate the area, we need to split the integral into 2 parts

$$\text{Area} = \int_1^2 -\left(x^2 - 2x\right)\,dx + \int_2^3 x^2 - 2x\,dx$$

$$= \left[x^2 - \frac{1}{3}x^3\right]_1^2 + \left[\frac{1}{3}x^3 - x^2\right]_2^3$$

$$= \left[\left(4 - \frac{8}{3}\right) - \left(1 - \frac{1}{3}\right)\right] + \left[(9-9) - \left(\frac{8}{3} - 4\right)\right]$$

$$= 2$$

15.13 Integrating composite functions

When we differentiate the composite function $y = f[g(x)]$,

we get $$\frac{dy}{dx} = f'\left[g(x)\right] \times g'(x)$$

This is a product, so integrating it could be quite a problem.

However, if $g(x)$ is a linear function, $g'(x)$ will be a constant and that is a much easier task.

Example 15.11

Solve the equation $y = \int (2x-3)^3\,dx$.

Solution:

This is a composite with outer function $f(x) = x^3$

and inner function $g(x) = 2x - 3$ which is linear.

We use the same process as before:

Step 1: Try $y = (2x-3)^4$

Step 2: $\frac{dy}{dx} = 4(2x-3)^3 \times (2) = 8(2x-3)^3$

Step 3: Not quite right. Correct power, wrong constant.

Step 4: Try $y = \frac{1}{8}(2x-3)^4$

Step 5: $\frac{dy}{dx} = \frac{1}{8} 4(2x-3)^3 \times (2) = (2x-3)^3$

Correct, so

$$y = \frac{1}{8}(2x-3)^4 + c$$

Don't forget the "c".

Example 15.12

A curve is such that $\frac{dy}{dx} = \frac{36}{(2x+1)^3}$ and it passes through the point (1, 4). Find its equation.

Solution:

Step 1: Try $y = \frac{1}{(2x+1)^2}$

Step 2: $\frac{dy}{dx} = \frac{-4}{(2x+1)^3}$

Step 3: Correct function, wrong constant.

Step 4: Try $y = \frac{-9}{(2x+1)^2}$

$$\frac{dy}{dx} = \frac{36}{(2x+1)^3}$$

Correct, so $y = \frac{-9}{(2x+1)^2} + c$

The curves passes through the point (1, 4)

$$4 = \frac{-9}{(2\times1+1)^2} + c$$

giving $c = 5$

and $y = \frac{-9}{(2x+1)^2} + 5$

Exercise 15.2

1 Evaluate each definite integral:

a $\int_0^2 x^3\,dx$ **b** $\int_{-\frac{3}{2}}^{\frac{3}{2}}(9-4x^2)\,dx$ **c** $\int_0^4 \sqrt{x^3}\,dx$

d $\int_4^5 (4x+3)\,dx$ **e** $\int_{-2}^3 (3x^2+2x)\,dx$ **f** $\int_1^8 \sqrt[3]{x}\,dx$

g $\int_1^3 y-y^2\,dy$ **h** $\int_1^4 \sqrt{y}\,dy$ **i** $\int_{-2}^0 y^3-y+6\,dy$

j $\int_1^2 1+\frac{1}{x^2}\,dx$ **k** $\int_1^4 1-\frac{1}{\sqrt{x}}\,dx$ **l** $\int_1^2 \frac{2}{x^3}\,dx$

m $\int_0^2 (2+3x)^2\,dx$ **n** $\int_{-1}^3 \sqrt{1+x}\,dx$ **o** $\int_{-2}^{-1} \frac{1}{(x-1)^2}\,dx$

2 Evaluate each definite integral:

a $\int_0^2 x(x^2-1)\,dx$ **b** $\int_1^3 x(x-1)^2\,dx$ **c** $\int_0^{\frac{1}{2}} 10x^3(2x-1)\,dx$

d $\int_{-2}^2 \left(\frac{x^3-4x}{x}\right)dx$ **e** $\int_1^3 \left(\frac{x^3-1}{x^2}\right)dx$ **f** $\int_1^9 \left(\frac{\sqrt{x}-3}{\sqrt{x}}\right)dx$

3 Find the area between each curve and the x-axis between the values of x shown:

a $y=1-x^2;\ 0\leqslant x\leqslant 1$ **b** $y=x^2+3;\ 1\leqslant x\leqslant 2$ **c** $y=\sqrt{x};\ 4\leqslant x\leqslant 9$

d $y=5x-x^2;\ 0\leqslant x\leqslant 5$ **e** $y=2x^2+3;\ -1\leqslant x\leqslant 2$ **f** $y=\frac{1}{x^2}+3;\ 2\leqslant x\leqslant 4$

4 **a** Sketch the curve $y=x(x-1)(x-2)$.

b Find the following integrals:

(i) $\int_0^1 x^3-3x^2+2x\,dx$ **(ii)** $\int_1^2 x^3-3x^2+2x\,dx$ **(iii)** $\int_0^2 x^3-3x^2+2x\,dx$

c Explain why the area between the curve and the x-axis in the interval $0\leqslant x\leqslant 2$ is not the answer to **b (iii)**. Find the area between the curve and the x-axis in the interval $0\leqslant x\leqslant 2$.

5 **a** Sketch the curve $y=16-x^2$.

b Find the area bounded by the curve and the x-axis.

6 **a** Sketch the curve $y=(x+1)(3-x)$.

b Find the area bounded by the curve and the x-axis.

7 **a** Sketch the curve $y=\sqrt{x+3}$.

b Find the area bounded by the curve, the x-axis and the y-axis.

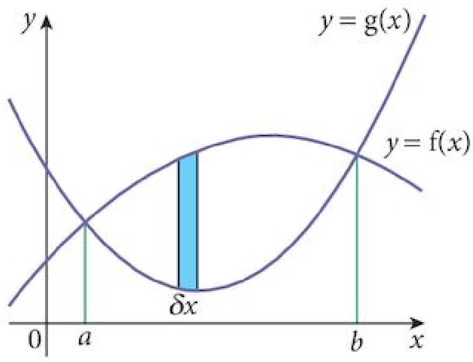
8 **a** Find the stationary points of the curve $y=x^2(x-1)(x+2)$ and state their nature.

b Sketch the curve.

c Find the total area bounded by the curve and the x-axis.

Summary

Definition	Integration is the inverse process of differentiation. However, producing a derived function is a many-to-one process.
The integration process	This is an iterative process where you make a guess at the correct function, then differentiate your guess to see how wrong you are. Change your guess and repeat until you get the correct solution. Remember: the majority of functions cannot be integrated but the ones you meet should be okay.
Rules for integration	Follow from the equivalent rules for differentiation.
Constant multiplier rule	If $y = \int k\, \mathrm{f}(x)\, \mathrm{d}x$ then $y = k\int \mathrm{f}(x)\, \mathrm{d}x$
Sum / difference rule	If $y = \int \mathrm{f}(x) \pm \mathrm{g}(x)\, \mathrm{d}x$ then $y = \int \mathrm{f}(x)\, \mathrm{d}x \pm \int \mathrm{g}(x)\, \mathrm{d}x$
Integrating powers of *x*	If $y = \int x^n\, \mathrm{d}x$ then $y = \dfrac{1}{n+1} x^{n+1}$
The constant of integration	Remember to add a constant of integration "*c*" whenever you integrate. This solves the many-to-one problem of the differentiating process. Note: If you have to integrate twice, use 2 different constants.
Using integration to find areas	We can use integration to find areas. However, an area must be positive, an integral need not be.
The area between two curves (definite integration)	$\mathrm{A} = \int_a^b [\mathrm{f}(x) - \mathrm{g}(x)]\, \mathrm{d}x$ Often the lower function is $y = 0$. This is only valid between the points where the graphs intersect. We do not need the constant, *c*. Areas below the *x*-axis will produce a negative integral but the area should be positive.
Integrating composite functions	Usually, the inside function, $\mathrm{g}(x)$, of $\mathrm{f}[\mathrm{g}(x)]$ will be linear.



Chapter 15 Summative Exercise

1 Find f(x) when:

a $f'(x) = 10x^4$ **b** $f'(x) = 8x^7$ **c** $f'(x) = 15x^{14}$

d $f'(x) = \dfrac{8}{x^5}$ **e** $f'(x) = \dfrac{10}{x^3}$ **f** $f'(x) = \dfrac{9}{x^4}$

2 Solve the differential equations.

a $\dfrac{dy}{dx} = 6\sqrt{(x)}$ **b** $\dfrac{dy}{dx} = 10\sqrt{x^3}$ **c** $\dfrac{dy}{dx} = 7\sqrt{x^5}$

d $\dfrac{dy}{dx} = 8\sqrt[3]{x}$ **e** $\dfrac{dy}{dx} = 10\sqrt[4]{x}$ **f** $\dfrac{dy}{dx} = 6\sqrt[5]{x}$

3 Solve the differential equations.

a $\dfrac{dy}{dx} = \dfrac{2}{\sqrt{x}}$ **b** $\dfrac{dy}{dx} = \dfrac{4}{\sqrt{x^3}}$ **c** $\dfrac{dy}{dx} = \dfrac{6}{\sqrt{x^5}}$

d $\dfrac{dy}{dx} = \dfrac{2}{\sqrt[3]{x}}$ **e** $\dfrac{dy}{dx} = \dfrac{6}{\sqrt[4]{x}}$ **f** $\dfrac{dy}{dx} = \dfrac{8}{\sqrt[5]{x}}$

4 The following curves have derivatives as shown and pass through the given points. Find their equations.

a $\dfrac{dy}{dx} = 2x - 3$ (2, 3) **b** $\dfrac{dy}{dx} = 4x - \dfrac{6}{x^2}$ (3, 16)

c $\dfrac{dy}{dx} = 6x^2 - 4x + 3$ (1, 4) **d** $\dfrac{dy}{dx} = 3x^2 - \dfrac{20}{(2x-1)^2}$ (−2, −6)

5 The equation of a curve has second derivative $\dfrac{d^2y}{dx^2} = -12x + 4$.

It passes through the point (3, 34), where it has gradient −12.

Find the equation of the curve.

6 The equation of a curve has second derivative $\dfrac{d^2y}{dx^2} = 12x^2 + 12x - 12$.

It passes through the point (−1, −11), where it has gradient 15.
Find the equation of the curve.

7 **a** Use the composite rule to differentiate the function $\dfrac{1}{3x+4}$.

b Use the result to find y if $\dfrac{dy}{dx} = \dfrac{12}{(3x-4)^2}$
given that the solution curve passes through the point (−2, 0).

8 Evaluate each of the following definite integrals.

a $\int_{-1}^{2} x^3 - x + 3\,dx$ b $\int_{1}^{3} x + \frac{1}{x^2}\,dx$ c $\int_{4}^{7} 2x + \frac{2}{(x-3)^2}\,dx$

d $\int_{0}^{4} \left(\frac{x^2 - 2x}{\sqrt{x}}\right) dx$ e $\int_{-1}^{3} x^3(2x+3)\,dx$ f $\int_{-2}^{3} (x+2)(x-1)(x-3)\,dx$

9 a Show that $\frac{3x^2 - 6x + 4}{x^2 - 2x + 1} = 3 + \frac{1}{(x-1)^2}$

b Hence find $\int_{-6}^{0} \frac{3x^2 - 6x + 4}{x^2 - 2x + 1}\,dx$

10 Find the area bounded by the curve $y = 6x^2 - 4x + 2$, the line $x = -1$ and the line $x = 1$.

11 The line $y = x + 7$ meets curve $y = x^2 - 4x + 1$ at the points A and B.

a Find the coordinates of the points A and B.

b Sketch the line and the curve on the same set of axes.

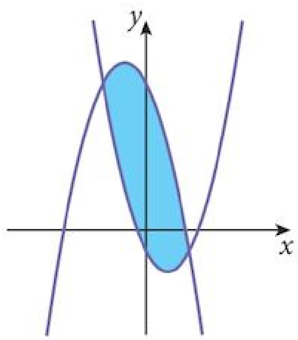
c Find the area between the line and the curve.

12 The diagram shows the curves

$y = x^2 - 2x - 1$

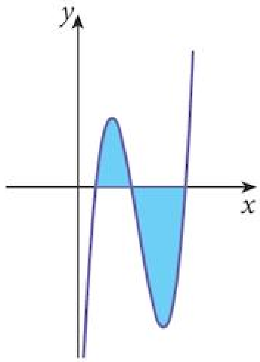
and $y = 7 - 2x - x^2$.

Find the area (shaded) between the curves.



13 The diagram shows the curve $y = x^3 - 10x^2 + 27x - 18$.

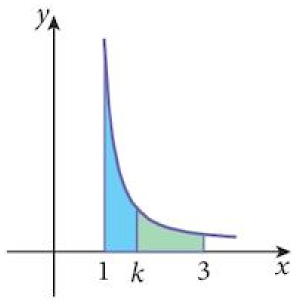
Find the total area (shaded) bounded by the curves and the x-axis.



14 The diagram show part of the curve $y = \frac{6}{x^2}$.

The area bounded by the curve and the lines $x = 1$ and $x = k$ (shaded blue) is equal to the area bounded by the curve and the lines $x = k$ and $x = 3$ (shaded green).

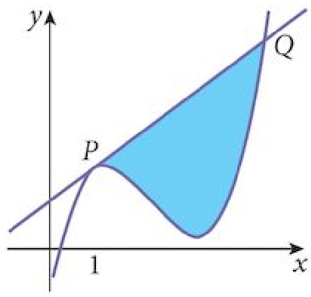
Find the value of k.



Chapter 15 Test

1 hour

1 Find $\int_0^3 \frac{12}{(2x+3)^2}\,dx$. [4]



2 The diagram shows the curve $y = x^3 - 7x^2 + 13x + 14$.

The tangent at the point P (where $x = 1$) cuts the curve again at Q. [4]

a Find the equation of the tangent at the point P.

b Find the coordinates of the point Q. [3]

c Find the shaded area bounded by the curve and the tangent line. [3]

3 **a** If $y = \frac{x+1}{\sqrt{2x+1}}$, find $\frac{dy}{dx}$.

b Use your result to find $\int_0^4 \frac{x}{\sqrt{(2x+1)^3}}\,dx$. [2]

4 **a** Find $\int \left(1 + \frac{12}{x^3}\right) dx$. [2]

b Hence find the values of k such that $\int_k^{2k} \left(1 + \frac{12}{x^3}\right) dx = \frac{7}{2}$. [4]

5 **a** Find $\int \sqrt{3x+1}\,dx$. [3]

b Hence evaluate $\int_4^8 \sqrt{3x+1}\,dx$. [2]

6 **a** Find $\int \left(4x - \frac{1}{(x-4)^2}\right) dx$. [2]

b Hence find the area of the region bounded by the curve, the line $x = 2$ and the line $x = 3$. [3]

7 **a** Find $\int \sqrt{5x-4}\,dx$. [3]

b Hence evaluate $\int_4^8 15\sqrt{5x-4}\,dx$. [2]

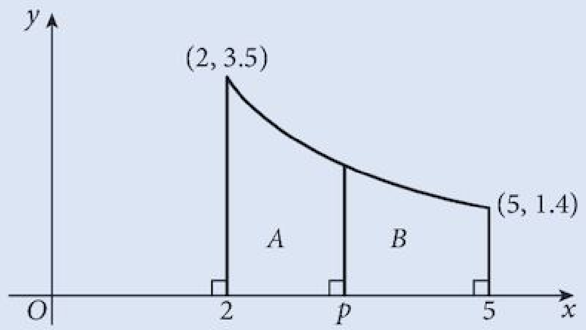
Examination Questions

1 A curve is such that $\dfrac{dy}{dx} = \dfrac{6}{(2x-3)^2}$.

Given that the curve passes through the point (3, 5), find the coordinates of the point where the curve crosses the x-axis. [6]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2002, P1, Qu 6]

2



The diagram shows part of a curve, passing through the points (2, 3.5) and (5, 1.4). The gradient of the curve at any point (x, y) is $-\dfrac{a}{x^3}$, where a is a positive constant.

(i) Show that $a = 20$ and obtain the equation of the curve. [5]

The diagram also shows lines perpendicular to the x-axis at $x = 2$, $x = p$ and $x = 5$.
Given that the areas of the regions A and B are equal,

(ii) find the value of p. [5]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2004, P1, Qu 11]

3 A curve is such that $\dfrac{d^2y}{dx^2} = 6x - 2$. The gradient of the curve at the point (2, –9) is 3.

(i) Express y in terms of x. [5]

(ii) Show that the gradient of the curve is never less than $-\dfrac{16}{3}$. [3]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2005, P2, Qu 10]

4 A curve is such that $\dfrac{dy}{dx} = \dfrac{6}{\sqrt{4x+1}}$, and (6, 20) is a point on the curve.

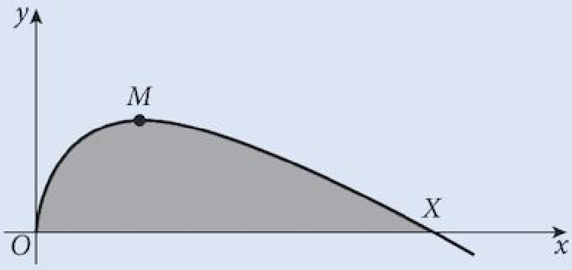
(i) Find the equation of the curve. [4]

A line with gradient $-\dfrac{1}{2}$ is a normal to the curve.

(ii) Find the coordinates at which this normal meets the coordinate axes. [4]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2007, P1, Qu 6]

5



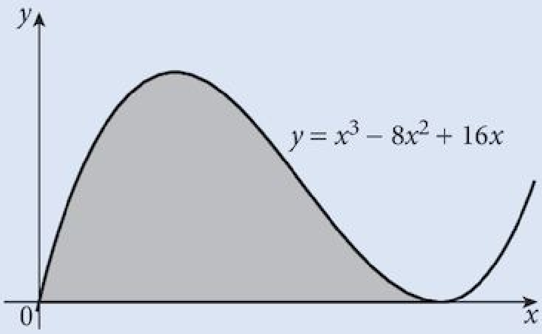
The diagram shows part of the curve $y = 4\sqrt{x} - x$. The origin O lies on the curve and the curve intersects the positive x-axis at X. The maximum point of the curve is at M. Find

(i) the coordinates of X and of M, [5]

(ii) the area of the shaded region. [4]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2007, P2, Qu 10]

6



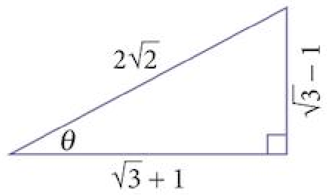
The diagram shows part of the curve $y = x^3 - 8x^2 + 16x$.

(i) Show that the curve has a minimum point at (4, 0) and find the coordinates of the maximum point. [4]

(ii) Find the area of the shaded region enclosed by the x-axis and the curve. [4]

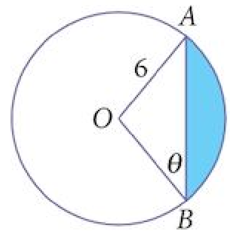
[Cambridge IGCSE Additional Mathematics 0606, Nov 2008, P2, Qu 10]

Term test 4A (Chapters 12–15) 1 hour



1 You are given the information shown in the upper diagram where $\theta = \dfrac{\pi}{12}$.

In the lower diagram, OAB is a sector of a circle, centre O, radius 6. Angle OBA, $\theta = \dfrac{\pi}{12}$.



Find:

a the length of AB [2]

b the angle AOB [1]

c the perimeter of the shaded region [1]

d the area of the shaded region. [3]

2 **a** Given that $y = \dfrac{x}{3+x^2}$, show that $\dfrac{dy}{dx} = \dfrac{f(x)}{(3+x^2)^2}$ where $f(x)$ is a function to be found. [3]

b Hence evaluate $\displaystyle\int_{-\sqrt{3}}^{\sqrt{3}} \frac{6-2x^2}{(3+x^2)^2}\,dx$. [2]

3 A curve is such that $\dfrac{dy}{dx} = 4 - ax$ where a is a constant.

The normal to the curve at the point $(1, 6)$ has gradient $\dfrac{1}{2}$.

a Find the equation of the curve. [5]

b Find the point of intersection of the tangent at $(1, 6)$ and the tangent at the point where $x = 3$. [5]

4 It is given that $\mathbf{A} = \begin{pmatrix} 4 & 3 \\ 3 & 2 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} -1 & -2 & 3 \\ 2 & -2 & -1 \end{pmatrix}$ and $\mathbf{C} = \begin{pmatrix} 4 \\ 6 \\ -2 \end{pmatrix}$.

a Calculate $\mathbf{AB}$. [2]

b Calculate $\mathbf{BC}$. [2]

c Find $\mathbf{A}^{-1}$. [2]

5 A curve is such that its gradient function $\frac{dy}{dx} = \frac{4}{\sqrt{2x+1}}$ for $x > -\frac{1}{2}$.

The curve passes through the point (4, 9).

a Find the equation of the curve. [4]

b Find the x-coordinate of the point on the curve where $y = 17$. [1]

6 **a** Given that $\mathbf{A} = \begin{pmatrix} 4 & -2 \\ 5 & -3 \end{pmatrix}$, find $\mathbf{A}^{-1}$.

b Using your answer to **a**, find the values of a, b, c and d such that [2]

$\mathbf{A}\begin{pmatrix} a & b \\ c & 2 \end{pmatrix} = \begin{pmatrix} 4 & d \\ 3 & -16 \end{pmatrix}$. [5]

16 Further trigonometry

Syllabus statements

- know the trigonometric functions of angles of any magnitude (secant, cosecant, cotangent)
- understand amplitude and periodicity and the relationship between the graphs of e.g. $\sin x$ and $\sin 2x$
- draw and use the graphs of

  $y = a\sin(bx) + c$

  $y = a\cos(bx) + c$

  $y = a\tan(bx) + c$

  where a and b are positive integers and c is an integer
- use the relationships

  $\frac{\sin A}{\cos A} = \tan A$ $\quad$ $\frac{\cos A}{\sin A} = \cot A$

  $\sin^2 A + \cos^2 A = 1$ $\quad$ $\sec^2 A = 1 + \tan^2 A$ $\quad$ $\mathrm{cosec}^2 A = 1 + \cot^2 A$

  and solve simple trigonometric equations involving the trigonometric functions and the above relationships (not including the general solution of triginometric equations)
- prove simple trigonometric identities

16.1 Introduction

In this chapter we return to trigonometry. We introduce the three reciprocal functions and extend your knowledge of the properties of amplitude and periodicity. We look at the graphs of more complicated trigonometric functions and see how they are related to the basic sine, cosine and tangent curves. Finally we look at trigonometric identities and their uses.

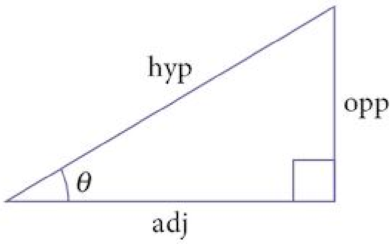
16.2 Reciprocal trigonometric functions

It is sometimes convenient to use the reciprocals of the sine, cosine and tangent functions. These are called the secant (sec), cosecant (cosec) and cotangent (cot) and are defined as:

$$\sec\theta = \frac{1}{\cos\theta} = \frac{\text{hyp}}{\text{adj}}$$

$$\operatorname{cosec}\theta = \frac{1}{\sin\theta} = \frac{\text{hyp}}{\text{opp}}$$

$$\cot\theta = \frac{1}{\tan\theta} = \frac{\text{adj}}{\text{opp}}$$



Be careful!

Secant is the reciprocal of cosine and cosecant is the reciprocal of sine.

This is confusing until you get used to it.

These definitions are true for angles of any size.

Problem 16.1

The diagram here shows the unit circle and the tangents at $x = 1$ and $y = 1$.

a State the lengths:

(i) OC

(ii) CB

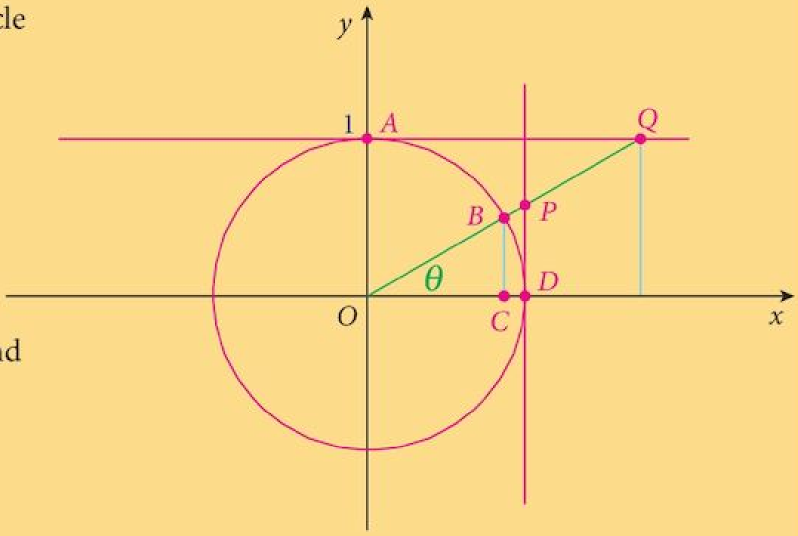
(iii) DP

b Using similar triangles and your knowledge of sine and cosine, find the lengths:

(i) OP

(ii) OQ

(iii) AQ



For most purposes, it is often easier to convert everything to sines, cosines and tangents but there are important cases where the reciprocal functions are useful.

Example 16.1

Solve the equation $\sec 2\theta = -4$, for $-180° \leqslant \theta \leqslant 180°$.

Solution:

$$\sec 2\theta = -4$$
$$\cos 2\theta = -0.25$$

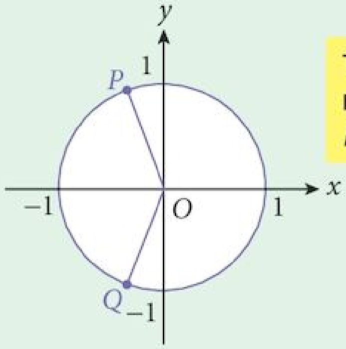
We know how to solve this equation.

$$2\theta \in \{-255.522°, 104.477°\}$$

or $\quad 2\theta \in \{-104.477°, 255.522°\}$

The required solutions are:

$$\theta \in \{-127.8°, -52.2°, 52.2°, 127.8°\}$$



The solutions are represented by points P and Q in the diagram.

Exercise 16.1

1 Find all solutions to each equation in the range $0° \leqslant \theta \leqslant 360°$, giving your answers correct to 0.1°.

a $\sec\theta = 2$ **b** $\cot\theta = 1.192$ **c** $\operatorname{cosec}\theta = 1.1547$

d $\sec\theta = -1.221$ **e** $\cot\theta = -1.5$ **f** $\operatorname{cosec}\theta = -1.25$

2 Find all solutions to each equation in the range $-180° \leqslant \theta \leqslant 180°$, giving your answers correct to 0.1°.

a $\sec\theta = 1.555$ **b** $\cot\theta = 0.577$ **c** $\operatorname{cosec}\theta = 1.0353$

d $\sec\theta = -1.667$ **e** $\cot\theta = -2.5$ **f** $\operatorname{cosec}\theta = -5$

3 Find all solutions to each equation in the range $-\pi \leqslant \theta \leqslant \pi$, giving your answers correct to 3 s.f.

a $\sec\theta = 2.5$ **b** $\cot\theta = 3$ **c** $\operatorname{cosec}\theta = 4$

d $\sec\theta = -3$ **e** $\cot\theta = -2$ **f** $\operatorname{cosec}\theta = -10$

4 Find all solutions to each equation in the range $0 \leqslant \theta \leqslant 2\pi$, giving your answers correct to 3 s.f.

a $\sec\theta = 3.5$ **b** $\cot\theta = 0.4$ **c** $\operatorname{cosec}\theta = 3$

d $\sec\theta = -1$ **e** $\cot\theta = -0.2$ **f** $\operatorname{cosec}\theta = -6$

5 Find all solutions to each equation in the range $0° \leqslant \theta \leqslant 360°$, giving your answers correct to 0.1°.

a $\sec 2\theta = 2.5$ **b** $\cot 3\theta = 3$ **c** $\operatorname{cosec} 2\theta = 4$

d $\sec 3\theta = -5$ **e** $\cot 4\theta = -0.8$ **f** $\operatorname{cosec} 4\theta = -8$

g $\sec(2\theta + 30°) = 2.5$ **h** $\cot(3\theta - 60°) = -3$ **i** $\operatorname{cosec}(2\theta + 10°) = 4$

6 Find all solutions to each equation in the range $0 \leqslant \theta \leqslant 2\pi$, giving your answers correct to 3 s.f.

a	$\sec 2\theta = 1.5$	**b**	$\cot 3\theta = 2$	**c**	$\operatorname{cosec} 2\theta = 2.5$
d	$\sec 3\theta = -3.5$	**e**	$\cot 4\theta = -0.4$	**f**	$\operatorname{cosec} 4\theta = -4$
g	$\sec(2\theta + 1) = -1.5$	**h**	$\cot(3\theta - 2) = 0.5$	**i**	$\operatorname{cosec}(2\theta + 3) = -2$

16.3 Identities

An **identity** is similar to an equation. However, there is an important difference.

An equation is a mathematical statement that is sometimes true.

It contains a variable (sometimes more than one), and the values of the variable that make the equation true are called **solutions**.

You have met the term **identity** before with a different meaning. This does not usually cause confusion.

For example, when the value $x = 2$ is substituted into the equation $3x + 5 = 11$, the result is a true statement.

If we substituted the value $x = 1$ (or any other value) into the equation, the result would not be true.

On the other hand, an **identity** is true for all values of the variable (or variables).

For example, $(x + 1)^2 = x^2 + 2x + 1$.

Problem 16.2

a Choose a value for x and substitute it into each side of the identity above. What happens? Does it make any difference which value you choose?

b What happens if you try to solve the identity?

We distinguish between equations and identities by using a different relationship symbol (≡).

So $(x+1)^2 \equiv x^2 + 2x + 1$ is an identity

but $x^2 - 5x + 4 = 0$ is an equation.

This equation has 2 solutions.

The symbol ≡ means "**is identically equal to**".

Unfortunately, while we should use the correct symbol in every case, we are often lazy about using this symbol! You should try to use it correctly, or, at least, recognise its significance when it should be used.

In fact, the definitions we began with are identities.

$$\sec\theta \equiv \frac{1}{\cos\theta} \equiv \frac{\text{hyp}}{\text{adj}} \quad [1]$$

$$\operatorname{cosec}\theta \equiv \frac{1}{\sin\theta} \equiv \frac{\text{hyp}}{\text{opp}} \quad [2]$$

$$\cot\theta \equiv \frac{1}{\tan\theta} \equiv \frac{\text{adj}}{\text{opp}} \quad [3]$$

and so is Pythagoras' Theorem $a^2 + b^2 \equiv c^2$ [4]

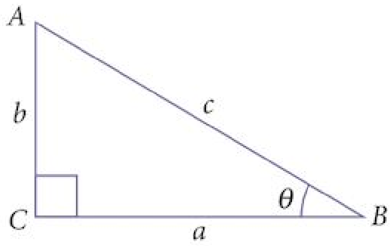
16.4 Trigonometric identities

16.4.1 Tangent, cotangent

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta} \qquad [5]$$

$$\cot\theta \equiv \frac{\cos\theta}{\sin\theta} \qquad [6]$$

16.4.2 Pythagorean identities



Starting with Pythagoras' Theorem

$$a^2 + b^2 \equiv c^2$$

we can divide both sides by either:

a^2 or b^2 or c^2

to get $\dfrac{a^2}{a^2} + \dfrac{b^2}{a^2} \equiv \dfrac{c^2}{a^2}$ $\qquad$ $\dfrac{a^2}{b^2} + \dfrac{b^2}{b^2} \equiv \dfrac{c^2}{b^2}$ $\qquad$ $\dfrac{a^2}{c^2} + \dfrac{b^2}{c^2} \equiv \dfrac{c^2}{c^2}$

or $1 + \left(\dfrac{b}{a}\right)^2 \equiv \left(\dfrac{c}{a}\right)^2$ $\qquad$ $\left(\dfrac{a}{b}\right)^2 + 1 \equiv \left(\dfrac{c}{b}\right)^2$ $\qquad$ $\left(\dfrac{a}{c}\right)^2 + \left(\dfrac{b}{c}\right)^2 \equiv 1$

$1 + \tan^2\theta \equiv \sec^2\theta$ $\qquad$ $\cot^2\theta + 1 \equiv \text{cosec}^2\theta$ $\qquad$ $\cos^2\theta + \sin^2\theta \equiv 1$

Thus we have created three more identities:

$$\sec^2\theta \equiv 1 + \tan^2\theta \qquad [7]$$

$$\text{cosec}^2\theta \equiv 1 + \cot^2\theta \qquad [8]$$

$$\sin^2\theta + \cos^2\theta \equiv 1 \qquad [9]$$

Of these three, [9] is the most useful, then [7]. [8] is not used all that often.

16.5 Using trigonometric identities

There are two main uses of identities:

1 to help simplify equations so that we can solve them

2 to create more identities.

16.5.1 Simplifying equations

Example 16.2

Solve the equation $2\sin\theta - \cos\theta = 0$, for $-180° \leqslant \theta \leqslant 180°$.

Solution:

$2\sin\theta - \cos\theta = 0$

Add $\cos\theta$ to both sides: $\quad 2\sin\theta = \cos\theta$

Divide both sides by $2\cos\theta$: $\quad \dfrac{\sin\theta}{\cos\theta} = \dfrac{1}{2}$

$\dfrac{\sin\theta}{\cos\theta} = \tan\theta$

Using identity [5]: $\quad \tan\theta = 0.5$

We know how to solve this equation.

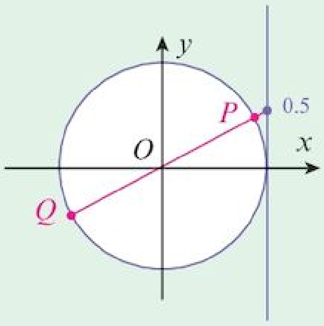
$$\tan\theta = 0.5$$

$$\theta \in \{26.57°, 206.57°, 386.57°, \ldots\}$$

or $\quad \theta \in \{-153.43°, -333.43°, \ldots\}$

The required solutions are:

$$\theta \in \{-153.43°, 26.57°\}$$



The solutions are represented by points P and Q in the diagram.

Note that the simplification technique is to substitutute part of the identity by another part. This is similar to solving a pair of simultaneous equations but it is not the same thing. This is an identity which is true for all values of the variable. We are replacing one expression by another that is exactly the same even though it looks different.

Example 16.3

Solve the equation $2\sin\theta - \tan\theta = 0$, for $-180° \leqslant \theta \leqslant 180°$.

Solution:

$$2\sin\theta - \tan\theta = 0$$

Using identity [5] $\quad 2\sin\theta - \dfrac{\sin\theta}{\cos\theta} = 0$

This reduces to $\quad 2\sin\theta\cos\theta - \sin\theta = 0$

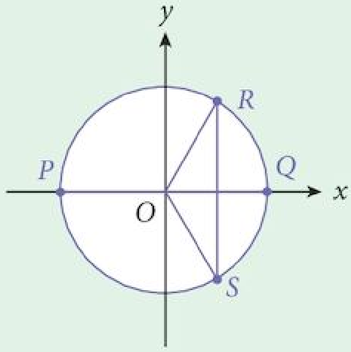
or $\quad \sin\theta(2\cos\theta - 1) = 0$

Note that you cannot divide both sides by $\sin\theta$ because it might be zero! However, $\cos\theta$ cannot be zero. Why not?

so, either $\sin\theta = 0$ or $\cos\theta = 0.5$

The required solutions are:

$\theta \in \{-180°, -60°, 0°, 60°, 180°\}$



Example 16.4

Solve the equation $4\cos^2\theta + 3\sin\theta = 4$, for $0° \leqslant \theta \leqslant 360°$.

Solution:

$$4\cos^2\theta + 3\sin\theta = 4$$

Using identity [9] $\quad 4(1-\sin^2\theta) + 3\sin\theta = 4$

Rearrange the equation $\quad 4\sin^2\theta - 3\sin\theta = 0$

$\sin^2\theta + \cos^2\theta \equiv 1$

This is just a quadratic equation:

$$\sin\theta(4\sin\theta - 3) = 0$$

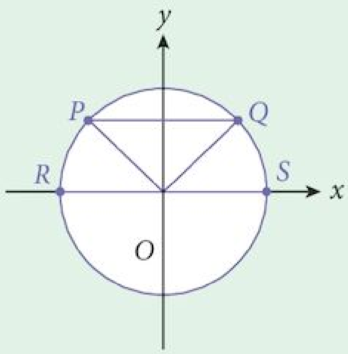
so, either $\sin\theta = 0$ or $(4\sin\theta - 3) = 0$

from which $\sin\theta = 0$ or $\sin\theta = \frac{3}{4}$

$\theta \in \{\ldots, 0°, 180°, 360°\ldots\}$

or $\theta \in \{\ldots, 48.6°, 131.4°, \ldots\}$

So $\theta \in \{0°, 48.6°, 131.4°, 180°, 360°\}$



16.5.2 Creating new identities

If you want to create a new identity (or prove one in a question) such as A ≡ B, you saw in Problem 16.2 that it is no use starting with A = B and trying to solve the equation.

Eventually, you will get to 0 = 0 and that, while true, is no help.

There are three techniques that are usually used:

(i) Starting with A	(ii) Starting with B	(iii) Starting at each end	
		A first	then B
A	B	A	B
$\equiv A_1$	$\equiv B_1$	$\equiv A_1$	$\equiv B_1$
$\equiv \ldots$	$\equiv \ldots$	$\equiv \ldots$	$\equiv \ldots$
$\equiv \ldots$	$\equiv \ldots$	$\equiv \ldots$	$\equiv \ldots$
$\equiv \ldots$	$\equiv \ldots$	$\equiv C$	$\equiv C$
$\equiv B$	$\equiv A$		

Technique (i) is the obvious starting point, unless you see something easier in technique (ii).

Technique (iii) is more usual if you start with (i) and then get stuck!

In that case, start at the other end and try to get to the same place.

At every step, you replace parts of the expression using other identities and then simplify what you have.

There are also two alternative techniques available:

(iv) Prove that $A - B \equiv 0$

(v) Prove that $\frac{A}{B} \equiv 1$

Example 16.5

Prove the identity $\tan\theta + \cot\theta \equiv \sec\theta \operatorname{cosec}\theta$.

Solution:

Using technique (i):

$$\tan\theta + \cot\theta$$

Using identities [5] and [6]: $\equiv \frac{\sin\theta}{\cos\theta} + \frac{\cos\theta}{\sin\theta}$

Put them over a common denominator: $\equiv \frac{\sin^2\theta + \cos^2\theta}{\cos\theta\sin\theta}$

Avoid doing this step in reverse!

Using identity [9]: $\equiv \frac{1}{\cos\theta\sin\theta}$

$$\equiv \frac{1}{\cos\theta} \times \frac{1}{\sin\theta}$$

Using identities [1] and [2]: $\equiv \sec\theta \operatorname{cosec}\theta$

Example 16.6

Prove the identity $\tan^2\theta - \cot^2\theta \equiv \sec^2\theta - \operatorname{cosec}^2\theta$.

Solution:

Using technique (iii):

LHS $\quad \tan^2\theta - \cot^2\theta$

Using identities [5] and [6]: $\quad \equiv \dfrac{\sin^2\theta}{\cos^2\theta} - \dfrac{\cos^2\theta}{\sin^2\theta}$

Put them over a common denominator: $\quad \equiv \dfrac{\sin^4\theta - \cos^4\theta}{\cos^2\theta\sin^2\theta}$

Using identity $\left[a^2 - b^2 \equiv (a+b)(a-b)\right]$: $\quad \equiv \dfrac{(\sin^2\theta + \cos^2\theta)(\sin^2\theta - \cos^2\theta)}{\cos^2\theta\sin^2\theta}$

Using identity [9]: $\quad \equiv \dfrac{\sin^2\theta - \cos^2\theta}{\cos^2\theta\sin^2\theta}$

RHS $\quad \sec^2\theta - \operatorname{cosec}^2\theta$

Using identities [5] and [6]: $\quad \equiv \dfrac{1}{\cos^2\theta} - \dfrac{1}{\sin^2\theta}$

Put them over a common denominator: $\quad \equiv \dfrac{\sin^2\theta - \cos^2\theta}{\cos^2\theta\sin^2\theta}$

We have reached the same expression from each end.

Thus $\quad \tan^2\theta - \cot^2\theta \equiv \sec^2\theta - \operatorname{cosec}^2\theta$

Exercise 16.2

1 Find all solutions to each equation in the range $0° \leqslant \theta \leqslant 360°$, giving your answers correct to $0.1°$ unless they can be written accurately.

a $\sin\theta - 2\cos\theta = 0$ **b** $\sec\theta - 3\operatorname{cosec}\theta = 0$ **c** $5\cos\theta - 3\sin\theta = 0$

d $3\sin^2\theta - \cos^2\theta = 0$ **e** $4\tan^2\theta - 3\sec^2\theta = 0$ **f** $\operatorname{cosec}^2\theta - 2\cot^2\theta = 0$

g $\operatorname{cosec}\theta - \cot^2\theta = 0$ **h** $2\sec^2\theta - \tan\theta - 5 = 0$ **i** $2\sin^2\theta - 2\cos^2\theta = 1$

j $\sec^2\theta - 7 = \tan\theta$ **k** $3\cos^2\theta - 5\sin\theta - 5 = 0$ **l** $3\operatorname{cosec}^2\theta + 8\cot\theta = 0$

2 Find all solutions to each equation in the range $-\pi \leqslant \theta \leqslant \pi$, giving your answers correct to 3 s.f. or as a rational multiple of π as appropriate.

a $\sin\theta + 3\cos\theta = 0$ **b** $2\sec\theta + \operatorname{cosec}\theta = 0$ **c** $\cos\theta + \sin\theta = 0$

d $\sin^2\theta - \cos^2\theta = 0$ **e** $4\tan^2\theta - \sec^2\theta = 0$ **f** $2\operatorname{cosec}^2\theta - 3\cot^2\theta = 0$

g $\cot^2\theta - \operatorname{cosec}\theta = 1$ **h** $\tan^2\theta - 2\sec\theta = 2$ **i** $6\sin^2\theta - 5\cos\theta - 7 = 0$

j $\sec^2\theta + \tan\theta - 3 = 0$ **k** $8\cos^2\theta + 2\sin\theta - 5 = 0$ **l** $\operatorname{cosec}^2\theta - \cot\theta - 2 = 0$

3 Prove these identities:

a $\operatorname{cosec}\theta - \sin\theta \equiv \cos\theta\cot\theta$

b $(\sin\theta + \cos\theta)^2 \equiv 1 + 2\sin\theta\cos\theta$

c $\sec\theta - \tan\theta \equiv \dfrac{1}{\sec\theta + \tan\theta}$

d $\operatorname{cossec}\theta + \cot\theta \equiv \dfrac{1}{\operatorname{cosec}\theta - \cot\theta}$

e $1 - 2\cos^2\theta \equiv 2\sin^2\theta - 1$

f $\tan^2\theta - \cot^2\theta \equiv (\sec\theta + \operatorname{cosec}\theta)(\sec\theta - \operatorname{cosec}\theta)$

g $\dfrac{\tan^2\theta - 1}{\tan^2\theta + 1} \equiv \sin^2\theta - \cos^2\theta$

h $\dfrac{\sin\theta}{1+\cos\theta} + \dfrac{1+\cos\theta}{\sin\theta} \equiv 2\operatorname{cosec}\theta$

i $\dfrac{1+\sin\theta}{\cos\theta} \equiv \dfrac{\cos\theta}{1-\sin\theta}$

j $\dfrac{1+\cos\theta}{1-\cos\theta} \equiv \dfrac{1+\sec\theta}{\sec\theta - 1}$

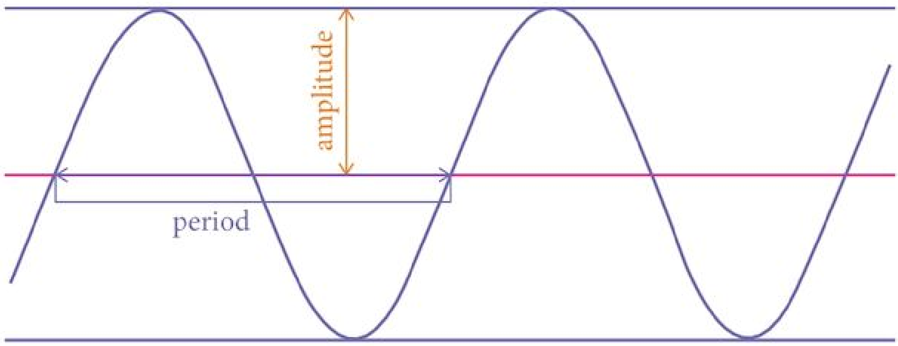
16.6 Graphs of trigonometric functions

You have already met the graphs of the trigonometric functions. The graphs of the sine and cosine functions are waves, and even though the graph of the tangent function does not look like a wave, it is still periodic.

We describe periodic functions by their amplitude and period.

The **amplitude** is the distance from the centre line to the extremes of the wave.

The **period** is the distance between any two equivalent points in the direction of the wave.



The graphs of

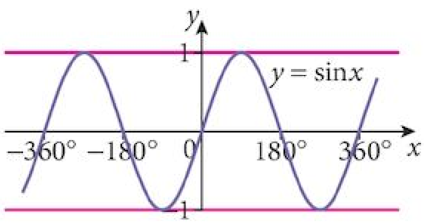
$y = \sin x$

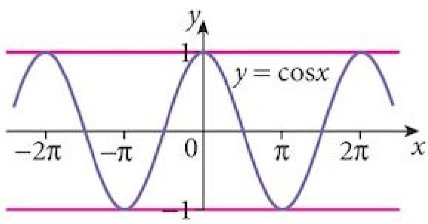
and $y = \cos x$

both have amplitude 1

and period 360°

or 2π radians.





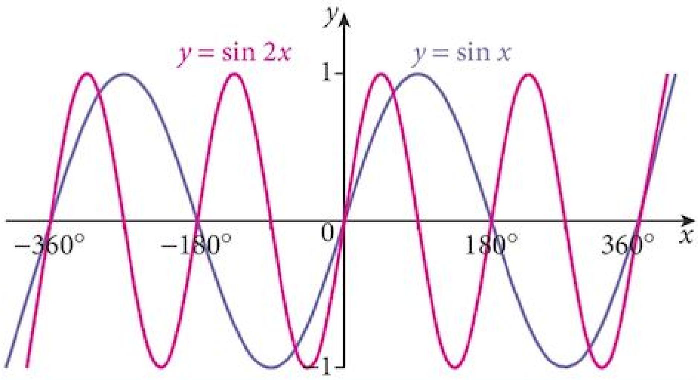
16.7 Transforming the graphs of trigonometric functions

16.7.1 Dilation in the x-direction

What happens when we replace x by $2x$ in the equation of the graph?

The result is shown opposite.

As you can see, the amplitude is unchanged but the period has been halved.



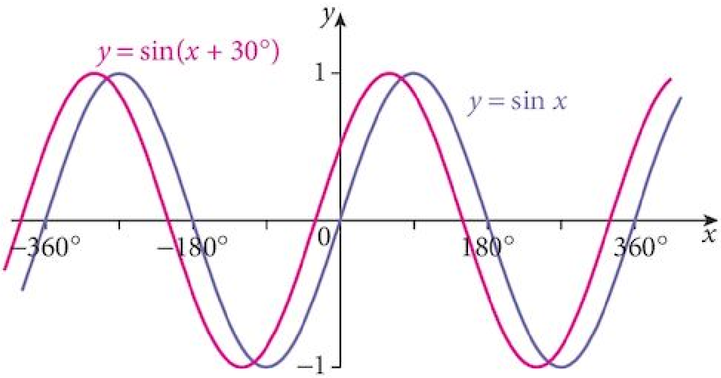
> ### *Problem 16.3*
>
> **a** Draw the graphs of $y=\sin 3x$, $y=\sin 4x$ and $y=\sin\frac{1}{2}x$ and compare them with the graph of $y=\sin x$.
>
> **b** Draw the graphs of $y=\cos 2x$, $y=\cos 3x$ and $y=\cos\frac{1}{2}x$ and compare them with the graph of $y=\cos x$.
>
> **c** Formulate a conjecture about the effect of replacing x by kx in the graphs of $y=\sin x$ and $y=\cos x$.

16.7.2 Translation in the x-direction

What happens when we replace x by $x+30°$ in the equation of the graph?

The result is shown opposite.

As you can see, the amplitude and period are unchanged but the graph has been translated to the left (negatively) by 30°.



> ### *Problem 16.4*
>
> **a** Draw the graphs of $y=\sin(x+60°)$, $y=\sin(x-30°)$ and $y=\sin(x+60°)$ and compare them with the graph of $y=\sin x$.
>
> **b** Draw the graphs of $y=\cos(x+60°)$, $y=\cos(x-30°)$ and $y=\cos(x+60°)$ and compare them with the graph of $y=\cos x$.
>
> **c** Formulate a conjecture about the effect of replacing x by $x+\theta$ in the graphs of $y=\sin x$ and $y=\cos x$.

16.7.3 Dilation in the y-direction

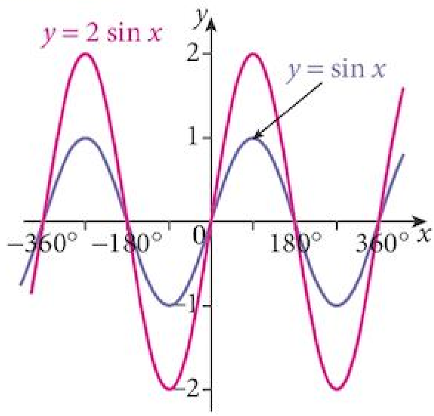
When we replace the "y" in an equation, we do not usually do just that.
It is more usual to transpose the equation so that it is still of the form $y = f(x)$.
However, we can simplify the rules if we remember that it is really a y-transformation.

So, for example, replacing y by $\frac{y}{2}$ to get $\frac{y}{2} = \sin x$ would normally be written as $y = 2\sin x$.

What happens when we replace y by $\frac{y}{2}$ in the equation of the graph?

The result is shown opposite.

This time, the amplitude has been doubled while the period is unchanged.



Problem 16.5

a Draw the graphs of $y = 3\sin x$, $y = 4\sin x$ and $y = \frac{1}{2}\sin x$ and compare them with the graph of $y = \sin x$.

b Draw the graphs of $y = 2\cos x$, $y = 3\cos x$ and $y = \frac{1}{2}\cos x$ and compare them with the graph of $y = \cos x$.

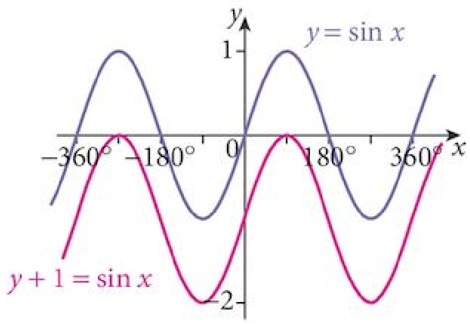
c Formulate a conjecture about the effect of replacing y by ky in the graphs of $y = \sin x$ and $y = \cos x$.

16.7.4 Translation in the y-direction

What happens when we replace y by $y + 1$ in the equation of the graph?

The result is shown opposite.

As you can see, the amplitude and period are unchanged but the graph has been translated downward (negatively) by 1 unit.



Problem 16.6

a Draw the graphs of $y+2=\sin x$, $y-1=\sin x$, $y-2=\sin x$ and compare them with the graph of $y=\sin x$. Rewrite the equations in the format $y=f(x)$.

b Draw the graphs of $y+1=\cos x$, $y-1=\cos x$, $y-2=\cos x$ and compare them with the graph of $y=\cos x$. Rewrite the equations in the format $y=f(x)$.

c Formulate a conjecture about the effect of replacing y by $y+b$ in the graphs of $y=\sin x$ and $y=\cos x$.

16.7.5 Sumarising the results

Starting with one of the equations

$$y=a\sin(bx)+c$$
$$y=a\cos(bx)+c$$
$$y=a\tan(bx)+c$$

This theory applies to all curves, not just trigonometric ones.

we can reorganise the equations to become

$$\frac{y-c}{a}=\sin(bx)$$
$$\frac{y-c}{a}=\cos(bx)$$
$$\frac{y-c}{a}=\tan(bx)$$

The three elements a, b and c have the following effects on the basic graphs $y=\sin x$, $y=\cos x$, $y=\tan x$:

Element	Effect	Magnitude
$a(>0)$	dilation in the y-direction	a
$b\ (>0)$	dilation in the x-direction	$\frac{1}{b}$
$c\ (>0)$	translation in the y-direction	positive (upwards) distance c
$c\ (<0)$	translation in the y-direction	negative (downwards) distance c

Notice that, in each case, the effect is the opposite to what might be expected.
Dividing y by a enlarges the graph in the y-direction by factor a.
Multiplying x by b squashes the graph in the x-direction.
Subtracting a positive quantity, c, from y moves the graph positively (upwards).
Adding a positive quantity (subtracting a negative), c, to y moves the graph negatively (downwards).

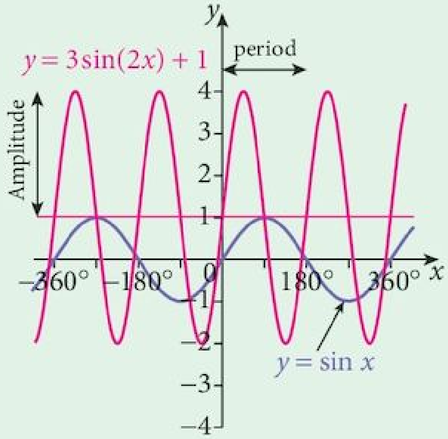
There are two transformations in the y-direction: dilation and translation.
We prefer to do the dilation first because that keeps the origin invariant.
If we translated first, the dilation would then change the size of the translation.
Remember: Dilate, then Translate.

Example 16.7

Sketch the curve $y = 3\sin(2x) + 1$.

Solution:

Transform the equation to get $\frac{y-1}{3} = \sin(2x)$



Comparing with the standard equation $y = \sin x$

We have two dilations: y-direction by factor 3

x-direction by factor 0.5

followed by a translation: y-direction by positive (upwards) 1 unit

The result is as shown.

The axis of the graph is the line $y = 1$ (this is from the y-direction translation)

The amplitude of the graph $= 3$ (this is the y-direction scale factor)

The period of the graph (0.5)) $= 180°$ (this is $360° \times$ the x-direction scale factor

or, in radians, $= \pi$ (this is $2\pi \times$ the x-direction scale factor (0.5))

16.8 Maximising trigonometric functions

The standard process to find a maximum or mininum of a function is to differentiate the function.

However, we have seen that there is an alternative process for quadratic functions – namely, completing the square.

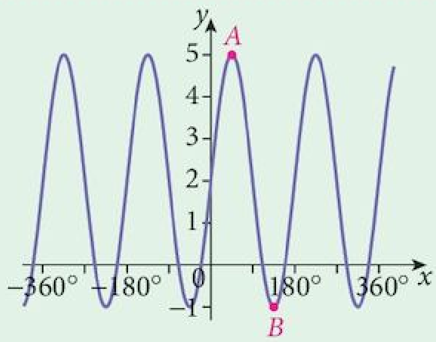
There is another alternative for sine and cosine functions since we know that they are bounded above and below.

Example 16.8

Find the maximum and minimum values of the function $y = 2 + 3\sin 2x$ and the smallest positive values of x necessary to attain a maximum or minimum value.

Solution:

The graph of the function $y = 2 + 3\sin 2x$ is shown.



It has amplitude 3 and period 180°.

The maximum value of the sine function is 1,

thus the maximum value of $2 + 3\sin 2x = 5$.

The smallest positive maximum is at point A.

This happens when $\sin 2x = 1$

or $2x = 90°$

$x = 45°$

The minimum value of the sine function is -1

thus the minimum value of $2 + 3\sin 2x = -1$.

The smallest positive minimum is at point B.

This happens when $\sin 2x = -1$

or $2x = 270°$

$x = 135°$

Exercise 16.3

1 You are given the quadratic equation $y = 2x^2 - 4x + 4$.

   a Write the equation in completed square format.

   b Show that this is equivalent to

   $$\frac{y-2}{2} = (x-1)^2.$$

   c Comparing this to the basic quadratic equation, $y = x^2$, describe the transformations needed to convert the basic graph into the graph of $y = 2x^2 - 4x + 4$.

   d Using this information, sketch the curve.

2 For each of the following graphs,

   (i) describe the transformations necessary to convert the equivalent basic graph into it

   (ii) sketch the graph

   (iii) specify the period of the graph in both degree and radian formats.

   (iv) in the case of a sine or cosine graph, state the amplitude of the curve.

   a $y = 2\sin(3x) + 1$ b $y = 3\cos(4x) - 2$ c $y = \tan(2x) + 3$

   d $y = 4\sin(4x) - 3$ e $y = 2\cos(2x) + 2$ f $y = 2\tan(3x)$

3 Sketch the graph of $y = |\sin x|$.

4 For each function, state the maximum and minimum values and find the smallest positive value of x for which the function attains these stationary values.

   a $y = 3\sin(2x) - 1$ b $y = 4\cos(3x) + 2$ c $y = \sin(3x) + 3$

   d $y = 2\sin(6x) + 4$ e $y = 3\cos(6x) - 2$ f $y = 2\cos(5x)$

5 On the same axes, sketch the graphs of $y = \sin x$ and $y = \cos x$.

What transformation will map the graph of $y = \sin x$ onto that of $y = \cos x$?

Hence make a conjecture about the relationship between the sine function and the cosine function.

(Note that there are many possible answers to this question. Try to choose the simplest.)

6 **a** Sketch the graph of $y = \cos x$.

b Sketch the result of applying a dilation of scale factor -1 in the y-direction to this graph.

Write down the equation of the resulting graph.

c What happens if you apply a dilation of scale factor -1 in the x-direction to this graph?

Write down the equation of the resulting graph.

Formulate a conjecture about the functions $\cos(x)$ and $\cos(-x)$.

Summary

Trigonometric identities $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$ $\cot\theta \equiv \dfrac{\cos\theta}{\sin\theta}$

The reciprocal trigonometric functions $\operatorname{cosec}\theta \equiv \dfrac{1}{\sin\theta}$ $\sec\theta \equiv \dfrac{1}{\cos\theta}$ $\cot\theta \equiv \dfrac{1}{\tan\theta}$

Pythagorean identities

$$\sec^2\theta \equiv 1 + \tan^2\theta$$
$$\operatorname{cosec}^2\theta \equiv 1 + \cot^2\theta$$
$$\sin^2\theta + \cos^2\theta \equiv 1$$

Using trigonometric identities Use trigonometric identities to simplify equations and prove other identities.

Transforming graphs Starting with $y = \sin x$

to get $y = a\sin(bx) + c$

we can reorganise the equations to become

$$\frac{y - c}{a} = \sin(bx)$$

The three elements a, b and c have the following effects:

Element	Effect	Magnitude
a (> 0)	dilation in the y-direction	a
b (> 0)	dilation in the x-direction	$\frac{1}{b}$
c (> 0)	translation in the y-direction	positive (upwards) distance c
c (< 0)	translation in the y-direction	negative (downwards) distance c

Maximising trigonometric functions

The function $y = a \sin(bx) + c$ has a maximum value of $a + c$ and a minimum value of $a - c$. These occur when bx is of the form $360°n + 90°$ and $360°n + 270°$.

Chapter 16 Summative Exercise

1 Find all solutions of the following equations in the range $0° \leqslant x \leqslant 360°$.

a $\sec x = 2.923$ **b** $\operatorname{cosec} x = 1.305$ **c** $\cot x = -1.192$

d $\sec 2x = -1.305$ **e** $\operatorname{cosec} 2x = -1.556$ **f** $\cot 2x = 2.747$

2 Find all solutions of the following equations in the range $-\pi \leqslant x \leqslant \pi$, expressing your answer as a rational multiple of π.

a $\sec x = 1.236$ **b** $\operatorname{cosec} x = 1.155$ **c** $\cot x = -0.414$

d $\sec 2x = -3.326$ **e** $\operatorname{cosec} 2x = -1.414$ **f** $\cot 2x = 1.376$

3 Find all solutions of the following equations in the range $-180° \leqslant x \leqslant 180°$.

a $\sec(x - 20°) = 1.035$ **b** $\operatorname{cosec}(x + 30°) = -1.556$ **c** $\cot(x - 50°) = -5.671$

d $\sec(3x + 10°) = -5.759$ **e** $\operatorname{cosec}(2x - 60°) = 1.155$ **f** $\cot(3x - 10°) = 0.839$

4 Find all solutions of the following equations in the range $-\pi \leqslant x \leqslant \pi$, expressing your answer either as a rational multiple of π or correct to 3 s.f.

a $3\cos x + \cot x = 0$ **b** $2\sin^2 x + 5\cos^2 x = 3$ **c** $\tan^2 x = \sec x + 1$

d $2\cos x + 3\sec x = 7$ **e** $4\tan^2 x - 3\sec^2 x = 1$ **f** $3\operatorname{cosec}^2 x - 5\cot^2 x = 1$

5 Solve the following equations for values of x in the range $0 \leqslant x \leqslant 2\pi$, expressing your answer either as a rational multiple of π or correct to 3 s.f.

a $2\sin^2 x - 1 + 2\cos x = 0$ **b** $\sec x(1 + \tan x) = 6\operatorname{cosec} x$

c $2\tan^2 x = 5\sec x + 1$ **d** $16\sin x - 8\sin^2 x = 5\cos^2 x$

6 Find all values in the range $0 < x < 10$ for which

a $2\sin\left(\dfrac{\pi x}{6}\right) = 1$ **b** $2\tan^2\left(\dfrac{\pi x}{3}\right) = 6$

7 If $p = \cos x + \sin x$ and $q = \cos x - \sin x$:

a show that $p^2 - q^2 = 4\sin x\cos x$

b find the value of $p^2 + q^2$

c find $\dfrac{p}{q}$ in terms of $\tan x$.

8 Prove the following identities.

a $\dfrac{1}{1+\sin x}+\dfrac{1}{1-\sin x} \equiv 2\sec^2 x$

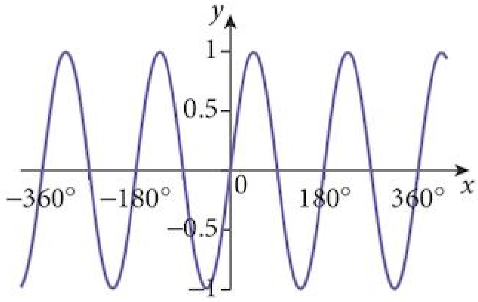
b $(1+\operatorname{cosec} x)(1-\sin x) \equiv \cos x \cot x$

c $\sqrt{\operatorname{cosec}^2 x-1}+\sqrt{\sec^2 x-1} \equiv \operatorname{cosec} x \sec x$

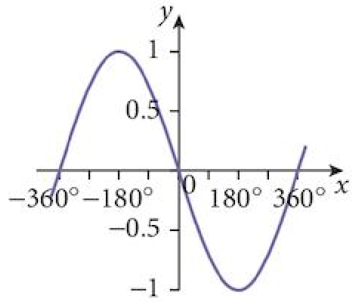
d $\dfrac{1-\sin x+\cos x}{\cos x} \equiv \dfrac{1+\sin x+\cos x}{1-\sin x}$

9 Write down the equations of the graphs shown below:

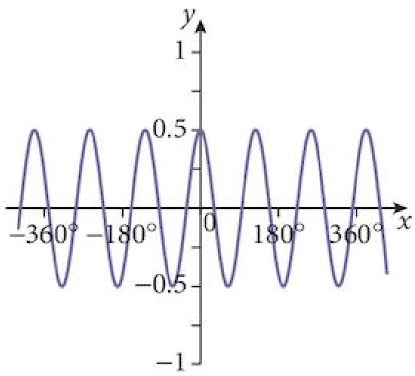
a



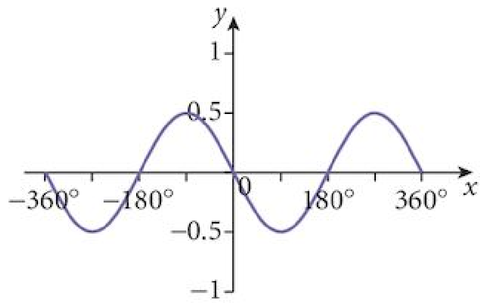
b



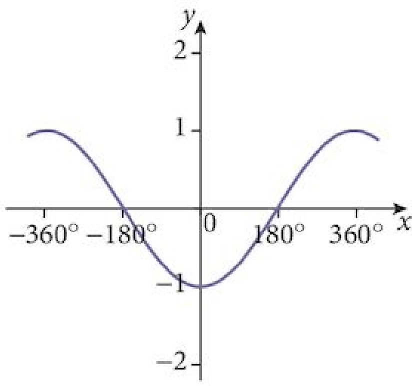
c



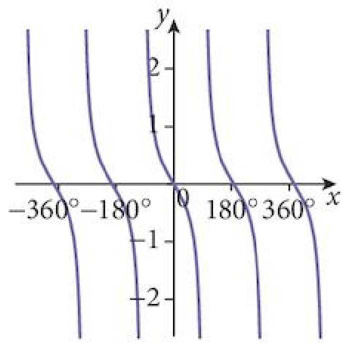
d



e



f



10 Sketch the following curves, indicating the maximum and minimum values and the period of the function.

a $y = 3 + 2\sin 3x$

b $y = 4\cos 2x - 1$

c $y = 2\sin \dfrac{1}{2}x$

d $y = 3\cos \dfrac{1}{3}x + 1$

Chapter 16 Test

1 hour

1 Prove that $(\operatorname{cosec} x + \sin x)(\operatorname{cosec} x - \sin x) \equiv \cot^2 x\,(1 + \sin^2 x)$. [4]

2 Solve the following equations for values of x in the range $0 \leqslant x \leqslant 4\pi$.

a $2\cos\left(\dfrac{x}{3} - \dfrac{\pi}{6}\right) = \sqrt{3}$ [5]

b $\sqrt{2}\operatorname{cosec}\left(\dfrac{x}{3} - \dfrac{\pi}{4}\right) = 2$ [5]

3 You are given the function $f(x) = 1 - 3\sin 2x$.

a Sketch the curve $y = f(x)$, for $0° \leqslant x \leqslant 360°$. [4]

b State the amplitude of the function. [1]

c State the period of the function. [1]

4 Solve the following equations for values of x in the range $0 \leqslant x \leqslant 2\pi$.

a $2\operatorname{cosec}^2 x - 5\cot x - 2 = 0$ [5]

b $5\sec 2x - 4\operatorname{cosec} 2x = 0$ [4]

5 Given that $\tan\theta = p$ and that θ is an acute angle, obtain an expression for

a $\sin\theta$ b $\sec\theta$. [4]

6 Solve the equation $2(\sin x + \cos x) = 3\sin x$ for $0 \leqslant x \leqslant 2\pi$, giving your answers to 3 d.p. [3]

7 Prove the identity $(\operatorname{cosec}\theta - \cot\theta)^2 \equiv \dfrac{1-\cos\theta}{1+\cos\theta}$ [4]

Examination Questions

1 Solve, for $0° \leqslant \theta \leqslant 360°$, the equation $4\sin\theta + 3\cos\theta = 0$. [3]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2002, P1, Qu 1]

2 **a** Solve, for $0° \leqslant \theta \leqslant 360°$, the equation $4\tan^2\theta + 8\sec\theta = 1$. [4]

b Given that $y < 4$, find the largest value of y such that $5\tan(2y+1) = 16$. [4]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2003, P1, Qu 9]

3 Show that $\cos\theta\left(\dfrac{1}{1-\sin\theta} - \dfrac{1}{1+\sin\theta}\right)$ can be written in the form $k\tan\theta$ and find the value of k. [4]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2003, P2, Qu 2]

4 The function f is defined, for $0° \leqslant x \leqslant 360°$, by $\mathrm{f}(x) = a\sin(bx) + c$, where a, b and c are positive integers. Given that the amplitude of f is 2 and the period of f is 120°,

(i) State the value of a and of b. [2]

Given further that the minimum value of f is -1,

(ii) State the value of c. [1]

(iii) Sketch the graph of f. [3]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2004, P1, Qu 4]

5 Given that x is measured in radians and $x > 10$, find the smallest value of x such that

$10\cos\left(\dfrac{x+1}{2}\right) = 3$. [4]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2004, P2, Qu 2]

6 Prove the identity $(1+\sec\theta)(\operatorname{cosec}\theta - \cot\theta) = \tan\theta$. [4]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2004, P2, Qu 4]

7 The function f is defined, for $0 < x < \pi$, by $\mathrm{f}(x) = 5 + 3\cos 4x$. Find

(i) the amplitude and period of f, [2]

(ii) the coordinates of the maximum and minimum points of the curve $y = \mathrm{f}(x)$. [4]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2004, P1, Qu 6]

8 Given that $x = 3\sin\theta - 2\cos\theta$ and $y = 3\cos\theta + 2\sin\theta$,

(i) find the value of the acute angle θ for which $x = y$, [3]

(ii) show that $x^2 + y^2$ is constant for all values of θ. [3]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2004, P2, Qu 6]

9 Given θ is acute and that $\sin\theta = \dfrac{1}{\sqrt{3}}$, express, without using a calculator,

$\dfrac{\sin\theta}{\cos\theta - \sin\theta}$ in the form $a + \sqrt{b}$, where a and b are integers. [5]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2005, P1, Qu 3]

10 Solve the equation

(i) $\tan 2x - 3\cot 2x = 0$, for $0° < x < 180°$, [4]

(ii) $\operatorname{cosec} y = 1 - 2\cot^2 y$, for $0° \leqslant y \leqslant 360°$, [5]

(iii) $\sec\left(z + \dfrac{\pi}{2}\right) = -2$, for $0 < z < \pi$ radians. [3]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2008, P1, Qu 11]

11 The function f is defined, for $0 \leqslant x \leqslant 2\pi$, by $f(x) = 3 + 5\sin 2x$. State

(i) the amplitude of f, [1]

(ii) the period of f, [1]

(iii) the maximum and minimum values of f. [2]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2008, P2, Qu 8]

12 Show that $\dfrac{1 - \cos^2\theta}{\sec^2\theta - 1} = 1 - \sin^2\theta$. [4]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2009, P1, Qu 3]

13 **a** Find all the angles between $0°$ and $360°$ which satisfy

(i) $2\sin x - 3\cos x = 0$, [3]

(ii) $2\sin^2 y - 3\cos y = 0$. [5]

b Given that $0 \leqslant z \leqslant 3$ radians, find, correct to 2 decimal places, all the values of z for which $\sin(2z + 1) = 0.9$. [3]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2009, P2, Qu 11]

14 a Show that $\tan\theta + \cot\theta = \operatorname{cosec}\theta\sec\theta$. [3]

b Solve the equation

(i) $\tan x = 3\sin x$, for $0° < x < 360°$ [4]

(ii) $2\cot^2 y + 3\operatorname{cosec} y = 0$, for $0 < y < 2\pi$ radians. [5]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2009, P1, Qu 11]

15 (i) Using graph paper, draw the curve $y = \sin 2x$, for $0° \leqslant x \leqslant 360°$. [3]

In order to solve the equation $1 + \sin 2x = 2\cos x$, another curve must be added to your diagram.

(ii) Write down the equation of this curve and add this curve to your diagram. [3]

(iii) State the number of values of x which satisfy the equation $1 + \sin 2x = 2\cos x$ for $0° \leqslant x \leqslant 360°$. [1]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2009, P2, Qu 7]

17 Further differentiation

Syllabus statements

- differentiate products and quotients of functions

17.1 Introduction

We return to the problem of making functions more complicated by combining simpler ones together. So far you have seen what happens when we multiply a function by a constant or add or subtract functions. In Chapter 13, you saw the effect of creating composite functions. Differentiating those gave us a product. Now we look at a more complicated pair of processes: multiplying and dividing functions. Unfortunately, the results are not as simple as we would have liked. First, try this problem using two very simple functions and you will see where the difficulty lies.

Problem 17.1

You are given the functions $u(x) = x^2$ and $v(x) = x^3$.

Find:

a $u'(x)$ **b** $v'(x)$ **c** $f(x) = u(x) \times v(x)$ **d** $f'(x)$ **e** $u'(x) \times v'(x)$

Is $f'(x) = u'(x) \times v'(x)$?

Do you get the answer you expected?

Now that we know that the simple answer is not the correct one, we need to find the correct formula.

Life would have been much simpler if these had produced the same result!

17.2 The product rule

The starting point, as usual, is with the definition of the derivative.

$$f'(x) = \underset{h \mapsto 0}{\text{Lt}}\left[\frac{f(x+h) - f(x)}{h}\right]$$

If $\quad f(x) = u(x)v(x)$

then $\quad f(x+h) = u(x+h)v(x+h)$

from which $\quad f(x+h) - f(x) = u(x+h)v(x+h) - u(x)v(x)$

We re-write this as $\quad = u(x+h)v(x+h) - u(x+h)v(x) + u(x+h)v(x) - u(x)v(x)$

which we can then factorize as $\quad = u(x+h)\left[v(x+h) - v(x)\right] + \left[u(x+h) - u(x)\right]v(x)$

We divide his by h $\quad = u(x+h)\left[\frac{v(x+h) - v(x)}{h}\right] + \left[\frac{u(x+h) - u(x)}{h}\right]v(x)$

Finally, we let $h \mapsto 0$: $\quad u(x+h) \mapsto u(x);$

$$\left[\frac{v(x+h) - v(x)}{h}\right] \mapsto v'(x);$$

$$\left[\frac{u(x+h) - u(x)}{h}\right] \mapsto u'(x);$$

and $v(x)$ is not affected by the value of h.

Thus $f'(x) = u(x)v'(x) + u'(x)v(x)$

Problem 17.2

Using the same two functions $u(x) = x^2$ and $v(x) = x^3$,
find and simplify $u(x) \times v'(x) + u'(x) \times v(x)$.
Is that better?

This is the **product rule**.

Example 17.1

If $f(x)=(2x^2-3x+1)(x^2+2x-1)$, find $f'(x)$.

Solution:

Let $u(x) = 2x^2-3x+1$ and $v(x) = x^2+2x-1$

then $u'(x) = 4x-3$ and $v'(x) = 2x+2$

giving $f'(x) = (2x^2-3x+1)(2x+2)+(4x-3)(x^2+2x-1)$

Example 17.2

If $y = x^2(x^2+2)^3$ find $\frac{dy}{dx}$.

Solution:

Let $u(x) = x^2$ and $v(x) = (x^2+2)^3$

Note that v(*x*) is a composite function.

then $u'(x) = 2x$ and $v'(x) = 3(x^2+2)^2 \times 2x$

$= 6x(x^2+2)^2$

giving $\frac{dy}{dx} = (x^2)(6x(x^2+2)^2)+(2x)((x^2+2)^3)$

Tidy up! $\frac{dy}{dx} = 6x^3(x^2+2)^2+2x(x^2+2)^3$

There are other things we could do to this expression. It would depend upon why we wanted to find the answer.

Example 17.3

Find the equation of the tangent to the curve $y=(x^3-3x)(x^2+1)^3$ at the point where $x=1$.

Solution:

We need to find **1:** the gradient when $x = 1$; this will use the product rule,

2: the position of the point when $x = 1$; we find this by substitution in the formula.

1: If $y=(x^3-3x)(x^2+1)^3$

$u(x) = (x^3-3x)$ and $v(x) = (x^2+1)^3$ v is a composite

$u'(x) = 3x^2-3$ and $v'(x) = 3(x^2+1)^2 \times 2x$

then $\frac{dy}{dx} = (x^3-3x)\times[3(x^2+1)^2\times 2x] + (3x^2-3)[(x^2+1)^3]$

When $x = 1$, $\frac{dy}{dx} = (-2)\times 3(2)^2\times 2 + (0)(2)^3$

$= -48$

2: When $x = 1$, $y = (-2)(2)^3$

$$= -16$$

So the equation of the tangent is

$$y + 16 = -48(x - 1)$$

Exercise 17.1

In questions **1–4**, use the product rule to differentiate each of the functions with respect to x. Do not simplify the results.

1 **a** $(x+3)(x-4)$ **b** $(2x-3)(5-4x)$

c $(2-3x)(6-2x)$ **d** $f(x)=(x+6)(x^2-4x)$

e $f(x)=(2x^3-3x^2)(1-2x)$ **f** $f(x)=(2-3x)(6-2x+x^2)$

g $y=(x^2-6x)(2x^2+3x)$ **h** $y=(2x-x^3)(4x-2x^2)$

i $y=(3x-2x^2)(4+2x)$

2 **a** $x^2(x-4)$ **b** $x^3(5-4x)^2$

c $x^4(6-2x)^3$ **d** $f(x)=3x^2(2x+5)^2$

e $f(x)=4x^3(1-2x)^4$ **f** $f(x)=5x^3(6-2x)^3$

g $y=4x^3(3x^2-4x+2)$ **h** $y=(2x-1)(4-2x)^2$

i $y=(3x-5)^2(2x-1)^4$

3 **a** $(x^2-4)^2(2x^3-3x)^2$ **b** $(2x-x^3)^2(4x-2x^2)^3$

c $(x^2-6x)^3(2x^2+3x)^4$ **d** $f(x)=(3x^2-2x)^3(2x+5)^2$

e $f(x)=(4x^3+2x)^2(x^2-2x^3)^4$ **f** $f(x)=(2x^2-5x)^4(6-2x)^3$

g $y=(x^2-4)^3(3x^2-4x+2)^5$ **h** $y=(x^2-3x+1)^5(3x-4x^2)^3$

i $y=(4x^2-6x)^6(3x^3-4x^2)^5$

4 **a** $x\sqrt{x-4}$ **b** $2x\sqrt{3-x}$

c $2x^2\sqrt{3x-1}$ **d** $f(x)=\sqrt{x-4}\sqrt{2x+3}$

e $f(x)=\sqrt{x^2-4}\sqrt{2x^3+3x}$ **f** $f(x)=(x^2-5x)^3\sqrt{6x^2-2x}$

g $y=(x^2-4)\sqrt[3]{2x^2+4x-2}$ **h** $y=(x^2-3x)^2\sqrt[4]{3x-4x^2}$

i $y=\left[(2x^2-3x)(4x^3-2x^2)\right]^5$

5 Find the equation of the tangent and the normal to the curve $y=x^3(1-x)^2$ at the point where $x=2$.

6 You are given the function $y=\left(x^2-2x\right)^3\left(3x-4\right)^2$.

When the value of x is increased by a small amount from 1 to $(1+p)$, what is the approximate change in the value of y?

7 At the point where $x=1$, the normal to the curve $y=2x\sqrt{(3+x)}$ meets the x-axis at the point P and the y-axis at the point Q. Find the area of the triangle OPQ.

8 A cuboid has sides of length $(x+4)$, $(x-1)$ and $(x-1)$.

a Write down the formulae for **(i)** the volume, V, of the cuboid

(ii) the total surface area, S, of the cuboid.

b Find the rate of change $\frac{dV}{dS}$ when $x=3$.

Problem 17.3

a Given that $f(x)=u(x)\times s(x)$, write down $f'(x)$.

b Given that $s(x)=v(x)\times w(x)$, write down $s'(x)$.

c Substitute in your result for **a** the formulae for $s(x)$ and $s'(x)$ to give a formula for $\left[u(x)\times v(x)\times w(x)\right]'$.

The result of Problem 17.3 is interesting.
Each term contains all three elements of the product $\left[u(x)\times v(x)\times w(x)\right]$ but in each term, a different element is differentiated.
Make a conjecture about what the result would be if there were four functions multiplied together.
Can you justify your conjecture?

17.3 The quotient rule

We could develop this rule, as usual, from the definition of the derivative.
However, it is a little easier if we use the product rule and the composite rule.

$$f(x)=\frac{u(x)}{v(x)}=u(x)\times\left[v(x)\right]^{-1}$$

Problem 17.4

a Use the composite rule to find $\frac{d}{dx}\left(\left[v(x)\right]^{-1}\right)$.

b Use this result and the product rule to find $\frac{d}{dx}\left(u(x)\times\left[v(x)\right]^{-1}\right)$.

We can write the final result in the format

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{u}(x)}{\mathrm{v}(x)}\right)=\frac{\mathrm{v}(x)\mathrm{u}'(x)-\mathrm{v}'(x)\mathrm{u}(x)}{\left[\mathrm{v}(x)\right]^2}$$

This is the **quotient rule**.

Notice that, as with the product rule, the terms in the numerator of the fraction contain each element of the quotient, in each case with one of the differentiated functions.
However, there is a big difference – the subtraction sign!

Thus, it matters in which order you write things, whereas order does not matter when we use the product rule. To help you get the correct order, it is suggested that you follow the following steps:

Step 1: Start with the denominator $\left[\mathbf{v}(x)\right]^2$

Step 2: Write $\mathbf{v}(x)$ (from the denominator) as the first part of the numerator.

Step 3: The rest should be no problem.

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\mathrm{u}(x)}{\mathrm{v}(x)}\right)=\frac{\overbrace{\mathbf{v}(x)}^{\text{Step 2}}\overbrace{\mathbf{u}'(x)-\mathbf{v}'(x)\mathbf{u}(x)}^{\text{Step 3}}}{\underbrace{\left[\mathbf{v}(x)\right]^2}_{\text{Step 1}}}$$

Do not forget the negative sign.

Example 17.4

Find $\frac{\mathrm{d}y}{\mathrm{d}x}$ if $y=\frac{x^2}{x^3+2x}$.

Solution:

Let $\mathrm{u}(x)=x^2$ and $\mathrm{v}(x)=x^3+2x$

then $\mathrm{u}'(x)=2x$ and $\mathrm{v}'(x)=3x^2+2$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x^2}{x^3+2x}\right)=\overbrace{\frac{}{\left(x^3+2x\right)^2}}^{\text{Step 1}}\mapsto\overbrace{\frac{\left(x^3+2x\right)}{\left(x^3+2x\right)^2}}^{\text{Step 2}}\mapsto\overbrace{\frac{\left(x^3+2x\right)(2x)-\left(x^2\right)\left(3x^2+2\right)}{\left(x^3+2x\right)^2}}^{\text{Step 3}}$$

and then simplify it if you need to.

Example 17.5

Find $\frac{dy}{dx}$ if $y = \frac{(x^2+2x)^3}{(x^3-x^2)^2}$.

Each element is a composite.

Solution 1: (Using the quotient rule)

Let $\quad u(x) = (x^2+2x)^3 \quad$ and $\quad v(x) = (x^3-x^2)^2$

then $\quad u'(x) = 3(x^2+2x)^2(2x+2) \quad$ and $\quad v'(x) = 2(x^3-x^2)(3x^2-2x)$

Step 1 $\quad \frac{d}{dx}\left[\frac{(x^2+2x)^3}{(x^3-x^2)^2}\right] = \frac{\qquad\qquad}{(x^3-x^2)^4}$

Step 2 $\quad \rightarrow \frac{(x^3-x^2)^2 \qquad\qquad}{(x^3-x^2)^4}$

Step 3 $\quad \rightarrow \frac{(x^3-x^2)^2\times 3(x^2+2x)^2(2x+2)-2(x^3-x^2)(3x^2-2x)\times(x^2+2x)^3}{(x^3-x^2)^4}$

This can be simplified by dividing top and bottom by (x^3-x^2) to give:

$$\frac{d}{dx}\left[\frac{(x^2+2x)^3}{(x^3-x^2)^2}\right] = \frac{3(x^3-x^2)(x^2+2x)^2(2x+2)-2(3x^2-2x)(x^2+2x)^3}{(x^3-x^2)^3}$$

Solution 2: (Writing the quotient as a composite and using the product rule)

$$y = (x^2+2x)^3 \times (x^3-x^2)^{-2}$$

Let $\quad u(x) = (x^2+2x)^3 \quad$ and $\quad v(x) = (x^3-x^2)^{-2}$

then $\quad u'(x) = 3(x^2+2x)^2(2x+2) \quad$ and $\quad v'(x) = -2(x^3-x^2)^{-3}(3x^2-2x)$

then

$$\frac{d}{dx}\left[\frac{(x^2+2x)^3}{(x^3-x^2)^2}\right] = \left[(x^2+2x)^3\right]\times\left[-2(x^3-x^2)^{-3}(3x^2-2x)\right]+\left[3(x^2+2x)^2(2x+2)\right]\times\left[(x^3-x^2)^{-2}\right]$$

or $\quad \frac{d}{dx}\left[\frac{(x^2+2x)^3}{(x^3-x^2)^2}\right] = \frac{-2(x^2+2x)^3(3x^2-2x)}{(x^3-x^2)^3} + \frac{3(x^2+2x)^2(2x+2)}{(x^3-x^2)^2}$

This can be written in the format of Solution 1 if necesary.

When we say "simplify", there is no absolute definition of what this means!
It is often a matter of personal opinion.
It may depend upon what we want to use the result for.

Example 17.6

Find the stationary points on the curve

$$y = \frac{x}{x^2 + 1}$$

and find the nature of those stationary points.

Solution:

$$\frac{dy}{dx} = \frac{(x^2+1)(1) - (x)(2x)}{(x^2+1)^2}$$

Using the quotient rule.

$$= \frac{1 - x^2}{(x^2+1)^2}$$

We require $\frac{dy}{dx} = 0$ The solutions for this are $x = -1$ and $x = 1$.

If a fraction is zero, the numerator must be zero.

When $x = -1,\ y = -\frac{1}{2}$

When $x = 1,\ y = \frac{1}{2}$

To find the nature of the stationary points, we need $\frac{d^2y}{dx^2}$

$$\frac{d^2y}{dx^2} = \frac{(x^2+1)^2[-2x] - [2(x^2+1)(2x)](1-x^2)}{(x^2+1)^4}$$

Using the quotient rule again.

When $x = -1$ $\quad \frac{d^2y}{dx^2} = \frac{4(2) - (2(2))(-2)(0)}{16} > 0$

When $x = 1$ $\quad \frac{d^2y}{dx^2} = \frac{4(-2) - (2(2))(2)(0)}{16} < 0$

Thus $\left(-1, -\frac{1}{2}\right)$ is a minimum and $\left(1, \frac{1}{2}\right)$ is a maximum.

Example 17.7

A function is defined by $y = \frac{x^2 - x}{x - 2}$.

a Find $\frac{dy}{dx}$.

b Hence find $\int_4^6 \frac{2x^2 - 8x + 4}{(x-2)^2}\,dx$.

Solution:

a If $y = \frac{x^2 - x}{x - 2}$

then $u(x) = x^2 - x$ and $v(x) = x - 2$

giving $u'(x) = 2x - 1$ and $v'(x) = 1$

from which
$$\frac{dy}{dx} = \frac{(x-2)(2x-1) - (x^2 - x)(1)}{(x-2)^2}$$
$$= \frac{x^2 - 4x + 2}{(x-2)^2}$$

b We know that $\int_4^6 \frac{x^2 - 4x + 2}{(x-2)^2}\,dx = \left[\frac{x^2 - x}{x - 2}\right]_4^6$

We know this only because we were given the function to differentiate.

So
$$\int_4^6 \frac{2x^2 - 8x + 4}{(x-2)^2}\,dx = \left[\frac{2(x^2 - x)}{x - 2}\right]_4^6$$
$$= (15) - (12)$$
$$= 3$$

Exercise 17.2

In questions **1–4**, use the quotient rule to differentiate each of the functions with respect to x. Do not simplify the results.

1 **a** $\frac{x+3}{x-4}$ **b** $\frac{2x-3}{5-4x}$ **c** $\frac{2-3x}{6-2x}$

d $f(x) = \frac{x+6}{x^2 - 4x}$ **e** $f(x) = \frac{2x^3 - 3x^2}{1 - 2x}$ **f** $f(x) = \frac{2-3x}{6 - 2x + x^2}$

g $y = \frac{x^2 - 6x}{2x^2 + 3x}$ **h** $y = \frac{2x - x^3}{4x - 2x^2}$ **i** $y = \frac{3x - 2x^2}{4 + 2x}$

2 **a** $\dfrac{x^2}{x-4}$ **b** $\dfrac{x^3}{(5-4x)^2}$ **c** $\dfrac{x^4}{(6-2x)^3}$

d $f(x)=\dfrac{3x^2}{(2x+5)^2}$ **e** $f(x)=\dfrac{4x^3}{(1-2x)^4}$ **f** $f(x)=\dfrac{5x^3}{(6-2x)^3}$

g $y=\dfrac{4x^3}{3x^2-4x+2}$ **h** $y=\dfrac{2x+x^2}{x^3+4x}$ **i** $y=\dfrac{x^2-6}{x^3-4}$

3 **a** $\dfrac{(x^2-4)^2}{(2x^3-3x)^2}$ **b** $\dfrac{(2x-x^3)^2}{(4x-2x^2)^3}$ **c** $\dfrac{(x^2-6x)^3}{(2x^2+3x)^4}$

d $f(x)=\dfrac{(3x^2-2x)^3}{(2x+5)^2}$ **e** $f(x)=\dfrac{(4x^3+2x)^2}{(x^2-2x^3)^4}$ **f** $f(x)=\dfrac{(2x^2-5x)^4}{(6-2x)^3}$

g $y=\dfrac{(x^2-4)^3}{(3x^2-4x+2)^5}$ **h** $y=\dfrac{(x^2-3x+1)^5}{(3x-4x^2)^3}$ **i** $y=\dfrac{(4x^2-6x)^6}{(3x^3-4x^2)^5}$

4 **a** $\dfrac{x}{\sqrt{x-4}}$ **b** $\dfrac{2x}{\sqrt{3-x}}$ **c** $\dfrac{2x^2}{\sqrt{3x-1}}$

d $f(x)=\dfrac{\sqrt{x-4}}{\sqrt{2x+3}}$ **e** $f(x)=\dfrac{\sqrt{x^2-4}}{\sqrt{2x^3+3x}}$ **f** $f(x)=\dfrac{(x^2-5x)^3}{\sqrt{6x^2-2x}}$

g $y=\dfrac{x^2-4}{\sqrt[3]{2x^2+4x-2}}$ **h** $y=\dfrac{(x^2-3x)^2}{\sqrt[4]{3x-4x^2}}$ **i** $y=\dfrac{x^2}{[(2x-3)(x^2+x)]^2}$

5 Find the points on the curve $y=\dfrac{x}{1+x}$ at which the gradient of the tangent is $\dfrac{1}{4}$.

6 Find the equation of the tangent and the normal to the curve $y=\dfrac{2x^2-x-2}{x-1}$ at the point where $x=2$.

7 Find and classify all the stationary points on these curves.

a $y=\dfrac{2x^2+x-2}{x-1}$ **b** $y=\dfrac{x^2-1}{x^4}$

8 You are given the function $y=\dfrac{x^3}{x^2-1}$.

When the value of x is increased by a small amount from 2 to $(2+p)$, what is the approximate change in the value of y?

9 At the point where $x=2$, the normal to the curve $y=\dfrac{x^2}{(1-x^2)}$ meets the x-axis at the point P and the y-axis at the point Q. Find the area of the triangle OPQ where O is the origin.

10 The profit, P, of an airline company depends upon the average number of passengers, N, travelling. The profit is given by the formula

$$P = \frac{200\sqrt{N}}{50 + N}.$$

The company calculates that the number of passengers is increasing at a rate of 5 per month. Calculate the equivalent rate of increase of the profit when

a $N = 25$

b $N = 100$.

Summary

Differentiation

The definition

$$f'(x) = \operatorname*{Lt}_{h \to 0}\left[\frac{f(x+h) - f(x)}{h}\right] \qquad [1]$$

Rules

The Constant Multiplier Rule

$$(kf(x))' = k(f'(x)) \qquad [2]$$

The Sum (Difference) Rule

$$[u(x) \pm v(x)]' = u'(x) \pm v'(x) \qquad [3]$$

The Composite Rule

$$(f(g(x))' = f'(g(x)) \times g'(x) \qquad [4]$$

The Product Rule

$$[u(x) \times v(x)]' = u'(x) \times v(x) + u(x) \times v'(x) \qquad [5]$$

The Quotient Rule

$$\left(\frac{u(x)}{v(x)}\right)' = \frac{u'(x) \times v(x) - u(x) \times v'(x)}{[v(x)]^2} \qquad [6]$$

Applications

Gradients
Tangents and Normals
Stationary Points
Connected Rates of Change
Small Increments
Maxima and Minima

You will be pleased to know that there are no more rules of differentiation but, one day, you might meet more applications.

Integration

The definition

Integration is the reverse process of differentiation.
If you can get a result by differentiating something, when you integrate that result, you obtain the original function.

The zero!

Do not forget to include the integral of zero (which gives a constant) every time you integrate something.

Applications Solving differential equations
Solving rate of change problems (these are differential equations)
Finding areas using definite integration (integration with limits)

Chapter 17 Summative Exercise

1 Differentiate these functions.

a $(3x + 4)(2x - 7)$ **b** $f(x) = (6 - 2x)(5 - 3x)$ **c** $y = (2x - x^2)(5x + x^3)$

d $(x^2 - 2x)(x^3 + 3x)$ **e** $f(x) = (x + \sqrt{x})(x^2 - \sqrt{x})$ **f** $y = (\sqrt[4]{x} - 3)(\sqrt[3]{x} + 4)$

2 Differentiate these functions.

a $(x^2 - 4x)^3(3x - 2)^2$ **b** $f(x) = [(x^2 + 2x)(x - 3)]^2$ **c** $y = [(3x - 8)^3(x^2 - 5x)]^3$

d $(\sqrt{x} + 4)^2(3 - \sqrt{x})^3$ **e** $f(x) = (x^3 + \sqrt{x})^2(\sqrt{x} + 4)^3$ **f** $y = (\sqrt[4]{x} + 3)^2(\sqrt[3]{x} - 4)^3$

3 Differentiate these functions.

a $\dfrac{(2x - 4)^2}{x - 1}$ **b** $f(x) = \dfrac{x^2 + 2x}{x - 3}$ **c** $y = \dfrac{(3x - 8)^3}{\sqrt{x^2 - 5x}}$

d $\dfrac{3x - 2}{\sqrt{x} + 2}$ **e** $f(x) = \dfrac{(x^3 + \sqrt{x})^2}{\sqrt{x} + 4}$ **f** $y = \dfrac{\left(\sqrt[4]{x} + 3\right)}{\left(\sqrt[3]{x} - 4\right)^2}$

4 Find the stationary points on these curves.

a $y = x(4 - x)^3$ **b** $y = \dfrac{x^2}{3x - 1}$ **c** $y = \dfrac{2x^3 + 3x^2 + 1}{2x^2}$

d $y = (x + 3)(3 - x)^2$ **e** $y = \dfrac{3x^2}{x - 1}$ **f** $y = \dfrac{4x - 8}{(x - 1)^2}$

5 You are given the function $y = \dfrac{x + 2}{x - 1}$, $x \neq 1$.

a Find $\dfrac{dy}{dx}$ when $x = 4$.

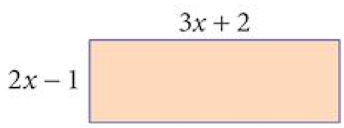
b Find the approximate change in the value of y as x increases from 4 to $(4 + p)$.

6 A curve has equation $y = \dfrac{3x + 2}{x + 2}$, $x \neq -2$.

Find the approximate increase in the value of y as x increases from 2 to $(2 + p)$.

7 A rectangle has sides of length $(2x - 1)$ and $(3x + 2)$.

Find the approximate change in the area of the rectangle as x increases from 3 to $(3 + p)$.



8 Find the gradient of these curves at the value of x indicated.

a $y = (x + 2)^2(x - 3)$ at $x = 1$ **b** $y = \dfrac{3x^2 - 4}{x - 1}$ at $x = 2$

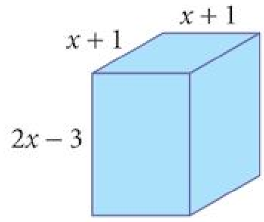
9 A box in the shape of a cuboid has a square cross-section of side $(x + 1)$ cm and a height of $(2x - 3)$ cm.

The current value of $x = 4$, but x is increasing at a rate of 0.1 cm s^{-1}.

Find the rate of increase of:

a the volume of the box

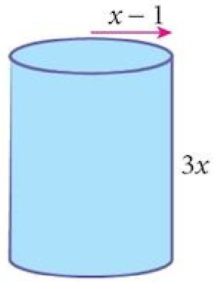
b the surface area of the box.



10 A solid cylinder has radius $(x - 1)$ cm and height $3x$ cm.

When the radius of the cylinder is 1 cm the total surface area is increasing at a rate of 0.2 $cm^2\ s^{-1}$.

Find the rate of increase of the volume of the cylinder at this time.



11 Find the equation of **a** the tangent and **b** the normal to the curve

$$y = \frac{4(x+1)}{3x-1}, \quad x \neq \frac{1}{3}$$

at the point where $x = 3$.

12 Find the equation of **a** the tangent and **b** the normal to the curve

$$y = \frac{4x+3}{2-x}, \quad x \neq 2$$

at the point where $x = -2$.

13 Find the equation of **a** the tangent and **b** the normal to the curve

$$y = \frac{x^2+3x}{x-3}, \quad x \neq 3$$

at the point where $x = 4$.

14 Given that $\frac{d}{dx}[(2x+3)^3(x-4)] = (2x+3)^2(ax+b)$, find the value of a and the value of b.

Chapter 17 Test

1 hour

1 Find the gradient of the curve $y = \frac{6-2x^2}{x-3}$ at the point where $x = 2$. [4]

2 a Find the equation of the normal to the curve $y = \frac{2x-1}{x-5}$ at the point where $x = 6$. [5]

b Find the coordinates of the point where the normal intersects the curve again. [3]

3 If $y = \dfrac{ax+b}{1-2x}$ passes through the point (3, 1), where its gradient is $\dfrac{1}{5}$, find the value of a and of b. [5]

4 A curve has equation $y = \dfrac{3-x^2}{2+x}$.

a Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$. [4]

b Find the coordinates of the stationary points on the curve. [2]

c Determine the nature of the stationary points. [2]

5 You are given the function $y = \dfrac{3x+2}{2-x}$.

Given that x is increasing at a rate of 0.2 units s^{-1} when $x = 4$, find the corresponding rate of change of y. [5]

6 Find the coordinates of the stationary points of the curve $y = \dfrac{x^2}{3x-1}$. [5]

7 A curve has equation $y = \dfrac{4x+3}{x-1}$. Find an expression in terms of p for the approximate increase in the value of y as the value of x increases from 3 to $(3+p)$, where p is small. [5]

Examination Questions

1 A curve has the equation $y = \dfrac{2x+4}{x-2}$.

(i) Find the value of k for which $\dfrac{dy}{dx} = \dfrac{k}{(x-2)^2}$. [2]

(ii) Find the equation of the normal to the curve at the point where the curve crosses the x-axis. [4]

A point (x, y) moves along the curve in such a way that the x-coordinate of the point is increasing at a constant rate of 0.05 units per second.

(iii) Find the corresponding rate of change of the y-coordinate at the instant that $y = 6$. [3]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2003, P2, Qu 10]

2 **(i)** Given that $y = (2x+3)\sqrt{4x-3}$, show that $\dfrac{dy}{dx}$ can be written in the form $\dfrac{kx}{\sqrt{4x-3}}$ and state the value of k. [5]

(ii) Hence evaluate $\displaystyle\int_1^7 \frac{x}{\sqrt{4x-3}}\,dx$. [3]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2003, P2, Qu 9]

3 Given that $y = \dfrac{3x - 2}{x^2 + 5}$, find

(i) an expression for $\dfrac{dy}{dx}$,

(ii) the x-coordinates of the stationary points. [4]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2004, P1, Qu 1]

4 It is given that $y = (x+1)(2x-3)^{\frac{3}{2}}$.

(i) Show that $\dfrac{dy}{dx}$ can be written in the form $kx\sqrt{2x-3}$ and state the value of k. [4]

Hence

(ii) find, in terms of p, an approximate value of y when $x = 6 + p$, where p is small, [3]

(iii) evaluate $\int_2^6 x\sqrt{(2x-3)}\,dx$. [3]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2004, P2, Qu 11]

5 A curve has equation $y = (x+2)\sqrt{x-1}$.

(i) Show that $\dfrac{dy}{dx} = \dfrac{kx}{\sqrt{x-1}}$, where k is a constant, and state the value of k. [4]

(ii) Hence evaluate $\displaystyle\int_2^5 \frac{x}{\sqrt{x-1}}\,dx$. [4]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2005, P1, Qu 8]

6 A curve has equation $y = (x-1)(2x-3)^8$. Find the gradient of the curve at the point where $x = 2$. [4]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2006, P1, Qu 1]

7 A curve has the equation $y = \dfrac{2x - 4}{x + 3}$.

(i) Obtain an expression for $\dfrac{dy}{dx}$ and hence explain why the curve has no turning points. [3]

The curve intersects the x-axis at the point P. The tangent to the curve at P meets the y-axis at the point Q.

(ii) Find the area of the triangle POQ, where O is the origin. [5]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2006, P1, Qu 9]

8 Find the equation of the normal to the curve $y = \dfrac{2x+4}{x-2}$ at the point where $x = 4$. [5]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2007, P1, Qu 2]

9 **a** Find

(i) $\displaystyle\int \frac{12}{(2x-1)^4}\,dx$ [2]

(ii) $\displaystyle\int x(x-1)^2\,dx$ [3]

b **(i)** Given that $y = 2(x-5)\sqrt{x+4}$, show that $\dfrac{dy}{dx} = \dfrac{3(x+1)}{\sqrt{x+4}}$. [3]

(ii) Hence find $\displaystyle\int \frac{(x+1)}{\sqrt{x+4}}\,dx$. [2]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2008, P2, Qu 10]

10 A curve has equation $y = \dfrac{2x}{x^2+9}$.

(i) Find the x-coordinates of each of the stationary points of the curve. [4]

(ii) Given that x is increasing at the rate of 2 units per second, find the rate of increase of y when $x = 1$. [3]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2009, P1, Qu 8]

11 **(i)** Given that $y = x\sqrt{4x+12}$, show that $\dfrac{dy}{dx} = \dfrac{k(x+2)}{\sqrt{4x+12}}$ where k is a constant to be found. [4]

(ii) Hence evaluate $\displaystyle\int_{-2}^{6} \frac{3x+6}{\sqrt{4x+12}}\,dx$ [3]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2009, P2, Qu 6]

18 Exponential and logarithmic functions

Syllabus statements

- know simple properties and graphs of the logarithmic and exponential functions
- know and use the laws of logarithms (including change of base of logarithms)
- solve equations of the form $a^x = b$
- transform given relationships, including $y = ax^n$ and $y = Ab^x$, to straight line form and hence determine unknown constants by calculating the gradient or intercept of the transformed graph

18.1 Introduction

In this chapter we introduce the final variety of function in this syllabus, the **exponential function** and its inverse, the **logarithmic function**. In fact, there are many of these functions, depending upon the base number chosen. They have many applications in real life including population growth rates and the way that invested money grows in value.

Until the advent of calculators (which was not all that long ago!) the logarithmic function was used for doing most calculations involving multiplication and division. School students, such as the author, spent many lessons learning how to use tables of logarithmic values. Now that you have a calculator, you can spend that time doing more interesting things.

18.2 Exponential functions

An **exponential function** is a power, or index function. Something like $y = 2^x$.
The number 2 is called the **base** of the function, but it could be any number.

Problem 18.1

a Copy and complete the table for the function $y = 2^x$.

x	0	1	2	3	4	5	6	7	8	9	10
2^x											

Any base would do but 2 gives us lots of integer values.

b What is the domain of this function.

c Plot the first seven values on graph paper.

How does the domain affect the appearance of your graph?

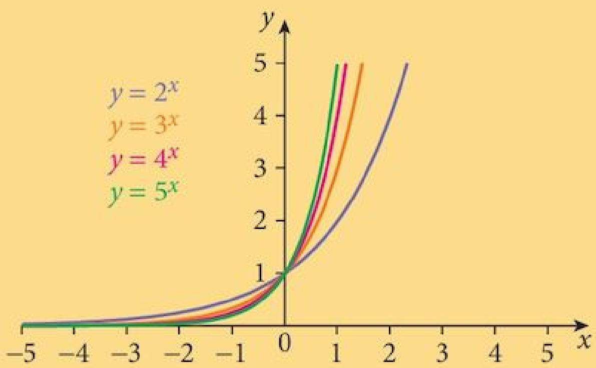
d If the domain of the function was extended to all real numbers, what would the graph look like?

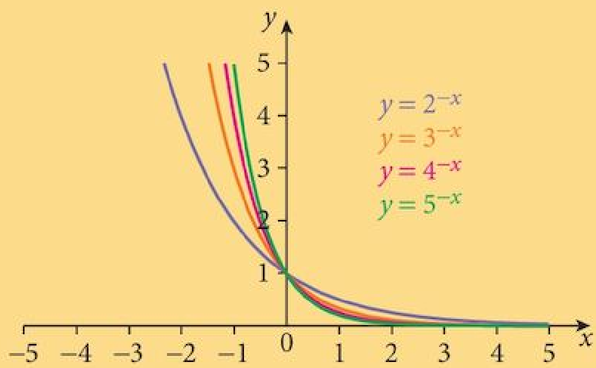
e What sort of function is f: $x \mapsto 2^x$?

You know what $2^{\frac{5}{3}}$ means but 2^{π} is a bit more of a problem. All we can say is that it is what we read from the graph. Fortunately you can get the result from your calculator.

If we were to look at other exponential functions, what would they have in common?

Below are the graphs of several exponential functions.
As you would expect, replacing x by $(-x)$ in the formula will reflect the curve in the y-axis.





f What happens to the curve when you increase the size of the base?

g Exponential curves all intersect the y-axis at the same point. What is that point?

h How would you answer part **e** in respect of these additional curves?

18.3 Logarithmic functions

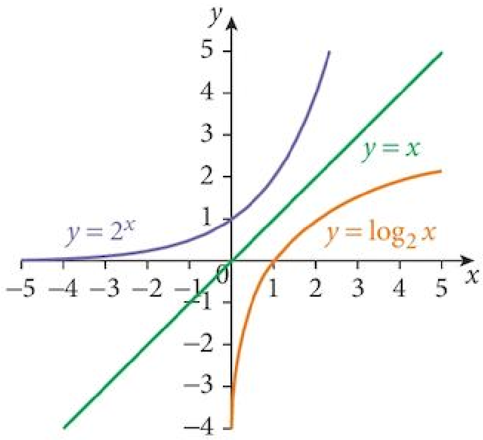
You have seen that all exponential functions are 1 : 1.
Thus, they have inverses.
We call the inverses of exponential functions **logarithmic functions**.

We have to specify the base of the logarithm in order to avoid confusion.

So if $f(x) = 2^x$, $f^{-1}(x) = \log_2 x$.

The graphs of $y = 2^x$ and $y = \log_2 x$ are shown.

From your table, $32 = 2^5$ so $\log_2 32 = 5$



Exercise 18.1

1 Find the values of the following:

a	$\log_2 8$	**b**	$\log_2 256$	**c**	$\log_2 1024$
d	$\log_2 0.25$	**e**	$\log_2 0.0625$	**f**	$\log_2 0.015625$
g	$\log_3 81$	**h**	$\log_4 64$	**i**	$\log_5 625$
j	$\log_{10} 1\,000\,000$	**k**	$\log_7 2401$	**l**	$\log_9 59\,049$

Note that since $a^x > 0$ you cannot find the log of a negative number. The domain of the log function is $\{x: x > 0\}$

2 Use your graph to find the following values.

a	$\log_2 50$	**b**	$\log_2 15$	**c**	$\log_2 20$

18.4 Laws of exponential and logarithmic functions

You already know the rules for calculating with indices:
For each of them there is an equivalent law for logarithms (logs for short).

	Indices	Logs	
The first law	$a^n \times a^m = a^{n+m}$	$\log_a(p \times q) = \log_a(p) + \log_a(q)$	[1]
The second law	$a^n \div a^m = a^{n-m}$	$\log_a(p \div q) = \log_a(p) - \log_a(q)$	[2]
The third law	$(a^n)^m = a^{n \times m}$	$\log_a(p^n) = n \times \log_a(p)$	[3]
	$a^0 = 1$	$\log_a(1) = 0$	[4]
	$a^{-n} = \dfrac{1}{a^n}$	$\log_a\left(\dfrac{1}{p}\right) = -\log_a(p)$	[5]

These laws allow us to perform complex calculations using mainly addition and subtraction.

Example 18.1

Calculate 16×32.

Solution:

	Using indices	Using logs
	$16 = 2^4$	$\log_2(16) = 4$
	$32 = 2^5$	$\log_2(32) = 5$
so	$16 \times 32 = 2^4 \times 2^5$	$\log_2(16 \times 32) = 4 + 5$
	$= 2^9$	$= 9$
	$= 512$	$2^9 = 512$

Notice how we need only work with the logs (the indices).

Exercise 18.2

1 Use your table of values from Problem 18.1, extended if necessary, to calculate these using $\log_2$:

a 4×128 **b** 16×64 **c** 8×256

d $1024 \div 16$ **e** $512 \div 8$ **f** $64 \div 256$

g 8^3 **h** 32^2 **i** 16^2

j $\dfrac{64 \times 128}{16 \times 4}$ **k** $\dfrac{256 \times 128}{32 \times 8}$ **l** $\dfrac{128 \times 512}{4 \times 1024}$

2 Choose an appropriate base and calculate these:

a $\dfrac{81 \times 81}{243 \times 9}$ **b** $\dfrac{125 \times 5}{625 \times 25}$ **c** $\dfrac{216 \times 6}{36 \times 36}$

3 Your calculator has a "log" function. This is short for $\log_{10}$.

Use this function, and its inverse "10^x" to calculate these:

a $\dfrac{80 \times 123}{79 \times 56}$ **b** $\dfrac{650 \times 525}{36 \times 24}$ **c** $\dfrac{7.142 \times 6.234}{2.86 \times 4.39}$

4 Express each of these as a single logarithm.

a $\log 3 + \log 5$ **b** $\log 81 - \log 3$ **c** $2 \log 4$

d $3 \log 2 + 2 \log 5$ **e** $5 \log 2 - 2 \log 4$ **f** $\dfrac{1}{2} \log 16 + 2 \log 3$

g $2 \log x + \log y$ **h** $3 \log xy - 2 \log y$ **i** $\dfrac{1}{2} \log x + 2 \log y$

18.5 Problem solving with logarithms

The equations $y = a^x$ and $x = \log_a y$ are equivalent.

If you cannot remember this, make it up with simple numbers.
Try $100 = 10^2$
and $2 = \log_{10} 100$

Example 18.2

Solve the following equations.

a $2^x = 10$ **b** $4^{2x-1} = 9^{x-1}$

c $2^{2x} - 12.2^x + 32 = 0$ **d** $3^{2x} - 3^{x+2} + 20 = 0$

Solution:

a $$2^x = 10$$

Take logs: $$x\log_{10} 2 = \log_{10} 10$$
$$x\log_{10} 2 = 1$$
$$x = \frac{1}{\log_{10} 2}$$
$$= 3.322$$

Remember: **Do the same thing to both sides!**

Use your calculator log function for this.

b $$4^{2x-1} = 9^{x-1}$$

Take logs: $$(2x-1)\log_{10} 4 = (x-1)\log_{10} 9$$
$$x(2\log_{10} 4 - \log_{10} 9) = \log_{10} 4 - \log_{10} 9$$
$$x = \frac{\log_{10} 4 - \log_{10} 9}{2\log_{10} 4 - \log_{10} 9} \quad \text{or} \quad \frac{\log_{10}\left(\frac{4}{9}\right)}{\log_{10}\left(\frac{16}{9}\right)}$$
$$= -1.41$$

c $$2^{2x} - 12.2^x + 32 = 0$$

This is $$(2^x)^2 - 12(2^x) + 32 = 0$$

Factorise: $$(2^x - 8)(2^x - 4) = 0$$

So $2^x = 8$ or $2^x = 4$

From which $x = 3$ or $x = 2$

This is just a quadratic equation in (2^x).

d $$3^{2x} - 3^{x+2} + 20 = 0$$

This can be written as

$$(3^x)^2 - 9(3^x) + 20 = 0$$

Factorise: $$(3^x - 5)(3^x - 4) = 0$$

Now it is a quadratic equation in (3^x).

So $3^x = 5$ or $3^x = 4$

Take logs: $x\log_{10} 3 = \log_{10} 5$ or $x\log_{10} 3 = \log_{10} 4$

$$x = \frac{\log_{10} 5}{\log_{10} 3} \quad \text{or} \quad x = \frac{\log_{10} 4}{\log_{10} 3}$$

$x = 1.46$ or $x = 1.26$

Any base would do but base 10 is on your calculator so that is the easiest to use.

18.6 Changing the base of logarithms

Starting with the equation $x = \log_a b$

we can change it to $b = a^x$.

From here, we can take logs to a different base

to give $\log_c b = x \log_c a$

and $$x = \frac{\log_c b}{\log_c a}$$

so $$\log_a b = \frac{\log_c b}{\log_c a} \qquad [6]$$

There is a special case of formula [6] when $b = c$.

In that case $\log_c b = 1$

and $$\log_a b = \frac{1}{\log_b a} \qquad [7]$$

Example 18.3

Solve the following equations:

a $\log_2 x = 16 \log_x 2$ **b** $\log_2 x = 6 \log_x 3$

Solution:

a $\log_2 x = 16 \log_x 2$

So, using [7] $\log_2 x = \dfrac{16}{\log_2 x}$

and $[\log_2 x]^2 = 16$

So $\log_2 x = 4$ or $\log_2 x = -4$

giving $x = 2^4$ or $x = 2^{-4}$

$x = 16$ or $x = 0.0625$

b $\log_2 x = 6 \log_x 3$

So, using [6] $\log_2 x = \dfrac{\log_{10} x}{\log_{10} 2}$

and $\log_x 3 = \dfrac{\log_{10} 3}{\log_{10} x}$

giving $$\frac{\log_{10} x}{\log_{10} 2} = \frac{6 \log_{10} 3}{\log_{10} x}$$

or $$\left[\log_{10} x\right]^2 = 6 \log_{10} 3 \log_{10} 2$$
$$= 0.861767$$

Proceeding as before, $x = 0.1179$ or $x = 8.478$

Exercise 18.3

In this exercise, if the answer cannot be written exactly, give it correct to 3 s.f.

1 Solve these equations:

a $3^x = 20$ **b** $4^x = 55$ **c** $5^x = 200$

d $6^x = 512$ **e** $7^x = 432$ **f** $12^x = 2000$

2 Solve these equations:

a $2^{x+2} = 3^{x-1}$ **b** $4^{2x} = 5^{x+1}$ **c** $5^{3x} = 10^{2x+3}$

d $6^{3x-1} = 4^{2x+4}$ **e** $7^{3x-1} = 9^{2x-3}$ **f** $12^x = 2^{5x-2}$

3 Solve these equations:

a $\log_3 x = 25 \log_x 3$ **b** $\log_4 x = 36 \log_x 4$ **c** $\log_2 x = 81 \log_x 2$

d $\log_2 x = 9 \log_x 5$ **e** $\log_3 x = 12 \log_x 4$ **f** $\log_2 x = 15 \log_x 3$

4 Solve these equations:

a $2^{2x} - 6 \times 2^x + 8 = 0$ **b** $3^{2x} - 10 \times 3^x + 9 = 0$ **c** $2^{2x} - 5 \times 2^x + 4 = 0$

d $3^{2x} - 3^{x+2} + 20 = 0$ **e** $4^{2x} - 5 \times 4^{x+1} + 4^3 = 0$ **f** $2^{2x} - 5 \times 2^{x+1} + 2^4 = 0$

g $3^{2x} - 3^{x+1} + 2 = 0$ **h** $2^{2x} - 2^{x+3} + 15 = 0$ **i** $5^{2x} - 7 \times 5^x + 12 = 0$

5 Solve these equations:

a $2^{2x} - 2^{x+1} - 3 = 0$ **b** $3^{2x} - 4 = 0$ **c** $5^{2x} - 5^x + 12 = 0$

d $4^{2x} - 4^{x+2} - 6 = 0$ **e** $4^{2x} - 4^{x+1} - 5 = 0$ **f** $3^{2x} + 3^{x+1} - 10 = 0$

6 Solve these equations:

a $2^{2x+1} + 2^x - 1 = 0$ **b** $6^{2x+1} - 7 \times 6^x + 2 = 0$ **c** $3 \times 4^{2x+1} - 13 \times 4^x + 3 = 0$

d $2^{2x+2} - 9 \times 2^x + 2 = 0$ **e** $3 \times 5^{2x} + 4 \times 5^x - 4 = 0$ **f** $10 \times 4^{2x} - 4^x - 3 = 0$

7 **a** Show that $(y-1)$ is a factor of $f(y) = y^3 - 6y^2 + 11y - 6$.

b Hence factorise $f(y) = y^3 - 6y^2 + 11y - 6$.

c Hence solve the equation $4^{3x} - 6 \times 4^{2x} + 11 \times 4^x - 6 = 0$.

8 Solve the simultaneous equations:

$$\log(xy) = 5$$

$$\log\left(\frac{x}{y}\right) = 1$$

18.7 Straightening curves

18.7.1 Drawing graphs of experimental data

When you collect the data from an experiment, you plot a graph and try to work out the formula that connects the input and output values (the independent and dependent variables).

If we have some idea of what the relationship might be, that helps a lot.
It is harder if we have no idea of the relationship, but we can often make an intelligent guess.

Example 18.4

The variables x and y are related by the formula $y = Ax^2 + Bx$, where A and B are constants.

a By dividing both sides of the equation by x, convert this formula into a form suitable for drawing a straight line graph.

b The graph is drawn.

(i) What do the X-and Y-axis represent?

(ii) What will the gradient of the graph?

(iii) What will be the Y-intercept?

Solution:

a $y = Ax^2 + Bx$

Dividing by x $\quad \dfrac{y}{x} = Ax + B$

This is the required form.

b Compare with $\quad Y = mX + c$

(i) On the Y-axis, we plot values of $\quad \dfrac{y}{x}$

On the X-axis, we plot values of $\quad x$

(ii) The gradient of the graph will be the value of $\quad A$

(iii) The Y-intercept will be the value of $\quad B$

Example 18.5

A stone is dropped from a bridge. The distance h (in metres) that the stone has fallen at time t (in seconds) is measured, and the results are given in the table below.

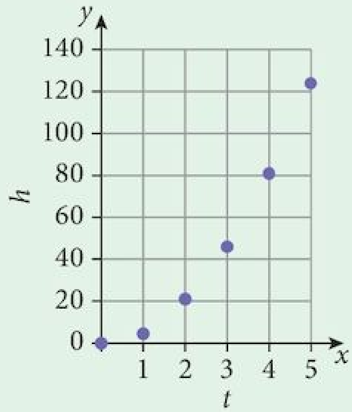
Find the relationship between h and t.

time (t)	0	1	2	3	4	5
distance (h)	0	4.5	21	46	81	124

Solution:

Step 1: Plot the points on a graph:

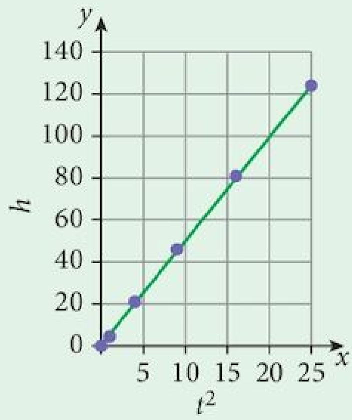
Idea! This looks like a quadratic.



Careful, it might be a cubic or some other power of t.

Step 2: Re-calculate the table in order to plot h against t^2.

t^2	0	1	4	9	16	25
distance (h)	0	4.5	21	46	81	124



This looks like a line passing through the origin.

The gradient of the line $= \dfrac{124}{25}$

≈ 5

The equation of the line is $Y \approx 5X$

but the y-axis represents h and the x-axis represents t^2.

So, the formula is $h \approx 5t^2$

Note that the points do not lie exactly on the line. This happens with experiments.

What would have happened if we were wrong about our idea?

If it had been some other power of t, the second graph would not have been a line and we would have spent a long time finding the correct power.

Fortunately, there is a way that deals with all powers. This technique uses logarithms.

18.7.2 Curves of the form $y = ax^n$

If the formula had been $h = kt^b$

We take logs of both sides $\log h = b \log t + \log k$

and compare this with the line $Y = mX + c$

$\log h = b\log t + \log k$

It does not matter which base we use.

If we plot $\log h$ on the y-axis and $\log t$ on the x-axis, we should get a line with gradient b (the unknown power) and y-intercept $\log k$.

Example 18.5 (revisited)

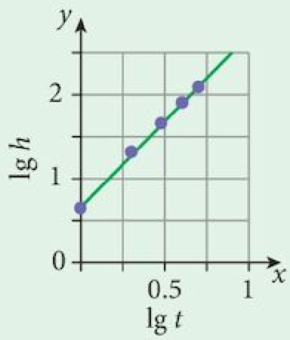
Solution:

Extend the table to include $\log t$ and $\log h$.

Time (t)	0	1	2	3	4	5
Height (h)	0	4.5	21	46	81	124
lg t	--	0	0.301	0.477	0.602	0.699
lg h	--	0.653	1.322	1.663	1.908	2.093

By convention, lg means $\log_{10}$. However your calculator uses log for this.

Then plot the graph.



Try to choose two points as far apart as possible on your line.

From the graph,

find the gradient: $\approx \dfrac{2.093 - 0.653}{0.699 - 0}$

$b = 2.06$

and the y-intercept: $\lg k = 0.653$

giving $k = 10^{0.653} = 4.50$

So $h = 4.50\,t^{2.06}$

Not quite the same result but this was not an accurate experiment.

18.7.3 Curves of the form $y = Ab^x$

A second type of formula that we can straighten by using logarithms is one where it is thought that:

$$y = Ab^x$$

When we take logs, we get $\log y = x\log b + \log A$

comparing this with the line $Y = mX + c$

$$\log y = x\log b + \log A$$

This time, we must plot a graph of $\log y$ against x.

If this is a line, the gradient will be $\log b$

and the y-intercept will be $\log A$.

Example 18.6

A colony of rabbits on an island with no predators and lots of food is monitored by scientists, and the number of rabbits (N) after time (t) years is given in the table.

Time (t)	1	2	3	4	5	6
Number (N)	60	80	100	120	150	190

It is assumed that the growth in the rabbit population is exponential and of the form:

$$N = Ar^t$$

a Find the values of the constants A and r.

b State what these constants represent.

Solution:

a

Using $\log_{10}$ again.

Step 1: Take logs: $\lg N = t\lg r + \lg A$

Compare with $Y = mX + c$

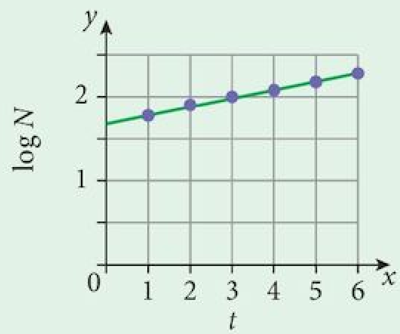
$$\lg N = t\lg r + \lg A$$

Step 2: Convert the values. We need to plot $\lg N$ against t.

Time (t)	1	2	3	4	5	6
Number (N)	60	80	100	120	150	190
lg N	1.778	1.903	2	2.079	2.176	2.279

Step 3: Plot the points

Step 4: Draw the best straight line with a ruler.



Step 5: Calculate the gradient

$$\lg r = \frac{2.279 - 1.778}{6 - 1}$$

$$= 0.1002$$

and find the y-intercept

$$\lg A = 1.7$$

giving $r = 1.26$

and $A = 50$

We cannot have 50.12 rabbits!

b $A = 50$ is the intial number of rabbits

$r = 1.26$ is the population growth rate (26% per annum)

Note that calculating the gradient and finding the y-intercept in this way is not very accurate, but you should try to get the best results you can. When you do A-level mathematics, you will find statistical techniques to give a more accurate result. Your calculator can probably do this calculation for you but you will need to read the manual to find out how. These advanced techniques are not required in this syllabus.

Exercise 18.4

1 Variables p and q are known to be connected by the relationship $q = Ap^2 + B$. Data is collected and the results are shown in the table:

p	1	2	3	4	5
q	7	19	39	67	103

By plotting a graph of q (on the y-axis) against p^2 (on the x-axis), find the values of A and B.

2 Variables m and n are thought to be connected by the relationship $m = An^3 + B$. Data is collected and the results are shown in the table:

m	6.33	15.67	41.0	90.33
n	1	2	3	4

By plotting a graph of m (on the y-axis) against n^3 (on the x-axis), find the values of A and B.

3 Variables s and t are known to be connected by the relationship $t = \frac{A}{s} + B$. Data is collected and the results are shown in the table:

s	1	2	3	4	6
t	14	8	6	5	4

a By plotting a graph of t (on the y-axis) against $\frac{1}{s}$ (on the x-axis), find the values of A and B.

b By plotting a graph of st (on the y-axis) against s (on the x-axis), find the values of A and B.

4 Variables x and y are known to be connected by the relationship $y = \frac{A}{x} + Bx$. Data is collected and the results are shown in the table:

x	2	3	4	5	6
y	66	49	42	39	38

a By multiplying both sides by x, convert the formula into the format $Y = mX + c$.

b Plot a suitable graph and use it to find the values of A and B.

c From your formula, calculate the value of y when $x = 8$.

d Find the positive value of x at the stationary point of the curve.

5 A ball is thrown vertically upwards. The height h (in metres) is measured at various times t (in seconds) giving the following results:

t (s)	1	2	3	4
h (m)	20	30	30	20

It is known that the relationship between h and t is given by $h = ut + \frac{1}{2}at^2$ where u is the initial velocity of the ball and a is its acceleration.

a By dividing both sides by t, convert the formula into the format $Y = mX + c$.

b Plot a suitable graph and use it to find the values of u and a.

6 Variables A and t are known to be connected by the relationship $A = Pr^t$. Equivalent values are shown in the table.

t	2	5	9	15
A	6050	8053	11 780	20 886

Plot a suitable graph and find the value of P and the value of r.

7 Variables N and x are known to be connected by the relationship $N = Ab^x$. Equivalent values are shown in the table.

x	3	6	9	12
N	1688	5695	19 222	64 873

a Plot a suitable graph and find the value of A and the value of b.

b Find the value of N when $x = 5$.

8 Variables G and t are known to be connected by the relationship $G = At^n$. Equivalent values are shown in the table.

t	2	4	6	8
G	102.3	3277	24 883	104 858

a Plot a suitable graph and find the value of A and the value of n.

b Find the value of G when $t = 5$.

9 Variables p and q are known to be connected by the relationship $q = Ab^p$. Equivalent values are shown in the table.

p	2	5	9	14
q	7.9	31.3	75.6	146.7

a Plot a suitable graph and find the value of A and the value of n.

b Find the value of q when $p = 4$.

10 Variables N and t are known to be connected by the relationship $N = Ab^t$.
Equivalent values are shown in the table.

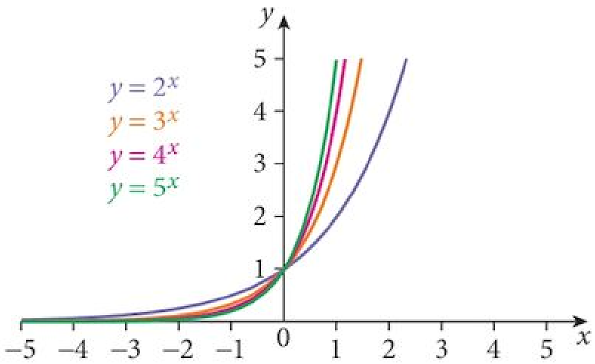
t	2	5	7	10
N	16.9	206.0	1089.5	13 256.5

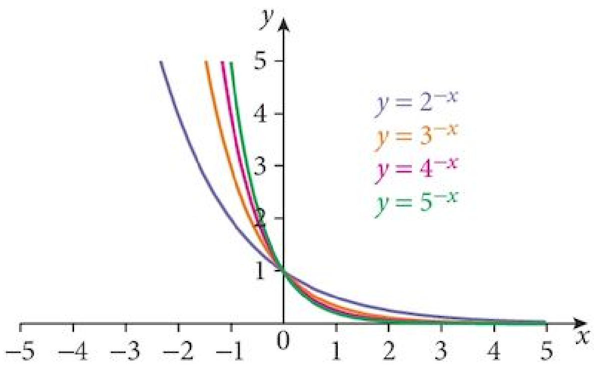
a Plot a suitable graph and find the value of A and the value of b.

b Find the value of N when $t = 6$.

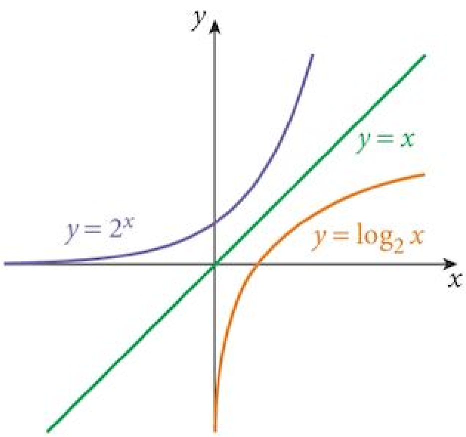
Summary

Exponential functions Exponential functions all have similar graphs:





Logarithmic functions Logarithmic functions are the inverses of exponential functions.



Laws of exponential and logarithmic functions

The first law	$a^n \times a^m = a^{n+m}$	$\log_a(p \times q) = \log_a(p) + \log_a(q)$
The second law	$a^n \div a^m = a^{n-m}$	$\log_a(p \div q) = \log_a(p) - \log_a(q)$
The third law	$(a^n)^m = a^{n \times m}$	$\log_a(p^n) = n \times \log_a(p)$
	$a^0 = 1$	$\log_a(1) = 0$
	$a^{-n} = \dfrac{1}{a^n}$	$\log_a\left(\dfrac{1}{p}\right) = -\log_a(p)$

Changing the base

$$\log_a(b) = \frac{\log_c(b)}{\log_c(a)} \qquad \log_a(b) = \frac{1}{\log_c(a)}$$

Straightening curves

Equation	re-write as	plot	against	gradient	y-intercept
$y = kx^2 + c$		y	x^2	k	c
$y = \dfrac{c}{x} + k$	$xy = c + kx$	xy	x	k	c
$y = ax^n$	$\log y = \log a + n \log x$	$\log y$	$\log x$	n	$\log a$
$y = AB^x$	$\log y = \log A + x \log B$	$\log y$	x	$\log B$	$\log A$

Chapter 18 Summative Exercise

1 Find the value of these logarithms.

a $\log_2 16$ **b** $\log_3 27$ **c** $\log_4 256$

d $\log_5 125$ **e** $\log_4 32$ **f** $\log_9 27$

2 Express each of these as a single logarithm.

a $\log 4 + \log 3 - \log 2$ **b** $\log 8 + \log 32 - 2 \log 4$ **c** $5 \log 3 - 2 \log 9$

d $6 \log 2 - \log 8 + 2 \log 3$ **e** $5\log 4 - 3 \log 2$ **f** $2 \log xy - \log x - \log y$

3 Solve these equations.

a $3^x = 4$ **b** $4^x = 6^{x-2}$ **c** $3^x = 2^{x-3}$

d $4^x = 10$ **e** $5^{x+1} = 0.75$ **f** $6^{2x} = 4^{x+2}$

4 Solve these equations.

a $9 \log (x - 1) = \log 16$ **b** $\lg (40y) - \lg (y - 6) = 2$

c $\log_4(3x) - \log_4(x - 1) = 1$ **d** $2\log_2 x + 2 = \log_2(9x - 2)$

e $2 + 2 \log_3(x + 1) = \log_3(3x + 7) + \log_3(3x + 1)$ **f** $2 + 2 \log_4(x + 1) = \log_4 5 + \log_4(3x + 5)$

g $1 + 2 \log_8 x = \log_8(9x - 1)$ **h** $\log x - \log(x - 1) = \log 3 - \log(x - 5)$

5 Solve the following pairs of simultaneous equations.

a $1 + \log_2 x = \log_2(y + 3)$
$\log_2(y + 2) - \log_2(x - 4) = \log_2(x + 4)$

b $9 \times 3^p = 27^q$
$\log_2(3p - q) = 1$

c $3^x \times 9^y = 6561$
$\log 4 + \log x = \log (11 - y)$

d $\log_2 x + 2 \log_2 y = 5 + \log_2 10$
$2 \log_2 x + \log_2 y = 1 + 2 \log_2 10$

e $2^x = 8 \times 2^y$
$\log(2x - 3y) = 1$

f $\log x + \log y = \log 2 + 2 \log 6$
$3 \log x + 2 \log y = 6 \log 6$

6 Solve the following equations.

a $4^{2x} - 4^x - 6 = 0$ **b** $2^{x+1} + 2^x = 9$

c $2^{2x} - 2^{x+3} + 7 = 0$ **d** $3^{2x} - 3^x = 12$

e $2^{2x+1} + 2^{x+2} = 12 - 2^x$ **f** $6^{2x+1} - 3 \times 6^{x+1} = 7 \times 6^x - 25$

7 The table shows experimental values of two variables, x and y.

x	1	2	3	4	5
y	1	10	35	95	200

It is known that x and y are connected by the equation $y = ax^3 + bx^2$.

a By dividing both sides of the equation by x^2, rewrite the equation in a form suitable for drawing a straight line graph.

b Draw the graph and use it to find the value of a and the value of b.

c Use your graph to find an approximate value for y when $x = 4.5$.

8 After an experiment, a graph is drawn, plotting $\log y$ against x.

This graph passes through the points (2, 1) and (6, 3) and is a straight line.

Find the gradient of the graph and use it to express y in terms of x.

9 The table shows experimental values of two variables, M and F.

M	1.5	2	2.5	3	3.5
F	1.8	2.1	2.4	2.6	2.8

It is known that M and F are connected by the equation of the form $F = AM^b$.

Plot a graph of $\log F$ against $\log M$ and use it to estimate the value of A and the value of b.

10 It is known that the variables x and y are connected by the equation $y = Ab^x$.
A graph is drawn from experimental values of $\log y$ against x and it produces a straight line passing through the points (2, 1.65) and (5, 3.08).
Use this information to estimate the value of A and the value of b.

11 The table shows the experimental values obtained from two variables that are known to be connected by the equation $P = Ax^b$.

x	1.5	2	3	4	5
P	78	107	167	230	294

Plot a graph of $\log P$ against $\log x$ and use it to estimate the value of A and the value of b.

12 When a graph is drawn from experimental values of x^2y against x, it produced a straight line passing through the points (2, 5) and (4, 1.75). Find the gradient and intercept of the graph and from these values, express y in terms of x.

Chapter 18 Test

1 hour

1 Given that $p = \log_4 x$ and that $q = \log_4 y$, express the following in terms of p and/or q.

a $\log_4\left(\dfrac{x^3}{\sqrt{y}}\right)$ b $\log_x 16 - \log_2 y^2$ c $\log_{xy} 16$ [3]

2 a Given that $\log_3 x = 2$, find the value of x. [1]

b Solve the equation $2\log_3 y - \log_3(y - 2) = 2$. [4]

3 If $\log_a(pq^2) = 12$ and $\log_a(p^3q^2) = 16$, find:

a $\log_a p$ [2]

b $\log_a q$ [2]

c $\log_p a + \log_q a$ [2]

4 a Sketch the graph of $y = \log x$ for $x > 0$.

b Express $x^2 = 10^{x-2}$ in the form $\log x = ax + b$, stating the values of a and of b.

c Add to your sketch the graph of an appropriate straight line required to solve the equation $x^2 = 10^{x-2}$. [7]

[Note: You are not required to solve the equation.]

5 It is believed that N and m are connected by the formula $N = Am^n$.
In an experiment, the following results were obtained:

m	2	4	6	8
N	50	400	1300	3100

a Rewrite the formula $N = Am^n$ in a format suitable for creating a straight line graph. [1]

b Using graph paper, plot the appropriate points and draw a straight line graph. [3]

Use your graph to estimate:

c the value of A and the value of n [4]

d the value of N when $m = 5$. [2]

6 Given that $A = \log_{2a} 8$, express $A(1 + \log_a 2)$ as a single logarithm. [4]

7 Solve the simultaneous equations:

$$\log_5 2 + \log_5(2y - 5) = 1 + \log_5 x$$
$$\log_3(x + 5y) = 3$$ [5]

Examination Questions

1

x	2	3	4	5	6
y	9.2	8.8	9.4	10.4	11.6

The table above shows experimental values of the variables x and y.
On graph paper, draw the graph of xy against x^2. [3]

Hence

(i) express y in terms of x. [4]

(ii) find the value of x for which $x = \dfrac{45}{y}$. [2]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2003, P1, Qu 10]

2 Solve the equation $\log_2 x - \log_4(x-4) = 2$. [4]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2003, P2, Qu 3]

3 Solve the equation $\log_{16}(3x-1) = \log_4(3x) + \log_4(0.5)$. [6]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2004, P2, Qu 5]

4 In order that each of the equations

(i) $y = ab^x$,

(ii) $y = Ax^k$,

(iii) $px + qy = xy$,

where a, b, A, k, p and q are unknown constants, may be represented by a straight line, they each need to be expressed in the form $Y = mX + c$, where X and Y are each functions of x and/or y, and m and c are constants. Copy the following table and insert in it an expression for Y, X, m, and c for each case.

	Y	X	m	c
$y = ab^x$				
$y = Ax^k$				
$px + qy = xy$				

[7]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2004, P2, Qu 9]

5 **a** Variables l and t are related by the equation $l = l_0(1+\alpha)^t$ where l_0 and α are constants. Given that $l_0 = 0.64$ and $\alpha = 2.5 \times 10^{-3}$, find the value of t for which $l = 0.66$. [3]

b Solve the equation $1 + \lg(8-x) = \lg(3x-2)$. [4]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2005, P1, Qu 7]

6

x	10	100	1000	10000
y	1900	250	31	4

The table above shows experimental values of the variables x and y which are related by an equation of the form $y = kx^n$, where k and n are constants.

(i) Using graph paper, draw the graph of $\lg y$ against $\lg x$. [3]

(ii) Use your graph to estimate the value of k and of n. [4]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2005, P1, Qu 8]

7 **a** Solve $\log_7(17y+15) = 2 + \log_7(2y-3)$. [4]

b Evaluate $\log_p 8 \times \log_{16} p$. [3]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2005, P1, Qu 7]

8 **a** Given that $u = \log_4 x$, find, in its simplest form in terms of u,

(i) x

(ii) $\log_4\left(\frac{16}{x}\right)$

(iii) $\log_x 8$. [5]

b Solve the equation $(\log_3 y)^2 + \log_3(y^2) = 8$. [4]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2006, P1, Qu 9]

9 **a** Solve the equation $\lg(x+12) = 1 + \lg(2-x)$. [3]

b Given that $\log_2 p = a$, $\log_8 q = b$ and $\frac{p}{q} = 2^c$, express c in terms of a and b. [4]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2006, P1, Qu 8]

10

x	0.100	0.125	0.160	0.200	0.400
y	0.050	0.064	0.085	0.111	0.286

The table above shows experimental values of the variables x and y.

(i) On graph paper, draw the graph of $\frac{1}{y}$ against $\frac{1}{x}$. [3]

Hence

(ii) express y in terms of x, [4]

(iii) find the value of x for which $y = 0.15$. [2]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2007, P1, Qu 9]

11 **(i)** Use the substitution $u = 2^x$ to solve the equation $2^{2x} = 2^{x+2} + 5$. [5]

(ii) Solve the equation $2\log_9 3 + \log_5(7y - 3) = \log_2 8$ [4]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2007, P1, Qu 7]

12

x	2	4	6	8	10
y	14.4	10.8	11.2	12.6	14.4

The table above shows experimental values of two variables x and y.

(i) Using graph paper, plot xy against x^2. [2]

(ii) Use the graph of xy against x^2 to express y in terms of x. [4]

(iii) Find the value of y for which $y = \frac{83}{x}$. [3]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2007, P1, Qu 9]

19 Calculus and trigonometry

Syllabus statements

- use derivatives of the standard functions $\sin x$, $\cos x$, $\tan x$ together with constant multiples, sums and composite functions of these.
- integrate functions of the form $\sin(ax + b)$, $\cos(ax + b)$

19.1 Introduction

In this chapter, we investigate the question of differentiating and integrating trigonometric functions. This process provides the justification for using the radian measure for angles. Without these, life would be so much more complicated.

We start by using matrices to develop formulae extra to the syllabus but which are needed to obtain the results that are in the syllabus.

19.2 $\sin(A + B)$, $\cos(A + B)$

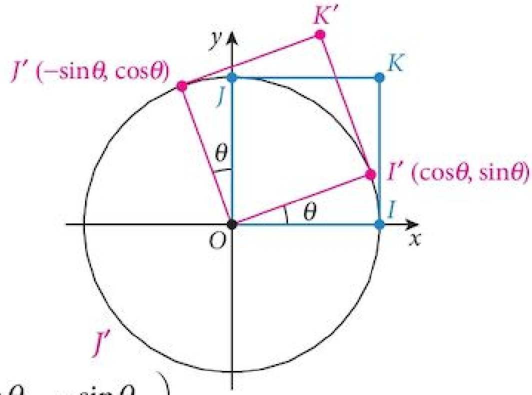
Before we can begin to differentiate the functions $y = \sin x$ and $y = \cos x$, we need formulae for $\sin(A + B)$ and $\cos(A + B)$.

Note that these formulae are not required for the syllabus. They are used only to prove a result that is on the syllabus.

We start with a transformation matrix.

The matrix $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

represents a rotation through the angle θ.



We can show this by transforming the unit square $OIKJ$ to give $OI'K'J'$.

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{array}{c} \begin{array}{cccc} O & I & K & J \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix} \end{array} = \begin{array}{c} \begin{array}{cccc} O & I' & K' & J' \end{array} \\ \begin{pmatrix} 0 & \cos\theta & \cos\theta - \sin\theta & -\sin\theta \\ 0 & \sin\theta & \sin\theta + \cos\theta & \cos\theta \end{pmatrix} \end{array}$$

If we have a rotation through angle A followed by a rotation through angle B, this is equivalent to a rotation through angle $(A + B)$.

In matrix terms this is $\quad R_{A+B} = R_B R_A$

or $$\begin{pmatrix} \cos(A+B) & -\sin(A+B) \\ \sin(A+B) & \cos(A+B) \end{pmatrix} = \begin{pmatrix} \cos B & -\sin B \\ \sin B & \cos B \end{pmatrix} \begin{pmatrix} \cos A & -\sin A \\ \sin A & \cos A \end{pmatrix}$$

$$= \begin{pmatrix} \cos B\cos A - \sin B\sin A & -\cos B\sin A - \sin B\cos A \\ \sin B\cos A + \cos B\sin A & \cos B\cos A - \sin B\sin A \end{pmatrix}$$

Then, comparing the terms:

$$\cos(A+B) = \cos A\cos B - \sin A\sin B \qquad [1]$$

and $$\sin(A+B) = \sin A\cos B + \cos A\sin B \qquad [2]$$

The A's and B's have been rearranged.

19.3 The derivative of sin *x*

Problem 19.1

a Sketch the curve $y = \sin x$.

b Below it, sketch the gradient function. Start with the points where the gradient is zero and then estimate what happens between these points.

c Make a conjecture about the gradient function.

To prove your conjecture, we return to the definition of the derivative:

$$f'(x) = \underset{h \to 0}{\mathrm{Lt}}\left[\frac{f(x+h)-f(x)}{h}\right]$$

We have $\quad f(x) = \sin x$

and $\quad f(x+h) = \sin(x+h)$

using [2] $\quad = \sin x \cos h + \cos x \sin h$

and so $\quad f(x+h) - f(x) = \sin x \cos h + \cos x \sin h - \sin x$

we can factorise this $\quad = \sin x(\cos h - 1) + \cos x \sin h$

Now we divide by h: $\quad \dfrac{f(x+h)-f(x)}{h} = \sin x\left[\dfrac{\cos h - 1}{h}\right] + \cos x\left[\dfrac{\sin h}{h}\right]$

We now have to reduce the value of h and see what happens as $h \to 0$.

Problem 19.2

Use your calculator to investigate what happens as $h \mapsto 0$ to:

a $\dfrac{\sin h}{h}$ **(i)** with h measured in degrees

(ii) with h measured in radians.

b $\dfrac{\cos h - 1}{h}$ **(i)** with h measured in degrees

(ii) with h measured in radians.

So, if we use radian measurement, we get

$$\frac{d}{dx}(\sin x) = \sin x \times 0 + \cos x \times 1 = \cos x$$

and, if we use degree measurement, we get

$$\frac{d}{dx}(\sin x) = \sin x \times 0 + \cos x \times 0.0174532\ldots = 0.0174532 \cos x$$

Which is easier to remember? For this reason, we never use degree measurement when doing calculus with trigonometric functions.

Remember: You must always use radians.

19.4 The derivative of cos *x*

Problem 19.3

Use formula [1] and repeat the steps above to find $\dfrac{d}{dx}(\cos x)$.

19.5 The derivative of tan x

We could use a similar technique to find $\frac{d}{dx}(\tan x)$, but it is easier to use the trigonometric identities from Chapter 16 and the quotient rule from Chapter 17.

Problem 19.4

Writing $\tan x = \frac{\sin x}{\cos x}$

Use the quotient rule and the derivatives of $\sin x$ and $\cos x$ to find

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

Now that we have those results, from Chapter 15, we also know the integrals of the results:

$\frac{d}{dx}(\sin x) = \cos x$	$\int \cos x \, dx = \sin x$
$\frac{d}{dx}(\cos x) = -\sin x$	$\int \sin x \, dx = -\cos x$
$\frac{d}{dx}(\tan x) = \sec^2 x$	$\int \sec^2 x \, dx = \tan x$

You will not need this last integral. Notice, however, that $\int \sec x \, dx$ is a much more difficult problem even though it looks simpler!

Example 19.1

Find $f'(x)$ for each of these functions:

a $f(x) = \tan 3x$ **b** $f(x) = \cos(2x+5)$ **c** $f(x) = \sin x^2$

Solution:

These are all composite functions so we must use the composite rule.

a $f(x) = \tan 3x$ $\quad f'(x) = \left[\sec^2(3x)\right] \times 3$

$= 3\sec^2(3x)$

b $f(x) = \cos(2x+5)$ $\quad f'(x) = \left[-\sin(2x+5)\right] \times 2$

$= -2\sin(2x+5)$

c $f(x) = \sin x^2$ $\quad f'(x) = \left[\cos x^2\right] \times (2x)$

$= 2x\cos x^2$

Solution **c**, above, gives another example of the difficulties of integration.

$\int 2x\cos x^2 \, dx = \sin x^2 + c$ but $\int \cos x^2 \, dx$ like most integrals, is impossible to solve!

However, do not worry. You will not be asked to solve an impossible integral.

Example 19.2

Find $f'(x)$ for each of the functions:

a $f(x) = \sin x \cos x$ **b** $f(x) = x^2 \sin(4x-2)$

Solution:

These functions are products so we must use the product rule and, in **b**, the composite rule as well.

a $f(x) = \sin x \cos x$

$$u(x) \times v'(x) + u'(x) \times v(x)$$

so $f'(x) = [\sin x] \times [-\sin x] + [\cos x] \times [\cos x]$

$= \cos^2 x - \sin^2 x$

b $f(x) = x^2 \sin(4x-2)$

$$u(x) \times v'(x) + u'(x) \times v(x)$$

so $f'(x) = [x^2] \times [4\cos(4x-2)] + [2x] \times [\sin(4x-2)]$

$= 4x^2 \cos(4x-2) + 2x \sin(4x-2)$

Example 19.3

If $y = x \cos x$,

a show that $x \sin x = \cos x - \dfrac{dy}{dx}$

b Hence find $\int x \sin x \, dx$

Solution:

a $y = x \cos x$

Using the product rule, $\dfrac{dy}{dx} = x(-\sin x) + \cos x$

So $x \sin x = \cos x - \dfrac{dy}{dx}$

b Integrating both sides w.r.t. x:

$$\int x \sin x \, dx = \int \cos x \, dx - \int \frac{dy}{dx} dx$$

$$= \sin x - y + c$$

$$= \sin x - x \cos x + c$$

w.r.t. = with respect to

$$\int \frac{dy}{dx} dx = \int dy = y$$

Exercise 19.1

1 Differentiate with respect to x:

a $\sin 2x$ b $\cos 3x$ c $\tan 4x$

d $3\sin 5x$ e $2\cos 7x$ f $4\tan 3x$

g $4\sin\frac{x}{2}$ h $6\cos\frac{x}{3}$ i $12\tan\frac{x}{4}$

j $2\sin\left(x+\frac{\pi}{3}\right)$ k $-2\cos(3x-2)$ l $2\tan\left(\frac{x+\pi}{4}\right)$

2 Find $\frac{dy}{dx}$ for each:

a $y=\sin x^2$ b $y=\sin^2 x$ c $y=\cos x^3$

d $y=3\cos(x^2-1)$ e $y=\tan^3 x$ f $y=3\tan(x^3+x^2)$

g $y=4\sin(x^2-2x)$ h $y=3\cos^2(2x-1)$ i $y=2\tan\left(x-\frac{\pi}{3}\right)$

3 Find $\frac{dy}{dx}$ for each:

a $y=\sin(\sqrt{x})$ b $y=\sqrt{\sin x}$ c $y=\cos(\sqrt{x^3})$

d $y=2\cos(\sqrt{x^2-1})$ e $y=\sin^4 6x$ f $y=\sqrt{1+\sin 2x}$

g $y=\frac{1}{\sin^2 x}$ h $y=\frac{1}{\sqrt{\cos 4x}}$ i $y=\frac{1}{\tan^2(x^3)}$

4 Find $\frac{dy}{dx}$ for each:

a $y=x\sin x$ b $y=x^2\cos x^2$ c $y=x\tan x$

d $y=(2x+1)\sin(x^2+x)$ e $y=\sin x\cos x$ f $y=\sin^2 x\cos^2 x$

g $y=\frac{x}{\sin x}$ h $y=\frac{2}{1+\cos 2x}$ i $y=\frac{1+\sin x}{1+\cos x}$

j $y=-\frac{\cos 3x}{x+2}$ k $y=-\frac{1+\cos 2x}{\sin 2x}$ l $y=x^2\sin x\cos x$

5 Find each of these integrals.

a $\int 8\sin 2x\,dx$ b $\int 6\cos 3x\,dx$ c $\int 6\sin(2x+1)dx$

d $\int 9\cos(3x+\pi)dx$ e $\int 4\sec^2 2x\,dx$ f $\int 4\sin(1-2x)dx$

6 Find each of these integrals.

a $\int 2x\cos x^2\,dx$ b $\int 3x^2\cos x^3\,dx$ c $\int 8x^3\sin(x^4+1)\,dx$

d $\int (4x+2)\sin(x^2+x)\,dx$ e $\int \frac{\cos(\sqrt{x})}{\sqrt{x}}\,dx$ f $\int 6\sec^2(2x+1)\,dx$

7 **a** Differentiate $y = \sin^3 x$.

b Use your result to find $\int 6\cos x \sin^2 x \, dx$.

8 **a** Differentiate $y = \cos^4 x$.

b Use your result to find $\int 12\sin x \cos^3 x \, dx$.

9 **a** Differentiate $y = \sin x^2$.

b Use your result to find $\int 6x \cos x^2 \, dx$.

10 **a** Differentiate $y = x - \sin x$.

b Use your result to find $\int 3(1-\cos x)(x-\sin x)^2 \, dx$.

11 **a** Differentiate $y = 4 - \cos x$.

b Use your result to find $\int \frac{2\sin x}{\sqrt{4-\cos x}} \, dx$.

12 **a** If $y = x\sin x$, show that $x\cos x = \frac{dy}{dx} - \sin x$.

b Integrate both sides to find $\int x\cos x \, dx$.

19.6 Applications

We have already covered the applications of calculus in previous chapters.

Here we apply them to trigonometric functions.

Example 19.4

Find the equation of the tangent to the curve $y = x\sin x$ at the point where $x = \pi$.

Solution:

$$y = x\sin x$$

So $\quad \frac{dy}{dx} = \sin x + x\cos x$

When $x = \pi$, $\quad \frac{dy}{dx} = 0 + \pi(-1) = -\pi$

and $\quad y = 0$

So the equation of the tangent is

$$y = -\pi(x - \pi)$$

Example 19.5

Find the area between the curve $y = \sin 2x$, the x-axis and the lines $x = \frac{\pi}{6}$ and $x = \frac{\pi}{3}$.

Solution:

$$\begin{aligned}\text{Area} &= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin 2x \, dx \\ &= \left[\frac{-1}{2}(\cos 2x)\right]_{\frac{\pi}{6}}^{\frac{\pi}{3}} \\ &= \left[\frac{-1}{2}\left(\frac{-1}{2}\right) - \left(\frac{-1}{2}\left(\frac{1}{2}\right)\right)\right] \\ &= \frac{1}{2}\end{aligned}$$

Example 19.6

Find the stationary points on the curve $y = 3\sin x + 4\cos x$ for $0 \leqslant x \leqslant 2\pi$ and identify their nature.

Solution:

$$y = 3\sin x + 4\cos x$$

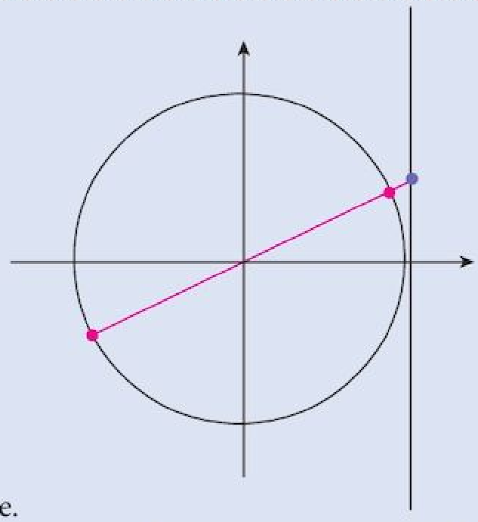
Differentiate both sides w.r.t. x: $\frac{dy}{dx} = 3\cos x - 4\sin x$

We need $\frac{dy}{dx} = 0$ $\quad 0 = 3\cos x - 4\sin x$

Solve this: $\tan x = \frac{3}{4}$

so, either $x = 0.644$ or $x = 3.785$

from which $y = 5$ or $y = -5$



To identify the nature of the points, find the second derivative.

Differentiate both sides w.r.t. x: $\frac{d^2y}{dx^2} = -3\sin x - 4\cos x$

When $x = 0.644$ $\quad \frac{d^2y}{dx^2} = -5$ Maximum

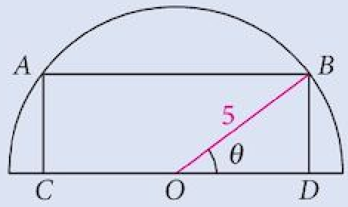
When $x = 3.785$ $\quad \frac{d^2y}{dx^2} = 5$ Minimum

So (0.644, 5) is a maximum and (3.785, –5) is a minimum.

Note that when $x = 0.644$:
$\tan x = 0.75$,
$\sin x = 0.6$
and $\cos x = 0.8$

Example 19.7

The figure shows a semi-circle, centre O, radius 5 units. A rectangle, $ABCD$ is drawn so that A and B lie on the semi-circle and C and D lie on the diameter as shown.



The angle $DOB = \theta$ radians.

a Show that the area A of the rectangle is given by

$$A = 50\sin\theta\cos\theta.$$

b As θ varies, find the maximum value of the area of the rectangle.

Solution:

a

$$OD = 5\cos\theta$$
$$CD = 10\cos\theta$$
$$BD = 5\sin\theta$$

Thus the area of the rectangle is

$$A = 5\sin\theta \times 10\cos\theta$$
$$= 50\sin\theta\cos\theta$$

b Differentiate w.r.t. θ $\quad \dfrac{dA}{d\theta} = 50\left[\sin\theta(-\sin\theta) + \cos\theta\cos\theta\right]$

$$= -50\left[\sin^2\theta - \cos^2\theta\right]$$

For a maximum $\quad 0 = 50\left[\sin^2\theta - \cos^2\theta\right]$

or $\quad 0 = (\sin\theta + \cos\theta)(\sin\theta - \cos\theta)$

So, either $\quad \sin\theta + \cos\theta = 0 \quad$ or $\quad \sin\theta - \cos\theta = 0$

giving $\quad \tan\theta = -1$ not a valid solution $\quad$ or $\quad \tan\theta = 1$

$$\theta = \frac{\pi}{4}$$

So the maximum area $\quad = 50 \times \dfrac{1}{\sqrt{2}} \times \dfrac{1}{\sqrt{2}}$

$$= 25$$

Note that this is obviously a maximum since $A = 0$ when $\theta = 0$ or $\dfrac{\pi}{2}$. For other values it is > 0.

Exercise 19.2

Tangents and normals:

1 Find the equation of the tangent and the normal to the curve $y = x\sin x$ at the point where $x = \dfrac{\pi}{6}$.

2 Find the equation of the tangent and the normal to the curve $y = \sin x + \cos x$ at the point where $x = \dfrac{\pi}{3}$.

3 Find the equation of the tangent and the normal to the curve $y = x\cos x$ at the point where $x = \frac{\pi}{3}$.

4 Find the equation of the tangent and the normal to the curve $y = \frac{1}{1 + \sin x}$ at the point where $x = \frac{\pi}{3}$.

5 Find the equation of the tangent and the normal to the curve $y = \frac{\cos x}{1 + \sin x}$ at the point where $x = \frac{\pi}{6}$.

6 Tangents to the curve $y = \sin 2x$ at the points where $x = \frac{\pi}{6}$ and $x = \frac{\pi}{3}$ meet at the point A, and meet the x-axis at the points B and C. Find the area of the triangle ABC.

Stationary points:

7 Find the x-coordinates of the stationary points on the curve $y = 2\sin x - 3\cos x$ that lie in the range $0 \leqslant x \leqslant 2\pi$.

8 Find the x-coordinates of the stationary points on the curve $y = x\sin x + \cos x$ that lie in the range $0 \leqslant x \leqslant 2\pi$.

9 A curve has equation $y = \frac{1}{\cos x}$.

a Find $\frac{dy}{dx}$.

b Hence find the x-coordinates of the stationary points of the curve in the range $0 \leqslant x \leqslant 2\pi$.

10 A curve has equation $y = \cos x(1 - \sin x)$.

a Show that $\frac{dy}{dx} = (2\sin x + 1)(\sin x - 1)$.

b Find the stationary points of the curve in the range $0 \leqslant x \leqslant 2\pi$.

c In the cases where $\sin x \neq 1$, identify the nature of the stationary points.

11 A curve has equation $y = \frac{x - \sin x}{1 + \cos x}$.

a Show that $\frac{dy}{dx} = \frac{x\sin x}{(1 + \cos x)^2}$.

b Hence find the x-coordinates of two stationary points of the curve in the range $0 \leqslant x < 2\pi$.

Small changes and rates of change:

12 You are given the function $y = \sin x - \cos x$.
Find the approximate increase in y if x is increased by 0.1 when it is $\frac{\pi}{4}$.

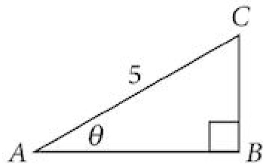
13 The variables x and y are connected by the relationship $y = x\cos x$.

When $x = \dfrac{3\pi}{2}$, it is increasing at a rate of 0.2 per second.

Find the corresponding rate of increase of the variable y.

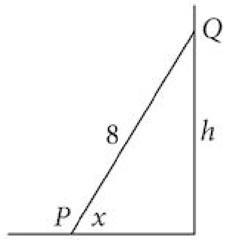
14 ABC is a right-angled triangle with hypoteneuse 5 units.

Find the approximate increase in the area of the triangle when θ is increased by 0.1 from $\dfrac{\pi}{6}$ radians.



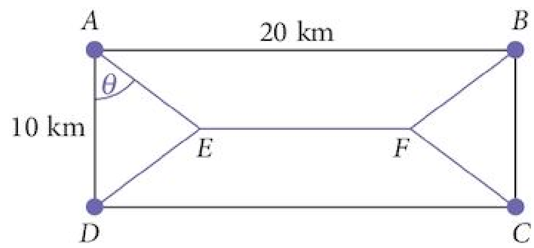
15 A ladder, PQ, of length 8 m, is leaning against a wall.

The bottom end, P, is moving toward the wall at a rate of 0.05 m per second. Find the rate at which the other end, Q, is moving up the wall when the angle, x, that the ladder makes with the ground is $\dfrac{\pi}{3}$ radians.



Maximum and minimum:

16 A road network is proposed connecting the four towns A, B, C and D. The towns form a rectangle as shown. In order to minimise construction costs, the network will be built as shown where AED and BFC are identical isosceles triangles with angle $DAE = \theta$.



a Show that the total length, L, of the roads to be constructed is given by

$$L = 20 - 10\tan\theta + \frac{20}{\cos\theta}.$$

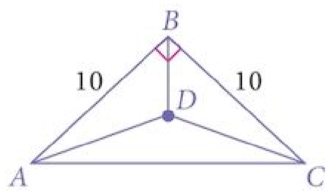
b Show that the total length of roads to be constructed can be minimised and find the value of θ that gives this minimum length.

17 ABC is an isosceles right-angled triangle.

$AB = BC = 10$ units.

Point D lies on the line of symmetry of the triangle such that the sum of lengths AD, BD and CD is minimised.

Find the length of BD.

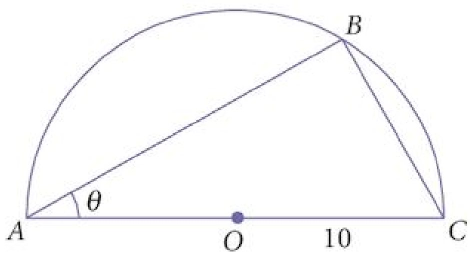


18 Triangle ABC is drawn in the semi-circle shown.

The radius of the semi-circle is 10 units.

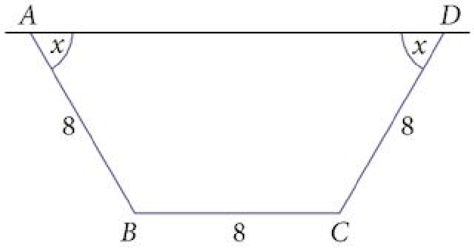
a If angle BAC is θ, show that the area, A, of the triangle is given by $A = 200\sin\theta\cos\theta$.

b Find the maximum area of the triangle as θ varies.



19 A farmer has three pieces of rigid fencing panels, each 8 m long.

He wants to construct a small temporary pen in the shape of an isosceles trapezium against a long fence in his field that runs through points A and D as shown. If the angle between the slanted panels and the field fence is x radians,



a show that the area, A, of the pen is given by

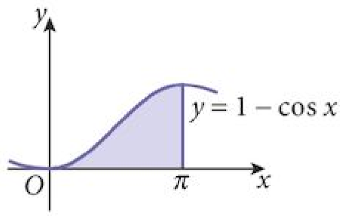
$$A = 64 \sin x(\cos x + 1)$$

b find the largest possible area of the pen.

Areas by integration:

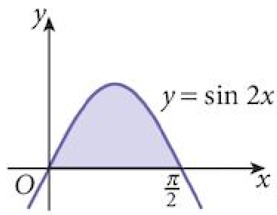
20 Find the shaded area given by:

$$\text{Area} = \int_0^{\pi} 1 - \cos x \, \mathrm{d}x$$



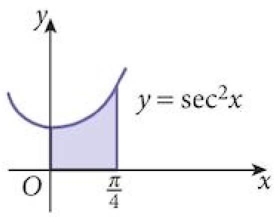
21 Find the shaded area given by:

$$\text{Area} = \int_0^{\frac{\pi}{2}} \sin 2x \, \mathrm{d}x$$



22 Find the shaded area given by:

$$\text{Area} = \int_0^{\frac{\pi}{4}} \sec^2 x \, \mathrm{d}x$$



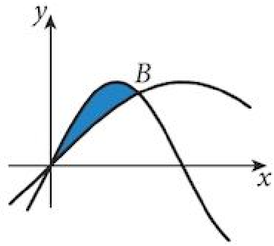
23 **a** Differentiate $y = \sin^2 x$.

b Hence find $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin x \cos x \, \mathrm{d}x$.

24 **a** Differentiate $y = \sin^3 x$.

b Hence find $\int_0^{\frac{\pi}{2}} \cos x \sin^2 x \, \mathrm{d}x$.

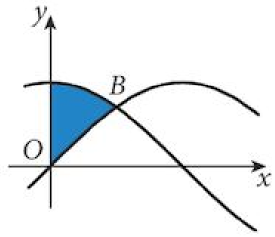
25 The curves $y = \sin x$ and $y = 2\sin x\cos x$ are as shown.



a Find the coordinates of the point B.

b Find the shaded area between the two curves.

26 The curves $y = \sin x$ and $y = \cos x$ are as shown.



a Find the coordinates of the point B.

b Find the shaded area between the two curves and the y-axis.

Summary

Angles When doing calculus, angles should always be measured in **radians**. Angles are dimensionless quantities (just numbers, really).

Derivatives

function	derivative	integral
$\sin x$	$\frac{d}{dx}\sin x = \cos x$	$\int \cos x \, dx = \sin x$
$\cos x$	$\frac{d}{dx}\cos x = -\sin x$	$\int \sin x \, dx = -\cos x$
$\tan x$	$\frac{d}{dx}\tan x = \sec^2 x$	

Applications You can apply trigonometric functions to any of the applications of differentiation and integration using the results above.

Chapter 19 Summative Exercise

1 Differentiate these functions.

a $\sin(3x+1)$ b $\cos(4x-2)$ c $\tan(2x+1)$

d $y = \sin(x^2+2x)$ e $y = \cos(\sqrt{x})$ f $y = \tan^2(2x-1)$

g $f(x) = \sqrt{1-\sin x^2}$ h $f(x) = \cos^3(x^2)$ i $f(x) = \dfrac{1}{\tan 3x}$

2 Differentiate these functions.

a $\sin x\cos x$ b $\sin 3x\cos 2x$ c $\sin^2 x\cos x$

d $y = \sin x\cos^2 x$ e $y = \sin^2 x\cos^2 x$ f $y = \sin x\cos x\tan x$

3 Find each of these integrals.

a $\int -6\sin 3x\,dx$ b $\int 8\cos 4x\,dx$ c $\int 6\sec^2 3x\,dx$

d $\int 4x\cos x^2\,dx$ e $\int (2-2x)\sin(x^2-2x)\,dx$ f $\int 2x\sec^2(x^2)\,dx$

4 By writing $\operatorname{cosec} x = \dfrac{1}{\sin x}$, $\sec x = \dfrac{1}{\cos x}$, and $\cot x = \dfrac{\cos x}{\sin x}$, differentiate:

a $f(x) = \operatorname{cosec} x$ **b** $f(x) = \sec x$ **c** $f(x) = \cot x$

5 Show that $y = A \sin x + B \cos x$ is a solution to the second order differential equation $\dfrac{d^2y}{dx^2} = -y$.

6 **a** If $y = x \cos x$ show that $x \sin x = \cos x - \dfrac{dy}{dx}$.

b Use your result to find $\int x \sin x \, dx$.

7 If $y = \dfrac{1}{3}\cos^3 x - \cos x$, show that $\dfrac{dy}{dx} = \sin^3 x$.

8 Evaluate $\displaystyle\int_0^{\frac{\pi}{3}} (\sin x - \cos 3x)\, dx$.

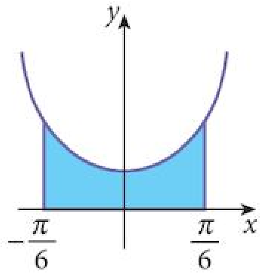
9 You are given the function $y = \dfrac{\sin x}{2 - \cos x}$.

Find the coordinates of the stationary point for values of x in the range $0 < x < \pi$.

10 **a** If $y = \tan 2x$, find $\dfrac{dy}{dx}$.

b The diagram shows the graph of $y = \sec^2 x$ for $-\dfrac{\pi}{6} \le x \le \dfrac{\pi}{6}$.

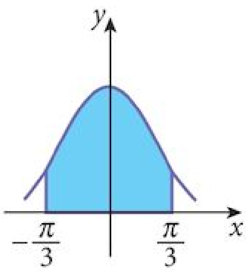
Find the shaded area.



11 **a** If $y = x + \dfrac{1}{2}\sin 2x$, find $\dfrac{dy}{dx}$.

b The diagram shows the graph of $y = 1 + \cos 2x$ for $-\dfrac{\pi}{3} \le x \le \dfrac{\pi}{3}$.

Find the shaded area.

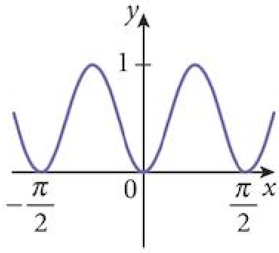


12 You are given the function $y = \sin^2 2x$.

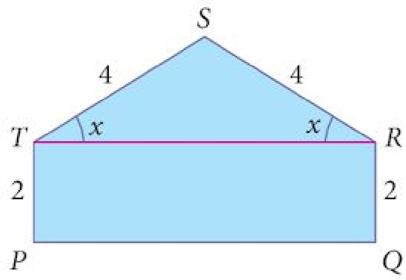
a Find $\dfrac{dy}{dx}$.

b Find the equation of the tangent to the curve at the point where $x = \dfrac{\pi}{3}$.

c Find, in terms of p, the approximate change in the value of y as x increases from $\dfrac{\pi}{3}$ to $\left(\dfrac{\pi}{3} + p\right)$.



13 A metal plate, $PQRST$, is in the form of a rectangle, surmounted by an isosceles triangle so that it is symmetrical about a line through S perpendicular to PQ. The length $ST = SR = 4$ units. The length $TP = QR = 2$ units. The angle $STR = SRT = x$ radians.



a Show that the area, A, of the plate is given by

$A = 16\cos x + 16\sin x\cos x$.

b Find $\dfrac{dA}{dx}$ and show that it has a stationary value and find the corresponding value of x.

c Find this stationary value of A, and determine its nature.

Chapter 19 Test

1 hour

1 Differentiate the following:

a $x^2\sin x$ [2]

b $\sin x\cos x$ [2]

c $\dfrac{\tan x}{x-1}$ [3]

2 Find the equation of the tangent to the curve $y = \dfrac{\sin x}{x}$ at the point where $x = \dfrac{\pi}{3}$. [5]

3 You are given the function $f(x) = 12 - 3\cos^2 x,\ 0 \leqslant x \leqslant \dfrac{\pi}{2}$.

The value of x is increasing at a rate of 0.6 radians s^{-1}.

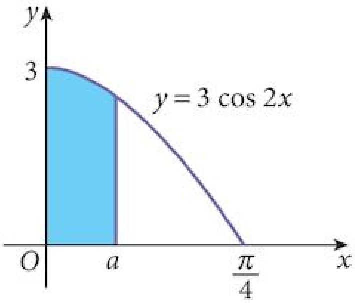
Find the corresponding rate of change of $f(x)$ when $f(x) = 10$. [6]

4 a Find the equation of the normal to the curve $y = 2\tan x - 3$ at the point where $x = \dfrac{\pi}{4}$. [5]

b Find the coordinates of the point where the normal intersects the x-axis. [1]

5 The diagram shows the curve $y = 3\cos 2x$.

Given that the shaded area is 1 unit², find the value of a. [6]



6 a Given that $y = \cos 3x$, find $\dfrac{dy}{dx}$. [1]

b Find the approximate change in the value of y as x increases from $\dfrac{\pi}{6}$ to $\left(\dfrac{\pi}{6} + p\right)$, where p is small. [2]

7 a Find $\dfrac{d}{dx}(\tan 2x)$. [2]

b Hence find $\displaystyle\int_{-\frac{\pi}{8}}^{\frac{\pi}{8}} (2 + \sec^2 2x)\,dx$. [5]

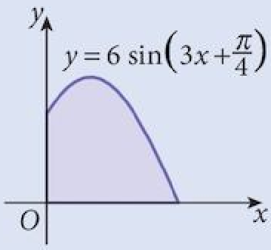
Examination Questions

1 (i) Differentiate $x \sin x$ with respect to x. [2]

(ii) Hence evaluate $\int_0^{\frac{\pi}{2}} x \cos x \, dx$ [4]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2002, P2, Qu 7]

2

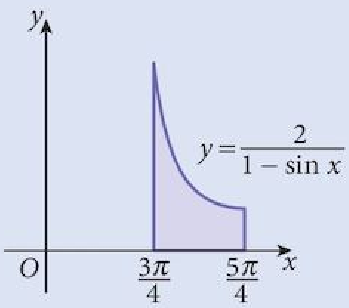


The diagram shows part of the curve $y = 6\sin\left(3x + \frac{\pi}{4}\right)$. Find the area of the shaded region bounded by the curve and the coordinate axes. [6]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2003, P1, Qu 5]

3 (i) Given that $y = \frac{1 + \sin x}{\cos x}$, show that $\frac{dy}{dx} = \frac{1}{1 - \sin x}$. [5]

(ii)



The diagram shows part of the curve $y = \frac{2}{1 - \sin x}$. Using the results from part **(i)**, find the area of the shaded region bounded by the curve, the x-axis and the lines $x = \frac{3\pi}{4}$ and $x = \frac{5\pi}{4}$. [3]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2006, P1, Qu 8]

4 Evaluate $\int_0^{\frac{\pi}{6}} \sin\left(2x + \frac{\pi}{6}\right) dx$. [4]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2006, P2, Qu 3]

Term test 5A (Chapters 16–19)

1 hour

1 Solve the following equations for values of x in the range $0 \leqslant x \leqslant 4\pi$.

a $\sin\left(x - \frac{\pi}{3}\right) = \frac{1}{2}$ [3]

b $2\operatorname{cosec}^2 x - 3\cot x - 4 = 0$ [4]

c $4\sin(x + 1.5) = 1$ [4]

2 **a** If $y = x\sqrt{3x - 2}$, find $\frac{dy}{dx}$. [3]

b Find the equation of the tangent at the point where $x = 2$. [2]

c Use your results to find $\int_4^6 \frac{18x - 8}{\sqrt{3x - 2}}\,dx$. [2]

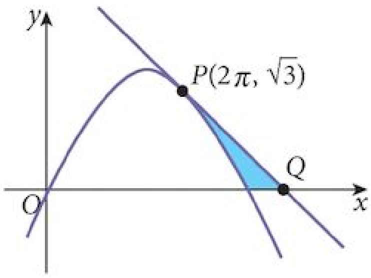
3 **a** The equation $y = \log_x 2$ is equivalent to the equation $a = b^c$.

Write down the values of a, b and c. [1]

b Express y as a logarithm in base 2. [1]

c Use the substitution $u = \log_2 x$ to solve the equation $\log_2 x = 8 - 15\log_x 2$. [4]

4



The diagram shows the graph of $y = 2\sin\frac{1}{3}x$.

The tangent at the point $P(2\pi, \sqrt{3})$ intersects the x-axis at the point Q.

Find:

a the equation of the tangent at P [3]

b the coordinates of the point Q [1]

c the size of the shaded area. [7]

5 When xy is plotted against x^2, a straight line is obtained which passes through the points (4, 12) and (16, 36). Express y in terms of x. [5]

20 The number e and its applications

Syllabus statements

- know the simple properties and graphs of $\ln x$ and e^x and graphs of $ke^{nx} + a$ and $k\ln(ax+b)$ where n, k, a and b are integers.
- use derivatives of the standard functions e^x, $\ln x$ together with constant multiples, sums and composite functions of these.
- integrate functions of the form e^{ax+b}.

20.1 Introduction

This chapter introduces one of the five most important numbers in mathematics. You have already met three of them: 0, 1 and π. The final one you will meet at A-level.

Problem 20.1

For each of the graphs drawn here:

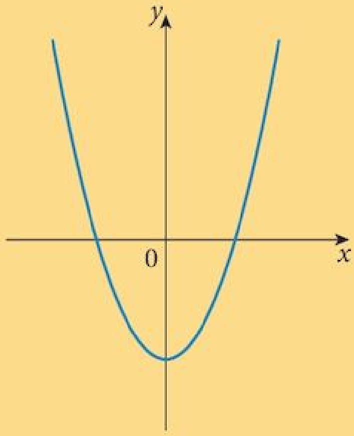
(i) Make a sketch copy of the graph.

(ii) Below it, sketch the gradient function of the graph. Start with the stationary points, where the gradient is zero and then try to work out what happens for the other values of x.

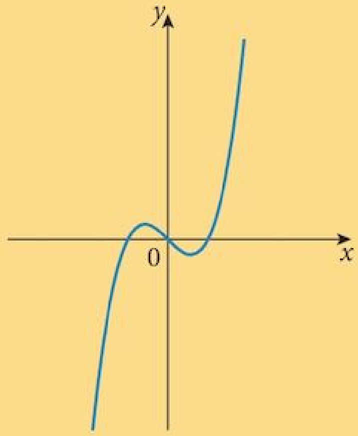
(iii) Compare each function with its own gradient function.

Which gradient functions are the same as the original function?

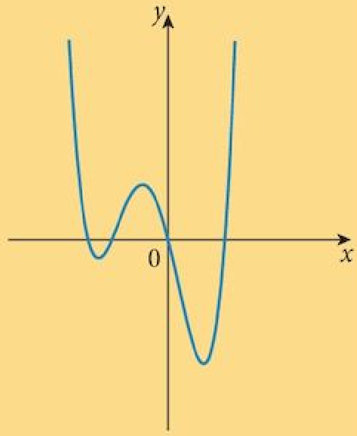
a $y = x^2 - 3$



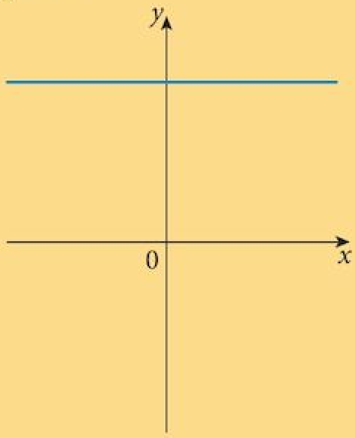
b $y = x^3 - x$



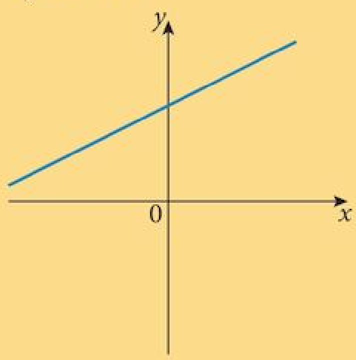
c $y = x^4 + 2x^3 - 2x^2 - 4x$



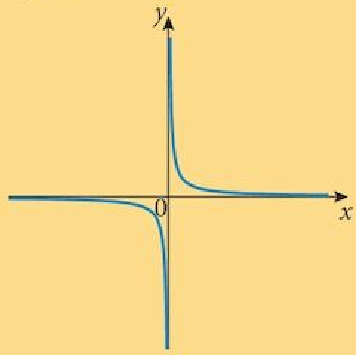
d $y = 4$



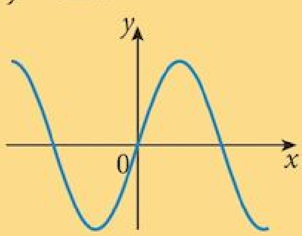
e $2y = x + 6$



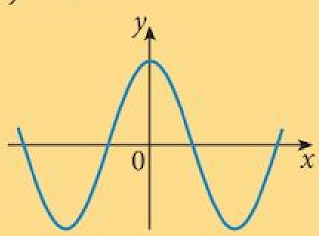
f $xy = 1$



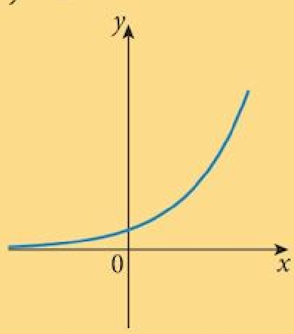
g $y = \sin x$



h $y = \cos x$



i $y = 2^x$



Our objective is to find a function which has a gradient function that is exactly the same as itself.

In other words, when we differentiate the function, we get the same result: we want to find $f'(x)$ such that: $f'(x)=f(x)$.

In fact there are two possibilities, one of which has not been drawn here.

The graphs of $y=\sin x$ and $y=\cos x$ come close to satisfying our requirements since the gradient functions are the same shape as the original functions, but they are not in the same place.

The one that really looks attractive is the exponential function $y=2^x$.
When the value of the function is small, so is the gradient.
As the value of the function increases, so does the gradient.

> *Problem 20.2*
>
> What other function does not change when you differentiate it?
> Its gradient graph is the same as the function graph.

20.2 The derivative of the exponential function $y=a^x$

Once again we return to the definition of the derivative:

$$f'(x)=\operatorname*{Lt}_{h\mapsto 0}\left[\frac{f(x+h)-f(x)}{h}\right]$$

We have $\qquad f(x)=a^x$

and $\qquad f(x+h)=a^{x+h}$

and so $\qquad f(x+h)-f(x)=a^{x+h}-a^x$

we can factorise this $\qquad =a^x\left(a^h-1\right)$

Divide the result by h: $\qquad \dfrac{f(x+h)-f(x)}{h}=a^x\dfrac{\left(a^h-1\right)}{h}$

Finally, we find out what happens as $h\mapsto 0$.

You can see that the derivative of an exponential function will be a multiple of the function.

The question is: What is the value of $\dfrac{a^h-1}{h}$ as $h\mapsto 0$?

> *Problem 20.3*
>
> **a** Use your calculator to find the value of $\dfrac{2^h-1}{h}$ as $h\mapsto 0$.
>
> **b** Copy and complete this table showing the results for different values of a.

a	$\dfrac{a^h-1}{h}$
1	0
2	
3	
4	

You can see from your table that, as a increases, so does this multiplying factor.

What we really need is to find the value of a that gives a multiplying factor $= 1$.

From your table, this is somewhere between $a = 2$ and $a = 3$.

We need $\quad \dfrac{a^h-1}{h}=1$

or $\quad a^h-1=h$

giving $\quad a=(1+h)^{\frac{1}{h}} \quad$ as $h \mapsto 0$

Using a calculator, this value is $a = 2.718281828\ldots$

So $\quad \dfrac{d}{dx}\left[(2.718281828)^x\right]=(2.718281828)^x$

This number, like π, is irrational.

Also, like π, it is so important that we give it a name: e.

So, $\quad e = 2.718281828\ldots$

and $\quad \dfrac{d}{dx}(e^x)=e^x$

This makes calculus easy!

Also $\quad \int e^x\,dx = e^x + c$

20.3 Natural logarithms

The exponential function $\quad f(x)=e^x$

has an inverse $\quad f^{-1}(x)=\log_e x$

All exponential functions are 1 : 1.

However, we usually shorten this to $\ln x$.

Logarithms to base "e" are called **natural logarithms**.

We also shorten $\log_{10} x$ to $\lg x$.

With all other bases we have to write them out in full, e.g. $\log_4 x$.

Historically, $\log_{10}$ was used for calculating.

When calculators were invented, $\log_{10}$ was no longer needed.

In practice, $\log_e$, or ln, are the only useful logarithms.

20.4 The derivative of the logarithmic function $y = \ln x$

If $\quad y=\ln x$

then $\quad x=e^y$

Using the definition of log.

Differentiating w.r.t. y: $\dfrac{dx}{dy} = e^y$

So $\dfrac{dy}{dx} = \dfrac{1}{e^y}$

$= \dfrac{1}{x}$

This is the power of x missing from our list of "polynomial" derivatives. (See section 15.5)

So, we have

$f(x)$	$f'(x)$	$\int f(x)\,dx$
e^x	e^x	e^x
$\ln x$	$\dfrac{1}{x}$	A-level

$\int \ln x\, dx$ can be found but it is beyond the scope of this syllabus. However, you could be asked to develop the result. See Ex 20.1, Question 6.

Example 20.1

For the functions f(x) below, find f′(x).

a $f(x) = e^{6x}$ **b** $f(x) = x^2e^{(x^2+4x)}$ **c** $f(x) = e^{2x}\cos x$ **d** $f(x) = \dfrac{e^x}{\cos x}$

e $f(x) = \ln(2x+1)$ **f** $f(x) = x^3\ln(x^2+2x)$ **g** $f(x) = \tan x \ln x$ **h** $f(x) = \dfrac{x^2}{\ln x}$

Solution:

a $f(x) = e^{6x}$

$f'(x) = 6e^{6x}$

Using the composite rule

b $f(x) = x^2e^{(x^2+4x)}$

$f'(x) = 2xe^{(x^2+4x)} + x^2(2x+4)e^{(x^2+4x)}$

Using the product rule and the composite rule

c $f(x) = e^{2x}\cos x$

$f'(x) = 2e^{2x}\cos x - e^{2x}\sin x$

Using the product rule and the composite rule

d $f(x) = \dfrac{e^x}{\cos x}$

$f'(x) = \dfrac{e^x\cos x - e^x\sin x}{\cos^2 x}$

Using the quotient rule

e $f(x) = \ln(2x+1)$

$f'(x) = \dfrac{2}{2x+1}$

Using the composite rule

f $f(x) = x^3\ln(x^2+2x)$

$f(x) = 3x^2\ln(x^2+2x) + \dfrac{x^3(2x+2)}{x^2+2x}$

Using the product rule

g $f(x) = \tan x \ln x$

$$f'(x) = \sec^2 x \ln x + \frac{\tan x}{x}$$

Using the product rule

h $f(x) = \dfrac{x^2}{\ln x}$

$$f'(x) = \frac{2x \ln x - x}{(\ln x)^2}$$

Using the quotient rule

Example 20.2

Integrate each of these functions w.r.t. x.

a $f(x) = e^{6x}$ **b** $f(x) = 6e^{-3x}$ **c** $f(x) = 6e^{3x+2}$ **d** $f(x) = e^{2x}\sqrt{1 + e^{2x}}$

Solution:

a $\int f(x)\,dx = \int e^{6x}\,dx$

Try $\dfrac{d}{dx}(e^{6x}) = 6e^{6x}$

Correct function, wrong constant

2nd attempt $\dfrac{d}{dx}\left(\dfrac{1}{6}e^{6x}\right) = e^{6x}$

Correct!

so $\int e^{6x}\,dx = \dfrac{1}{6}e^{6x} + c$

b $\int f(x)\,dx = \int 6e^{-3x}\,dx$

Try $\dfrac{d}{dx}(e^{-3x}) = -3e^{-3x}$

Correct function, wrong constant

2nd attempt $\dfrac{d}{dx}(-2e^{-3x}) = 6e^{-3x}$

Correct!

so $\int 6e^{-3x}\,dx = -2e^{-3x} + c$

c $\int f(x)\,dx = \int 6e^{3x+2}\,dx$

Try $\dfrac{d}{dx}(e^{3x+2}) = 3e^{3x+2}$

Correct function, wrong constant

2nd attempt $\dfrac{d}{dx}(2e^{3x+2}) = 6e^{3x+2}$

Correct!

so $\int 6e^{3x+2}\,dx = 2e^{3x+2} + c$

d $$\int f(x)\,dx = \int e^{2x}\sqrt{1+e^{2x}}\,dx$$

Try $$\frac{d}{dx}\left(\sqrt{(1+e^{2x})^3}\right) = \frac{3}{2}\left[\sqrt{1+e^{2x}}\right][2e^{2x}]$$

Using the composite rule

$$= 3e^{2x}\sqrt{1+e^{2x}}$$

Correct function, wrong constant

2nd attempt $$\frac{d}{dx}\left(\frac{1}{3}\sqrt{(1+e^{2x})^3}\right) = e^{2x}\sqrt{1+e^{2x}}$$

Correct!

so $$\int e^{2x}\sqrt{1+e^{2x}}\,dx = \frac{1}{3}\sqrt{(1+e^{2x})^3} + c$$

Example 20.3

a If $y = \ln(1+x^2)$, find $\frac{dy}{dx}$.

b Hence find $\int \frac{4x}{1+x^2}\,dx$

Solution:

a $$y = \ln(1+x^2)$$

so $$\frac{dy}{dx} = \frac{2x}{1+x^2}$$

b Hence $\int \frac{4x}{1+x^2}\,dx = 2\ln(1+x^2) + c$

Exercise 20.1

1 Find $\frac{dy}{dx}$ for each:

a $y = e^{4x}$ **b** $y = 4e^{3x}$ **c** $y = e^{x^2}$ **d** $y = e^{\sqrt{x}}$

e $y = e^{3x-1}$ **f** $y = e^{\frac{1}{x}}$ **g** $y = e^{x^2+3x}$ **h** $y = \frac{1}{e^{2x}}$

2 Find $\frac{dy}{dx}$ for each:

a $y = xe^{4x}$ **b** $y = e^{3x}\sin x$ **c** $y = \frac{e^{x^2}}{x}$ **d** $y = e^{\sqrt{x}}\cos x$

e $y = (x^2+2x)e^{3x-1}$ **f** $y = e^{\frac{1}{x}}\sin x$ **g** $y = e^{x^2+3x}\tan x$ **h** $y = \frac{\sin^2 x}{e^{2x}}$

3 Find $\frac{dy}{dx}$ for each:

a $y = \ln x$ **b** $y = \ln 3x$ **c** $y = \ln x^2$ **d** $y = 2\ln x$

e $y = (\ln x)^2$ **f** $y = x\ln x$ **g** $y = x^2\ln x$ **h** $y = \sin x\ln x$

4 **a** In question **3**, why do you get the same answer in both parts **a** and **b**?

Hint: expand $\ln 3x$ using the identity for $\log(ab)$.

b In question **3**, why do you get the same answer in both parts **c** and **d**?

5 Integrate each of these:

a e^{3x} **b** $4e^{2x}$ **c** e^{-4x} **d** $-12e^{-3x}$

e $4xe^{x^2}$ **f** $6e^x(1+e^x)^2$ **g** $6e^x\sqrt{1+e^x}$ **h** $\dfrac{e^x}{\sqrt{1+e^x}}$

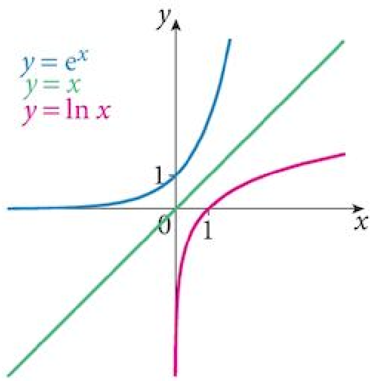
6 **a** If $y = x\ln x$, show that $\ln x = \dfrac{dy}{dx} - 1$.

b Integrate both sides of the equation with respect to x to obtain an expression for $\int \ln x\,dx$.

7 **a** If $y = \ln(e^x + 3)$, find $\dfrac{dy}{dx}$.

b Hence find $\displaystyle\int \frac{e^x}{e^x+3}\,dx$.

20.5 Graphs of exponential and logarithmic functions



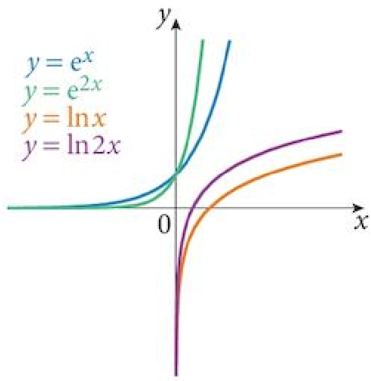
As we have seen, the functions $y = e^x$ and $y = \ln x$ are inverses of each other.

The graph of each is a reflection of the other in the line $y = x$, as shown.

They cut the axes respectively at (0, 1) and (1, 0), as you can see.

We can transform the graphs just like any others and with the same results.

20.5.1 $y = e^{kx}$ and $y = \ln kx$



Replacing x by kx in the equation will squash the graph in the x-direction by a factor of $\dfrac{1}{k}$ about the y-axis.

We can also write $\quad y = \ln kx$

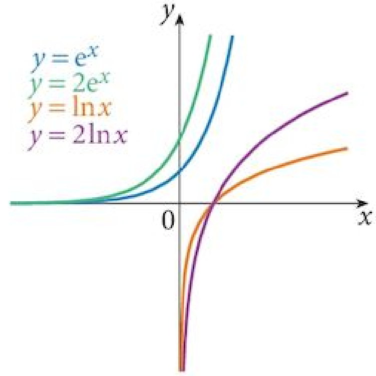
as $\quad y = \ln k + \ln x$.

This alternative is equivalent to a translation of the graph in the y-diection by $\ln k$.

20.5.2 $y = A\mathrm{e}^x$ and $y = A\ln x$

These transformations are equivalent to replacing y by $\frac{y}{A}$.

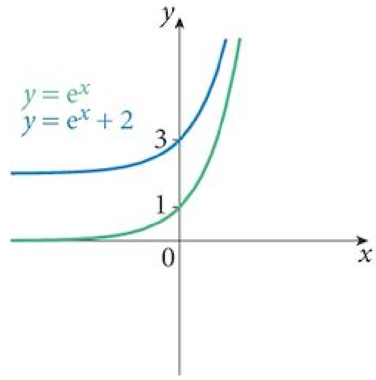
The effect will be to stretch the graphs in the y-direction by a factor of A.
The x-axis is the centre of the stretch and so it is invariant.



20.5.3 $y = \mathrm{e}^x + a$

This transformation replaces y by $(y - a)$.

This is just a translation by a units in the y-direction.
In this case, the graph has been moved upwards by 2 units.

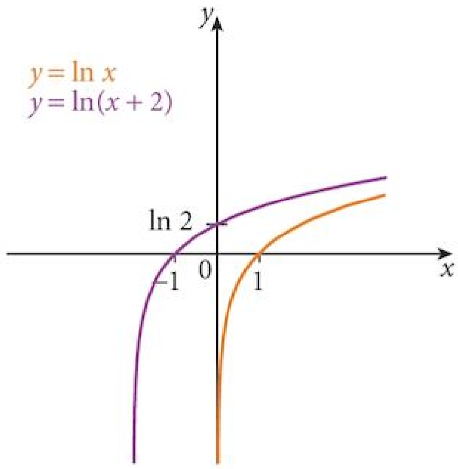


20.5.4 $y = \ln(x + b)$

This transformation replaces x by $(x + b)$.

This is a translation in the x-direction by $-b$ units.

In this case, the graph has been moved to the left by 2 units.
It cuts the x-axis at -1 instead of 1 and the y-axis at $\ln 2$.



Example 20.4

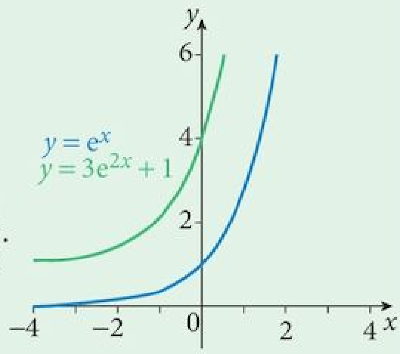
Sketch the following graphs:

a $y = 3\mathrm{e}^{2x} + 1$ **b** $y = 3\ln(2x + 1)$

Solution:

a $$y = 3e^{2x} + 1$$

or $$\frac{y-1}{3} = e^{2x}$$



Starting from $y = e^x$

x-direction: replace x by $2x \Leftrightarrow$ Squash (dilation) by factor $\frac{1}{2}$.

y-direction: replace y by $\frac{y}{3} \Leftrightarrow$ Dilation by factor 3

followed by: replace y by $(y - 1) \Leftrightarrow$ Translation by 1 unit in the y-direction.

b $$y = 3\ln(2x+1)$$

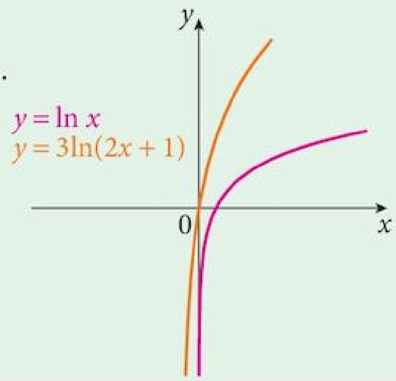
or $$\frac{y}{3} = \ln[2(x+0.5)]$$

Starting from $y = \ln x$

x-direction: replace x by $2x \Leftrightarrow$ Squash (dilation) by factor $\frac{1}{2}$.

followed by: replace x by $(x + 0.5) \Leftrightarrow$ Translation by −0.5 unit in the x-direction.

y-direction: replace y by $\frac{y}{3} \Leftrightarrow$ Dilation by factor 3.



Note that the point (1, 0) first moves to (0.5, 0) and then to (0, 0).

Example 20.5

a Express $x^2 = e^{x-2}$ in the form $\ln x = ax + b$.

b Sketch the graph of $y = \ln x$.

c On your sketch, add the second graph necessary to graphically solve the equation $x^2 = e^{x-2}$.

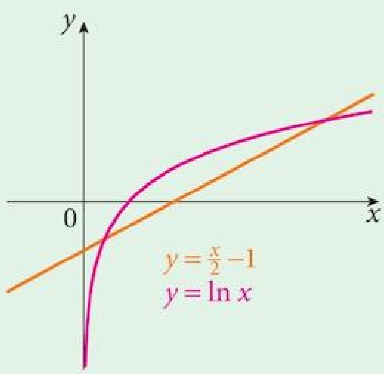
Solution:

a $$x^2 = e^{x-2}$$

Take logs $2\ln x = x - 2$

So $$\ln x = \frac{x}{2} - 1$$

b



c Required graph is $y = \dfrac{x}{2} - 1$

Exercise 20.2

1 Draw each of the following graphs, taking the domain as $-2 \leqslant x \leqslant 2$.
Take x intervals of 0.5 and find the point where each graph crosses the y-axis.

a $y = 2e^{3x}$ **b** $y = 3e^{-x}$ **c** $y = 3e^{3x} + 1$
d $y = 2e^{-x} - 2$ **e** $y = 3e^{-2x} - 3$ **f** $y = -2e^{3x} + 2$

2 Draw each of the following graphs, taking the domain as $0 \leqslant x \leqslant 4$.
Take x intervals of 0.5 and find the point where each graph crosses the x-axis (if any):

a $y = \ln 3x$ **b** $y = 3\ln 2x + 1$ **c** $y = 3\ln(2x+1)$
d $y = \ln(4-x)$ **e** $y = -2\ln x$ **f** $y = -3\ln(2x-1)$

3 **a** Draw the graph of $y = \ln x$ for $0 \leqslant x \leqslant 8$.
b Show that the equation $x^2 = e^{x-3}$ can be written as $\ln x = ax + b$, where the values of a and b are to be found.
c Add the graph of $y = ax + b$ that you found in **b** to your drawing and use it to find the solutions of $x^2 = e^{x-3}$.

4 **a** Draw the graph of $y = e^{2x} - 2$ for $-5 \leqslant x \leqslant 1$.
b On the same axes draw the graph of $y = x + 2$.
c Find the x-coordinates of the points of intersection of the two graphs and write down the equation of which they are the solution.

5 **a** Draw the graph of $y = 2\ln x + 1$ for $0 \leqslant x \leqslant 5$.
b On the same axes draw the graph of $y = 2x - 2$.
c Find the x-coordinates of the points of intersection of the two graphs.
d Show that the equation for which the x-coordinates of your points are the solution can be written as $x^2 = e^{2x-3}$.

6 **a** Draw the graph of $y = e^{-2x} - 4$ for $-2 \leqslant x \leqslant 3$.
b On the same axes draw the graph of $x + y + 1 = 0$.
c Find the x-coordinates of the points of intersection of the two graphs.
d Show that the equation for which the x-coordinates of your points are the solution can be written as $\ln(3-x) = -2x$.

20.6 Applications of e^x and $\ln x$

We have considered many applications in chapters:

2 Indices and surds
4 Functions
13 Applications of the derivative
15 Integration
17 Further differentiation
18 Exponential and logarithmic functions

and possibly others to which we can apply e^x and/or $\ln x$.

Example 20.6

Solve the equation $3\ln(x-1)=\ln 8$.

Solution:

$$3\ln(x-1)=\ln 8$$

So $$\ln\left[(x-1)^3\right]=\ln 8$$

$$(x-1)^3=8$$

$$x-1=2$$

$$x=3$$

Example 20.7

Solve the equation $\ln 20y-\ln(y+8)=2$.

Solution:

$$\ln 20y-\ln(y+8)=2$$

So $$\ln\left[\frac{20y}{y+8}\right]=2$$

$$\frac{20y}{y+8}=e^2$$

$$20y=e^2(y+8)$$

$$y(20-e^2)=8e^2$$

$$y=\frac{8e^2}{20-e^2}$$

$$=4.69$$

Example 20.8

Solve the equations $(e^x)^2 - (e^y)^5 = 0$ **[1]**

$\ln 2 + \ln(x-4) = \ln(8-3y)$ **[2]**

Solution:

Step 1: Simplify the equations $e^{2x} = e^{5y}$ **[1]**

$\ln(2x-8) = \ln(8-3y)$ **[2]**

Step 2: Eliminate the functions $2x = 5y$ **[1]**

$2x - 8 = 8 - 3y$ **[2]**

Step 3: Solve the simultaneous equations

$x = 5$ and $y = 2$

Example 20.9

By using the substitution $u = e^{2x}$ solve the equation $e^{2x} - 5e^{-2x} = 4$.

Solution:

$$e^{2x} - 5e^{-2x} = 4$$

Substitute $u = e^{2x}$ $\quad u - \dfrac{5}{u} = 4$

or $\quad u^2 - 5 = 4u$

giving $\quad u^2 - 4u - 5 = 0$

Factorise $\quad (u-5)(u+1) = 0$

So $\quad u = 5$ or $u = -1$

$e^{2x} = 5$ or $e^{2x} = -1$

not valid

e^x is always positive.

Giving $\quad 2x = \ln 5$

$$x = \frac{1}{2}\ln 5$$

$$= 0.805$$

Example 20.10

It is given that $y = e^x \cos 2x$.

a Find $\frac{dy}{dx}$.

b Hence determine the x-coordinates of the stationary points of the graph for which $0 \leqslant x \leqslant 2\pi$.

Solution:

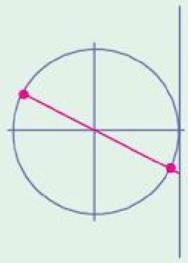
a $y = e^x \cos 2x$

Using product rule

Differentiate $\frac{dy}{dx} = e^x(-2\sin 2x) + e^x \cos 2x$

b We need $\frac{dy}{dx} = 0$

$$0 = e^x(-2\sin 2x + \cos 2x)$$
$$0 = -2\sin 2x + \cos 2x$$
$$\tan 2x = 0.5$$
$$2x \in \{0.46, 3.60, 6.75, 9.89\}$$
$$x \in \{0.23, 1.80, 3.37, 4.94\}$$



Example 20.11

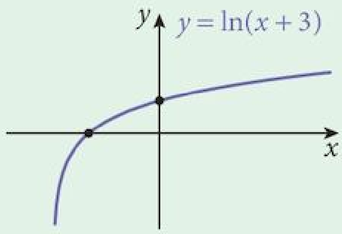
a Draw the curve whose equation is $y = \ln(x+3)$.

b By drawing a suitable line, find an approximate solution to the equation $e(e^x)^2 - 3 = x$.

Solution:

a The curve is shown opposite.

It cuts the y-axis at $(0, \ln 3)$ and the x-axis at $(-2, 0)$.



b If $e(e^x)^2 - 3 = x$

then $e^{2x+1} = x + 3$

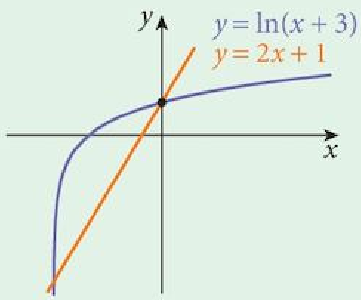
Taking logs: $2x + 1 = \ln(x+3)$

Thus the required line is $y = 2x + 1$

This has been added to the graph opposite.

Thus there are two solutions to the equation:

$x \approx -3$ or $x \approx 0$ (in fact -2.99317 or 0.0590526)



Exercise 20.3

Exponential equations

1 Solve these equations.

a $e^x(e^x-1)=6$ b $e^x(6-e^x)=8$ c $17-6e^x=12e^{-x}$

2 By using the substitution given, or otherwise, solve these equations.

a $e^x-5e^{2x}+6=0$; $u=e^x$ b $e^{6x}-4=0$; $u=e^{3x}$ c $6e^{8x}-13e^{4x}+6=0$; $u=e^{4x}$

d $e^{2x}+4e^{-2x}=4$; $u=e^{2x}$ e $e^{2x}-e^x-6=0$; choose your own.

3 Solve the equation $2\ln x+\ln 4=\ln(9x-2)$.

4 It is given that $e^xe^y=e^7$ and $\ln(2x+y)=\ln 5+\ln 2$.
Calculate the values of x and y.

5 It is given that $(e^x)^3(e^y)^2=e^4$ and $\ln(4y-2x)=3\ln 2+\ln 3$.
Calculate the values of x and y.

6 It is given that $4^x2^y=128$ and $\ln(4y-x)=\ln 2+\ln 5$.
Calculate the values of x and y.

Graphs

7 A curve has the equation $y=3xe^{-x}$.

a Show that the curve has a stationary point when $x=1$.

b Copy and complete this table.

x	0	0.5	1	1.5	2	3
y						

c Draw the graph of $y=3xe^{-x}$ for $0\leqslant x\leqslant 3$.

d By drawing a suitable line, find an approximate solution to the equation $x+1=12xe^{-x}$.

8 A curve has the equation $y=\ln(x-2)$.

a Copy and complete this table.

x	2.5	3	3.5	4	4.5	5
y						

b Draw the graph of $y=\ln(x-2)$ for $0\leqslant x\leqslant 5$.

c By drawing a suitable line, find an approximate solution to the equation $\sqrt{x-2}=e^{3-x}$.

9 a Draw the graph of $y=e^{-x}$ for $-2\leqslant x\leqslant 1$.

b By drawing a suitable line, find an approximate solution to the equation $e^{-x}=4x$.

Functions

10 A function is defined by $f: x\mapsto e^x-1$ for the domain $x\geq 0$.

a Evaluate $f^2(1)$.

b Obtain an expression for f^{-1}.

c State the domain and the range of f^{-1}.

d On the same axes, sketch the graphs of f and f^{-1}, showing the relationship between them.

11 A function is defined by $\mathrm{f}: x \mapsto 5+3\mathrm{e}^{-x}$ for the domain $x \geqslant 0$.

a State the range of f.

b Evaluate $\mathrm{f}^2(1)$.

c Obtain an expression for f^{-1}.

d State the domain and the range of f^{-1}.

e On the same axes, sketch the graphs of f and f^{-1}, showing the relationship between them.

12 A function is defined by $\mathrm{f}: x \mapsto 3\mathrm{e}^{2x}$ for the domain $x \in \mathbb{R}$.

a Sketch the curve, showing the points of intersection with the axes.

b State the range of f.

c Obtain an expression for f^{-1}, stating the domain and the range.

d Add the graph of f^{-1} to your sketch , showing the relationship between the two graphs.

13 Functions f and g are defined by $\mathrm{f}: x \mapsto 3\mathrm{e}^{-2x}$ for the domain $x \in \mathbb{R}$.

$\mathrm{g}: x \mapsto 4-x$ for the domain $x \in \mathbb{R}$.

a Find fg(2).

b Find gf(1).

c Obtain an expression for f^{-1}, stating the domain and the range.

d On the same axes, sketch the graphs of f and f^{-1}, showing the relationship between them.

Gradients

14 If $y=\dfrac{\ln x}{x}$, find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ when $x=1$.

15 The curves $y=\mathrm{e}^{2x-3}$ and $y=\mathrm{e}^{3-x}$ meet at the point P.

Find:

a the coordinates of P

b the gradient of each curve at P.

16 The equation of a curve is $y=\ln\left(x^2+2x\right)$.

Find the coordinates of the point on the curve where the gradient is 1.

17 If $y=3\ln x-\ln 2$:

a Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

b Solve the equation $\ln 32=3\ln x-\ln 2$.

c Find the gradient when $x=3$.

18 **a** Sketch the curve $y=\mathrm{e}^{3x}-3$.

b Find the gradient of the curve at the point where it crosses the x-axis.

Tangents and normals

19 Find an equation of the tangent to the curve $y = x^2 - e^x$ at the point where $x = 0$.

20 Find the point on the curve $y = e^x$ where the gradient is 1.

21 A curve has equation $y = e^{kx}$.

a Show that $\frac{dy}{dx} = ky$.

b Find the value of x, in terms of k, required for the gradient to be 1.

It is required that the value of k be chosen so that the curve touches the line $y = x$.

c Find the y-coordinate on the graph $y = x$ for this value of x.

d Equate this value with the y-coordinate on the graph of $y = e^{kx}$ for the same value of x and hence find the value of k required for the line $y = x$ to be a tangent to $y = e^{kx}$.

Stationary points

22 If $y = x^2 e^{-x}$,

a find $\frac{dy}{dx}$.

b Show that the curve has two stationary points. Find and classify them.

23 Find and classify the stationary point on the curve $y = 3 - 2x + e^{2x}$.

24 Show that the curve $y = e^x - 2e^{-x}$ has no stationary points.

25 Find the position of, and classify, the stationary point on the curve $y = x - \ln x$.

26 Find the position of, and classify, the stationary point on the curve $y = \frac{e^x}{x - 1}$.

27 If $y = e^x \cos x$,

a find $\frac{dy}{dx}$.

b Find the values of x in the range $0 < x < \pi$ for which the curve is stationary.

28 If $y = x^2 e^{3x}$, find $\frac{dy}{dx}$. Identify and classify any stationary points.

29 A curve has equation $y = 10(x + 2)e^x$.

a Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$.

b Find and classify the stationary points of the curve.

30 Find the position of, and classify, the stationary point on the curve $y = e^{3x}\cos x$ for $0 \leqslant x < \pi$.

31 Determine the position and nature of the stationary point on the curve $y = x\ln x - 2x$.

Straight line graphs

32 When a graph of $\ln y$ is plotted against $\ln x$, it is a line with a gradient 3 and crosses the $\ln y$ axis at 1.792. Express y in terms of x.

33 When a graph of $\ln y$ is plotted against x, it is a line with a gradient 1.61 and crosses the $\ln y$ axis at 4.095. Express y in terms of x.

Small changes and rates of change

34 A bacteria population, N, is modelled by the equation $N = 2000\,\mathrm{e}^{0.02t}$ where t is the time in days.

a Find the approximate rate of change of N when $t = 2$.

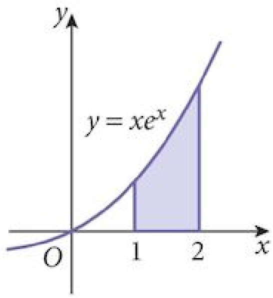
b Find the approximate increase in N when t increases by 5% from the value 2.

Area by integration

35 If $y = x\mathrm{e}^x$

a Show that $x\mathrm{e}^x = \dfrac{\mathrm{d}y}{\mathrm{d}x} - \mathrm{e}^x$.

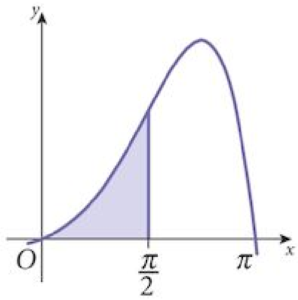
b Hence find $\displaystyle\int_1^2 x\mathrm{e}^x\,\mathrm{d}x$.



36 If $y = \mathrm{e}^x \sin x$

a Find $\dfrac{\mathrm{d}y}{\mathrm{d}x}$.

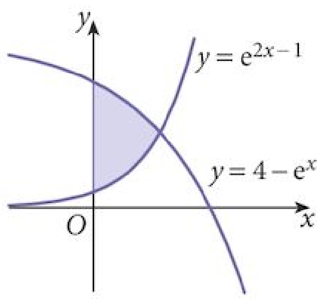
b Hence find $\displaystyle\int_0^{\frac{\pi}{2}} \mathrm{e}^x \sin x\,\mathrm{d}x$.



37 The diagram shows the graphs of $y = 4 - \mathrm{e}^x$ and $y = \mathrm{e}^{2x-1}$.

a Find the coordinates of the point of intersection of the graphs.

b Find the area shown, bounded by the two graphs and the y-axis.

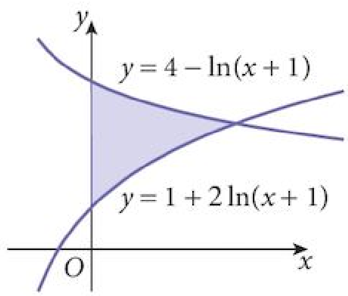


38 The diagram shows the graphs of $y = 1 + 2\ln(x+1)$ and $y = 4 - \ln(x+1)$.

a Find the coordinates of the point of intersection of the graphs.

b Differentiate $(x+1)\ln(x+1)$.

c Use your result to find the shaded area, bounded by the two curves and the y-axis.



Summary

Definition

The number **e** is unique. It arises from the search for a function which, when differentiated, gives itself.

The function $y = e^x$ is the only function with this property, other than $y = 0$.

So, $\dfrac{d}{dx}(e^x) = e^x$ and $\int e^x \, dx = e^x + c$

Value

$e = 2.718281828\ldots$

e is an irrational number.

ln *x*

$\ln x$ is the shorthand way of writing $\log_e x$.

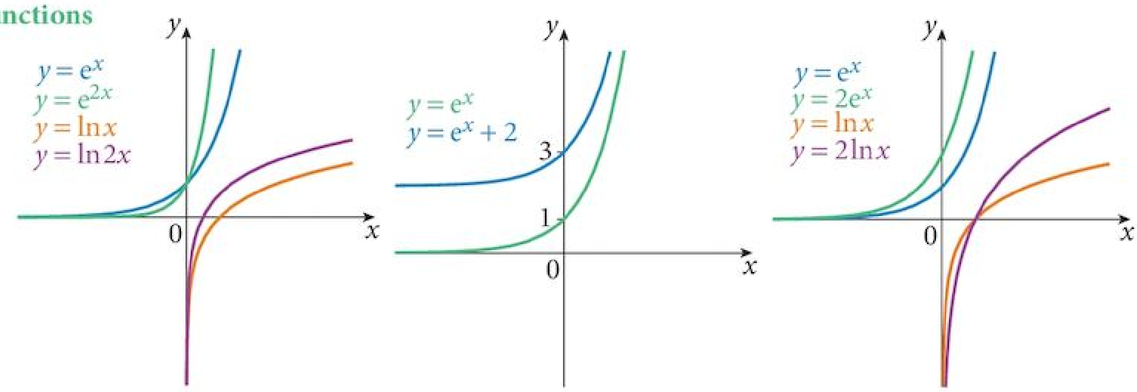
The function $y = \ln x$ is the inverse of $y = e^x$.

$$\frac{d}{dx}(\ln x) = \frac{1}{x}$$

Other exponential functions

$$\frac{d}{dx}(a^x) = (\ln a)\, a^x$$

Graphs of exponential and logarithmic functions



Applications

You can apply exponential and logarithmic functions to any of the applications of differentiation and integration using the results above.

Chapter 20 Summative Exercise

1 Use the $\ln x$ and e^x buttons on your calculator to find the values of:

a $\ln 10$ **b** $\ln 20$ **c** $\ln 30$ **d** $\ln 100$

e e^3 **f** e^5 **g** e^{-3} **h** e^{-7}

2 Differentiate the following functions.

a $y = e^{2x}$ **b** $y = 4e^{x^2}$ **c** $y = 3e^{\sqrt{x}}$ **d** $y = e^{2x^3 - x^2}$

e $y = x\,e^x$ **f** $y = (2x + 1)e^x$ **g** $y = e^x \sin x$ **h** $y = e^{-x} \sin x$

3 Differentiate these functions.

a $y = \ln(x^2)$ b $y = x^2 \ln x$ c $y = \cos x \ln x$ d $y = (\ln x)^3$

4 Find each of these integrals:

a $\int e^{2x}\, dx$ b $\int e^{-4x}\, dx$ c $\int 2x(1+e^{x^2})\, dx$ d $\int e^x(1+e^x)^2\, dx$

5 a If $y = x\,e^x$, show that $x\,e^x = \dfrac{dy}{dx} - e^x$.

b Hence find $\int x e^x\, dx$.

6 a If $y = x^2 \ln x$, show that $x \ln x = \dfrac{1}{2}\left(\dfrac{dy}{dx} - x\right)$.

b Hence find $\int x \ln x\, dx$.

7 Sketch the graph of $y = e^x \sin x$ for $x \geqslant 0$.

8 Solve the following equations:

a $e^{2x} + 4e^{-2x} = 4$ b $e^{2x} - 3e^x - 2 = 0$ c $1 - 8e^{-x} - 2e^{-2x} = 0$

9 Find the x-coordinate, for $0 < x < \dfrac{\pi}{2}$, of the stationary point on the curve $y = e^{3x} \cos x$.

10 Given that $e^x e^y = e^5$ and that $\ln(2x + y) = \ln 4 + \ln 3$, find the value of x and the value of y.

11 a Express $x^2 = e^{x-2}$ in the form $\ln x = ax + b$.

b On graph paper, draw the graph of $y = \ln x$.

c Add to your drawing the graph needed to solve the equation $x^2 = e^{x-2}$.

d Hence find the solution of the equation $x^2 = e^{x-2}$.

12 Draw the curve whose equation is $y = e^{-x}$ for $-2 \leqslant x \leqslant 1$.

By drawing an appropriate straight line graph, obtain an approximate solution to the equation $e^{-x} - 5x = 0$.

13 a Show that the function $f: x \mapsto e^{3x}$ is 1 : 1.

b Find the value of f(2).

c Find $f^{-1}(x)$.

d Solve the equation $f(x) = 50$.

14 You are given the function $y = 12(x-3)e^x$.

Find $\dfrac{dy}{dx}$ and hence find the coordinates of the stationary point on the curve.

Identify the nature of the stationary point.

15 The equation of a curve is $y = \ln(x^2 - 4x)$.

Find the point on the curve where the gradient is 6.

16 The curve $y = 4 - e^x$ meets the x-axis at A and the y-axis at B.

a Find the coordinates of the points A and B.

b Draw the curve for values of x in the range $-2 \leqslant x \leqslant 3$.

c Find the equation of the straight line needed to add to your graph in order to solve the equation $x = \ln(3 - x)$.

d Add this line to your graph and hence obtain an approximate solution to the equation.

17 Show that the curve $y = x^3 e^{-x}$ has a stationary value other than $x = 0$.

Find this stationary value of y and identify its nature.

Chapter 20 Test

1 hour

1 Differentiate with respect to x.

a $x^3 e^{4x}$ [2]

b $\ln(3 - \sin x)$ [2]

c $e^{2x} \tan x$ [2]

2 Variables x and y are related by the equation $y = 2x - 1 - 3e^{-x}$.

a Find $\dfrac{dy}{dx}$. [3]

b Find the approximate increase in y when x increases by k from 0, where k is small. [2]

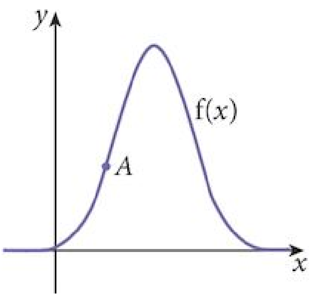
3 A curve has the equation $y = \dfrac{\ln x}{x}$ where $x > 0$.

a Show that the curve has a stationary point at $\left(e, \dfrac{1}{e}\right)$. [4]

b Find $\dfrac{d^2y}{dx^2}$. [4]

c Determine the nature of the stationary point of the curve. [2]

4



The diagram above shows a sketch of the function $f(x)$.
Its derivative is $f^{-1}(x) = -12(x - 2)\, e^{-(x-2)^2}$. The curve passes through the point $A\left(1, \dfrac{6}{e}\right)$.

a Differentiate $y = e^{-(x-2)^2}$. [3]

b Hence find the equation of the curve. [2]

c Find the coordinates of the stationary point on the curve. [2]

d Find the x-coordinate of the point where the tangent at the point A cuts the x-axis. [2]

5 The population of a colony of insects with a restricted food supply is recorded and is modelled by the equation $N = 50 + 30e^{\frac{t}{100}}$, where t days is the time after the start of observations and N is the number of insects in the colony.

a Find the size of the population at the start of the observation period. [1]

b Find the size of the population after 30 days. [2]

c Find how long would it take for the population to reach 120 insects. [3]

d Find the value of N when $\frac{dN}{dt} = 0.5$. [4]

Examination Questions

1 Solve the equation

(i) $e^x(2e^x - 1) = 10$ [3]

(ii) $\log_5(8y - 6) - \log_5(y - 5) = \log_4 16$ [4]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2002, P1, Qu 8]

2 (i) Sketch the graph of $y = \ln x$. [2]

(ii) Determine the equation of the straight line which would need to be drawn on the graph of $y = \ln x$ in order to obtain a graphical solution of the equation $x^2e^{x-2} = 1$. [4]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2002, P2, Qu 8]

3 A curve has the equation $y = xe^{2x}$.

(i) Find the x-coordinate of the turning point of the curve. [4]

(ii) Find the value of k for which $\frac{d^2y}{dx^2} = ke^{2x}(1 + x)$. [3]

(iii) Determine whether the turning point is a maximum or a minimum. [2]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2003, P1, Qu 11]

4 Given that $y = \frac{\ln x}{2x + 3}$, find

(i) $\frac{dy}{dx}$ [3]

(ii) the approximate change in y as x increases from 1 to $(1 + p)$, where p is small. [2]

(iii) the rate of change of x at the instant when $x = 1$, given that y is changing at the rate of 0.12 units per second at this instant.

[Cambridge IGCSE Additional Mathematics 0606, Nov 2003, P1, Qu 8]

5 Functions f and g are defined for $x \in \mathbb{R}$ by

$$f : x \mapsto e^x,$$
$$g : x \mapsto 2x - 3.$$

(i) Solve the equation $fg(x) = 7$. [2]

Function h is defined as gf.

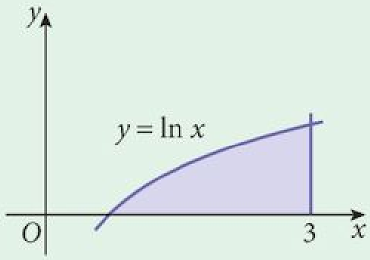
(ii) Express h in terms of x and state its range. [2]

(iii) Express h^{-1} in terms of x. [2]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2004, P2, Qu 7]

6 **(i)** Differentiate $x \ln x - x$ with respect to x. [2]

(ii)



The diagram shows part of the graph of $y = \ln x$. Use your result from part **(i)** to evaluate the area of the shaded region bounded by the curve, the line $x = 3$ and the x-axis. [4]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2005, P2, Qu 5]

7 A curve has equation $y = \dfrac{e^{2x}}{\sin x}$, for $0 < x < \pi$.

(i) Find $\dfrac{dy}{dx}$ and show that the x-coordinate of the stationary point satisfies $2\sin x - \cos x = 0$. [4]

(ii) Find the x-coordinate of the stationary point. [2]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2005, P2, Qu 6]

8 **(i)** Given that $y = 1 + \ln(2x - 3)$, obtain an expression for $\dfrac{dy}{dx}$. [2]

(ii) Hence find, in terms of p, the approximate value of y when $x = 2 + p$, where p is small. [3]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2005, P2, Qu 3]

9 **a** Differentiate $e^{\tan x}$ with respect to x. [2]

b Evaluate $\int_0^{\frac{1}{2}} e^{1-2x}\, dx$. [4]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2006, P2, Qu 4]

10 The equation of a curve is $y = xe^{-\frac{x}{2}}$.

(i) Show that $\frac{dy}{dx} = \frac{1}{2}(2-x)e^{-\frac{x}{2}}$. [3]

(ii) Find an expression for $\frac{d^2y}{dx^2}$. [2]

The curve has a stationary point at M.

(iii) Find the coordinates of M. [2]

(iv) Determine the nature of the stationary point at M. [2]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2006, P2, Qu 11]

11 **(i)** Differentiate $x^2 \ln x$ with respect to x. [2]

(ii) Use your result to show that $\int_1^e 4x \ln x \, dx = e^2 + 1$. [4]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2007, P2, Qu 6]

12 A curve is such that $\frac{d^2y}{dx^2} = 4e^{-2x}$. Given that $\frac{dy}{dx} = 3$ when $x = 0$ and that the curve passes through the point $(2, e^{-4})$, find the equation of the curve. [6]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2008, P1, Qu 8]

13 **(i)** Differentiate $x \ln x$ with respect to x. [2]

(ii) Hence find $\int \ln x \, dx$. [3]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2008, P2, Qu 4]

14 **(i)** Find $\frac{d}{dx}\left(x e^{3x} - \frac{e^{3x}}{3}\right)$. [3]

(ii) Hence find $\int x e^{3x} \, dx$. [3]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2009, P1, Qu 7]

15 A function f is defined by $f : x \mapsto e^{x-1}$, where $x > 0$.

(i) State the range of f. [1]

(ii) Find an expression for f^{-1}. [2]

(iii) State the domain of f^{-1}. [1]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2009, P2, Qu 1]

21 Vectors

Syllabus statements

- use vectors in any form, e.g. $\begin{pmatrix} a \\ b \end{pmatrix}$, $\overrightarrow{AB}$, $\mathbf{p}$, $a\mathbf{i} + b\mathbf{j}$
- know and use position vectors and unit vectors
- find the magnitude of a vector; add and subtract vectors and multiply vectors by scalars
- compose and resolve velocities
- use relative velocity, including solving problems on interception (but not closest approach)

21.1 What is a vector?

A **vector** is a quantity with two properties: **magnitude** (size) and **direction**.

Many physical quantities are vectors.

A quantity that has magnitude only is called a **scalar**.

For convenience, we often have a scalar associated with a vector.

Vector	Associated Scalar
velocity	speed
position	distance
acceleration	acceleration/deceleration
weight	mass

Some quantities have no association with the other variety.

Vector
Force
Translation

Scalar
number
angle

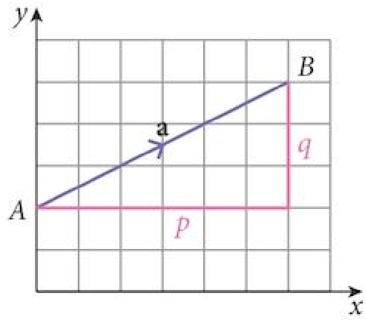
Note that we sometimes consider angle to have a direction.

When we develop the algebra of vectors, any results apply to all types of vectors.

In order to simplify our work it is convenient to develop results using translations.

21.2 Describing translations (vectors)

A translation is a transformation in which every point of an object moves exactly the same distance in the same direction. The object and its image are directly congruent.



The translation from A to B can be written in several ways:

$\overrightarrow{AB}$ denoting its end points

a giving it a name.

On a two dimensional grid we can also use grid measurements as shown.

$$\begin{pmatrix} p \\ q \end{pmatrix}$$

Note that when you handwrite vectors, you cannot easily or practically write them in bold characters.

We use one of several alternatives to denote a vector when handwriting:

$\overrightarrow{AB}$ $\overline{AB}$ $\underline{AB}$ $\underset{\sim}{AB}$ $\underset{\sim}{a}$ $\underline{a}$ $\vec{a}$

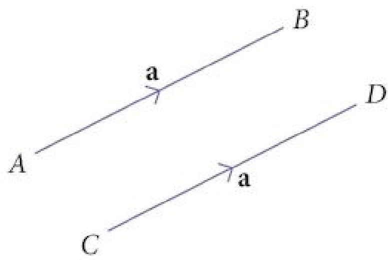
$\overrightarrow{AB}$ is the notation used in the syllabus. You could use any notation as long as your intention is clear.

21.2.1 Equality of translations

Two translations are equal if they have the same magnitude and the same direction.

Two vectors are equal if they have the same magnitude and the same direction.

So, $\overrightarrow{AB} = \overrightarrow{CD} = \mathbf{a}$



21.2.2 The magnitude of a translation (vector)

When we want to talk about the magnitude of a vector, we use the notation $|\mathbf{a}|$.

If we use grid notation, we can calculate this using Pythagoras' theorem.

If $\quad \mathbf{a} = \begin{pmatrix} p \\ q \end{pmatrix}$

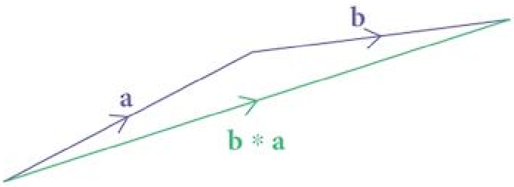
$$|\mathbf{a}| = \sqrt{p^2 + q^2}$$

21.2.3 Combining translations

We combine transformations (vectors) by first doing one and then doing a second: **a** "followed by" **b**.

Transformations are functions so **a** "followed by" **b** would be written as $\mathbf{b} * \mathbf{a}$.

We can use $*$ for "followed by"



Problem 21.1

a By drawing, show that $\mathbf{b} * \mathbf{a} = \mathbf{a} * \mathbf{b}$.

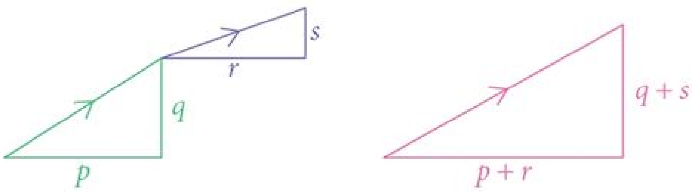
b By drawing, show that $\mathbf{c} * (\mathbf{b} * \mathbf{a}) = \mathbf{c} * (\mathbf{b} * \mathbf{a})$.

$\mathbf{b} * \mathbf{a} = \mathbf{a} * \mathbf{b}$ is the **commutative** law

$\mathbf{c} * (\mathbf{b} * \mathbf{a}) = \mathbf{c} * (\mathbf{b} * \mathbf{a})$ is the **associative** law

Because the combination of translations is both commutative and associative (just like when adding or multiplying numbers), we can change the order in which we do things.

Returning to the combination problem, if we write the problem in grid notation, we get:



$$\begin{pmatrix} r \\ s \end{pmatrix} * \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} p+r \\ q+s \end{pmatrix}$$

Because we are adding the corresponding elements, we use the addition sign (+) instead of $*$ for the "followed by" combination.

So, **a** followed by **b** is written **b** + **a**

and $\quad \mathbf{a}+\mathbf{b}=\mathbf{b}+\mathbf{a} \quad$ [1]

$(\mathbf{a}+\mathbf{b})+\mathbf{c}=\mathbf{a}+(\mathbf{b}+\mathbf{c}) \quad$ [2]

21.2.4 Scalar multiplication

What do we get if we have **a** + **a**?

As you can see, the result is a vector that is double the magnitude of **a**, but in the same direction.

In grid notation, we get

$$\begin{pmatrix} p \\ q \end{pmatrix}+\begin{pmatrix} p \\ q \end{pmatrix}=\begin{pmatrix} 2p \\ 2q \end{pmatrix}$$

In ordinary algebra, we would want to write this as 2**a**.

As this does not cause any problems, that is what we do.

Hence, the vector k**a** is a vector that is in the same direction as **a** and has its magnitude multiplied by k.

Example 21.1

The vectors **a** and **b** are defined as $\mathbf{a}=\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix} 5 \\ 12 \end{pmatrix}$.

a Find 3**a** + 2**b**.

b Find **(i)** |**a**| **(ii)** |**b**|

Note that "3" and "2" are not bold.

Solution:

a $3\mathbf{a}+2\mathbf{b}=3\begin{pmatrix} 3 \\ 4 \end{pmatrix}+2\begin{pmatrix} 5 \\ 12 \end{pmatrix}$

$=\begin{pmatrix} 9 \\ 12 \end{pmatrix}+\begin{pmatrix} 10 \\ 24 \end{pmatrix}$

$=\begin{pmatrix} 19 \\ 36 \end{pmatrix}$

b **(i)** $|\mathbf{a}|=\sqrt{3^2+4^2}$

$=5$

(ii) $|\mathbf{b}|=\sqrt{5^2+12^2}$

$=13$

21.2.5 The zero vector

Just as in other areas of mathematics, we need a **zero vector**. This we write as **0**, or $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

This is the vector that represents no movement.

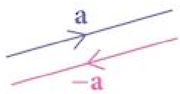
If you are not moving, it does not matter in which direction you go.

Thus the zero vector takes every direction.

21.2.6 The vector −a

What do we get if we have $\mathbf{a} - \mathbf{a}$?

Logic tells us that we ought to get $\mathbf{0}$.



Algebra also tells us that $\mathbf{a} - \mathbf{a} = \mathbf{a} + (-\mathbf{a})$.

So it seems sensible to define the vector $-\mathbf{a}$ as a vector which has the same magnitude as $\mathbf{a}$, but the opposite direction. This also agrees with our ideas about scalar multiplication. $-\mathbf{a} = (-1)\mathbf{a}$.

Thus we can give a meaning to "subtraction".

Example 21.2

The vectors **a** and **b** are defined as $\mathbf{a} = \begin{pmatrix} 1 \\ -10 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -1 \\ -3 \end{pmatrix}$.

a Find $3\mathbf{a} - 5\mathbf{b}$.

b Find $|3\mathbf{a} - 5\mathbf{b}|$.

Solution:

a
$$3\mathbf{a} - 5\mathbf{b} = 3\begin{pmatrix} 1 \\ -10 \end{pmatrix} - 5\begin{pmatrix} -1 \\ -3 \end{pmatrix}$$
$$= \begin{pmatrix} 3 \\ -30 \end{pmatrix} - \begin{pmatrix} -5 \\ -15 \end{pmatrix}$$
$$= \begin{pmatrix} 8 \\ -15 \end{pmatrix}$$

b
$$|3\mathbf{a} - 5\mathbf{b}| = \sqrt{8^2 + 15^2}$$
$$= 17$$

Exercise 21.1

Vectors are defined as follows:

$$\mathbf{a} = \begin{pmatrix} 2 \\ -1 \end{pmatrix} \quad \mathbf{b} = \begin{pmatrix} -1 \\ 3 \end{pmatrix} \quad \mathbf{c} = \begin{pmatrix} -2 \\ -5 \end{pmatrix} \quad \mathbf{d} = \begin{pmatrix} 3 \\ 2 \end{pmatrix} \quad \mathbf{e} = \begin{pmatrix} 4 \\ 7 \end{pmatrix}$$

Find the following vector combinations.

1 **a** $3\mathbf{a}$ **b** $-2\mathbf{b}$ **c** $4\mathbf{c}$ **d** $-3\mathbf{d}$ **e** $2\mathbf{e}$

f $\mathbf{a} - \mathbf{b}$ **g** $\mathbf{c} + \mathbf{a}$ **h** $\mathbf{b} + \mathbf{d}$ **i** $\mathbf{d} - \mathbf{e}$ **j** $\mathbf{e} - \mathbf{c}$

k $2\mathbf{a} + 3\mathbf{b}$ **l** $4\mathbf{c} - 2\mathbf{a}$ **m** $3\mathbf{b} + 2\mathbf{d}$ **n** $4\mathbf{d} - 3\mathbf{e}$ **o** $2\mathbf{e} + 3\mathbf{c}$

Find the magnitude of each vector.

2	**a**	$\lvert\mathbf{a}\rvert$	**b**	$\lvert\mathbf{b}\rvert$	**c**	$\lvert\mathbf{c}\rvert$	**d**	$\lvert\mathbf{d}\rvert$	**e**	$\lvert\mathbf{e}\rvert$
	f	$\lvert\mathbf{a}-\mathbf{c}\rvert$	**g**	$\lvert\mathbf{b}+\mathbf{e}\rvert$	**h**	$\lvert\mathbf{b}+\mathbf{d}\rvert$	**i**	$\lvert\mathbf{d}-\mathbf{c}\rvert$	**j**	$\lvert\mathbf{e}-\mathbf{a}\rvert$
	k	$\lvert 5\mathbf{a}+2\mathbf{b}\rvert$	**l**	$\lvert 3\mathbf{c}-4\mathbf{a}\rvert$	**m**	$\lvert 2\mathbf{b}+3\mathbf{d}\rvert$	**n**	$\lvert 3\mathbf{d}-2\mathbf{e}\rvert$	**o**	$\lvert 4\mathbf{e}+5\mathbf{c}\rvert$

21.3 Unit vectors

A **unit vector** is a vector with a magnitude of 1 unit.

An easy way to create a unit vector in a particular direction is to divide a vector by its own magnitude. This will not change the direction of the vector.

Example 21.3

Vectors **u** and **v** are defined as $\mathbf{u}=\begin{pmatrix}6\\-8\end{pmatrix}$ and $\mathbf{v}=\begin{pmatrix}-1\\2\end{pmatrix}$.

Find a unit vector in the direction of:

a **u** **b** **v** **c** 2**u** + 5**v**.

Solution:

a $|\mathbf{u}|=\sqrt{6^2+(-8)^2}=10$

A unit vector in the direction of **u** is $\begin{pmatrix}0.6\\-0.8\end{pmatrix}$

b $|\mathbf{v}|=\sqrt{5}$

A unit vector in the direction of **v** is $\begin{pmatrix}-\frac{1}{\sqrt{5}}\\\frac{2}{\sqrt{5}}\end{pmatrix}$

You could rationalise the surds.

c $2\mathbf{u}+5\mathbf{v}=\begin{pmatrix}7\\-6\end{pmatrix}$

$|2\mathbf{u}+5\mathbf{v}|=\sqrt{85}$

A unit vector in the direction of 2**u** + 5**v** is $\begin{pmatrix}\frac{7}{\sqrt{85}}\\\frac{-6}{\sqrt{85}}\end{pmatrix}$

21.4 Base vectors

A **base vector** is a unit vector in the direction of the standard axes.

Thus $\mathbf{i}=\begin{pmatrix}1\\0\end{pmatrix}$ and $\mathbf{j}=\begin{pmatrix}0\\1\end{pmatrix}$.

Any vector can be written in terms of base vectors. They create yet another way of writing vectors.

So, for example $\begin{pmatrix}p\\q\end{pmatrix}$ can be written as $p\mathbf{i}+q\mathbf{j}$.

In some situations, it is easier to solve problems by writing vectors in terms of base vectors.

On other occasions the column (grid) notation is easier.

Example 21.4

The vectors $\mathbf{u}$ and $\mathbf{v}$ are defined as $\mathbf{u} = 3\mathbf{i} - 4\mathbf{j}$ and $\mathbf{v} = 5\mathbf{i} + 12\mathbf{j}$.

Find: **a** $|\mathbf{u}|$ **b** $|\mathbf{v}|$ **c** $|\mathbf{u}+\mathbf{v}|$

Solution:

a $|\mathbf{u}| = \sqrt{3^2 + 4^2} = 5$

b $|\mathbf{v}| = \sqrt{5^2 + 12^2} = 13$

c $\mathbf{u}+\mathbf{v} = (3\mathbf{i} - 4\mathbf{j}) + (5\mathbf{i} + 12\mathbf{j}) = (8\mathbf{i} + 8\mathbf{j})$

$|\mathbf{u}+\mathbf{v}| = \sqrt{8^2 + 8^2} = 8\sqrt{2}$

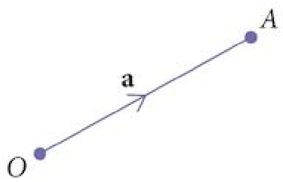
21.5 Position vectors

Even though vectors do not have position, there are times when we want to use vectors to talk about the position of a point. We can do this only if we have an origin.

The vector from the origin to a point A will be called $\mathbf{a}$.

Such a vector is called a **position vector**.

Position vectors are very useful when doing geometry.



21.6 Vector geometry

21.6.1 The vector $\overrightarrow{AB}$

The position vector of a point, A, relative to an origin, O, is $\mathbf{a}$.

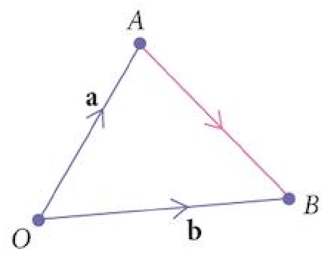
Similarly, the vector of a point, B, relative to the origin, O, is $\mathbf{b}$.

In order to find the vector $\overrightarrow{AB}$ (from A to B), we find an alternative route.

Thus $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$

$= -\mathbf{a} + \mathbf{b}$

or $\boxed{\overrightarrow{AB} = \mathbf{b} - \mathbf{a}}$ **[1]**



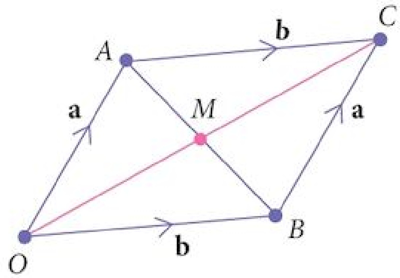
Think: second – first.

21.6.2 The mid-point of $\overrightarrow{AB}$

Quadrilateral $OBCA$ is a parallelogram.

If M is the mid-point of AB, then $\overrightarrow{OM} = \mathbf{m}$ and M is also the mid-point of OC.

$\overrightarrow{OC} = \mathbf{a} + \mathbf{b}$



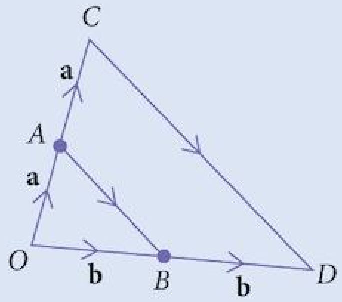
and $\overrightarrow{OM} = \frac{1}{2}(\mathbf{a}+\mathbf{b})$

$$\boxed{\mathbf{m} = \frac{1}{2}(\mathbf{a}+\mathbf{b})} \quad [2]$$

Example 21.5 The Mid-Point Theorem

In the triangle OCD, A is the mid point of OC, B is the mid point of OD. Prove that AB is parallel to CD and that $|CD| = 2|AB|$.

Solution:



Let $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{OC} = \mathbf{c}$, $\overrightarrow{OD} = \mathbf{d}$.

Then $\mathbf{c} = 2\mathbf{a}$ and $\mathbf{d} = 2\mathbf{b}$

$$\begin{aligned}\overrightarrow{CD} &= \mathbf{d} - \mathbf{c}\\ &= 2\mathbf{b} - 2\mathbf{a}\\ &= 2(\mathbf{b} - \mathbf{a})\end{aligned}$$

so $\overrightarrow{CD} = 2\,\overrightarrow{AB}$

Using [1]: AB = b − a

This means that [1] $\overrightarrow{AB}$ is parallel to $\overrightarrow{CD}$ Direction

and [2] $|\overrightarrow{CD}| = 2|\overrightarrow{AB}|$ Magnitude

Every vector equation tells you two things.

Example 21.6

Vectors **a** and **b** are defined as $\mathbf{a} = 2\mathbf{i} + \mathbf{j}$ and $\mathbf{b} = \mathbf{i} + 2\mathbf{j}$.

The position vectors of points P, Q and R are given by:

$$\mathbf{p} = -6\mathbf{a} + 9\mathbf{b}$$
$$\mathbf{q} = \mathbf{a} + 4\mathbf{b}$$
$$\mathbf{r} = 15\mathbf{a} - 6\mathbf{b}$$

Show that the points P, Q and R are collinear.

Solution:

$$\begin{aligned}\overrightarrow{PQ} &= \mathbf{q} - \mathbf{p}\\ &= 7\mathbf{a} - 5\mathbf{b}\\ \overrightarrow{PR} &= \mathbf{r} - \mathbf{p}\\ &= 21\mathbf{a} - 15\mathbf{b} = 3(7\mathbf{a} - 5\mathbf{b})\end{aligned}$$

So $\overrightarrow{PR} = 3\overrightarrow{PQ}$

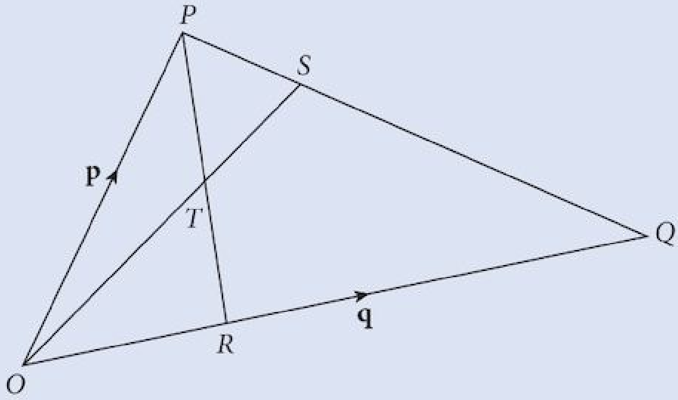
Hence [1] PR || PQ

[2] Point P lies on both lines

and points PQR are collinear.

We have not used the length in this proof. That is additional information.

Example 21.7



In the diagram, $\boldsymbol{OP} = \mathbf{p}$, $\boldsymbol{OQ} = \mathbf{q}$, $PS = \frac{1}{3}PQ$ and $OR = \frac{1}{3}OQ$

a Express $\boldsymbol{PR}$ and $\boldsymbol{OS}$ in terms of $\mathbf{p}$ and $\mathbf{q}$.

b Given that $\boldsymbol{OT} = \lambda OS$, express OT in terms of λ, $\mathbf{p}$ and $\mathbf{q}$.

c Given that $\boldsymbol{PT} = \mu PR$, express OT in terms of μ, $\mathbf{p}$ and $\mathbf{q}$.

d Hence find the value of λ and of μ.

Solution:

a
$$\mathbf{PR} = -\mathbf{p} + \frac{1}{3}\mathbf{q}$$
$$= \frac{1}{3}(\mathbf{q} - 3\mathbf{p})$$
$$\mathbf{OS} = \mathbf{p} + \frac{1}{3}(\mathbf{q} - \mathbf{p})$$
$$= \frac{1}{3}(2\mathbf{p} + \mathbf{q})$$

b
$$\mathbf{OT} = \frac{\lambda}{3}(2\mathbf{p} + \mathbf{q})$$

c
$$\mathbf{PT} = \frac{\mu}{3}(\mathbf{q} - 3\mathbf{p})$$
and $\mathbf{OT} = \mathbf{p} + \frac{\mu}{3}(\mathbf{q} - 3\mathbf{p})$
$$= (1 - \mu)\mathbf{p} + \frac{\mu}{3}\mathbf{q}$$

d The two expressions for **OT** must be identical.
The vectors **p** and **q** are not parallel, so we can compare coefficients.

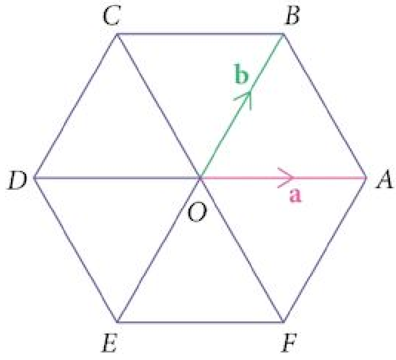
p: $\frac{2\lambda}{3} = 1 - \mu$

q: $\frac{\lambda}{3} = \frac{\mu}{3}$

Solving these equations give $\lambda = \mu = \frac{3}{5}$

Exercise 21.2

1 *ABCDEF* is a regular hexagon. The position vector of the point *A* is **a** and the position vector of the point *B* is **b**.



Express the following in terms of **a** and **b**:

a **BA**   **b** **DA**   **c** **AF**

d **DE**   **e** **EA**   **f** **CA**

g **FC**   **h** **CE**   **i** **BE**

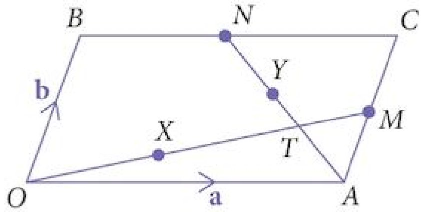
2 *OACB* is a parallelogram in which $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

The point *M* is the mid-point of *AC* and the point *N* is the mid-point of *BC*.

The lines *OM* and *AN* meet at *T*.

The point *X* lies on *OM* such that $\overrightarrow{OX} = \lambda\overrightarrow{OM}$.

Similarly, the point *Y* lies on *AN* such that $\overrightarrow{AY} = \mu\overrightarrow{AN}$.



a Express these vectors in terms of **a** and **b**.

i $\overrightarrow{OM}$ **ii** $\overrightarrow{OX}$

iii $\overrightarrow{AN}$ **iv** $\overrightarrow{AY}$ **v** $\overrightarrow{OY}$

b The values of λ and μ can vary and, in doing so, the points *X* and *Y* move along their respective lines. By equating the vectors $\overrightarrow{OX}$ and $\overrightarrow{OY}$, solve the equation to find the values of λ and μ and find the vector $\overrightarrow{OT}$.

3 **The medians of a triangle**

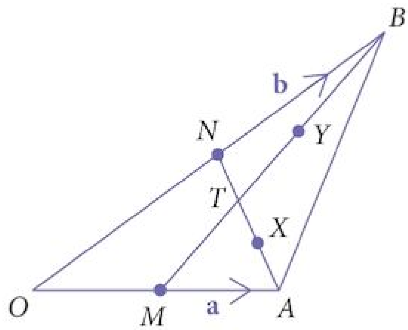
OAB is a triangle in which $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

The mid-point of *OA* is *M* and the mid-point of *OB* is *N*.

The medians, *AN* and *BM* meet at *T*.

The point *X* lies on *AN* such that $\overrightarrow{AX} = \lambda\overrightarrow{AN}$.

Similarly, the point *Y* lies on *BM* such that $\overrightarrow{BY} = \mu\overrightarrow{BM}$.



a Express each vector in terms of **a** and **b**.

i $\overrightarrow{AN}$ **ii** $\overrightarrow{AX}$ **iii** $\overrightarrow{OX}$

iv $\overrightarrow{BM}$ **v** $\overrightarrow{BY}$ **vi** $\overrightarrow{OY}$

b The values of λ and μ can vary and, in doing so, the points *X* and *Y* move along their respective lines. By equating the vectors $\overrightarrow{OX}$ and $\overrightarrow{OY}$, solve the equation to find the values of λ and μ and find the vector $\overrightarrow{OT}$.

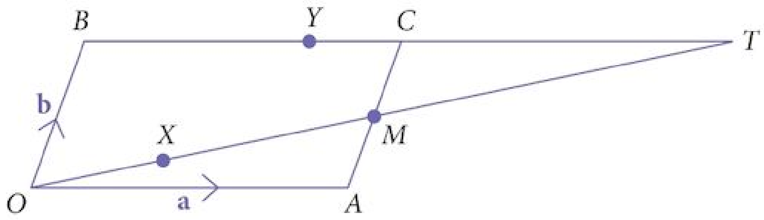
c If the mid-point of *AB* is *P*, show that the median *OP* also passes through *T* and that the ratios *OT* : *TP*, *BT* : *TM* and *AT* : *TN* are all equal.

4 $OACB$ is a parallelogram.

M is the mid-point of AC.

The line segment OM is extended until it meets the line BC (extended).

The point X lies on OM such that $\overrightarrow{OX} = \lambda\overrightarrow{OM}$.



Similarly, the point Y lies on BC such that $\overrightarrow{BY} = \mu\overrightarrow{BC}$.

a Express each vector in terms of **a** and **b**.

i $\overrightarrow{OM}$ **ii** $\overrightarrow{OX}$

iii $\overrightarrow{BY}$ **iv** $\overrightarrow{OY}$

b The values of λ and μ can vary and, in doing so, the points X and Y move along their respective lines. By equating the vectors $\overrightarrow{OX}$ and $\overrightarrow{OY}$, solve the equation to find the values of λ and μ and find the vector $\overrightarrow{OT}$.

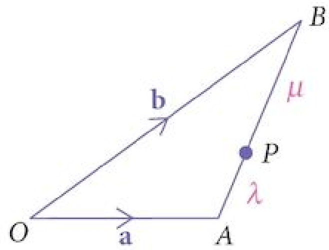
c What is the relationship between the points B, C and T?

5 OAB is a triangle in which $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

The point P lies on AB such that $AP : PB = \lambda : \mu$.

Express each vector in terms of **a** and **b**.

a $\overrightarrow{AB}$ **b** $\overrightarrow{AP}$ **c** $\overrightarrow{OP}$



21.7 Velocities

Velocities are vectors and so any result that we can prove for vectors can be applied to velocities.

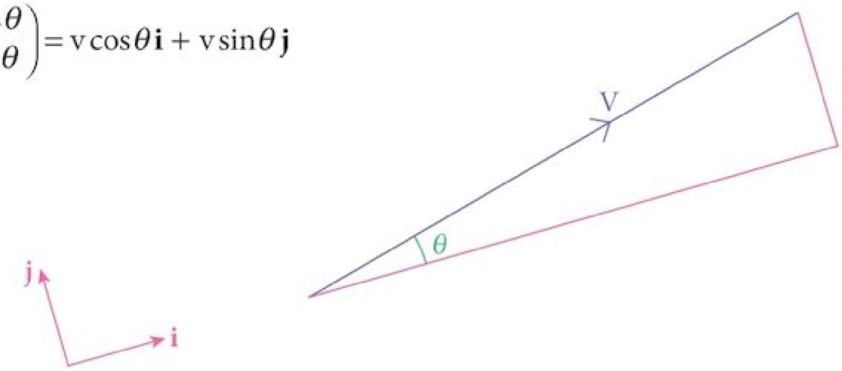
In particular, they can be added and subtracted.

We can also split a complicated velocity into components, or combine components to find the actual velocity of an object.

Given a velocity and the direction of the base vectors, the two components of the velocity are found using trigonometry. They are $\mathrm{v}\cos\theta$ in the **i** direction and $\mathrm{v}\sin\theta$ in the **j** direction (where v is the magnitude of **v**).

We can put the base vectors in any convenient directions as long as they are perpendicular.

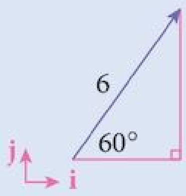
Thus $\mathbf{v} = \begin{pmatrix} \mathrm{v}\cos\theta \\ \mathrm{v}\sin\theta \end{pmatrix} = \mathrm{v}\cos\theta\,\mathbf{i} + \mathrm{v}\sin\theta\,\mathbf{j}$



Example 21.8

A billiard ball is hit with a velocity of 6 ms^{-1} in a direction that makes an angle of 60° with the edge of the table.

Find the components of the velocity in the directions parallel to the edges of the table.



Solution:

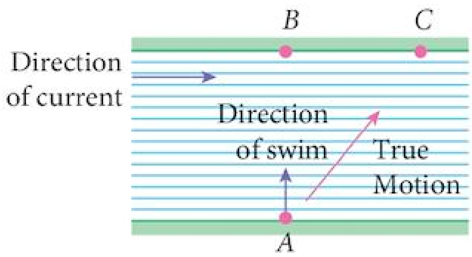
The components will be 6 cos 60° and 6 sin 60°.

Thus $\mathbf{v} = 3\mathbf{i} + 5.2\mathbf{j}$ $(3\sqrt{3}\mathbf{j}$ is more exact.)

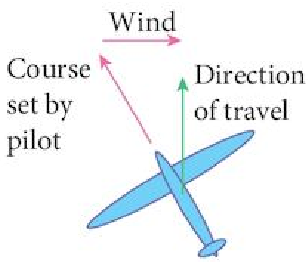
21.8 Currents and winds

Imagine that you are trying to swim across a river.

You set out from point A and swim towards point B on the opposite bank. While you are swimming in that direction, the water will carry you downstream and you will land maybe at point C.



Exactly the same thing will happen to an aircraft, except that it will be pushed by the wind.



Situations like these can be modelled using vectors.

Example 21.9

A river, 200 m wide, is flowing with a velocity of 1.5 ms^{-1} from left to right.

A swimmer at point A heads towards point B, swimming at 2 ms^{-1}.

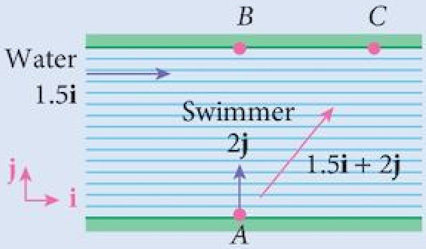
a What is the true speed of the swimmer?

b How far downriver from point B will she land?

c In which direction must she swim in order to land at point B?

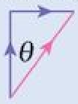
Solution:

If we choose base vectors **i** along the river bank and **j** across the river, the velocity of the swimmer will be $2\mathbf{j}$ ms^{-1} and the velocity of the river will be $1.5\mathbf{i}$ ms^{-1}.



a The true velocity of the swimmer will be the sum of these.

$$\mathbf{v} = 1.5\mathbf{i} + 2\mathbf{j}$$



Thus, the true speed, $|\mathbf{v}| = 2.5\ \text{ms}^{-1}$

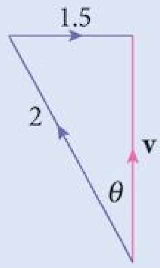
b The time taken to cross the river can be found in several ways. The simplest is to use the swimmer's speed and the distance she has to go.

$$\text{Time} = \frac{200\ \text{m}}{2\ \text{ms}^{-1}} = 100\ \text{s}$$

From which, the distance travelled will be

$$100\,(1.5\mathbf{i} + 2\mathbf{j})$$
$$= 150\mathbf{i} + 200\mathbf{j}$$

Hence, she will travel 150 m down stream before she lands.



c In order to travel directly across the river, she must head upstream so that the current pushes her back just far enough.

Using Pythagoras and trigonometry,

$$\text{v} = 1.32\ \text{ms}^{-1}$$
$$\theta = 48.6^\circ$$

Notice how much longer it takes!

21.9 Relative velocity

All velocities are measured relative to some reference point.

In the previous example, the swimmer's speed was measured relative to the water. The water's speed is measured relative to the bank.

The "true" velocity of the swimmer was the velocity relative to the bank.

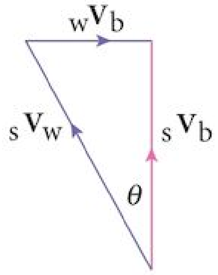
In addition, we can solve problems such as the relative motion and speeds of two ships sailing at sea. Their speeds are relative to the water, which is also moving, but we can find out how one is travelling relative to the other.

While you are sitting there reading this, your velocity relative to the earth is zero. However, if you are at the equator, you are travelling at 1670 km h^{-1} around the centre of the earth and the earth is travelling at 106 700 km h^{-1} relative to the sun. (Velocities are approximate.)

A useful notation is ${}_A\mathbf{V}_B$ which means "the velocity of A relative to B".

In the previous example, the vector addition becomes:

$${}_s\mathbf{V}_w + {}_w\mathbf{V}_b = {}_s\mathbf{V}_b$$



s = swimmer
w = water
b = bank

In general: $${}_A\mathbf{V}_B + {}_B\mathbf{V}_C = {}_A\mathbf{V}_C$$

So, the velocity of the swimmer relative to the water plus the velocity of the water relative to the bank equals the velocity of the swimmer relative to the bank.

This relationship will always be true. All we need to do is to identify which objects are A, B and C.

Notice how similar this diagram is to the one we use when multiplying matrices, working out the orders.

As we saw, velocities are vectors which do not have position.

If we want to completely solve a problem, we will need two diagrams: one for the velocities and one for the position.

Do not forget, a position vector is measured relative to an origin.

However, we can also measure it relative to something that itself is moving.

If you are running along a track with your friend by your side, his position relative to you will be constant. All these ideas can be included in questions.

Example 21.10

A boat, M, leaves a port, P, in still water and travels with a velocity, $\mathbf{v} = 6\mathbf{i} + 8\mathbf{j}$ km h^{-1}.

At the same time, another boat, N, leaves a port, Q, whose position relative to P is $\mathbf{r} = 50\mathbf{i} + 20\mathbf{j}$ where the base vectors $\mathbf{i}$ km and $\mathbf{j}$ km are measured in the easterly and northerly directions respectively.

The velocity of boat N is $-4\mathbf{i} + 4\mathbf{j}$ km h^{-1}.

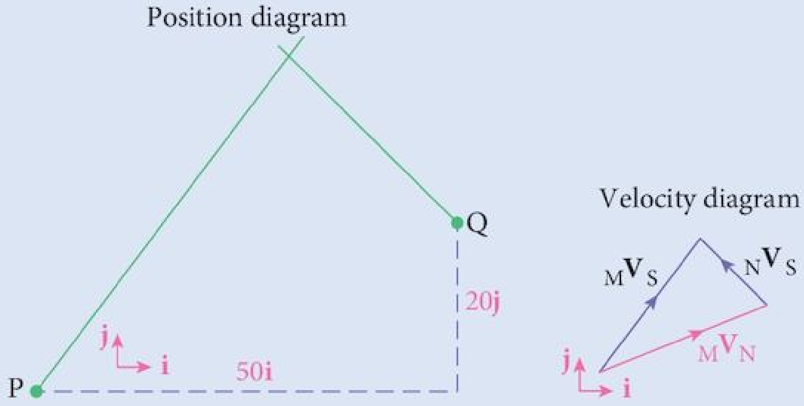
a What is the velocity of boat M relative to boat N?

b What will be the position vector of boat M relative to port P after t hours?

c What will be the position vector of boat N relative to port P after t hours?

d Show that boat M will intercept boat N and find the time taken to do so.

Solution:



M ≡ boat M
N ≡ boat N
S ≡ Sea

a The relative velocity equation is $\quad {}_A\mathbf{V}_B + {}_B\mathbf{V}_C = {}_A\mathbf{V}_C$

We choose A ≡ M; C ≡ N; leaving B ≡ S;

So $\quad {}_M\mathbf{V}_S + {}_S\mathbf{V}_N = {}_M\mathbf{V}_N$

or $\quad {}_M\mathbf{V}_S - {}_N\mathbf{V}_S = {}_M\mathbf{V}_N$

Note: ${}_B\mathbf{V}_A = -{}_A\mathbf{V}_B$
This is always true.

This is shown in the velocity diagram.

$${}_M\mathbf{V}_N = 10\mathbf{i} + 4\mathbf{j}$$

The position of Q relative to P is $\quad 50\mathbf{i} + 20\mathbf{j} = 5(10\mathbf{i} + 4\mathbf{j})$

This means that to an observer on boat N, as it moves along, boat M appears to be heading directly towards him.

b The position vector of boat M relative to port P after t hours $\quad = t(6\mathbf{i} + 8\mathbf{j})$

c The position vector of boat N relative to port P after t hours $\quad = t(-4\mathbf{i} + 4\mathbf{j}) + 50\mathbf{i} + 20\mathbf{j}$

This uses a relative position equation similar to the velocity equation
$${}_N\mathbf{r}_P = {}_N\mathbf{r}_Q + {}_Q\mathbf{r}_P$$

d The position diagram shows that the paths of the two boats will cross.

However, in order to intercept, they must be in the same place **at the same time**.

To check this, we can equate the two equations from parts **b** and **c**, which both show position relative to port P.

$$t(6\mathbf{i} + 8\mathbf{j}) = t(-4\mathbf{i} + 4\mathbf{j}) + 50\mathbf{i} + 20\mathbf{j}$$

Separating the equations:

i: $\quad 6t = -4t + 50 \quad$ [1]

j: $\quad 8t = 4t + 20 \quad$ [2]

Solving the equations: [1] $\quad t = 5$

[2] $\quad t = 5$

Two equations with one unknown variable – may not be consistent.

Thus the two boats intercept each other after 5 hours of travel.

Note that it is important that these two equations give consistent solutions.
If they were not consistent, it would mean that one boat arrived before the other.

Exercise 21.3

1 Oliveira and Raissa are swimming in a large pool. Oliveira has a velocity of $2\mathbf{i} + \mathbf{j}$ m s^{-1} relative to a corner of the pool and Raissa has a velocity of $\mathbf{i} - 3\mathbf{j}$ m s^{-1} relative to the same corner. As Oliveira swims along, how fast does Raissa appear to be travelling?

2 A ship, A, has a speed of 25 km h^{-1} on a bearing of 040°. A speedboat, B, has a speed of 40 km h^{-1} on a bearing of 310°.

To an observer on ship A, how fast does the speedboat appear to be moving and in which direction?

Remember, with bearings:
- a measured from North
- b always have 3 digits
- c clockwise is positive

3 An aircraft sets a course on a bearing of 330° and flies at a speed of 400 km h^{-1}. The wind is blowing at 100 km h^{-1} on a bearing of 060°. Find the speed and direction of travel of the aircraft as seen from an observer on the ground.

4 A canoe is launched into a river flowing at 5 m s^{-1} (assumed constant across the whole width). The canoe is steered straight across the river at a speed of 12 ms^{-1} (relative to the water). The river is 60 m wide.

a How far down stream will the canoe land and how long will it take to reach the far bank?

b In which direction must it be steered in order to land directly opposite its starting point?

c If it lands directly opposite its starting point, how long will it take to cross?

5 In order to travel due north, a ship with a speed of 15 km h^{-1} has to set a course on a bearing of 020°. The speed of the ship is measured at 20 km h^{-1} by an observer on land. Find the direction and speed of the current.

6 A passenger in a railway carriage travelling at a speed of 80 km h^{-1} notices that the rain, which is falling with a speed of 24 km h^{-1}, appears to be falling at an angle of 75° from the vertical in his direction. Find the apparent speed of the rain and the angle at which it is falling relative to the ground.

7 A swimmer can swim at a speed of 1.7 ms^{-1} in still water. She wishes to swim across a river that is flowing at 1.5 ms^{-1}. The river is 24 m wide.

a If she swims straight across the river, how far down stream will she land and how long will it take to reach the far bank?

b If she wishes to land directly opposite her starting point,

(i) in which direction must she swim

(ii) how long will it take to cross the river?

8 Two ships, P and Q set out from port O at the same time. The velocity of P is $4\mathbf{i} + 2\mathbf{j}$ km h^{-1} and the velocity of Q is $\mathbf{i} + 5\mathbf{j}$ km h^{-1}, where $\mathbf{i}$ and $\mathbf{j}$ are unit vectors due east and due north respectively.

a Find the velocity of P relative to Q.

b Find the velocity of Q relative to P.

c Find the distance between the two ships after 1, 2 and 3 hours.

9 A ferry can travel at a speed of 6 ms^{-1} in still water.
The river on which the ferry heads onto is 300 m wide and is flowing at a rate of 4 ms^{-1}.

a If it heads straight across the river,
   (i) find its velocity relative to the bank
   (ii) how long will it take to reach the far bank and how far down stream will it land?

b If it must arrive directly across the river,
   (i) find the direction in which it must steer
   (ii) find how long will it take to reach the far bank.

c If it must arrive at a point 100 m upstream from its starting point,
   (i) find the direction in which it must steer
   (ii) find how long will it take to reach the far bank.

10 An aircraft that can fly with an air speed (speed in still air) of 150 km h^{-1} wishes to fly between two towns, A and B. The position of B relative to A is $200\mathbf{i} + 150\mathbf{j}$ where $\mathbf{i}$ and $\mathbf{j}$ (km) are unit vectors measured in the easterly and northly directions respectively.
There is a wind blowing due east with a speed of 50 km h^{-1}.

For the journey from A to B,

a find the course that should be set

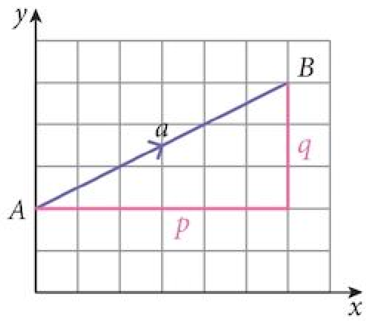
b find the time taken for the journey.

For the return journey from B to A,

c find the course that should be set

d find the time taken for the journey.

Summary



Notation $\boldsymbol{a} = \begin{pmatrix} p \\ q \end{pmatrix}$

The magnitude of a vector $|\boldsymbol{a}| = \sqrt{p^2 + q^2}$

Equality of vectors If $\mathbf{a} = \begin{pmatrix} p \\ q \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} m \\ n \end{pmatrix}$ and $\mathbf{a} = \mathbf{b}$, then $p = m$ and $q = n$.

Scalar multiplication The vector $k\mathbf{a}$ has the same direction as $\mathbf{a}$ and k times the magnitude of $\mathbf{a}$.

The zero vector Has magnitude 0 and can take any direction.

The vector -a Has the same magnitude as **a** but the opposite direction.

Unit vectors Have magnitude 1.

To create a unit vector in the direction of **a**, divide **a** by its own magnitude.

Base vectors $\mathbf{i} = \begin{pmatrix}1\\0\end{pmatrix}$ and $\mathbf{j} = \begin{pmatrix}0\\1\end{pmatrix}$

are unit vectors in the directions of the standard axes.

Position vectors We can specify the position of the point A by defining the vector $\overrightarrow{OA} = \mathbf{a}$.

Vector geometry $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$

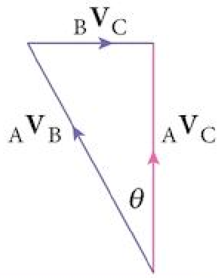
If M is the mid-point of AB then

$$\mathbf{m} = \frac{1}{2}(\mathbf{a} + \mathbf{b}).$$

Components of a vector $\mathbf{v} = \begin{pmatrix}\text{v} \cos\theta\\ \text{v} \sin\theta\end{pmatrix} = \text{v} \cos\theta\,\mathbf{i} + \text{v} \sin\theta\,\mathbf{j}$

Adding velocities Velocities are vectors and can added in the same way.

Relative velocity ${}_A\mathbf{V}_B + {}_B\mathbf{V}_C = {}_A\mathbf{V}_C$



Chapter 21 Summative Exercise

$\mathbf{r} = \begin{pmatrix}2\\1\end{pmatrix}$ $\quad\mathbf{s} = \begin{pmatrix}5\\3\end{pmatrix}$ $\quad\mathbf{t} = \begin{pmatrix}-1\\2\end{pmatrix}$ $\quad\mathbf{u} = \begin{pmatrix}-2\\3\end{pmatrix}$ $\quad\mathbf{v} = \begin{pmatrix}-3\\-2\end{pmatrix}$ $\quad\mathbf{w} = \begin{pmatrix}-4\\-3\end{pmatrix}$

1 Find the following:

a $|3\mathbf{r} - 2\mathbf{s}|$ **b** $|2\mathbf{t} - 4\mathbf{u}|$ **c** $|32\mathbf{v} - 18\mathbf{w}|$ **d** $|9\mathbf{r} + 2\mathbf{v}|$

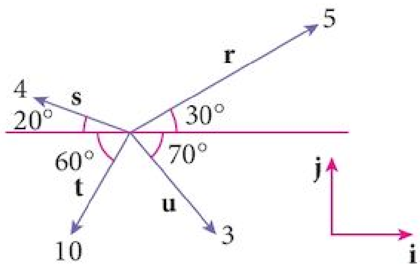
e $|27\mathbf{s} + 37\mathbf{v}|$ **f** $|13\mathbf{s} + 5\mathbf{w}|$ **g** $|6\mathbf{u} + 4\mathbf{v}|$ **h** $|9\mathbf{t} + 29\mathbf{v}|$

2 Find a unit vector in the direction of the following, expressing your results in the format $a\mathbf{i} + b\mathbf{j}$, where the vectors $\mathbf{i}$ and $\mathbf{j}$ are base vectors:

a $\mathbf{r}$ **b** $\mathbf{s}$ **c** $\mathbf{v}$ **d** $\mathbf{w}$

3 In the diagram, the vectors $\mathbf{r}$, $\mathbf{s}$, $\mathbf{t}$ and $\mathbf{u}$ have magnitudes 5, 4, 10 and 3 respectively, and the directions of these vectors are as shown.

Write each vector in component form $a\mathbf{i} + b\mathbf{j}$, where $\mathbf{i}$ and $\mathbf{j}$ are the base vectors shown.

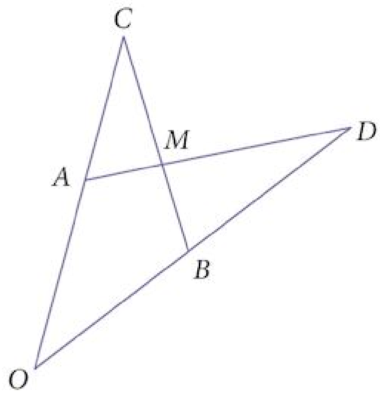


4 The points P and Q have position vectors $\mathbf{p}$ and $\mathbf{q}$ relative to an origin O.

The vector $\mathbf{p}$ has magnitude 26 units and acts in the direction $-5\mathbf{i} + 12\mathbf{j}$.
The vector $\mathbf{q}$ has magnitude 20 units and acts in the direction $3\mathbf{i} + 4\mathbf{j}$.

Write down the vectors $\overrightarrow{OP}$ and $\overrightarrow{OQ}$ and find the magnitude of $\overrightarrow{PQ}$ correct to 3 s.f.

5 In the diagram, the position vectors of the points A and B relative to the origin, O, are $\mathbf{a}$ and $\mathbf{b}$ respectively.



The point C is such that $\overrightarrow{AC} = \frac{1}{3}\mathbf{a}$ and the point D is such that $\overrightarrow{BD} = \lambda\mathbf{b}$. The mid-point of BC is M.

a Find the position vector of M.

b Write $\overrightarrow{AM}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

c Write $\overrightarrow{MD}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.

d Find the value of λ such that AMD is a straight line.

6 The point P has position vector $3\mathbf{i} + 4\mathbf{j}$. The point Q has position vector $x\mathbf{i} + 2x\mathbf{j}$.

The vector $\overrightarrow{PQ}$ is a unit vector. Find the possible values of x.

7 A cyclist is travelling on a road that heads directly northward. He is travelling at a speed of 20 km hr^{-1}.

He notices that the wind appears to be heading from the direction 070° with a speed of 10 km hr^{-1}.

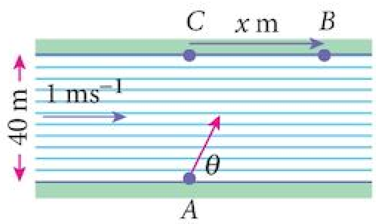
Find the true speed and direction of the wind relative to the ground.

8 A plane is travelling at a speed of 320 km hr^{-1} from airport A to airport B. Airport B is 140 km directly east of airport A. The wind is blowing from the direction 060° with a speed of 120 km hr^{-1}.

a Find the course set by the pilot and the time taken to travel from A to B.

b Find the course set by the pilot on the return journey from B to A and the time taken for this leg of the journey.

9 A swimmer can swim at a speed of 0.6 ms^{-1} in still water.



He leaves point A on one bank of a river to swim to point B on the opposite bank, as shown in the diagram.

The river is 40 m wide and is flowing at a speed of 1 ms^{-1}.

The swimmer sets a course making an angle θ with the river bank and takes 90 s to cross the river from A to B.

Find the size of the angle θ and the distance, x m.

10 A ship, A, travelling on a course of 045° at a speed of 20 km hr^{-1} notices a second ship, B, 2 km east of A that appeared to be travelling due north at a speed of 3 km hr^{-1}.

a Find the true velocity and direction of ship B.

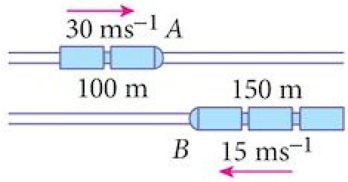
b Find, also, the distance between the two ships 30 minutes later.

11 A ship, A, is travelling at 16 km hr^{-1} on a course of 300°. A second ship, B, is 30 km due west of A and is travelling at 20 km hr^{-1}. Given that ship B intercepts ship A, find the direction that it is travelling and the time taken to intercept A.

12 Two trains, A and B, are travelling toward each other on parallel tracks.

Train A has a speed of 30 ms^{-1} and is 100 m in length.
Train B has a speed of 15 ms^{-1} and is 150 m in length.

Find the velocity of train A relative to train B and hence find the time taken for the trains to pass each other completely.



Chapter 21 Test

1 hour

1 The position vectors of the points P and Q are given by $\mathbf{p} = \begin{pmatrix} 7 \\ -3 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} 2 \\ 9 \end{pmatrix}$.

a Find a unit vector parallel to $\overrightarrow{PQ}$. [3]

b Find the position vector of the point R such that $\overrightarrow{PR} = 2\overrightarrow{PQ}$. [3]

2 A plane is flying on a course of 060° in still air at a speed of 300 km hr^{-1}, when it passes a control tower, A.

a Using $\mathbf{i}$ and $\mathbf{j}$ as the unit vectors due east and due north respectively, express the position vector of the plane relative to the control tower in terms of t, the time in hours since passing A. [2]

The next control at which the plane must report, B, has a position $300\sqrt{3}\mathbf{i} + 150\mathbf{j}$ relative to A.

b Show that, after 90 minutes of travel, the plane is 150 km away from B. [2]

c Find the length of time that the plane is less than 150 km from B. [2]

3 The position vectors of the points A and B relative to an origin are $\mathbf{a}$ and $\mathbf{b}$ respectively. The point C is such that $2\mathbf{c} = \mathbf{a}$. The point D is such that $\mathbf{d} = 2\mathbf{b}$. The lines AB and CD meet at the point R.

If $\overrightarrow{CR} = \lambda\overrightarrow{CD}$ and $\overrightarrow{AR} = \mu\overrightarrow{AB}$:

a Express $\boldsymbol{OR}$ in terms of $\mathbf{a}$, $\mathbf{b}$ and λ. [2]

b Express $\boldsymbol{OR}$ in terms of $\mathbf{a}$, $\mathbf{b}$ and μ. [2]

c Hence find the values of λ and μ. [3]

4 Find the values of p for which the vector $\begin{pmatrix} 2p-4 \\ 2p+3 \end{pmatrix}$ has magnitude 17. [4]

5 In this question, $\mathbf{i}$ is a unit vector due east and $\mathbf{j}$ is a unit vector due north.

At 12:00, a ship leaves a port P and travels at a speed of 15 km h^{-1} in a direction $3\mathbf{i} - 4\mathbf{j}$.

a Write the velocity of the ship in the format $a\mathbf{i} + b\mathbf{j}$. [2]

b Write down the position vector of the ship, relative to P, at 14:00. [1]

c Write down the position vector of the ship, relative to P, t hours after 14:00. [1]

At 14:00, a patrol boat leaves a port, Q, which is 63 km due east of P. It intercepts the ship 3 hours later.

d Find the position vector of the point of interception. [1]

e Find the velocity of the patrol boat in the format $a\mathbf{i} + b\mathbf{j}$. [2]

6 The position vectors of the points P and Q are given by $\mathbf{p} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} k \\ 2.5 \end{pmatrix}$.

If $\overrightarrow{PQ}$ is a unit vector, find the possible values of k. [4]

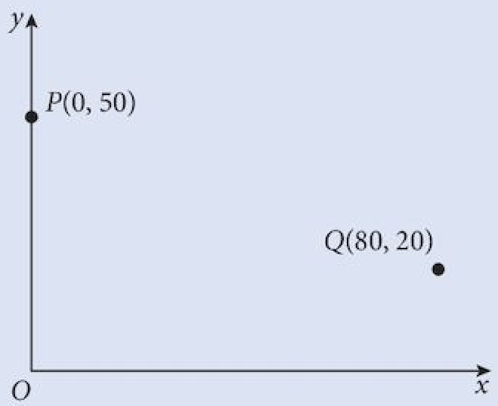
7 The position vectors of two points, A and B, relative to an origin, O, are $\mathbf{a} = \mathbf{i} + 9\mathbf{j}$ and $\mathbf{b} = 10\mathbf{i} + 6\mathbf{j}$. The point C lies on AB and is such that $\overrightarrow{AB} = 3\overrightarrow{AC}$.

a Find the position vector, $\mathbf{c}$ of point C. [4]

b Find a unit vector parallel to $\overrightarrow{OC}$. Leave surds in your answer. [2]

Examination Questions

1



At 1200 hours, ship P is at the point with position vector $50\mathbf{j}$ km and ship Q is at the point with position vector $(80\mathbf{i} + 20\mathbf{j})$ km, as shown in the diagram. Ship P is travelling with velocity $(20\mathbf{i} + 10\mathbf{j})$ km h^{-1} and ship Q is travelling with velocity $(-10\mathbf{i} + 30\mathbf{j})$ km h^{-1}.

(i) Find an expression for the position vector of P and of Q at time t hours after 1200 hours. [3]

(ii) Use your answers to part **(i)** to determine the distance apart of P and Q at 1400 hours. [3]

(iii) Determine, with full working, whether or not P and Q will meet. [2]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2002, P1, Qu 10]

2 The position vectors of the points A and B, relative to an origin O, are $\mathbf{i} - 7\mathbf{j}$ and $4\mathbf{i} + k\mathbf{j}$ respectively, where k is a scalar. The unit vector in the direction of $\overrightarrow{AB}$ is $0.6\mathbf{i} + 0.8\mathbf{j}$. Find the value of k. [4]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2004, P2, Qu 1]

3 An ocean liner is travelling at 36 km h^{-1} on a bearing of 090°. At 0600 hours the liner, which is 90 km from a lifeboat and on a bearing of 315° from the lifeboat, sends a message for assistance. The lifeboat sets off immediately and travels in a straight line at constant speed, intercepting the liner at 0730 hours. Find the speed at which the lifeboat travels. [5]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2003, P1, Qu 4]

4 The position vectors of points A and B, relative to an origin O, are $6\mathbf{i} - 3\mathbf{j}$ and $15\mathbf{i} + 9\mathbf{j}$ respectively.

(i) Find the unit vector parallel to $\overrightarrow{AB}$. [3]

The point C lies on AB such that $\overrightarrow{AC} = 2\overrightarrow{CB}$.

(ii) Find the position vector of C. [3]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2003, P2, Qu 2]

5 To a cyclist travelling due south on a straight horizontal road at 7 ms^{-1}, the wind appears to be blowing from the north-east. Given that the wind has a constant speed of 12 ms^{-1}, find the direction from which the wind is blowing. [5]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2004, P1, Qu 7]

6 In this question, $\mathbf{i}$ is a unit vector due east and $\mathbf{j}$ is a unit vector due north.

A plane flies from P to Q. The velocity, in still air, of the plane is $(280\mathbf{i} - 40\mathbf{j})$ km h^{-1} and there is a constant wind blowing with velocity $(50\mathbf{i} - 70\mathbf{j})$ km h^{-1}. Find

(i) the bearing of Q from P, [4]

(ii) the time of flight, to the nearest minute, given that the distance PQ is 273 km. [2]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2003, P1, Qu 6]

7 A motor boat travels in a straight line across a river which flows at 3 ms^{-1} between straight parallel banks 200 m apart. The motor boat, which has a top speed of 6 ms^{-1} in still water, travels directly from a point A on one bank to a point B, 150 m downstream of A, on the opposite bank. Assuming that the motor boat is travelling at top speed, find, to the nearest second, the time it takes to travel from A to B. [7]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2004, P2, Qu 8]

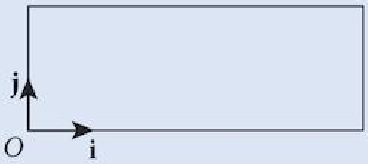
8 The position vectors of points A and B relative to an origin O are $-3\mathbf{i} - \mathbf{j}$ and $\mathbf{i} + 2\mathbf{j}$ respectively. The point C lies on AB and is such that $\overrightarrow{AC} = \frac{3}{5}\overrightarrow{AB}$. Find the position vector of C and show that it is a unit vector. [6]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2005, P1, Qu 4]

9 A plane, whose speed in still air is 300 km h^{-1}, flies directly from X to Y. Given that Y is 720 km from X on a bearing of 150° and that there is a constant wind of 120 km h^{-1} blowing towards the west, find the time taken for the flight. [7]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2005, P2, Qu 9]

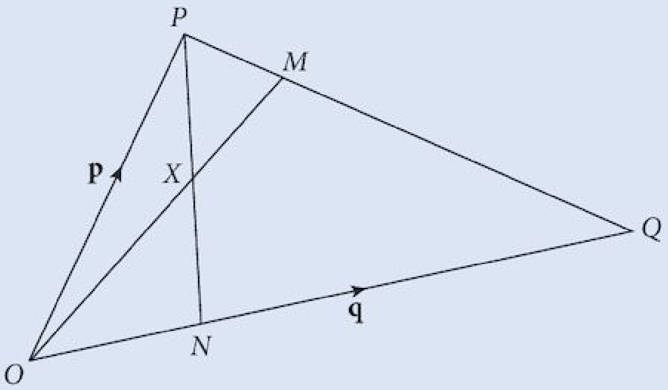
10 The diagram, which is not drawn to scale, shows a horizontal rectangular surface. One corner of the surface is taken as the origin O and $\mathbf{i}$ and $\mathbf{j}$ are unit vectors along the edges of the surface.



A fly, F, starts at the point with position vector $(\mathbf{i} + 12\mathbf{j})$ cm and crawls across the surface with a velocity of $(3\mathbf{i} + 2\mathbf{j})$ cm s^{-1}. At the instant the fly starts crawling, a spider, S, at the point with position vector $(85\mathbf{i} + 5\mathbf{j})$ cm, sets off across the surface with a velocity of $(-5\mathbf{i} + k\mathbf{j})$ cm s^{-1}, where k is a constant. Given that the spider catches the fly, calculate the value of k. [6]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2005, P1, Qu 5]

11



In the diagram, $\overrightarrow{OP} = \mathbf{p}$, $\overrightarrow{OQ} = \mathbf{q}$, $\overrightarrow{PM} = \frac{1}{3}\overrightarrow{PQ}$ and $\overrightarrow{ON} = \frac{2}{5}\overrightarrow{OQ}$

(i) Given that $\overrightarrow{OX} = m\overrightarrow{OM}$, express $\overrightarrow{OX}$ in terms of m, $\mathbf{p}$ and $\mathbf{q}$. [2]

(ii) Given that $\overrightarrow{PX} = n\overrightarrow{PN}$, express $\overrightarrow{OX}$ in terms of n, $\mathbf{p}$ and $\mathbf{q}$. [3]

(iii) Hence evaluate m amd n. [2]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2005, P2, Qu 7]

12 The position vectors, relative to an origin O, of three points P, Q and R are $\mathbf{i} + 3\mathbf{j}$, $5\mathbf{i} + 11\mathbf{j}$ and $9\mathbf{i} + 9\mathbf{j}$ respectively.

(i) By finding the magnitude of the vectors $\overrightarrow{PR}$, $\overrightarrow{RQ}$ and $\overrightarrow{QP}$, show that angle PQR is 90°. [4]

(ii) Find the unit vector parallel to $\overrightarrow{PR}$. [2]

(iii) Given that $\overrightarrow{OQ} = m\overrightarrow{OP} + n\overrightarrow{OR}$, where m and n are constants, find the value of m and of n. [3]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2007, P2, Qu 9]

13 The position vectors of points A and B, relative to an origin O, are $2\mathbf{i} + 4\mathbf{j}$ and $6\mathbf{i} + 10\mathbf{j}$ respectively. The position vector of C, relative to O, is $k\mathbf{i} + 25\mathbf{j}$, where k is a positive constant.

(i) Find the value of k for which the length of BC is 25 units. [3]

(ii) Find the value of k for which ABC is a straight line. [3]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2007, P2, Qu 7]

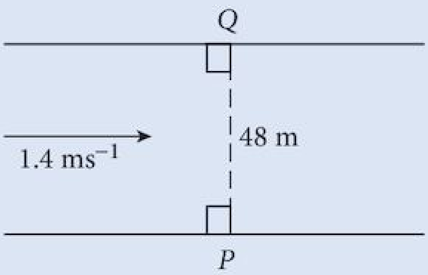
14 Given that $\mathbf{a} = 5\mathbf{i} - 12\mathbf{j}$ and that $\mathbf{b} = p\mathbf{i} + \mathbf{j}$, find

(i) the unit vctor in the direction of $\mathbf{a}$, [2]

(ii) the values of the constants p and q such that $q\mathbf{a} + \mathbf{b} = 19\mathbf{i} - 23\mathbf{j}$. [3]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2008, P1, Qu 5]

15



The diagram shows a river with parallel banks. The river is 48 m wide and is flowing with a speed of 1.4 m s^{-1}. A boat travels in a straight line from a point P on one bank to a point Q which is on the other bank directly opposite P. Given that the boat takes 10 seconds to cross the river, find

(i) the speed of the boat in still water, [4]

(ii) the angle to the bank at which the boat should be steered. [2]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2008, P2, Qu 7]

16 At 1000 hours, a ship P leaves a point A with position vector $(-4\mathbf{i} + 8\mathbf{j})$ km relative to an origin O, where $\mathbf{i}$ is a unit vector due East and $\mathbf{j}$ is a unit vector due North. The ship sails north-east with a speed of $10\sqrt{2}$ km h^{-1}. Find

(i) the velocity vector of P, [2]

(ii) the position vector of P at 1200 hours. [2]

At 1200 hours, a second ship Q leaves a point B with position vector $(19\mathbf{i} + 34\mathbf{j})$ km travelling with velocity vector $(8\mathbf{i} + 6\mathbf{j})$ km h^{-1}.

(iii) Find the velocity of P relative to Q. [2]

(iv) Hence, or otherwise, find the time at which P and Q meet and the position vector of the point where this happens. [3]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2009, P1, Qu 9]

22 Kinematics

Syllabus statements

- apply differentiation and integration to kinematics problems that involve displacement, velocity and acceleration of a particle moving in a straight line with variable or constant acceleration, and the use of x-t and v-t graphs

22.1 What is kinematics?

Kinematics is the branch of mathematics concerned with the mechanics of motion.

Thus, we are interested in displacement, velocity and acceleration.

These quantities are vectors, and many of the results we obtain are vector equations which can be extended to multiple dimensions.

However, we will be interested in straight line (one-dimensional) motion only and we usually do not need vectors for this.

What we must have is an origin to measure the quantities relative to, and a positive (and negative) direction so that we can specify where objects are and which way they are going.

22.2 Rates of change

You already know that velocity is the rate of change of displacement. Similarly, acceleration is the rate of change of velocity.

You also know how to find rates of change by differentiation.

We could investigate the rate of change of acceleration but we have no reason to do so.

Hence, we have a link between kinematics and calculus.

We can derive the formula for any of the three quantities from any other one by either differentiating or integrating.

Differentiate Differentiate

Starting from	Displacement x	Velocity v	Acceleration a
x	x	$\dfrac{dx}{dt}$	$\dfrac{d^2x}{dt^2}$
v	$\int v\,dt$	v	$\dfrac{dv}{dt}$
a	Integrate a twice	$\int a\,dt$	a

Integrate Integrate

22.3 Mathematical modelling

Many things that we meet in everyday life turn out to be very complicated.

As a result, developing a mathematical formula to tell us what is going to happen is impossible.

Because of this, we simplify many situations until we can find an equation covering the situation.

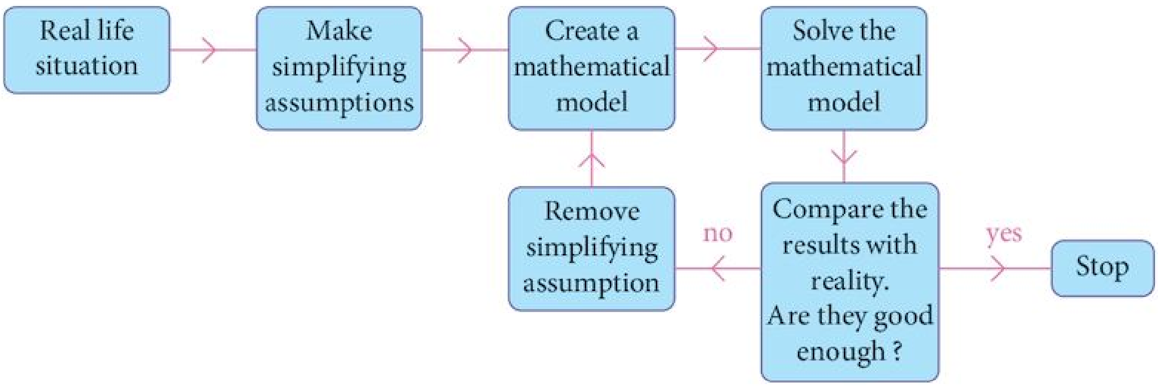
This equation is the **mathematical model**.

We then solve the equation and see how good the results are.

If the results are no good, we try to remove some of our simplifications to get a more complicated model that we can still solve.

This process continues until we cannot solve the latest equations, or until the results we get are good enough for the purpose for which we want them.

If the results are no good, we try to find another model.



You will meet many of these situations during your study of mathematics.

For our current purposes, two major simplifications are made:

1 All objects are particles.

 This means that we do not need to worry about them turning during their motion. Even a train would be considered to be a particle.

2 There is no air resistance.

 The particle model helps here.

 Particles do not have size, and so there is nothing for the air to resist.

With these simplifying assumptions, we can solve many problems.

Example 22.1

A particle travelling along a straight wire has a velocity of 4 $\mathrm{m\,s^{-1}}$ as it passes a point O on the wire.

Time is measured in seconds from this point so that $t = 0$ when the particle is at O.

The acceleration of the particle is given by $a = 6t$.

a Find the velocity of the particle when $t = 2, 3, 4$ seconds.

b Find the position of the particle when $t = 2, 3, 4$ seconds.

Solution:

a $v = \int a\,\mathrm{d}t \quad = \int 6t\,\mathrm{d}t$

$= 3t^2 + c$ where c is a constant.

When $t = 0$, $v = 4$

giving $c = 4$

Thus $v = 3t^2 + 4$

When $t = 2$, $v = 16\ \mathrm{ms^{-1}}$

When $t = 3$, $v = 31\ \mathrm{ms^{-1}}$

When $t = 4$, $v = 52\ \mathrm{ms^{-1}}$

b $x = \int v\,\mathrm{d}t \quad = \int (3t^2 + 4)\,\mathrm{d}t$

$= t^3 + 4t + k$ where k is a constant.

When $t = 0$, $x = 0$

giving $k = 0$

Thus $x = t^3 + 4t$

When $t = 2$, $x = 16$ m from O.

When $t = 3$, $x = 39$ m from O.

When $t = 4$, $x = 80$ m from O.

Example 22.2

A car is travelling along a straight road between two sets of traffic lights.

It is stationary at the first set of lights when they change to green, and it comes to rest when it reaches the second set of lights.

The velocity of the car is modelled as $v = \frac{1}{5}(10t - t^2)\,\mathrm{m\,s^{-1}}$.

a Find the time taken to reach the second set of traffic lights.

b Find the distance between the traffic lights.

c Find the maximum velocity of the car.

Solution:

a

$$v = \frac{1}{5}(10t - t^2)$$

$$= \frac{1}{5}t(10 - t)$$

The car is stationary when $v = 0$

i.e. $\quad t = 0 \quad$ and $\quad t = 10$

Thus it takes 10 seconds to reach the second set of traffic lights.

b

$$x = \int v\,\mathrm{d}t$$

$$= \int \frac{1}{5}(10t - t^2)$$

$$= \frac{1}{5}\left(5t^2 - \frac{1}{3}t^3\right) + k \quad \text{where } k \text{ is a constant.}$$

When $t = 0$, $x = 0$, so $k = 0$

giving $\quad x = \frac{1}{5}\left(5t^2 - \frac{1}{3}t^3\right)$

When $t = 10$ $\quad x = \frac{1}{5}\left(500 - \frac{1000}{3}\right)$

$$= 33.3\,\mathrm{m}$$

c The maximum velocity will be when $a = 0$

$$a = \frac{\mathrm{d}v}{\mathrm{d}t}$$

$$= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{5}(10t - t^2)\right)$$

$$= \frac{1}{5}(10 - 2t)$$

When $a = 0$ $\quad t = 5$

and $\quad v = 5\,\mathrm{m\,s^{-1}}$

Example 22.3

A particle is travelling along a straight line such that its distance from a fixed point is modelled by the equation $x = 4 \sin 3t$.

a Find an expression for v, the velocity of the particle, and a, the acceleration of the particle at time t.

b Find the velocity of the particle when it reaches O again.

c Find the maximum distance the particle travels from O.

Solution:

a $x = 4 \sin 3t$

so $v = \dfrac{dx}{dt}$ $\quad v = 12 \cos 3t$

and $a = \dfrac{dv}{dt}$ $\quad a = -36 \sin 3t$

b When $x = 0$, $\quad 3t = 0, \pi, 2\pi, \ldots$

So the particle will next pass O when $t = \dfrac{\pi}{3}$

and the velocity will be $\quad v = 12\cos\pi$

$= -12\,\mathrm{m\,s^{-1}}$

c When $v = 0$, $\quad 3t = \dfrac{\pi}{2}, 3\dfrac{\pi}{2}, \ldots$

and $\quad x = 4$ or -4 m.

Exercise 22.1

1 A particle travelling in a straight line starts from rest at point O and has an acceleration (in $\mathrm{m\,s^{-2}}$) modelled by the equation $a = 24 - 12t$, where t is measured in seconds.

- **a** Find an expression for the velocity of the particle at time t.
- **b** Find an expression for the position of the particle at time t.
- **c** Find its furthest distance from O in a positive direction during the subsequent motion.
- **d** Find the time taken for the particle to return to O, and its velocity at that time.
- **e** Describe the motion of the particle after it passes through O for the second time.

2 A particle moving in a straight line starts from a point A, at a distance p from the origin O. Its displacement from O (in m) is modelled by the equation $x = t^3 - 7t^2 + 14t - 8$, where t is measured in seconds.

- **a** Find the distance OA.
- **b** Find the times when the particle is at O.

c Find an expression for the velocity of the particle at time t.

d Find the velocity of the particle each time it passes through O.

e Find the maximum distance the particle is from O on each stage of its journey.

3 A particle, P, is moving in a straight line. Its dispacement, x m, from a fixed point O at time t is given by $x = t(t^2 - 9)$.

a Find the times when the particle is at O.

b Find an expression for the velocity of the particle.

c Find when the particle is at rest.

d Find where the particle is at rest.

e Find an expression for the acceleration of the particle.

f Find the acceleration of the particle when it is at rest.

4 A bird leaves its nest to find food to bring back for its chicks. It flies in a straight line to a source of food and its path is modelled by the equation $x = 20t - t^2$ where x m is the distance from the nest at time t s.

The bird collects the food and returns to its nest where the chicks are waiting open mouthed.

a Find an expression for the velocity of the bird.

b Find how far away from the nest the food source is.

c Find the length of time of the bird's journey.

d What is the main problem with this model?

5 A particle moves in a straight line with a displacement, x m, measured from a fixed point, O, modelled by the equation $x = 5 \sin 2t$.

a Find an expression for the velocity of the particle.

b Find an expression for the acceleration of the particle.

c Find the maximum velocity of the particle and the position of the particle when it attains this maximum.

d Find the maximum displacement of the particle from O and the period of the motion.

e Find an expression for the acceleration of the particle in terms of x.

f What does the result of part **e** tell you about the acceleration of the particle?

6 A particle moves in a straight line with a velocity, v m s^{-1}, modelled by the equation $v = \sin t + t \cos t$, where t s is measured from when the particle passes a fixed point, O.

a Differentiate $x = t \sin t$ with respect to t.

b Hence find an expression for the displacement, x, at time t.

c Find an expression for the acceleration, a, at time t.

d Find the period of the motion of the particle.

e What happens to the amplitude of the motion as time progresses?

22.4 Motion with uniform acceleration

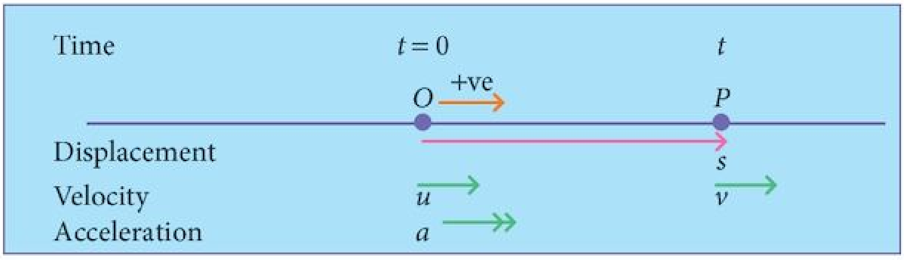
Motion with uniform (constant) acceleration is a special case of the types of motion we have been looking at, where, in general, the acceleration has not been constant.

Using the ideas from the previous section, we can develop a series of formulae that apply **only** in this special case.

Before we start, we need to establish the notation that we are going to use.

A particle will move in a straight line starting at a point O when $t = 0$.

> It is important that you know whether or not the acceleration is constant.
>
> One of the biggest mistakes is to assume that it is constant when really it isn't.



At a time t seconds later it will be at a point P, where the displacement, $OP = s$ m.

The initial velocity (at O) will be u m s^{-1} (fixed) and the velocity at P will be v m s^{-1} (variable).

At all times, the acceleration will be a m s^{-2} (uniform).

The positive direction will be to the right.

Problem 22.1

1 By integrating $\frac{dv}{dt} = a$ with respect to t, find the general solution for v in terms of t.

> Remember: a is a constant

2 Substitute the intitial values $t = 0$, $v = u$ to find the value of the constant of integration and the particular solution for v in terms of t. Call this equation [1].

3 By integrating $\frac{ds}{dt} = v$ with respect to t, find the general solution for s in terms of t.

> Using your particular solution for v from **step 2**.

4 Substitute the intitial values $t = 0$, $s = 0$ to find the value of the constant of integration and the particular solution for s in terms of t. Call this equation [2].

5 You now have two equations that are applicable to this special case.

  a Make a list of the quantities involved in the problem. How many are there?

  b Which quantities are involved in each of your equations [1] and [2]?
  In each case, which quantity is missing from the equation?

6 Make a the subject of equation [1].

  Substitute this expression into equation [2] and simplify your result, making s the subject.

  Call this equation [3].

  Which quantities are involved in equation [3] and which quantity is missing from the equation?

7 Make t the subject of equation [1] and use this expression to eliminate t from equation [2].

  Simplify your result, making v^2 the subject of the equation.

  Call this equation [4].

8 Make u the subject of equation [1] and use this expression to eliminate u from equation [2].

  Simplify your result, making s the subject of the equation.

  Call this equation [5].

You now have a set of five equations in the five quantities s, u, v, a and t.

Each equation has one of the quantities missing.

However, remember that there are only two independent equations: any pair would do.

The other three equations can be obtained by substituting to eliminate whichever quantity you do not want.

$$v = u + at \qquad [1]$$

$$s = ut + \frac{1}{2}at^2 \qquad [2]$$

$$s = \frac{(u+v)}{2}t \qquad [3]$$

$$v^2 = u^2 + 2as \qquad [4]$$

$$s = vt - \frac{1}{2}at^2 \qquad [5]$$

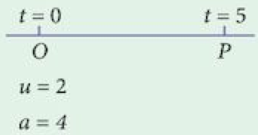
In terms of usefulness, [1], [2] and [4] are the most useful (probably in that order). Form [3] is not used much and [5] is hardly ever used.

Reminder: These equations can be used **only** if the acceleration is **constant**.

Example 22.4

A particle is moving in a straight line, starting from a point O with a velocity of $2\ \text{m s}^{-1}$. It has a uniform acceleration of $4\ \text{m s}^{-2}$.

After 5 seconds, it passes a point P. Find:



a the distance OP

b the velocity with which it passes the point P.

Solution:

a We know the values of u, a and t.

We need equation [2] $\quad s = ut + \frac{1}{2}at^2$

$$s = 2\times5+\frac{1}{2}\times4\times5^2$$
$$= 60\,\text{m}$$

b We need equation [1] $\quad v = u + at$

$$v = 2+4\times5$$
$$= 22\,\text{m s}^{-1}$$

Example 22.5

A ball is dropped from rest from the top of a tower, 45 m high.

Ignoring air resistance, it has a uniform acceleration of $10\ \text{m s}^{-2}$.

Find:

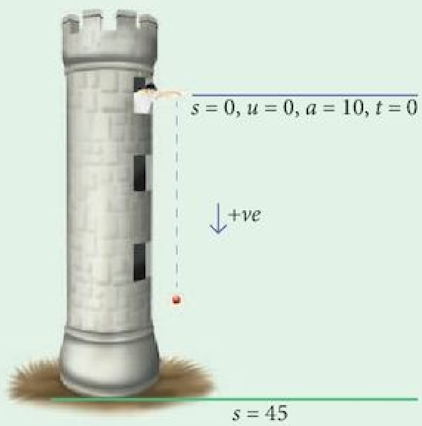
a the time taken for it reach the ground

b the velocity with which it hits the ground.

$10\ \text{ms}^{-2}$ is usually accepted as the acceleration due to gravity in Mathematics.

Solution:

For this question, we will take the place where the ball is released as the origin. We will also define downwards to be the positive direction.



a We know the values of u, a and s.

We need equation [2] $\quad s = ut + \frac{1}{2}at^2$

$$45 = 0+\frac{1}{2}\times10\times t^2$$
$$t = 3\,\text{s}$$

b We need equation [1] $\quad v = u + at$

$$v = 0+10\times3$$
$$= 30\,ms^{-1}$$

Example 22.6

A ball is thrown vertically upwards with a velocity of 10 ms^{-1} from the top of a tower, 40 m high. Ignoring air resistance, find:

a the time taken for the ball to reach the ground

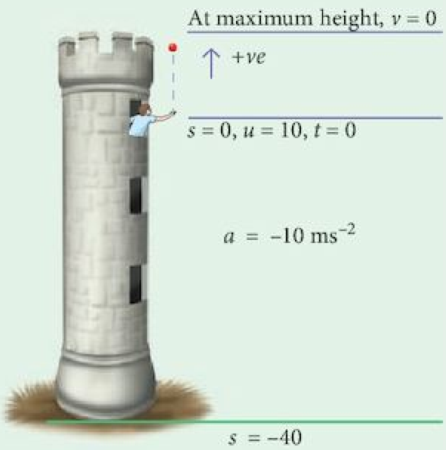
b the velocity with which it hits the ground

c the time taken for it to reach maximum height

d the maximum height reached by the ball

e the velocity with which it passes the person who threw the ball on its journey downwards.

Solution:



For this question, we will take the place where the ball is thrown upwards as the origin.

We will also define upwards to be the positive direction.

This means that the uniform acceleration will be -10 ms^{-2}.

The position of the ground will also be a negative quantity.

a We know the values of u, a and s.

We need equation **[2]**

$$s = ut + \frac{1}{2}at^2$$

$$-40 = 10t + \frac{1}{2} \times (-10)t^2$$

$$5t^2 - 10t - 40 = 0$$

$$t^2 - 2t - 8 = 0$$

$$(t+2)(t-4) = 0$$

$$t = 4s$$

The solution $t = -2$ is not valid. It does not satisfy the conditions of the problem.

b We need equation **[1]**

$$v = u + at$$

$$v = 10 - 10 \times 4$$

$$v = -30\,\text{ms}^{-1}$$

$-$ve is downwards

c When the ball is at maximum height, its velocity is zero.

We need equation **[1]**

$$v = u + at$$

$$0 = 10 - 10t$$

$$t = 1\,\text{s}$$

d We need equation **[2]**

$$s = u + \frac{1}{2}at^2$$

$$s = 10 \times 1 + \frac{1}{2} \times (-10) \times 1^2$$

$$= 5\ \text{m}$$

e We need equation **[4]**

$s = 0$

$$v^2 = u^2 + 2as$$

$$v^2 = 10^2 + 0$$

$$v = 10 \quad \text{or} \quad -10$$

$v = 10$ is the speed, u with which it was thrown.

On the way down it will have a speed of -10 m s^{-1}.

Exercise 22.2

1 A train leaves a station and travels along a straight track with a uniform acceleration of 2 ms^{-2}.

  a What is its velocity 40 seconds later?

  b How far has it travelled in this time?

2 A stone is dropped from the top of a tower. It takes 6 seconds to reach the ground.

  a How high is the tower?

  b What is its velocity when it reaches the ground?

3 A train leaves a station and travels along a straight track.

  After 1 minute it has a velocity of 18 ms^{-1}.

  a Find the acceleration of the train.

  b Find the distance it has travelled.

4 A car passes a speed check at 90 km h^{-1}. Seeing a police roadblock ahead, the driver slows down with uniform acceleration to stop after 100 m.

  a Find the acceleration of the car during this period.

  b Find the time taken to stop.

Negative acceleration is called deceleration.

5 A stone is thrown vertically upwards from a bridge over a river with a speed of 10 ms^{-1}.

  It hits the water with a speed of 40 ms^{-1}.

  Find:

  a the height of the bridge above the river

  b the time taken for the stone to hit the water

  c the total distance the stone has travelled on its journey.

6 A train leaves a station and travels with uniform acceleration. It attains its maximum speed of 15 ms^{-1} after 3.75 km. It then slows down with a uniform acceleration of -0.125 ms^{-2} to stop at the next station. Find the distance between the stations and the total time of the journey.

7 A car, A, starts from rest and drives with a uniform acceleration of 2 ms^{-2}.

  After 7.5 seconds, a second car, B, sets off from the same point in pursuit of car A.

  Car B has a uniform acceleration of 8 ms^{-2}.

  Find:

  a how long it take car B to catch up with car A

  b the distance from the starting point at which car B catches up with car A

  c the speed of the two cars when car B catches up with car A.

8 A lift descends from rest at the top floor of a skyscraper with a uniform acceleration of 0.4 ms^{-2}. After 4 seconds it slows with a uniform acceleration of -0.6 ms^{-2} to stop at another floor.

Find:

a the maximum speed of the lift

b the total time taken for the journey

c the distance between the two floors.

9 The safe speed for a car to cross a speed hump is 4 $m\,s^{-1}$. A car crosses one speed hump and then accelerates at 2 $m\,s^{-2}$ for 5 seconds before slowing down to cross the next speed hump.

The total distance between the speed humps is 105 m.

Find:

a the maximum speed of the car

b the (negative) acceleration of the car as it slowed down

c the total time taken for the journey.

10 Two boys are playing with a ball. One boy is standing on the ground and the other is leaning out of a window at a height of 4 m.

The boy on the ground throws the ball up so that the boy leaning out of the window can catch it.

He throws it with a velocity of 12 ms^{-1}. Unfortunately, the second boy misses the ball on the way up but manages to catch it on the way down.

Find:

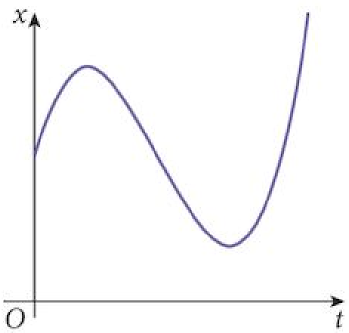
a the speed of the ball when it is caught

b the total time of flight of the ball

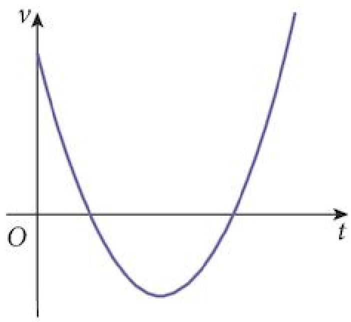
c the maximum height that the ball reaches.

22.5 Displacement-time and velocity-time graphs

Using our knowledge of the applications of calculus, we can derive results from the graphs of kinematics. The most common graphs are the displacement-time and velocity-time graphs.



displacement-time
x-t

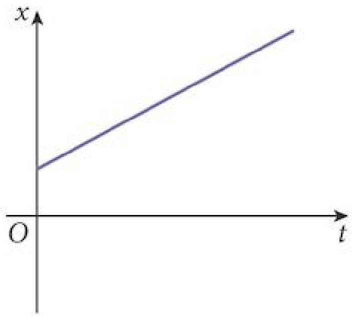


velocity-time
v-t

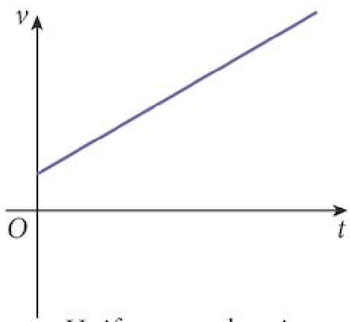
Graph	Gradient	Area below
x-t	$\frac{dx}{dt} = v$	- - -
v-t	$\frac{dv}{dt} = a$	$\int v\,dt = x$

If the velocity is uniform, the x-t graph will be a straight line.

If the acceleration is uniform, the v-t graph will be a straight line.



Uniform velocity



Uniform acceleration

If the displacement is negative, the x-t graph will be below the axis.

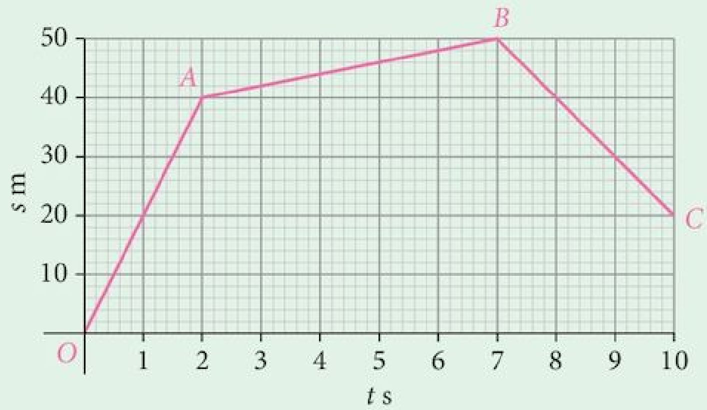
If the velocity is negative, the v-t graph will be below the axis and the area (integral-displacement) will be negative, indicating that the displacement of the particle is being reduced (or becoming more negative).

Example 22.7

The diagram shows the displacement-time graph for a particle P moving in a straight line with a displacement s m from a fixed point at time t s.

a Find the velocity of the particle in the three sections of its journey.

b Find the total distance travelled by the particle.



Solution:

a The velocity is given by the gradient of the displacement-time graph.

For the section OA, the velocity is $\dfrac{40}{2} = 20\,\text{m}\,\text{s}^{-1}$

For the section AB, the velocity is $\dfrac{10}{5} = 2\,\text{m}\,\text{s}^{-1}$

For the section BC, the velocity is $\dfrac{-30}{3} = -10\,\text{m}\,\text{s}^{-1}$

Velocity is negative but speed is positive.

b The distance travelled by the particle is

$$40 + 10 + 30 = 80\text{ m}$$

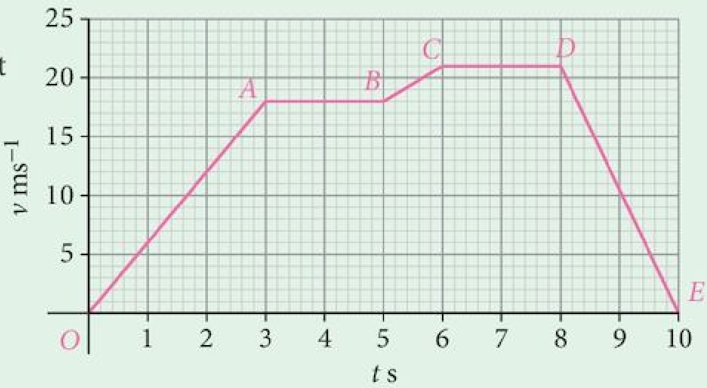
The section BC is returning towards O.

Example 22.8

The diagram shows the velocity-time graph for a particle Q moving in a straight line with velocity $v\text{ m}\,\text{s}^{-1}$, t seconds after leaving a fixed point, O.

a Find the acceleration of the particle in the five sections of its journey.

b Find the displacement of the particle from O.



Solution:

a The acceleration is given by the gradient of the velocity-time graph.

For the section OA, the acceleration is $\dfrac{18}{3} = 6\,\text{ms}^{-2}$

For the section AB, the acceleration is $= 0\text{ ms}^{-2}$

For the section BC, the acceleration is $\dfrac{3}{1} = 3\,\text{ms}^{-2}$

For the section CD, the acceleration is $= 0\text{ ms}^{-2}$

For the section DE, the acceleration is $\dfrac{-21}{2} = -10.5\,\text{ms}^{-2}$

b The displacement is given by the area below the velocity-time graph.

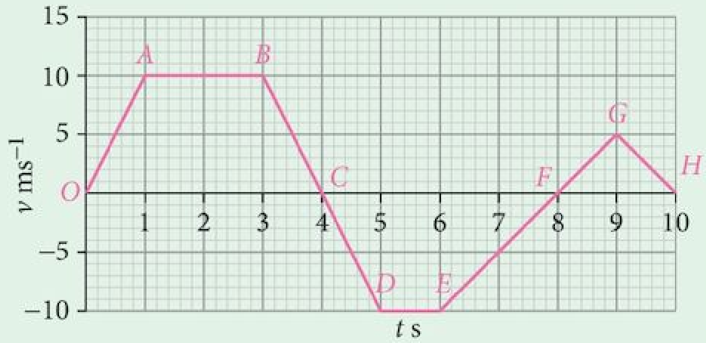
This is best found by dividing the area up into triangles, trapeziums and rectangles.

$$\begin{aligned}\text{Displacement} &= 27 + 36 + 19.5 + 42 + 21 \\ &= 145.5\,\text{m}\end{aligned}$$

The velocity is always positive so the particle is always moving away from the origin.

Example 22.9

The diagram shows the velocity-time graph for a particle moving in a straight line with velocity v ms^{-1}, t seconds after leaving a fixed point, O.



a Find the final displacement of the particle from O.

b Find the total distance travelled by the particle.

Solution:

a The displacement is given by the area between the velocity-time graph and the t-axis.

However, the section $CDEF$ indicates a negative velocity so the particle will be returning towards O.

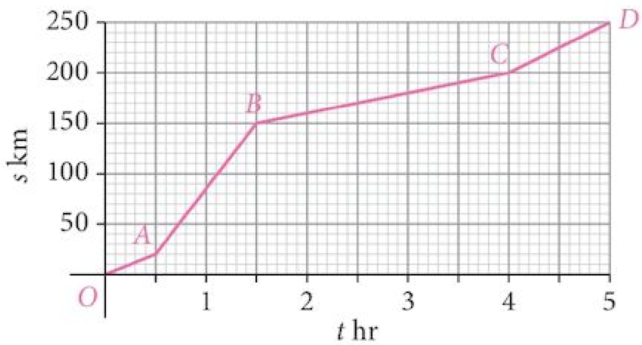
The displacements are:

$$OABC = 30$$
$$CDEF = -25$$
$$FGH = 5$$
$$\text{Final displacement} = 10\,\text{m from}\,O.$$

b Total distance travelled = 60 m

Exercise 22.3

1 The diagram shows the displacement-time graph for a particle on a journey modelled as a straight line with a displacement s km from a fixed point O at time t hr.



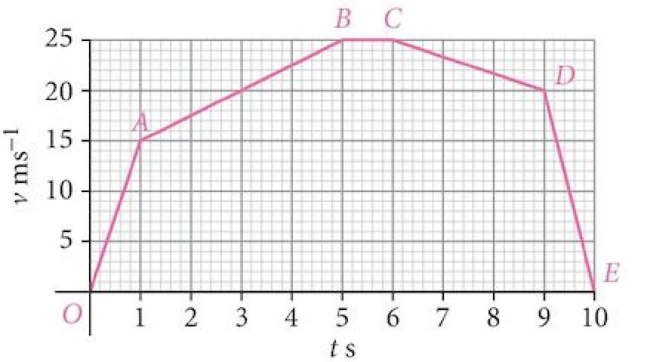
a Find the velocity of the particle in the four sections of its journey.

b Describe a journey that might produce a graph of this type.

2 The diagram shows the velocity-time graph for a particle moving in a straight line with a velocity v ms^{-1} at a time t s after passing a fixed point O.

a Find the acceleration of the particle in each section of its journey.

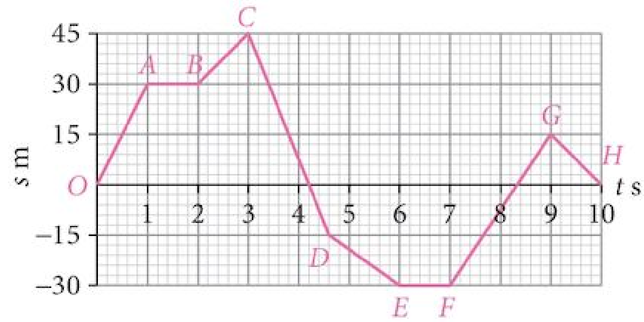
b Find the final displacement of the particle from the fixed point.



3 The diagram shows the displacement-time graph for a particle moving in a straight line with a displacement s km from a fixed point O at time t seconds.

a Find the velocity of the particle in the each section of its journey.

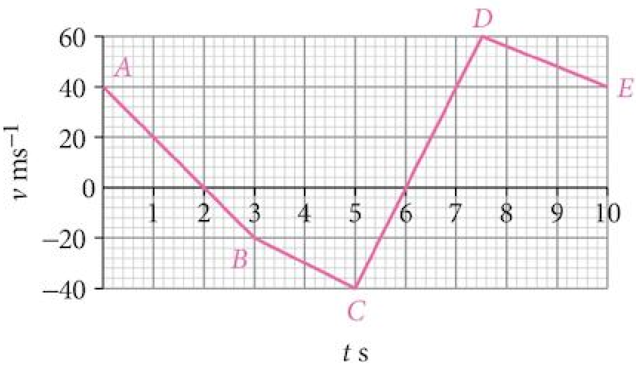
b Draw a velocity-time graph for the particle.



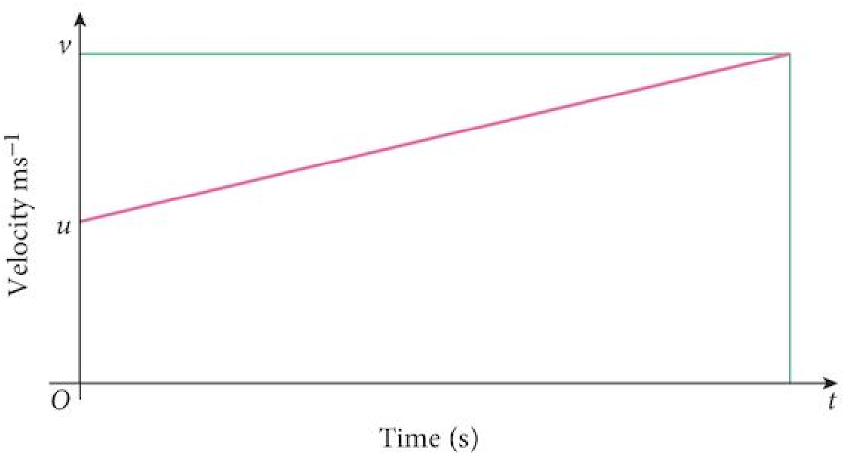
4 The diagram shows the velocity-time graph for a particle moving in a straight line with a velocity v m^{-1} at a time t s after passing a fixed point, A.

a Find the acceleration of the particle in the four sections of its journey.

b Find the final displacement of the particle from A.



5 The diagram below shows the velocity-time graph of a particle moving with constant acceleration, a ms^{-2}. Its initial velocity is u ms^{-1} and after a time of t s, it has a velocity of v ms^{-1}.



a Write down an expression for the acceleration, a, of the particle.

b Write down an expression for the distance, s m, travelled by the body during this time.

c Using your equations from parts **a** and **b,** derive, using substitution and elimination, three more equations for the motion of the body, each with one of the quantities s, u, v, a and t missing. Compare your results with those from Problem 22.1.

Summary

Definition

Kinematics is the branch of mathematics that is concerned with the mechanics of motion.

Rates of change

Velocity is the rate of change of position.

Acceleration is the rate of change of velocity.

Differentiate Differentiate

Starting from	Displacement x	Velocity v	Acceleration a
x	x	$\dfrac{dx}{dt}$	$\dfrac{d^2x}{dt^2}$
v	$\int v\,dt$	v	$\dfrac{dv}{dt}$
a	Integrate a twice	$\int a\,dt$	a

Integrate Integrate

Motion with uniform acceleration

Important: do not use these equations if the acceleration is not uniform.

$$v = u + at \qquad [1]$$

$$s = ut + \frac{1}{2}at^2 \qquad [2]$$

$$s = \frac{(u+v)t}{2} \qquad [3]$$

$$v^2 = u^2 + 2as \qquad [4]$$

$$s = vt - \frac{1}{2}at^2 \qquad [5]$$

Graphs

Graph	Gradient	Area below
$x - t$	$\frac{\mathrm{d}v}{\mathrm{d}t} = v$	---
$v - t$	$\frac{\mathrm{d}v}{\mathrm{d}t} = a$	$\int v\,\mathrm{d}t = x$

Chapter 22 Summative Exercise

1 A particle moving in a straight line has a displacement, t seconds after passing a fixed point O, given by the formulae below. Find:

(i) the velocity, in ms^{-1}, of the particle when $t = 3$ s

(ii) the acceleration of the particle when $t = 2$ s.

a $s = t^3 - 3t^2 + 2t$ **b** $s = 4\cos\left(\frac{\pi t}{6}\right)$ **c** $s = te^{-0.1t}$

2 The velocity of a particle, in ms^{-1}, moving in a straight line, t s after passing a fixed point, O, is given by the equation $v = 48t - 6t^2$. Find:

a the time at which the velocity reaches a maximum

b the time at which the particle comes instantaneously to rest

c the distance from O that the particle comes instantaneously to rest

d the speed of the particle when it passes O again.

3 The acceleration of a particle, in ms^{-2}, moving in a straight line, t s after passing a fixed point, O, is given by the equation $a = 8 - 6t$. The particle passes O with a velocity of 35 ms^{-1}. Find:

a the value of t when it comes to the point where it is instantaneously at rest

b the distance from O to the point where it comes instantaneously to rest.

4 A particle travels in a straight line such that, t s after passing a fixed point, O, its velocity is given by $v = 16\cos\left(\frac{t}{4}\right)$. Find:

a the value of t when $v = 8$ for the first time

b the acceleration when $t = 4$

c the distance from O when $v = 0$ for the first time.

5 A train starts from rest at a station and accelerates uniformly at 0.15 ms^{-2} to reach a maximum speed in 3 minutes. The train travels at this speed for 10 minutes and is then brought to rest at the next station with a uniform acceleration of $-$ 0.5 ms^{-2}.

a Sketch the velocity-time graph for the motion.

b Find the maximum speed of the train.

c Find the disance between the two stations.

6 A particle moves in a straight line so that, t s after passing a fixed point, O, its velocity, v ms^{-1} is given by $v = \frac{72}{(2t+6)^2}$.

a Find the velocity as it passes through O.

b Find the acceleration when $t = 3$.

c Find an expression for the displacement of the particle from O, t s after it has passed through O.

7 A particle, P, moving in a straight line with a uniform acceleration of 0.4 ms^{-2} passes a point, O, with a speed of 2 ms^{-1}. Ten seconds later, a second particle, Q, moving along the same straight line and in the same direction as P passes O with a speed v ms^{-1} and a uniform acceleration of 0.6 ms^{-2}.

When Q catches up with P they are both s m from O and the speed of P is 10 ms^{-1}.

Find the distance of the particles from O when they meet, the value of v and the speed of Q at that point.

8 A particle, P, is projected vertically upwards with a speed of 30 ms^{-1}. Two seconds later, a second particle, Q, is also projected vertically upwards with a speed of 30 ms^{-1}.

P and Q collide when P has been in motion for t s. The acceleration due to gravity is 10 ms^{-2} vertically downwards. Find:

a the value of t

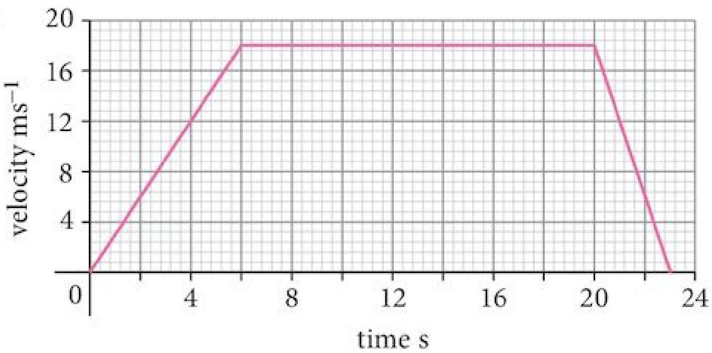
b the height above the launch point at which the particles collide

c the velocity of the particles at the point of collision.

9 The velocity-time graph represents the motion of a particle moving in a straight line.

a Find the acceleration during the first 6 seconds.

b Find the total distance travelled during the 23 seconds of motion.



Chapter 22 Test

1 hour

1 A point moves in a straight line such that, t seconds after passing a fixed point, O, its displacement, x m, is given by $x = 36 \sin\left(\frac{t}{3}\right)$.

a Find the velocity of the particle when $t = 2$ s. [3]

b Find the time taken to reach the point furthest from O for the first time during its motion. [3]

2 A point moves in a straight line such that, t seconds after passing a fixed point, O, its velocity, v ms^{-1}, is given by $v = 60t - 6t^2$ for $t \geqslant 0$.

a Find:

(i) the time at which the particle attains its maximum velocity [2]

(ii) the times at which the particle is instantaneously at rest [2]

(iii) the distance from O that the particle comes to instantaneous rest [3]

(iv) the speed of the particle as it passes O again. [4]

b Describe the subsequent motion of the particle. [1]

3 A particle moves in a straight line such that its displacement, s m, from a fixed point O on the line at time t seconds is given by $s = 2 + \dfrac{36}{t+1}$, $t \geqslant 0$.

Find:

a the distance from O of the particle at the start of its motion [1]

b the time at which the particle is 8 m from O [2]

c the velocity of the particle when $t = 2$ s [2]

d the distance from O that the particle approaches after a long time [1]

e an expression for the acceleration of the particle [2]

f the acceleration of the particle when $t = 2$ s. [1]

4 The acceleration of a particle, t seconds after passing a fixed point, O, is given by $a = 10 - 6t$ for $t \geqslant 0$. It passes O with a velocity of 24 ms^{-1}.

Find:

a the time at which the particle has its maximum velocity [2]

b the time at which the particle is instantaneously at rest [2]

c the distance from O that the particle comes to instantaneous rest [3]

d the time taken for the particle to return to O [2]

e the speed of the particle as it passes O again. [4]

Examination Questions

1 A car moves on a straight road. As the driver passes a point A on the road with a speed of $20\ \text{m s}^{-1}$, he notices an accident ahead at a point B. He immediately applies the brakes and the car moves with an acceleration of $a\ \text{m s}^{-2}$, where $a = \frac{3t}{2} - 6$ and t s is the time after passing A. When $t = 4$, the car passes the accident at B. The car then moves with a constant acceleration of $2\ \text{m s}^{-2}$ until the original speed of 20m s^{-1} is regained at a point C. Find

(i) the speed of the car at B, [4]

(ii) the distance AB, [3]

(iii) the time taken for the car to travel from B to C, [2]

Sketch the velocity-time graph for the journey from A to C. [2]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2002, P2, Qu 11]

2 The speed $v\ \text{ms}^{-1}$ of a particle travelling from A to B, at time t s after leaving A, is given by $v = 10t - t^2$. The particle starts from rest at A and comes to rest at B. Show that the particle has a speed of $5\ \text{ms}^{-1}$ or greater for exactly $4\sqrt{5}$ s. [5]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2002, P1, Qu 3]

3 A motorcyclist travels on a straight road so that, t seconds after leaving a fixed point, his velocity, $v\ \text{ms}^{-1}$, is given by $v = 12t - t^2$. On reaching his maximum speed at $t = 6$, the motorcyclist continues at this speed for another 6 seconds and then comes to rest with a constant deceleration of $4\ \text{ms}^{-2}$.

(i) Find the total distance travelled. [6]

(ii) Sketch the velocity-time graph for the whole of the motion. [2]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2003, P2, Qu 9]

4 A particle travels in a straight line so that, t s after passing a fixed point A, its speed, $v\ \text{ms}^{-1}$, is given by $v = 40(e^{-t} - 0.1)$.
The particle comes to rest instantaneously at B. Calculate the distance AB. [6]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2003, P2, Qu 6]

5 A particle travels in a straight line so that, t seconds after passing a fixed point A on the line, its acceleration, $a\ \text{ms}^{-2}$, is given by $a = -2 - 2t$. It comes to rest at a point B when $t = 4$.

(i) Find the velocity of the particle at A. [4]

(ii) Find the distance AB. [3]

(iii) Sketch the velocity-time graph for the motion from A to B. [1]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2005, P1, Qu 11]

6 A particle moves in a straight line so that, t seconds after leaving a fixed point O, its velocity, v ms^{-1} is given by $v = pt^2 + qt + 4$, where p and q are constants. When $t = 1$, the acceleration of the particle is 8 ms^{-2}. When $t = 2$, the displacement of the particle from O is 22 m. Find the value of p and of q. [7]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2006, P1, Qu 7]

7 A particle, moving in a straight line, passes through a fixed point O with a velocity 14 ms^{-1}. The acceleration, a ms^{-2}, of the particle, t seconds after passing through O, is given by $a = 2t - 9$. The particle susequently comes to instantaneous rest, firstly at A and later at B. Find

(i) the acceleration of the particle at A and at B, [4]

(ii) the greatest speed of the particle as it travels from A to B, [2]

(iii) the distance AB. [4]

[Cambridge IGCSE Additional Mathematics 0606, Jun 2007, P2, Qu 11]

8 A particle travels in a straight line so that, t s after passing through a fixed point O, its speed, v ms^{-1} is given by $v = 8\cos\left(\frac{t}{2}\right)$.

(i) Find the acceleration of the particle when $t = 1$. [3]

The particle first comes to instantaneous rest at the point P.

(ii) Find the distance OP. [4]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2007, P2, Qu 9]

9 A particle moves in a straight line so that, t seconds after passing through a fixed point O, its velocity, v ms^{-1}, is given by $v = \dfrac{20}{(2t+4)^2}$. Find

(i) the velocity of the particle at O, [1]

(ii) the acceleration of the particle when $t = 3$, [3]

(iii) the distance travelled by the particle in the first 8 seconds. [4]

[Cambridge IGCSE Additional Mathematics 0606, Nov 2009, P2, Qu 9]

Term test 6A (Chapters 20–22) 1 hour

1 Differentiate with respect to x:

a $(3 - x^2)\ln(2x - 3)$ [3]

b $\dfrac{\sin x}{1 - e^{2x}}$ [4]

2 The vectors $\mathbf{i} = \begin{pmatrix}1\\0\end{pmatrix}$ and $\mathbf{j} = \begin{pmatrix}0\\1\end{pmatrix}$ represent a unit vector due east and due north respectively, the units being km.

A patrol boat moves with a velocity $\begin{pmatrix}6\\7\end{pmatrix}$ km h^{-1} to intercept a suspected smuggler.

The path of the smuggler's boat is given by $\begin{pmatrix}2+5t\\4+2t\end{pmatrix}$, where t hours is the time since it was first noticed.

All measurements are made relative to the coastguard station at O (0, 0).

The patrol boat was initially resting at the point $\begin{pmatrix}11\\3\end{pmatrix}$ km.

a Find the speed of **(i)** the patrol boat, and **(ii)** the smuggler's boat. [1]

b Express, as a vector, the position of the patrol boat at time u hours after the time that it was first noticed. [1]

c **(i)** Find the time taken for the patrol boat to reach the path of the smuggler's boat. [2]

(ii) Find the position of the smuggler's boat at that time. [1]

(iii) How long will the patrol boat have to wait until the other boat reaches this position ? [2]

3 A particle moves in a straight line so that, t seconds after leaving a fixed point O, its velocity, v ms^{-1} is given by $v = 12 \cos 4t$.

a Find an expression for the acceleration of the particle at time t s. [2]

b Find an expression for the displacement of the particle from O at time t s. [2]

c Find the time taken for the particle first to come to instantaneous rest. [2]

d Find the amplitude of the motion. [1]

4 **a** If $y = xe^x$, show that $xe^x = \dfrac{dy}{dx} - e^x$. [2]

b Hence evaluate $\int xe^x \, dx$. [4]

5 The points A, B and C have position vectors $\mathbf{a} = 5\mathbf{i} - 2\mathbf{j}$, $\mathbf{b} = 7\mathbf{i} + 6\mathbf{j}$, and $\mathbf{c} = 3\mathbf{i} + 4\mathbf{j}$, relative to an origin, O.

a Find $|\,\mathbf{a} + \mathbf{b} + \mathbf{c}\,|$. [2]

b Find the values of λ and μ such that $\lambda\mathbf{a} + \mu\mathbf{b} = \mathbf{c}$. [3]

6 A particle starts from rest at a point, A, and moves with a constant acceleration in a straight line. It passes point B with a speed of 16 ms^{-1}. At a third point, C, its velocity is 24 ms^{-1}. Given that $BC = 120$ m, find:

a the time taken to travel from B to C [2]

b the acceleration of the particle [1]

c the total distance from A to C [3]

d the total time taken for the journey. [2]

Revision practice paper

Examination advice:

1 Write clearly and neatly. You do not want your examiner to misread what you have written.

2 Use **black** ink. Dark blue is acceptable but black is better.

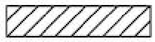
3 You can use coloured ink and pencils for drawings but **do not** use **red**.
Use a ruler for straight lines.

4 Make sure that you answer the question that is asked.

5 Coordinates should be written as (3, 4) and not $x = 3$, $y = 4$. These are equations of lines.

6 Underline your answers to make them easier to find. Do not hide them away in the middle of rough working.

7 You are **not** allowed to use corrector fluid, pens or tape. If you know that something is wrong, cross it out neatly. Maybe draw a box around it and cross through it, but don't waste time doing it. Make sure it is obvious which bits are crossed out and which are not. If you have to use extra paper, make a note on the correct page telling the examiner where your new answer is.



8 You should always write a calculated answer more accurately than required and then rewrite it with the correct number of decimal places or significant figures.

9 Do not use an approximate result (a previously rounded answer) in a subsequent calculation. Make use of your calculator memory facility to save calculated results so that you do not have to key them in again. You could easily make an error in your keying in.

10 If an answer is given in the question, you will need more explanation than normal of how you got the result.

11 Make sure you have all the equipment that you need. You will not be allowed to borrow anything for the examination. Keep a checklist of the items you need to take with you. Also note those items that you should not have with you.

12 Remember that your pencils, pens, ruler, etc. should be in a see-through package (maybe even a plastic bag) with no writing on the outside. Coloured pencil cases or boxes are not allowed.

13 The examination lasts for 2 hours. Try not to waste time going to the toilets during the examination.

14 Read the instructions carefully.

Finally, good luck and enjoy the experience.

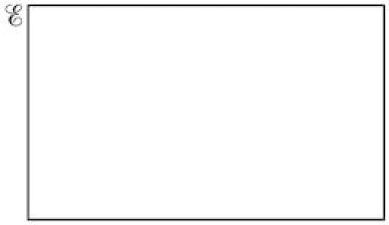
Practice paper 1

2 hours

1 Show that $\dfrac{\sin A}{1+\cos A} + \dfrac{1+\cos A}{\sin A}$ can be written in the form $p \operatorname{cosec} A$ where p is an integer to be found. [4]

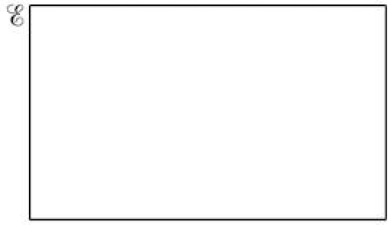
2 **a** Copy the Venn diagrams below. Add two sets, A and B as indicated.

(i)



$B \subset A$

(ii)



$A \cap B = \varnothing$

[2]

b The universal set, ℰ, and sets S and T are such that $n(\mathscr{E}) = 30$, $n(S \cap T') = 9$, $n(S' \cap T') = 10$ and $n(S' \cap T) = 7$.

Find :

(i) $n(T)$ [1]

(ii) $n(S \cup T)$ [1]

(iii) $n(S \cap T)$ [1]

3 **a** Sketch the graph of $y = |(x-1)(x+3)|$ for $-5 \leqslant x \leqslant 5$, showing the coordinates of the points where the graph meets the x-axis. [3]

b Find the range of values of k for which the equation $|(x-1)(x+3)| = k$ has 4 solutions. [3]

4 The region enclosed by the curve $y = 3\cos 2x$, the x-axis, the y-axis and the line $x = a$ is 1 square unit. Find the value of a. [6]

5 Solve the simultaneous equations:

$$27^x \times 9^y = 1$$
$$32^x \times 16^y = 4$$

[7]

6 **a** Matrices **A**, **B** and **C** are given by

$$\mathbf{A} = \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} -1 & 0 & -2 \\ 2 & 3 & -1 \end{pmatrix} \qquad \mathbf{C} = \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix}$$

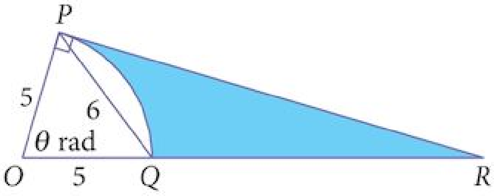
Write down, but do not evaluate, all products of the matrices **A**, **B** and **C** that are possible to find. [2]

b Matrices **P** and **Q** are such that

$$\mathbf{Q} = \begin{pmatrix} -5 & -3 \\ 3 & 2 \end{pmatrix} \qquad \mathbf{PQ} = \begin{pmatrix} 19 & 12 \\ -11 & -7 \end{pmatrix}$$

Find the matrix **P**. [5]

7



The diagram shows a sector of a circle of radius 5 cm.

The chord PQ has length 6 cm and angle OPR is a right angle.

a Show that $\theta = 1.29$ rad. [2]

b Find the perimeter of the shaded region. [3]

c Find the area of the shaded region. [3]

8 **a** **(i)** How many numbers in the range $1000 < x < 10\,000$ can be formed from the digits 1, 2, 3, 5, 8, 9 if no digit is repeated? [1]

(ii) How many of these are odd? [1]

(iii) How many of these odd numbers are greater then 4000? [3]

b Six members of a class of 12 will be selected to go on a trip in the school van.

In how many ways can this be done if Sally will not travel with Ahmed (given that both Sally and Ahmed are part of the six members selected)? [3]

9 A cylinder has a radius r cm and a volume of 5000 cm^3.

a Find the total surface area of the cylinder and express it in terms of r only. [3]

b Given that r can vary, show that the surface area has a minimum value and find that minimum. [6]

10 A ship leaves a port, A, and travels at a speed of 34 km per hour in the direction $8\mathbf{i} + 15\mathbf{j}$, where $\mathbf{i}$ is the unit vector due East and $\mathbf{j}$ is the unit vector due North.

a Write the velocity of the ship in terms of $\mathbf{i}$ and $\mathbf{j}$. [2]

b Three hours later, the ship passes a lighthouse. Find the position vector of the lighthouse relative to P. [1]

c Find the position vector of the ship t hours after passing the lighthouse. [2]

When the ship passes the lighthouse, word is sent to a group of pirates waiting in a speedboat in port B, whose position relative to port A is $128\mathbf{i} + 130\mathbf{j}$. They leave with a velocity of $-24\mathbf{i} + 10\mathbf{j}$ km per hour to intercept the ship.

d Find the position of the pirates t hours after leaving port B. [1]

e Find the value of the time, t, at which the pirates intercept the ship and the position vector, relative to port A, of the point of interception. [4]

11 Solve the equations:

a $3\cos^2 x + 2\cos x = 0$ for $0° \leqslant x \leqslant 180°$ [3]

b $5\sin^2 y - 2\cos y + 1 = 0$ for $0° \leqslant y \leqslant 360°$ [4]

c $\tan\left(3z - \dfrac{\pi}{4}\right) = 1$ for $0 \leqslant z \leqslant 2\pi$ radians [3]

Answers

Chapter 1

Exercise 1.1

1 **e** $\{2, 3, 5, 7, 11, 13, 17, 19\}$ **f** $\{1, 2, 3, 4, 6, 9, 12, 18, 36\}$ **g** $\{225, 256, 289, 324, 361, 400\}$

2 **a** $A = \{x : x \text{ is a positive even number less than } 12\}$ **b** $B = \{x : x \text{ is a factor of } 12\}$

c $C = \{x : x \text{ is a triangle number less than } 16\}$ **d** $D = \{x : x \text{ is a square number}\}$

e $E = \{x : x \text{ is an odd number between } 10 \text{ and } 20\}$ **f** $F = \{(x, y): x + y = 5, x, y \in \mathbb{N}\}$

g $G = \{(x, y, z) : x < 9 \text{ and } (x, y, z) \text{ is a Pythagorean Triple}\}$

3 **a** $A = \{1\}$ **b** $B = \{\text{Prime Numbers}\}$ **c** $C = \{\text{Square Numbers}\}$

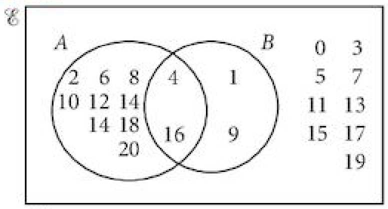
4 **a** $A = \{2, 4, 6, 12, 18, 36\}$ **b** $B = \{-4, -3, -2, -1, 0, 1, 2, 3, 4\}$ **c** $C = \{1, 3, 5, 7, 9, 15\}$

d $D = \{17, 20, 23, 26, 29\}$

5 **a** T **b** F **c** T **d** T **e** F

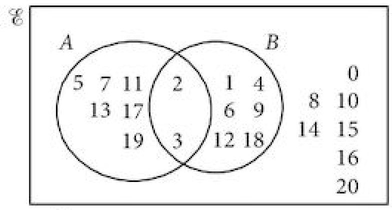
Exercise 1.2

1 **a**



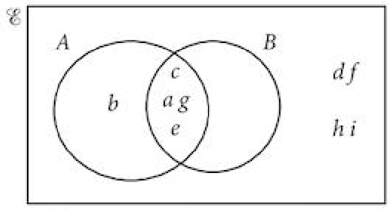
(i) 12
(ii) 2
(iii) 11
(iv) 17

b



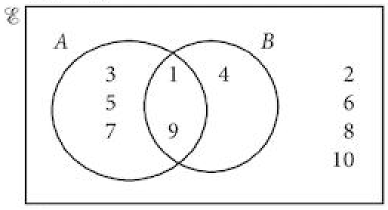
(i) 14
(ii) 2
(iii) 13
(iv) 13

c



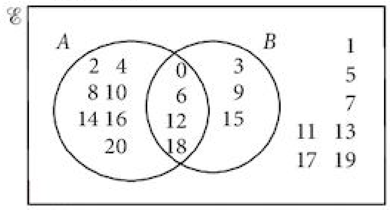
(i) 5
(ii) 4
(iii) 4
(iv) 5

d



(i) 6
(ii) 2
(iii) 5
(iv) 7

e



(i) 14
(ii) 4
(iii) 10
(iv) 14

2

	n(ℰ)	n(A)	n(B)	n($A \cap B$)	n($A \cup B$)	n($A \cup B$)′
a	14	5	6	2	9	5
b	15	3	5	0	8	7
c	24	14	13	9	18	6
d	28	18	16	10	24	4
e	48	24	29	9	44	4
f	30	12	9	0	21	9
g	40	16	24	6	34	6
h	32	12	19	5	26	6

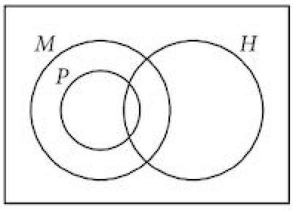
3 **a** $n(A) + n(B) = n(A \cap B) + n(A \cup B)$

Exercise 1.3

1 **a** $a \in P \cap Q' \cap R'$ **b** $b \in P \cap Q \cap R'$ **c** $c \in P' \cap Q \cap R'$ **d** $d \in P \cap Q' \cap R$

e $e \in P \cap Q \cap R$ **f** $f \in P' \cap Q \cap R$ **g** $g \in P' \cap Q' \cap R$ **h** $h \in P' \cap Q' \cap R'$

2 ℰ



3 **a** $H \subset E$ **b** $H \cap M = \varnothing$ **c** $n(M \cap E) = 32$

d Students who study Economics but not Mathematics nor History

e Students who study Economics or Mathematics but not History

4 **a** $(D \cap J) \subset C$

b $D \cap C = \varnothing$

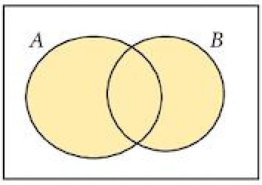
c $n(C) > n(D \cap J)$

d Shoppers who are dog owners but they neither drive a car nor jog

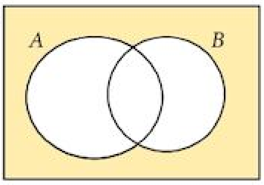
e Joggers who drive a car, or dog owners

f All shoppers who do not drive a car are joggers.

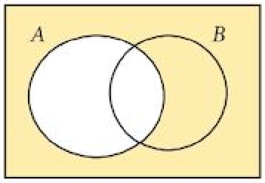
5 **a** **(i)** ℰ



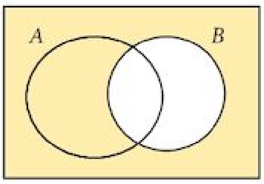
(ii) ℰ



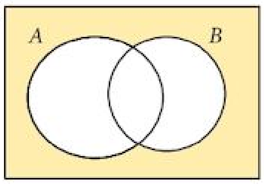
b **(i)** ℰ



(ii) ℰ

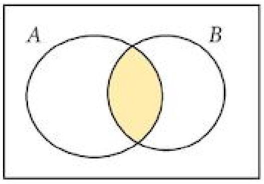


(iii) ℰ

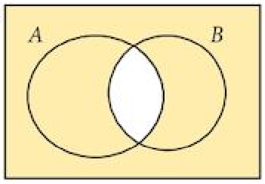


c They are the same.

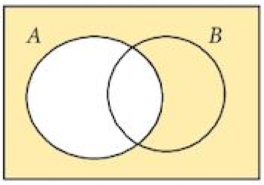
6 **a** **(i)** ℰ



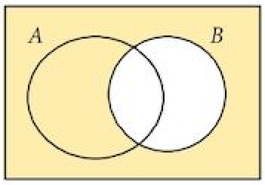
(ii) ℰ



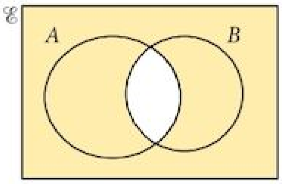
b **(i)** ℰ



(ii) ℰ

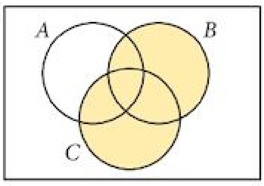


(iii) ℰ

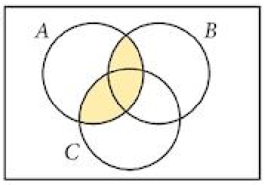


c They are the same.

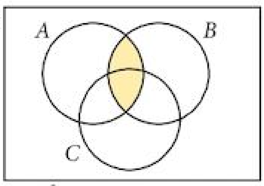
7 **a** **(i)** ℰ



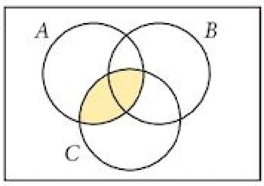
(ii) ℰ



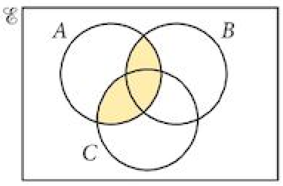
b **(i)** ℰ



(ii) ℰ

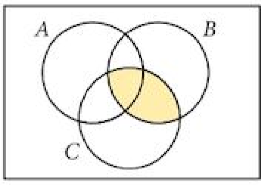


(iii) ℰ

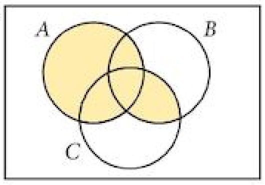


c They are the same.

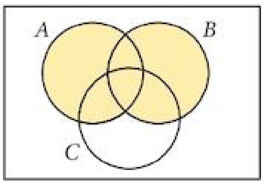
8 a (i) $\mathscr{E}$



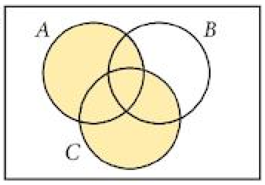
(ii) $\mathscr{E}$



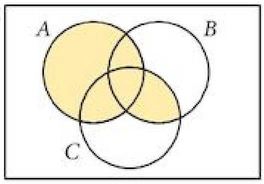
b (i) $\mathscr{E}$



(ii) $\mathscr{E}$



(iii) $\mathscr{E}$



c They are the same.

Exercise 1.4

1 8 **2** 2 **3** 10 **4** 25 **5** 10 **6** 17 **7** 19

8 a 45 **b** 19 **c** 30 **9 a** 15 **b** 35 **c** 45

10 a greatest = 16, smallest = 6

b greatest = 10, smallest = 0

11 a $y = x - 13$

b Max $(x) = 25$ Min $(x) = 13$

Max $(y) = 12$ Min $(y) = 0$

12 a $y = 2x$

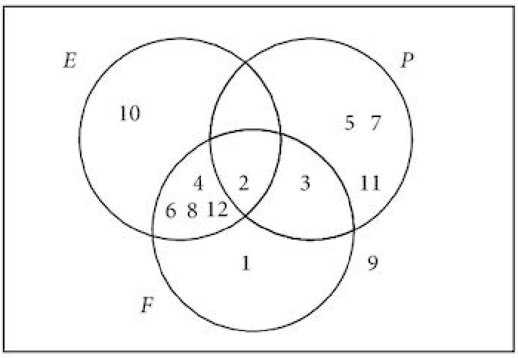
b Max $(n(A)) = 42$ Min $(n(A)) = 26$

Chapter 1 Summative Exercise

1 a {23, 29, 31, 37} **b** {81, 100, 121, 144, 169, 196} **c** {1, 2, 3, 4, 6, 9, 12, 18, 36}

2 a {6, 7, 8, 9, 10, 11, 12, 13, 14, 15} **b** 10 **c** {7, 11, 13}

3 $\mathscr{E}$



4 a $B \subset C$ **b** $A \cap B = \varnothing$ **c** $B' \cap C = \varnothing$

5 Examples:

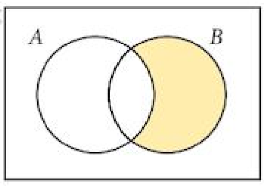
a All carrots are vegetables.

b There are no Siberian flying squirrels playing for the Calgary Flames hockey team.

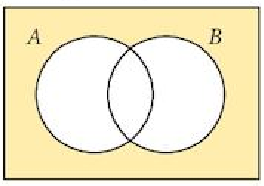
c Every student in the school is either a girl or a boy.

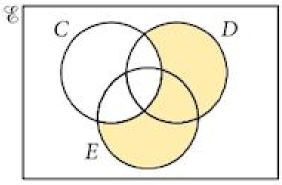
d In a mixed school, the number of pupils who are not girls is greater than the number of teachers.

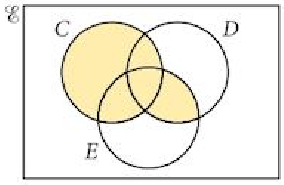
6 a $\mathscr{E}$

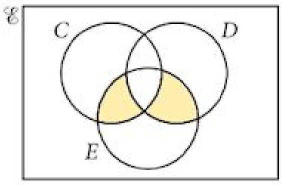


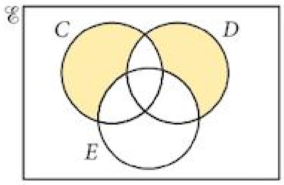
b $\mathscr{E}$



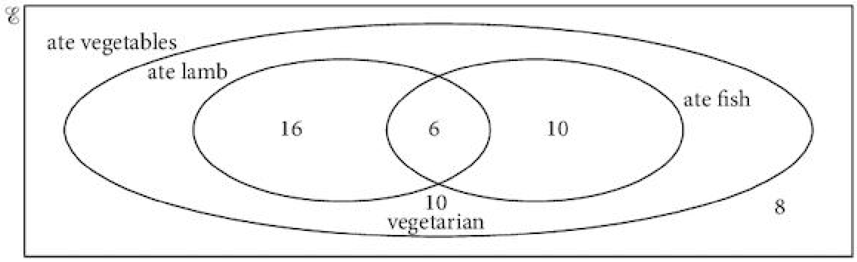
7 **a**


b


c


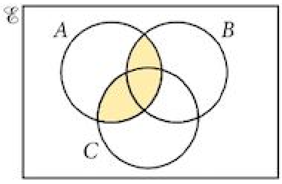
d


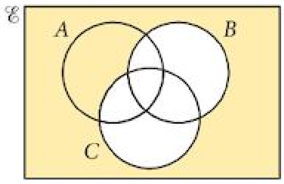
8 **a** 3 **b** 4 **c** 4

9


a 6 **b** 16 **c** 16

Chapter 1 Test

1 **a**


b


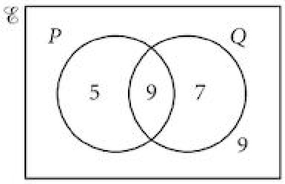
2 **a** $\{54, 60, 66\}$ **b** $\{50, 51, 52, 54, 56, 57, 58, 60, 62, 63, 64, 66, 68, 69, 70\}$ **c** 8

3 $(A' \cap B') \cup (A \cap B)$ or $(A' \cup B) \cap (A \cup B')$

4 **a** $n(\mathscr{E}) = 20$, $n(P) = 12$, $n(E) = 14$, $n(P' \cap E') = 3$ **b** $n(P \cap E) = 9$

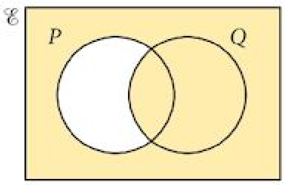
5 **a** 115 **b** 40 **c** 60 **d** 260

6 **a** $43 \in P$ **b** $65 \in S'$ or $65 \notin S$ **c** $S \cap P = \varnothing$ or $n(S \cap P) = 0$ **d** $n(P \cap N) = 10$

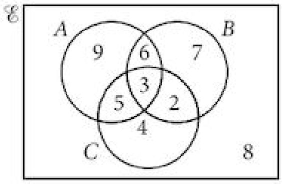
7 **a**


b 23

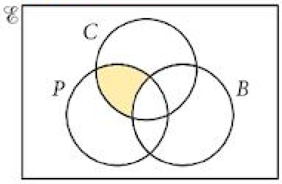
8 **a** $n(\mathscr{E}) = 86$ **b** $C \subset (B \cap P)$ **c** $n(B \cap C \cap P) = 30$ **d** $n(P \cap B') = 15$

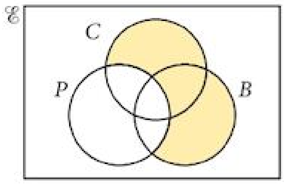
9 **a**


10 78

11 

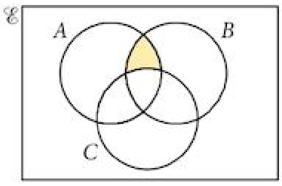
Examination Questions

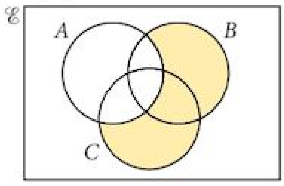
1 a (i) 

(ii) 

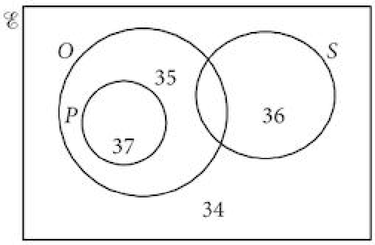
b (i) Max 10; Min 6 (ii) Max 20; Min 16

2 a (i) $A' \cap B$ (ii) $A' \cup B$

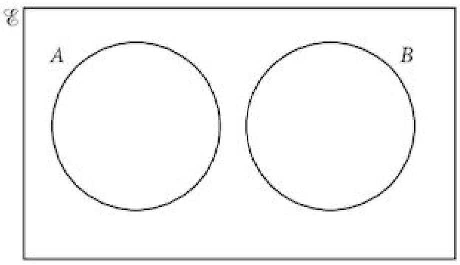
b (i) 

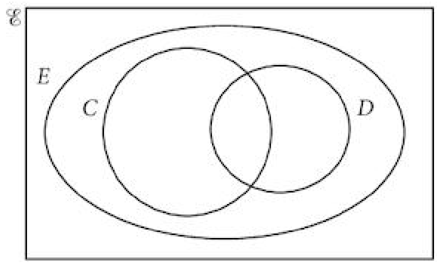
(ii) 

3 (i) Some vegetarian students are over 180 cm tall.
(ii) Students who are over 180 cm tall do not cycle.
(iii) $(B \cap C) \subset A'$

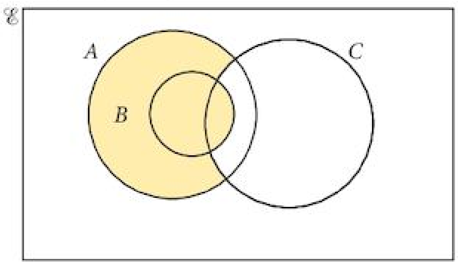
4 (i)
(ii) 

(iii) $n(O \cap S) = 4$ $n(O \cup S) = 54$

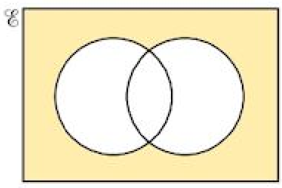
5 a (i) 

(ii) 

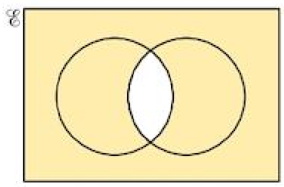
b $(X \cap Y') \cup (X' \cap Y)$

6 a (i) 

(ii) $A \cap C \cap B'$

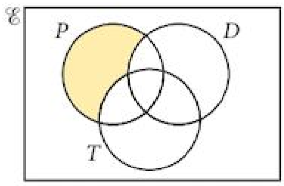
b 

$(X \cup Y)'$

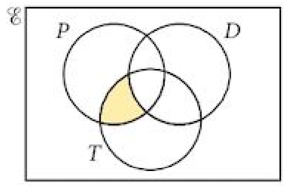


$X' \cup Y'$

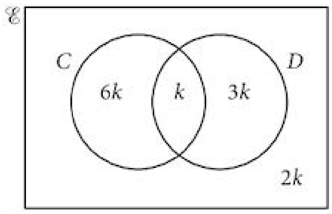
7 **(i)** $P \cap D' \cap T'$



(ii) $P \cap D' \cap T$

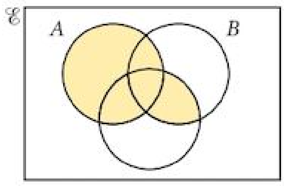


8 **(i)** $x \notin A$ **(ii)** $n(B') = 16$ **(iii)** $(C \cap D) = \varnothing$

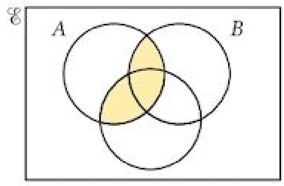
9 **a** 

b 180 000

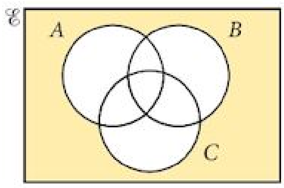
10 **(i)** $A \cup (B \cap C)$



(ii) $A \cap (B \cup C)$



(iii) $(A \cup B \cup C)'$



Chapter 2

Exercise 2.1

1	**a** 5	**b** 4	**c** 3	**d** 5	**e** 2	**f** 5
2	**a** $\frac{1}{9}$	**b** $\frac{1}{3}$	**c** $\frac{1}{2}$	**d** $\frac{1}{5}$	**e** $\frac{1}{7}$	**f** $\frac{1}{8}$
3	**a** 8	**b** 25	**c** 49	**d** $\frac{1}{216}$	**e** $\frac{1}{27}$	**f** $\frac{1}{16}$
4	**a** $\frac{16}{25}$	**b** $\frac{25}{36}$	**c** $\frac{27}{125}$	**d** $\frac{64}{343}$	**e** $\frac{32}{243}$	**f** $\frac{216}{125}$
5	**a** $\frac{256}{81}$	**b** $\frac{512}{125}$	**c** $\frac{64}{27}$	**d** $\frac{81}{49}$	**e** $\frac{343}{729}$	**f** $\frac{27}{8}$
6	**a** 2^{12}	**b** 3^{12}	**c** 5^{14}	**d** 2^{-8}	**e** 3^{12}	**f** 5^{10}
	g 2^{6}	**h** 3^{7}	**i** 5^{-1}	**j** 2^{-16}	**k** 3^{-20}	**l** 5^{12}

7 a 2^{3x+3} b 3^{8x+6} c 5^{14+6x} d 2^{-8x} e 3^{12x+2} f 5^{14x+6}
g $2^{-26x-14}$ h 3^{27x-18} i 5^{21-14x} j $2^{-51-33x}$ k 3^{20-20x} l 5^{4+15x}

8 a $\frac{3}{2}$ b $-\frac{1}{2}$ c 0 d 1 e $\frac{5}{2}$ f $-\frac{13}{2}$
g $-\frac{2}{3}$ h $\frac{2}{3}$ i 3 j 2 k 1 l $-\frac{4}{15}$
m $\frac{2}{3}$ n $\frac{1}{2}$ o 1 or −3

9 a $2^{-2}5^{3}$ b $3^{-6}5^{0}$ c $2^{-10}3^{-6}$
d $2^{8-16x}5^{-9}$ e $3^{12x-4}5^{14x-12}$ f $2^{-16x}3^{2x+19}$
g $2^{-16x-8}5^{4}$ h $3^{23x-14}5^{-3x-8}$ i $2^{8-16x}3^{-14+8x}$

10 a $x=5, y=-1$ b $x=2, y=4$ c $x=3, y=-2$ d $x=1, y=5$

11 a $x=2, y=1$ b $x=5, y=2$ c $x=4, y=3$

Exercise 2.2

1 a $2\sqrt{6}$ b $5\sqrt{3}$ c $7\sqrt{2}$ d $6\sqrt{3}$
e $8\sqrt{3}$ f $9\sqrt{5}$ g $16\sqrt{3}$ h $15\sqrt{5}$

2 a $11\sqrt{6}$ b $5\sqrt{3}$ c $11\sqrt{2}$ d $9\sqrt{3}$
e $20\sqrt{3}$ f $-3\sqrt{5}$ g $20\sqrt{3}$ h $\sqrt{7}$

3 a $\frac{\sqrt{15}}{3}$ b $\frac{\sqrt{2}}{2}$ c $\frac{2\sqrt{30}}{\sqrt{5}}$ d $\frac{2\sqrt{7}}{7}$
e $\sqrt{15}$ f $6\sqrt{2}$ g 10 h $3\sqrt{7}$

4 a $\frac{-1-\sqrt{3}}{2}$ b $\frac{3-\sqrt{2}}{7}$ c $-2-\sqrt{5}$ d $\frac{3+\sqrt{7}}{2}$
e $\frac{-5-2\sqrt{3}}{2}$ f $\frac{-12+11\sqrt{2}}{14}$ g $\frac{-21-11\sqrt{3}}{3}$ h $\frac{13-5\sqrt{7}}{2}$

5 a $2\sqrt{2}$ b $4+\sqrt{5}$ c $2-\sqrt{2}$ d $3\sqrt{5}-2$ e $3+2\sqrt{2}$

6 $\sqrt{34}$ cm 7 $\sqrt{3}$ cm 8 $\sqrt{29}$ m

9 a $2+6\sqrt{3}$ b $4+3\sqrt{3}$

10 a $\frac{\sqrt{5}-1}{2}$ b 1.618033989 c 0.618033989
The decimal part is exactly the same.

Chapter 2 Summative Exercise

1 a 2^{20} b 3^{3} c 5^{1} d 2^{-3} e 3^{-9} f 5^{-10}

2 a 3 b 5 c 2

3 a $p=3x, q=8-2x$ b $p=6x-18, q=2x-6$ c $p=11x, q=2x-8$

4 a $2\sqrt{2}$ b $3\sqrt{3}$ c $4\sqrt{5}$ d $7\sqrt{3}$

5 a $\frac{\sqrt{2}}{2}$ b 4 c $9\sqrt{3}$ d $4\sqrt{7}$

6 a $2-\sqrt{3}$ b $-\frac{1}{3}(2+\sqrt{7})$ c $\frac{1}{2}(3+\sqrt{5})$ d $\frac{1}{3}(7\sqrt{3}-12)$

7 a $\sqrt{2}$ b $\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2}$

8 a $\frac{\sqrt{3}}{2}$ b $\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}$

9 **a** $[(5-\sqrt{2})-(3+\sqrt{2})]^2 = 22-12\sqrt{2}$ **b** $4\times\frac{1}{2}(5-\sqrt{2})(3+\sqrt{2}) = 22-2\sqrt{2}$ **c** $44-14\sqrt{2}$

d Using Pythagoras, the length of the diagonal of the rectangle is $\sqrt{44-14\sqrt{2}}$

10 **c** $2(1+\sqrt{5})+2 = (4+2\sqrt{5})$ **d** Scale Factor $=\frac{1}{2}(3+\sqrt{5})$; $CD = (3+\sqrt{5}) = EA$

e $c=1, d=1, e=4$

Chapter 2 Test

1 3 **2** $7-4\sqrt{3}$

3 **b** $\sqrt{2}(1+\sqrt{3})$ **c** $\frac{\sqrt{2}(1+\sqrt{3})}{4}$

4 $n=\frac{3}{2}, m=\frac{11}{2}$ **5** $x=4, y=3$ **6** $4+\sqrt{3}$

7 **b** $12+4\sqrt{3}$ **8** $62-35\sqrt{3}$ **9** $x=2, y=-1$

Examination Questions

1 $\left(\sqrt{8}-\sqrt{3}\right)$ m **2** $\frac{9}{2}$ **3** **(i)** $4\sqrt{3}-2\sqrt{5}$ **(ii)** $76-14\sqrt{15}$

4 $18\sqrt{2}-25$ **5** $y=-2$

6 **a** $x=2$ **b** $p=-2, q=1$ **7** $x=\frac{8}{5}$ **8** $2\sqrt{2}+\sqrt{3}$

9 $x=5, y=9$ **10** **(i)** $2+\sqrt{3}$ **(ii)** $2\sqrt{3}$

11 **a** $a+b^2$ **b** $\frac{4}{5}$

12 **(i)** 3^{2x+2} **(ii)** 3^{2x} **(iii)** $\frac{2}{3}$

13 $a=19; b=-8$ **14** $3+2\sqrt{3}$ **15** $x=7$

Chapter 3

Exercise 3.1

1 **a** 5040 **b** 3 628 800 **c** 1 307 674 368 000 **d** $2.43290200817\times10^{18}$ **e** 2.652528598×10^{32}

f 30 240 **g** 6720 **h** 360 **i** 7920 **j** 2730

k 35 **l** 36 **m** 924 **n** 2002 **o** 455

2 **a** 13! **b** 69! Your calculator may give different results.

3 **a** $6! = 6\cdot5\cdot4\cdot3\cdot2\cdot1$ $5\times2=10$ so there is one zero.

b 20! has 5, 10, 15, 20 so there are 4 zeros.

c 30! has 5, 10, 15, 20, 25, 30 so there are 7 zeros (25 is 5^2 so this provides 2 zeros)

4 **a** 60 **b** 120 **c** 120 **5** **a** 6720 **b** 20 160 **c** 40 320

6 1200 **7** **a** 210 **b** 600 **c** 2160

8 8640 **9** **a** 48 **b** 72 **10** **a** 12 **b** 3

Exercise 3.2

1 **a** 60 **b** 1680 **c** 151 200 **d** 15 120

e 42 **f** 12 **g** 19 958 400

2 336 **3** 729 **4** 720 **5** 40 320 **6** 24

7 **a** 360 **b** 240 **c** 240

8 **a** 2184 **b** 1248 **c** 120 **d** 1008

9 **a** 10 **b** 28 **c** 210 **d** 126

e 21 **f** 6 **g** 495

10 210 **11** **a** 3003 **b** 1176 **c** 546

12 **a** 720 **b** 36 **c** 6

13 **a** 56 **b** 6 **14** 42 336

Chapter 3 Summative Exercise

1	**a** 720		**b** 40 320		**c** 362 880		**d** 479 001 600		**e** 6 227 020 800	
2	**a** 60 480		**b** 151 200		**c** 95 040		**d** 1 764 322 560		**e** 27 907 200	
	f 60 480		**g** 151 200		**h** 95 040		**i** 1 764 322 560		**j** 27 907 200	
3	**a** 56		**b** 210		**c** 1716		**d** 125 970		**e** 86 493 225	
	f 56		**g** 210		**h** 1716		**i** 125 970		**j** 86 493 225	

4 e.g. The number of ways of choosing 3 students from 8 to go on a trip is the same as the number of ways of choosing 5 students from the 8 to leave behind.

5 2	**6 a** 336	**b** 1680	**c** 6720		
7 315	**8 a** Combination	**b** Permutation	**c** Combination		
9 1	**10 a** 360	**b** 180	**c** 180	**d** 96	
11 a 10!	**b** 5184	**c** 864			
12 a 2	**b** 70	**c** 40			
13 a 30	**b** 12	**c** 30 edges, 12 vertices			

Chapter 3 Test

1 a 18 564	**b** 6720	**c** 8106		
2 a 360	**b** 120	**c** 180	**d** 60	
3 a 362 880	**b** 2880	**c** 5760	**d** 282 240	
4 a 120	**b** 60	**c** 20	**d** 48	
5 a 74 613	**b** 63 669	**c** 4389		
6 a 3003	**b** 1050	**c** 3002		

Examination Questions

1 (i) 720	**(ii)** 120	**(iii)** 48	**(iv)** 15	**(v)** 10
2 (i) 56	**(ii)** 27	**3 a** 720	**b** 50	
4 (i) 210	**(ii)** 95	**5 (i)** 252	**(ii)** 66	
6 (i) 126	**(ii)** 36	**(iii)** 72		
7 a 60	**b** 200	**8 (i)** 210	**(ii)** 7	**(iii)** 175
9 a 322560	**b** 40	**10 a (i)** 120	**(ii)** 36	**b** 96; 60
11 a 3024	**b** 910	**12 (i)** 1890	**(ii)** 1050	
13 a 450	**b** 240	**14 (i)** 1287	**(ii)** 531	

Chapter 4

Exercise 4.1

1	**a** 17	**b** 10	**c** −21	**d** 180	**e** 3	**f** 6
2	**a** 11	**b** 13	**c** 17	**d** 384	**e** 8	**f** 3
3	**a** 10	**b** 6	**c** 0	**d** 14	**e** 3	**f** 9

4 **a** $\mathbb{R}$ **b** $-11 \leqslant f(x) \leqslant 7$ **c** $g(x) \in \{\ldots, -5, -2, 1, 4, \ldots\}$
d $g(x) \in \{-11, -8, -5, -2, 1, 4, 7\}$ **e** $y \in \{1, 27, 125\}$ **f** $-6 \leqslant y \leqslant 14$
g $y \in \{2, 3, 4, 6, 12\}$ **h** $2 \leqslant y \leqslant 9$ **i** $h(x) \in \{0, 3, 4\}$ **j** $-9 \leqslant h(x) \leqslant 16$

Exercise 4.2

1 a (i) −8	**(ii)** $fg : x \mapsto 10 - 6x$	**b (i)** −26	**(ii)** $f^2 : x \mapsto 9x - 8$
c (i) 4	**(ii)** $fh : x \mapsto \frac{36}{x} - 2, x \neq 0$	**d (i)** − 8	**(ii)** $f^2g : x \mapsto 28 - 18x$
e (i) 32	**(ii)** $gf : x \mapsto 8 - 6x$	**f (i)** −12	**(ii)** $g^2 : x \mapsto 4x - 4$
g (i) 0	**(ii)** $gh : x \mapsto 4 - \frac{24}{x}, x \neq 0$	**h (i)** 4	**(ii)** $h^2 : x \mapsto x, x \neq 0$

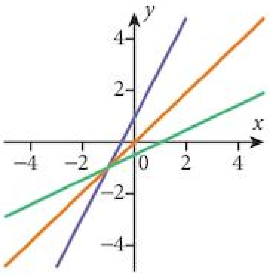
2 **(i)** $hf\left(\frac{2}{3}\right) = h(0)$: 0 is not in the domain of h.
(ii) $hg(2) = h(0)$: 0 is not in the domain of h.

3 **a** $fgfg(1) = -14$ $f^2g^2(1) = -8$ **b** $fgfg : x \mapsto 36x - 50$ $f^2g^2 : x \mapsto 36x - 44$

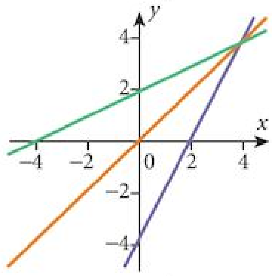
4 **a** $x = 5$ **b** $x = 3$ **c** $x = 7$ **d** $x = 6$

Exercise 4.3

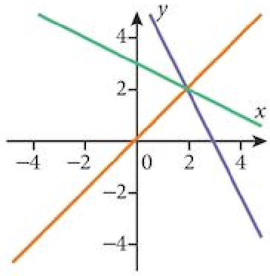
a **1** $f^{-1}: x \mapsto \frac{1}{2}(x-1)$



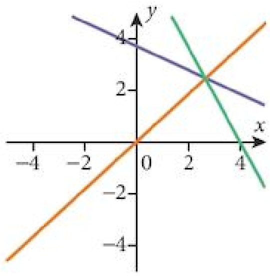
2 $f^{-1}: x \mapsto \frac{1}{2}(x+4)$



3 $g^{-1}: x \mapsto \frac{1}{2}(6-x)$

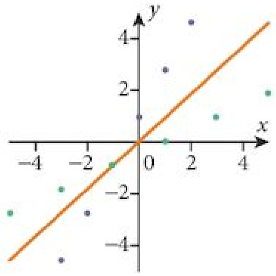


4 $g^{-1}: x \mapsto 2(4-x)$



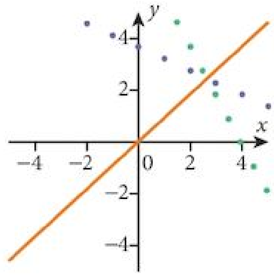
5 $h^{-1}: x \mapsto \frac{1}{2}(x-1)$

$x \in \{\text{odd numbers}\}$



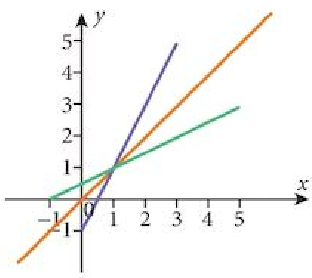
6 $h^{-1}: x \mapsto 2(4-x)$

$x \in \left\{\ldots, \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots\right\}$



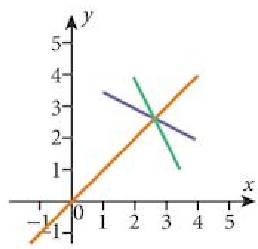
7 $f^{-1}: x \mapsto \frac{1}{2}(x+1)$

$-1 \leqslant x \leqslant 5$

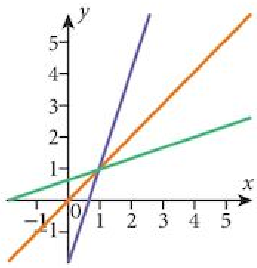


8 $h^{-1}: x \mapsto 8-2x$

$2 \leqslant x \leqslant 3.5$

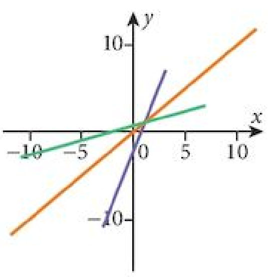


9 $g^{-1}: x \mapsto \frac{1}{3}(x+2)$



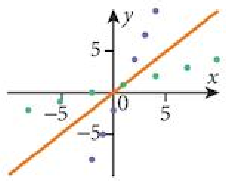
10 $g^{-1}: x \mapsto \frac{1}{3}(x+2)$

$-11 \leqslant x \leqslant 7$



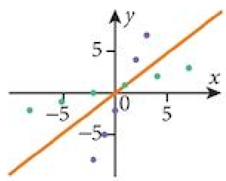
11 $h^{-1}: x \mapsto \frac{1}{3}(x+2)$

$x \in \{\ldots, -2, 1, 4, 7, \ldots\}$

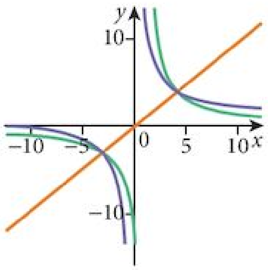


12 $h^{-1}: x \mapsto \frac{1}{3}(x+2)$

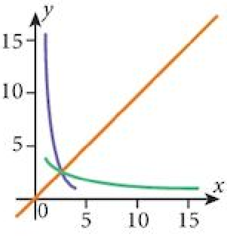
$x \in \{-11, -8, -5, -2, 1, 4, 7\}$



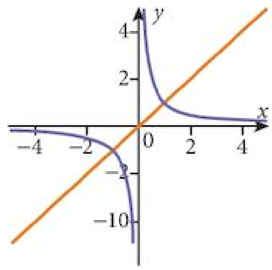
13 $y = \dfrac{12}{x-1}$
$3 \leqslant x \leqslant 13$



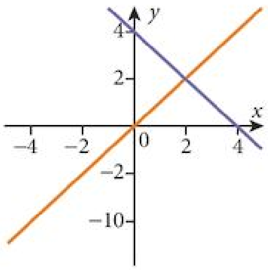
14 $y = \dfrac{4}{\sqrt{x}}$
$1 \leqslant x \leqslant 16$



15 $g^{-1} : x \mapsto \dfrac{1}{x}$



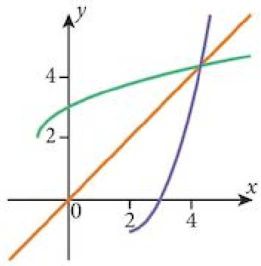
16 $g^{-1} : x \mapsto 4 - x$



Note that in Questions 15 and 16, the functions are self inverse: $g^{-1} = g$

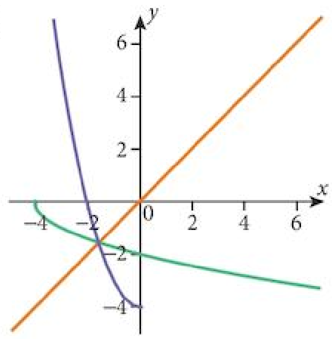
Exercise 4.4

1 **a** $f(1) = f(3) = 0$
b $f(x) \geqslant -1$
c $a = 2$
d



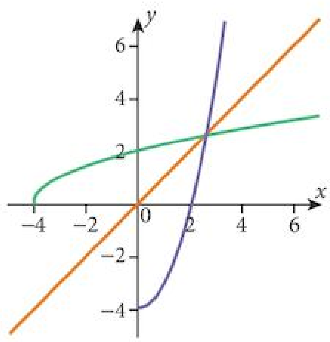
e g^{-1} is a reflection of g in the line $y = x$.

2 **a** $f(-3) = f(3) = 5$
b $f(x) \geqslant -4$
c $a = 0$
d

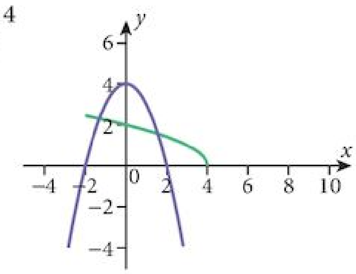


e g^{-1} is a reflection of g in the line $y = x$.

3 **a** $h^{-1} : x \mapsto \sqrt{x+4},\ x \geqslant -4$
b h^{-1} is a reflection of h in the line $y = x$.



4 **a** $f^{-1} : x \mapsto \sqrt{4-x},\ x \leqslant 4$
b f^{-1} is a reflection of f in the line $y = x$.



Exercise 4.5

1 a $(\mathrm{fg})^{-1}(x) = \frac{1}{6}(11 - x)$

2 a 7 **b** $\mathrm{jf}: x \mapsto 9x^2 - 6x + 4$ **c** $x \geqslant 3$

3 a $\mathbb{R} - \{1\}$ **b** $\mathbb{R} - \{0\}$ **c** $(\mathrm{hf})^{-1}: x \mapsto \frac{1}{3x} + 1$ Domain: $\mathbb{R} - \{0\}$ Range: $\mathbb{R} - \{1\}$

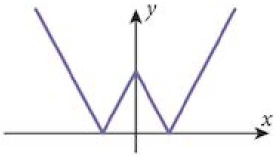
4 a $\mathbb{R} - \{2\}$ **b** $\mathbb{R} - \{0\}$ **c** $0 < x \leqslant 1$ **d** $(\mathrm{hj})^{-1}: x \mapsto \sqrt{\frac{1}{x} - 1}$ for $0 < x \leqslant 1$

5 $(\mathrm{jg})^{-1}: x \mapsto \sqrt{\frac{1}{x-3}} + 2$, for $x > 3$

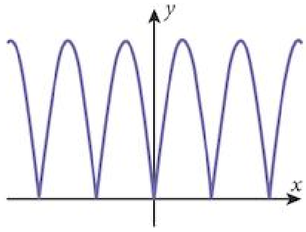
6 a $\mathrm{jg}(2) = 3$ **b** $\mathrm{g}^{-1}: x \mapsto \frac{1}{2}(4 - x)$; $\mathrm{j}^{-1}: x \mapsto \sqrt{x-3}$, $x \geqslant 3$

Exercise 4.6

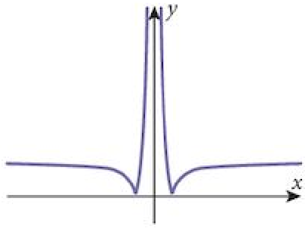
1 a



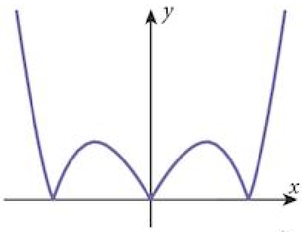
b



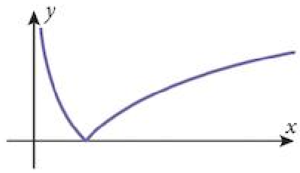
c



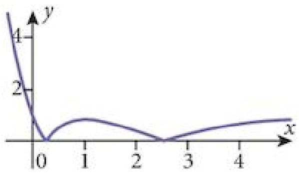
d



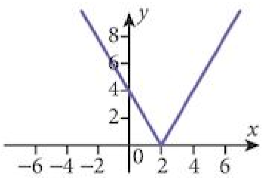
e



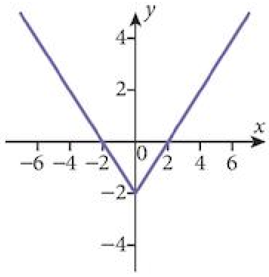
f



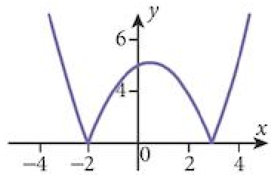
2 a



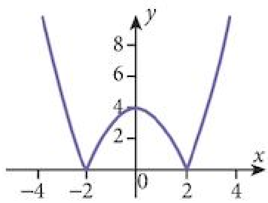
b



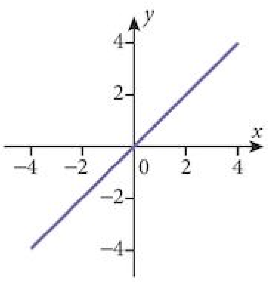
c



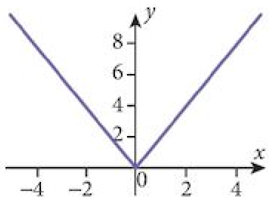
d



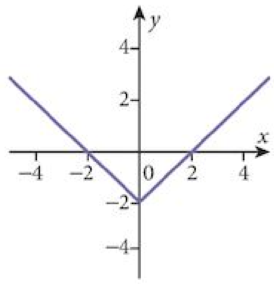
3 a



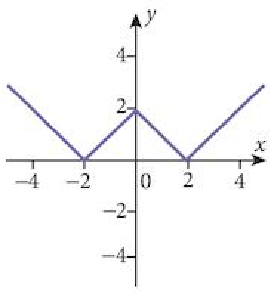
b



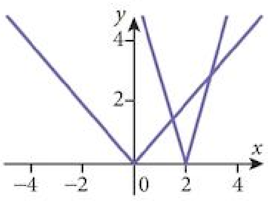
c



d



4 a



b 2 solutions, $x \in \{1.5, 3\}$

Chapter 4 Summative Exercise

1 **a** 5 **b** 7 **c** 2 **d** $-\frac{7}{4}$

2 **a** $f(x) \geqslant -4$ **b** $f(x) \in \mathbb{R}$

3 **a** $f(x);\ x \neq 1$ $g(x);\ x \neq 0$ **b** $f(x) \neq 1.5;\ g(x) \in \mathbb{R}$

4 **a** **(i)** $\text{fh} : x \mapsto 2(x^2 + 1) - 1$ **(ii)** $\text{fg}: x \mapsto \dfrac{2}{2-x} - 1$

(iii) $\text{hg} : x \mapsto \left(\dfrac{1}{2-x}\right)^2 + 1$ **(iv)** $\text{hf} : x \mapsto (2x - 1)^2 + 1$

$\text{fh}(x) > 1$ $\text{fg}(x) \neq -1$ $\text{hg}(x) > 1$ $\text{hf}(x) \geqslant 1$

b **(i)** $f(1) = 2$ This is not in the domain of g. **(ii)** $h(1) = 2$ This is not in the domain of g.

c **(i)** $f^2 : x \mapsto 2(2x - 1) - 1$ **(ii)** $h^2 : x \mapsto (x^2 + 1)^2 + 1$

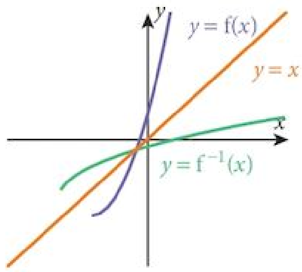
d **(i)** $\dfrac{3}{2}$ **(ii)** $g^2 : x \mapsto \dfrac{2-x}{3-2x};\ g^2(x) \neq \dfrac{1}{2}$

5 **a** $x = 4$ **b** $x = 2$

6 When it is 1 : 1

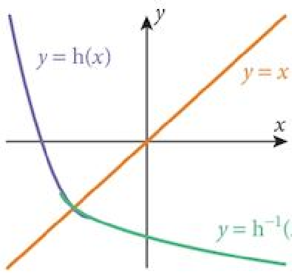
7 $f(-4) = f(0) \mapsto$ not 1 : 1

8 **a** -2 **b** $f^{-1} : x \mapsto \sqrt{x+3} - 2$ **c**

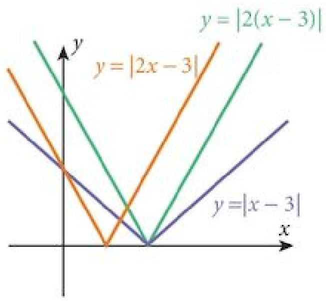


The graph of f^{-1} is a reflection of the graph of f
In the line $y = x$.

9 **a** $h^{-1} : x \mapsto -\sqrt{x+3} - 2$ **b**

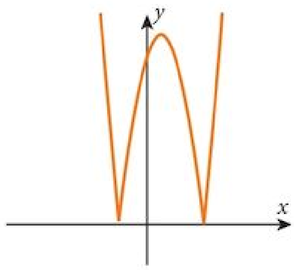


10 **a**



11 **a** $x = -2;\ x = 4$ **b** $-1 < x < 3$

12 **a**

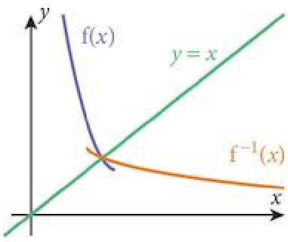


b $0 < k < 9$

13 **a** $f(13) = 5$ **b** $x = 25$ **c** $f^{-1} : x \mapsto \dfrac{1}{2}(x^2 + 1);\ x > 0;\ f^{-1}(x) > \dfrac{1}{2}$

Chapter 4 Test

1 **a/d**



b $f(x) \geqslant 2$

c $f^{-1}(x) = 3 - \sqrt{\dfrac{x-2}{2}}$

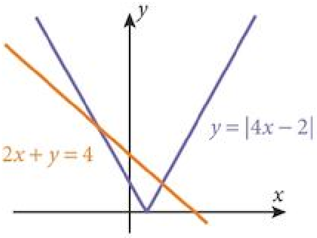
2 **a** $f(2) = 3$ – Not in the domain of g.

b $fg = \dfrac{2}{x-3} - 1$

c $\mathbb{R} - \{0\}$

d $g^{-1}(x) = 3 + \dfrac{1}{x}$, domain $\mathbb{R} - \{0\}$, range $\mathbb{R} - \{3\}$

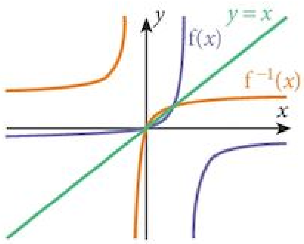
3 **a/b**



c $-1, 1$

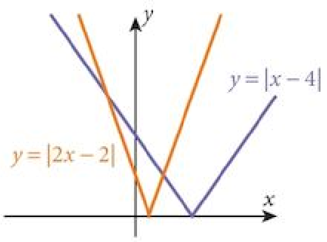
4 **a** $f^{-1} : x \mapsto \dfrac{3x}{1+x}$

b



c $a = 4, b = -3$

5 **a**



b $-2 < x < 2$

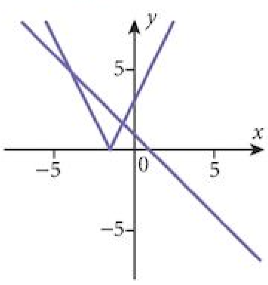
6 **a** $\dfrac{8x}{x+2}$

b $a = -2$

c $\mathbb{R} - \{-8, 4\}$

Examination Questions

1 **(i)**



(ii) $x \in \left\{-4, -\dfrac{2}{3}\right\}$

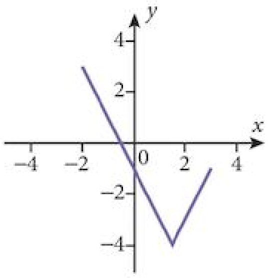
2 **(i)** $f^{-1} : x \mapsto \dfrac{3x+11}{x-3}, x \neq 3$ The graph is symmetrical about the line $y = x$.

(ii) $x \in \{-2, 5\}$

(iii) -2

3 **(i)** $-7 \leqslant f(x) \leqslant 8$ **(ii)** $0 \leqslant g(x) \leqslant 8$ **(iii)** $-7 \leqslant h(x) \leqslant 2$ Only f has an inverse

4 **(i)**



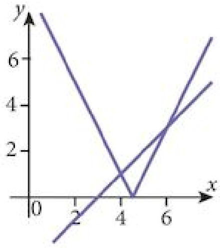
(ii) $-4 \leqslant f(x) \leqslant 3$ **(iii)** $x \in \{0.5, 2.5\}$

(iv) $\frac{3}{2}$ **(v)** $g^{-1} : x \mapsto -2x - 1$

5 **(i)** gf **(ii)** $g^{-1}f$ **(iii)** $f^{-1}g$

6 **(i)** $f^{-1} : x \mapsto \frac{1}{3}(x+2)$; $g^{-1} : x \mapsto \frac{a+x}{7-x}$ **(ii)** 8 **(iii)** $x = 3$

7 **(i)**



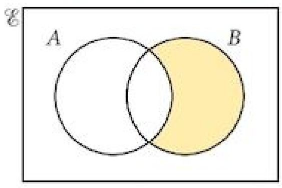
(ii) $x \in \{4, 6\}$

8 **(i)** $fg(x) = \frac{2x+6}{x+2}$ **(ii)** $-\frac{7}{4}$

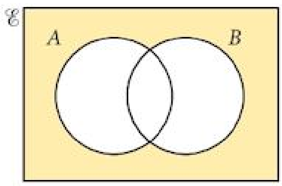
9 **(i)** $f(x) \geqslant 2$ **(ii)** $f^{-1} : x \mapsto (x-2)^2 + 3; x \geqslant 3$ **(iii)** 4.4

Term test 1A (Chapters 1–4)

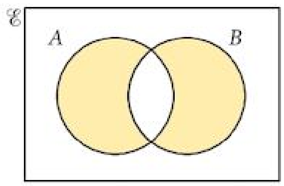
1 a



b



c

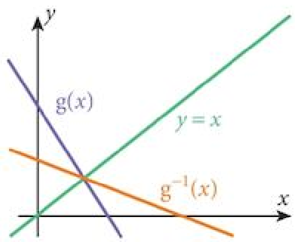


2 $2 + 2\sqrt{5}$

3 a 330 b 250 c 168

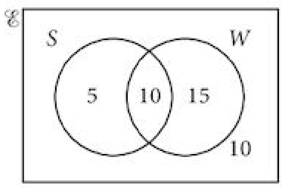
4 a $f(x) \geqslant -1$ b $fg(x) = (5 - 2x)^2 - 1$ c $-1, 6$

d f is not 1 : 1 e $g^{-1} : x = \frac{1}{2}(5 - x)$ f



5 a $n(\mathscr{E}) = 50$, $n(S \cap W') = 5$, $n(W) = 35$, $n(S \cap W) = 20$

b



6 $x = \frac{2}{5}$, $y = \frac{3}{5}$

7 a 462 b 281

Chapter 5

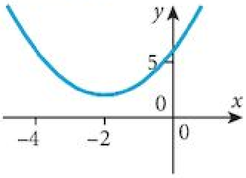
Exercise 5.1

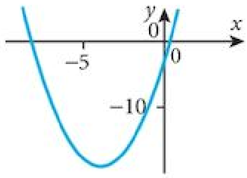
1 $(x+2)^2+2$	**2** $(x+4)^2-19$	**3** $(x+3)^2-16$	**4** $(x-3)^2-7$	**5** $(x-5)^2-40$
6 $(x-1)^2+4$	**7** $(x-2)^2+5$	**8** $(x+6)^2-33$	**9** $(x-8)^2+6$	**10** $2(x+3)^2-27$
11 $3(x+2)^2+3$	**12** $4(x+2)^2-19$	**13** $2(x-2)^2-3$	**14** $3(x-3)^2+2$	**15** $4(x-2)^2-4$
16 $-(x+2)^2+7$	**17** $-(x-1)^2-1$	**18** $-(x+3)^2+14$	**19** $-2(x-1)^2-9$	**20** $-3(x-3)^2+7$
21 $-5(x+6)^2+105$				

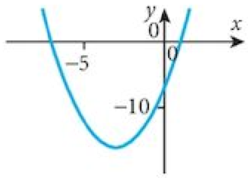
Exercise 5.2

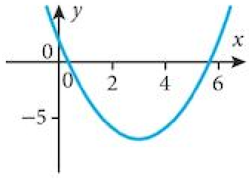
1 min 2; $x=-2$	**2** min -19; $x=-4$	**3** min -16; $x=-3$	**4** min -7; $x=3$	**5** min -40; $x=5$
6 min 4; $x=1$	**7** min 5; $x=2$	**8** min -33; $x=-6$	**9** min 6; $x=8$	**10** min -27; $x=-3$
11 min 3; $x=-2$	**12** min -19; $x=-2$	**13** min -3; $x=2$	**14** min 2; $x=3$	**15** min -4; $x=2$
16 max 7; $x=-2$	**17** max -1; $x=1$	**18** max 14; $x=-3$	**19** max -9; $x=1$	**20** max 7; $x=3$
21 max 105; $x=-6$				

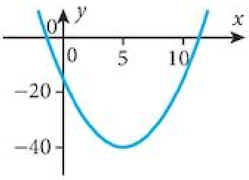
Exercise 5.3

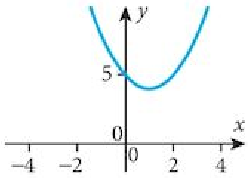
1


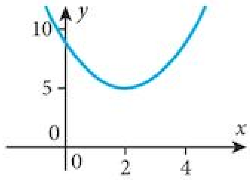
2


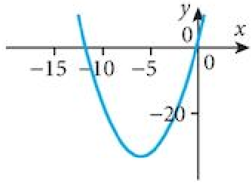
3


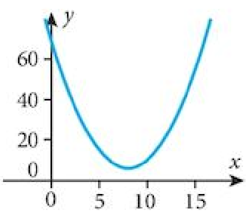
4


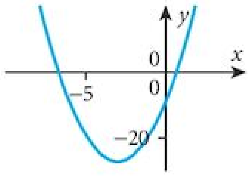
5


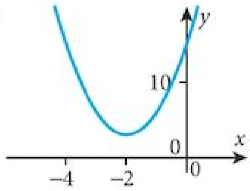
6


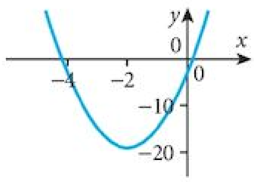
7


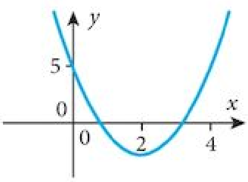
8


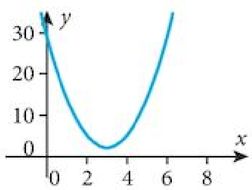
9


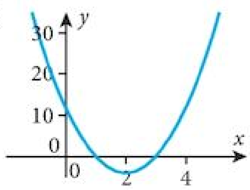
10


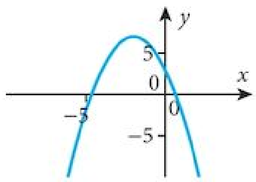
11


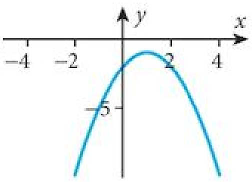
12


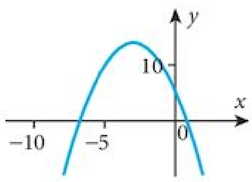
13


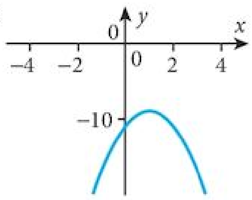
14


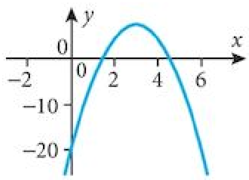
15


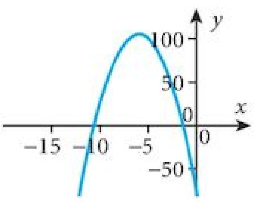
16


17


18


19


20


21


22 $k=-3$

$y \geqslant -9$

23 a 4.9m; 1 s **b** 2 s

Exercise 5.4

1 $x \in \{-4, -3\}$ 2 $x \in \{-5, -4\}$ 3 $x \in \{-6, -5\}$ 4 $x \in \{-5, 3\}$ 5 $x \in \{-6, 2\}$
6 $x \in \{-4, 2\}$ 7 $x \in \{2, 4\}$ 8 $x \in \{3, 5\}$ 9 $x \in \{4, 5\}$ 10 $x \in \{-3, 5\}$
11 $x \in \{-1, 6\}$ 12 $x \in \{-2, 6\}$ 13 $x \in \left\{-3, -\frac{1}{2}\right\}$ 14 $x \in \left\{-\frac{2}{3}, -\frac{1}{2}\right\}$ 15 $x \in \left\{-4, -\frac{4}{3}\right\}$
16 $x \in \left\{\frac{1}{3}, 4\right\}$ 17 $x \in \left\{\frac{1}{2}, \frac{2}{3}\right\}$ 18 $x \in \left\{\frac{2}{5}, 2\right\}$ 19 $x \in \{-5, -1\}$ 20 $x \in \{-5, -3\}$
21 $x \in \{-5, -2\}$ 22 $x \in \{-5, 2\}$ 23 $x \in \{-4, 3\}$ 24 $x \in \{-6, 4\}$ 25 $x \in \{3, 4\}$
26 $x \in \{1, 5\}$ 27 $x \in \{3, 6\}$ 28 $x \in \{-2, 5\}$ 29 $x \in \{-2, 4\}$ 30 $x \in \{-1, 4\}$
31 $x \in \left\{-2, -\frac{1}{3}\right\}$ 32 $x \in \left\{-2, -\frac{3}{2}\right\}$ 33 $x \in \left\{-1, -\frac{1}{4}\right\}$ 34 $x \in \left\{\frac{1}{2}, 3\right\}$ 35 $x \in \left\{\frac{4}{3}, 2\right\}$
36 $x \in \left\{-\frac{2}{5}, 3\right\}$ 37 $x \in \{-5, -1\}$ 38 $x \in \{-5, -3\}$ 39 $x \in \{-5, -2\}$ 40 $x \in \{-5, 2\}$
41 $x \in \{-4, 3\}$ 42 $x \in \{-6, 4\}$ 43 $x \in \{3, 4\}$ 44 $x \in \{1, 5\}$ 45 $x \in \{3, 6\}$
46 $x \in \{-2, 5\}$ 47 $x \in \{-2, 4\}$ 48 $x \in \{-1, 4\}$ 49 $x \in \left\{-2, -\frac{1}{3}\right\}$ 50 $x \in \left\{-2, -\frac{3}{2}\right\}$
51 $x \in \left\{-1, -\frac{1}{4}\right\}$ 52 $x \in \left\{\frac{1}{2}, 3\right\}$ 53 $x \in \left\{\frac{4}{3}, 2\right\}$ 54 $x \in \left\{-\frac{2}{5}, 3\right\}$

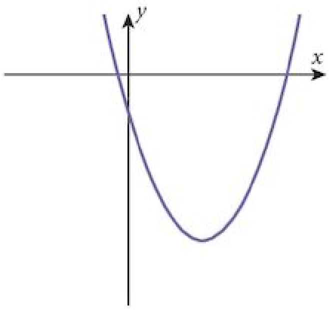
Exercise 5.5

1 −8; 0 2 0; 2 equal 3 76; 2 unequal 4 64; 2 unequal 5 28; 2 unequal
6 0; 2 equal 7 0; 2 equal 8 160; 2 unequal 9 −16; 0 10 −20; 0
11 0; 2 equal 12 132; 2 unequal 13 0; 2 equal 14 −24; 0 15 216; 2 unequal
16 0; 2 equal 17 −36; 0 18 304; 2 unequal 19 24; 2 unequal 20 0; 2 equal
21 −24; 0 22 64; 2 unequal 23 28; 2 unequal 24 −4; 0 25 0; 2 equal
26 56; 2 unequal 27 −72; 0 28 84; 2 unequal 29 800; 2 unequal 30 2100; 2 unequal

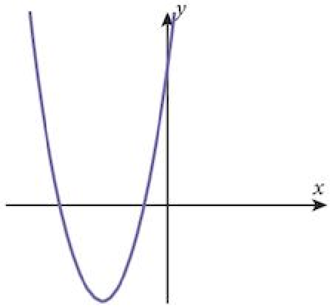
Chapter 5 Summative Exercise

1 a $(x-4)^2 - 11$ b $(x+6)^2 - 27$ c $(x-5)^2 - 32$
d $2(x-3)^2 - 7$ e $3(x+3)^2 - 19$ f $5(x-1)^2 - 17$
g $-2(x+3)^2 + 29$ h $3(x+2)^2 - 5$ i $-3(x+1)^2 + 7$

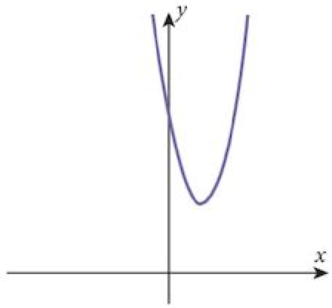
2 a Min -12, $x = -3$ b Max 14, $x = -2$ c Min -12.25, $x = 0.75$
d Max $\frac{40}{3}$, $x = -\frac{4}{3}$ e Min $-\frac{41}{8}$, $x = -\frac{9}{4}$ f Min $-27\frac{1}{3}$, $x = \frac{8}{3}$
g Max -1, $x = 1$ h Max 8, $x = 1$ i Max $\frac{7}{3}$, $x = \frac{2}{3}$

3 a


(3, −23)

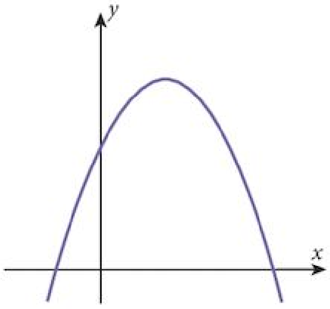
b


(−2, −5)

c


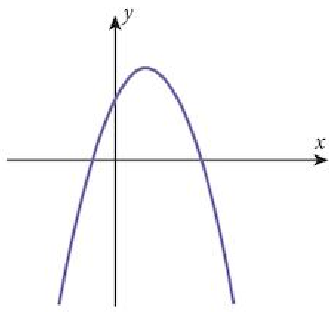
(1, 4)

d



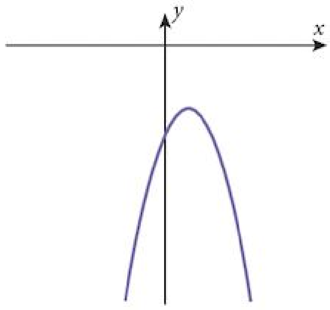
$(2, 11)$

e



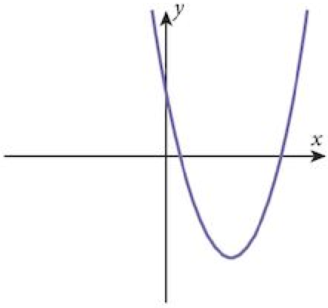
$\left(\frac{2}{3}, -3\frac{2}{3}\right)$

f



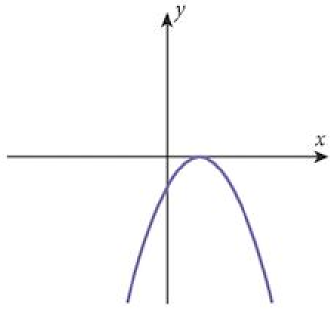
$(2, -7)$

g



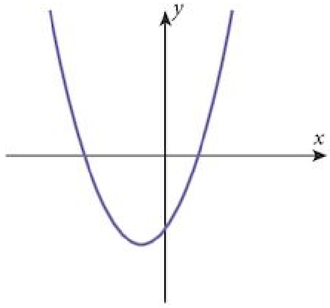
$(1, 0)$

h



$\left(-\frac{3}{4}, 6\frac{1}{8}\right)$

i



$\left(1\frac{1}{2}, 6\frac{1}{4}\right)$

4 **a** 13; 2 distinct roots **b** 0; 2 equal roots **c** 1; 2 distinct roots
d 128; 2 distinct roots **e** 28; 2 distinct roots **f** −7; no real roots
g 1; 2 distinct roots **h** 73; 2 distinct roots **i** 89; 2 distinct roots

5 **a** $-3, 2$ **b** $-3, -\frac{3}{2}$ **c** $-2, \frac{5}{3}$
d $-\frac{7}{3}, 3$ **e** $-1, \frac{5}{2}$ **f** $1, \frac{3}{2}$
g $-3, \frac{5}{4}$ **h** $-\frac{6}{5}, 1$ **i** $-4, -\frac{5}{4}$

6 **a** $1, 5$ **b** $-\frac{3}{2}, 2$ **c** $-\frac{5}{3}, -2$
d $\frac{1}{2}, \frac{2}{3}$ **e** $-\frac{3}{2}, \frac{1}{2}$ **f** $\frac{1}{2}, \frac{7}{2}$
g $\frac{2}{3}, \frac{10}{3}$ **h** $\frac{1}{2}, \frac{11}{2}$ **i** $-3 \pm \sqrt{2}$

7 **a** 0.268, 3.732 **b** −1.667, 1 **c** −1.721, 0.387
d −0.194, 0.860 **e** −1.380, 0.580 **f** 0.326, 7.674
g −3.186, −0.314 **h** −0.443, 1.693 **i** −1.935, −0.287

8 **a** $-3 < x < 2$ **b** $x < -3$ or $x > 4$ **c** $-\frac{2}{3} \leqslant x \leqslant \frac{5}{2}$
d $x < -\frac{5}{2}$ or $x > \frac{3}{2}$

9 3 cm by 7 cm

10 **b** $A = \frac{25}{12}x(4 - x)$ **c** Max $A = \frac{25}{3}, x = 2$

11 $12, x = 2$

12 **a** $A = 100x - \frac{x^2}{2}$ **b** Max $A = 5000\ \text{m}^2, x = 100$

Chapter 5 Test

1 $f(x) \leqslant 10$

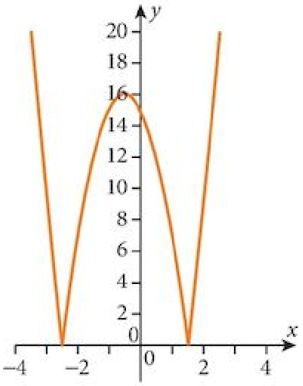
2 **a** **(i)** 2 **(ii)** 3 **(iii)** 0

b **(i)** $\{-3, -1, 2\}$ **(ii)** $\{-4, -3, -1, 2\}$

c $Z' = \{\text{all real numbers}\}$

3 **a** $4\left(x+\frac{1}{2}\right)^2 - 16$

b



c $0 < k < 16$

4 $-1 < k < 7$

5 **a** $f^2 : x \rightarrow \dfrac{9x}{x+1}$ **b** $f^{-1} : x \rightarrow \dfrac{2x}{x-6}$ **c** 0, 8

6 $-2-2\sqrt{3} < k < -2+2\sqrt{3}$

7 $k < -24$, or $k > 0$

8 **a** 2 **b** 0

Chapter 6

Exercise 6.1

1 Touches at (2, 6)

2 Intersects at (−1, 12), (2, 15)

3 Intersects at (−2, 5), (−1, 4)

4 Intersects at (−3, −4), (2, 6)

5 Misses

6 Touches at (−4, 6)

7 Misses

8 Touches at (1, 4)

9 Intersects at (−5, 8), (3, 0)

10 Intersects at (1, 6), $\left(9, \frac{2}{3}\right)$

11 Intersects at (−2, 1), (3, 11)

12 Misses

13 5

14 $k < -9\frac{9}{16}$

15 $k \in \{-3, 5\}$

16 **a** 14 ms^{-1} **b** $\frac{10}{7}$ s

17 8, 9

18 16 cm, 9 cm

19 **a** $p + q = 15$, $pq = 36$ **b** 3, 12 **c** $x^2 - 15x + 36 = 0$

Exercise 6.2

1

		3	
$x-3$	−	0	+

2

		−2	
$x+2$	−	0	+

3

		4	
$4-x$	+	0	−

4

		−2		2	
$x+2$	−	0	+	+	+
$x-2$	−	−	−	0	+
$f(x)$	+	0	−	0	+

5

		−3		4	
$x+3$	−	0	+	+	+
$x-4$	−	−	−	0	+
$f(x)$	+	0	−	0	+

6

		−1		4	
$x+1$	−	0	+	+	+
$4-x$	+	+	+	0	−
$f(x)$	−	0	+	0	−

7

		2		5	
$x-2$	−	0	+	+	+
$x-5$	−	−	−	0	+
$f(x)$	+	0	−	0	+

8

		−4		−3	
$x+4$	−	0	+	+	+
$x+3$	−	−	−	0	+
$f(x)$	+	0	−	0	+

9

		3		6	
$3-x$	−	0	+	+	+
$6-x$	−	−	−	0	+
$f(x)$	+	0	−	0	+

10

		−2		−1		3	
$x+2$	−	0	+	+	+	+	+
$x-3$	−	−	−	−	−	0	+
$x+1$	−	−	−	0	+	+	+
$f(x)$	−	0	+	∞	−	0	+

11

		−3		−1		5	
$x+1$	−	−	−	0	+	+	+
$x+3$	−	0	+	+	+	+	+
$x-5$	−	−	−	−	−	0	+
$f(x)$	−	∞	+	0	−	∞	+

12

		−5		−3		3		4	
$3-x$	−	−	−	−	−	0	+	+	+
$4-x$	−	−	−	−	−	−	−	0	+
$x+3$	−	−	−	0	+	+	+	+	+
$x+5$	−	0	+	+	+	+	+	+	+
$f(x)$	+	∞	−	∞	+	0	−	0	+

13 $x < 2$ or $x > 5$
14 $-5 < x < 1$
15 $x < -6$ or $x > -1$
16 $x \leqslant -1$ or $x \geqslant 4$
17 $x \leqslant -2$ or $x \geqslant 3$
18 $-4 \leqslant x \leqslant 2$
19 $x \leqslant -2$ or $x \geqslant 3$
20 $x < 2$ or $x > 4$
21 $x \leqslant -5$ or $x \geqslant 1$
22 $x < -\frac{2}{3}$ or $x > \frac{1}{2}$
23 $x \leqslant -2$ or $x \geqslant \frac{3}{4}$
24 $x < -\frac{3}{5}$ or $x > 2$
25 $x \leqslant 3 - \sqrt{13}$ or $x \geqslant 3 + \sqrt{13}$
26 $\frac{-5-\sqrt{41}}{2} < x < \frac{-5+\sqrt{41}}{2}$
27 $x \leqslant 3 - \sqrt{8}$ or $x \geqslant 3 + \sqrt{8}$
28 $x < \frac{2-\sqrt{10}}{2}$ or $x > \frac{2+\sqrt{10}}{2}$
29 $\frac{9-\sqrt{129}}{6} \leqslant x \leqslant \frac{9+\sqrt{129}}{6}$
30 $\frac{5-\sqrt{13}}{4} \leqslant x \leqslant \frac{5+\sqrt{13}}{4}$
31 $10 - \sqrt{60} <$ width < 10

Chapter 6 Summative Exercise

1 **a** Intersects **b** Intersects **c** Does not intersect
d Intersects **e** Touches **f** Intersects
g Does not Intersect **h** Touches

2 **a** (−2, 3); (1, 12) **b** (−2, 8); (1, 5) **c** (1, −9); (6, 11)
d (−1, −8); (2, −2) **e** (−3, 12); (4, 19) **f** They do not intersect
g (−5, −2); (2, 5) **h** (−2, −8); (4, 4)

3 **a** (1, −2) **b** (2, 1) **c** (5, −11)
d (5, 15) **e** (−2, 8) **f** (−1, 11)
g (4, 3) **h** (− 4, −5)

4 **a** $x < -4$ or $x > 5$ **b** $-3 < x < 6$ **c** $-\frac{1}{2} \leqslant x \leqslant \frac{2}{3}$
d $x \leqslant -\frac{2}{3}$ or $x \geqslant 5$ **e** $x \leqslant -\frac{3}{2}$ or $x \geqslant 4$ **f** $-\frac{5}{2} \leqslant x \leqslant 3$
g $1 < x < \frac{5}{2}$ **h** $x < \frac{4}{3}$ or $x > 4$

5 **a**

		−2		3	
$x-3$	−	−	−	0	+
$x+2$	−	0	+	+	+
$f(x)$	+	∞	−	0	+

b

		−1		6	
$6-x$	+	+	+	0	−
$x+1$	−	0	+	+	+
$f(x)$	−	∞	+	0	−

c

		−2		−1		3	
$x+1$	−	−	−	0	+	+	+
$x-3$	−	−	−	−	−	0	+
$x+2$	−	0	+	+	+	+	+
$f(x)$	−	∞	+	0	−	0	+

d

		−3		1		2	
$x+3$	−	0	+	+	+	+	+
$2-x$	+	+	+	+	+	0	−
$x-1$	−	−	−	0	+	+	+
$f(x)$	+	0	−	∞	+	0	−

e

		-3		-1		2	
$x-2$	$-$	$-$	$-$	$-$	$-$	0	$+$
$x+3$	$-$	0	$+$	$+$	$+$	$+$	$+$
$x+1$	$-$	$-$	$-$	0	$+$	$+$	$+$
$f(x)$	$-$	∞	$+$	∞	$-$	0	$+$

f

		-4		-1		2	
$x+1$	$-$	$-$	$-$	0	$+$	$+$	$+$
$x+4$	$-$	0	$+$	$+$	$+$	$+$	$+$
$x-2$	$-$	$-$	$-$	$-$	$-$	0	$+$
$f(x)$	$-$	∞	$+$	0	$-$	∞	$+$

g

		-2		-1		3		4	
$x-3$	$-$	$-$	$-$	$-$	$-$	0	$+$	$+$	$+$
$x+2$	$-$	0	$+$	$+$	$+$	$+$	$+$	$+$	$+$
$x+1$	$-$	$-$	$-$	0	$+$	$+$	$+$	$+$	$+$
$x-4$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	0	$+$
$f(x)$	$+$	0	$-$	∞	$+$	0	$-$	∞	$+$

h

		-4		-3		2		4	
$x+3$	$-$	$-$	$-$	0	$+$	$+$	$+$	$+$	$+$
$x-2$	$-$	$-$	$-$	$-$	$-$	0	$+$	$+$	$+$
$x+4$	$-$	0	$+$	$+$	$+$	$+$	$+$	$+$	$+$
$x-4$	$-$	$-$	$-$	$-$	$-$	$-$	$-$	0	$+$
$f(x)$	$+$	∞	$-$	0	$+$	0	$-$	∞	$+$

6 $c > -30$
7 $(-3, -5); (5, 9)$
8 $(-6, -5); (6, 5)$
9 $p \leqslant 0$ or $p \geqslant 1$
10 6, 9
11 7 cm, 12 cm
12 $-5, 7; x^2 - 2x - 35 = 0$
13 **a** $x = 8, y = 5$ **b** $x = -1.5, y = 14.5$
14 $2 \leqslant x \leqslant 6$
15 **a** $2 \leqslant$ shorter edge $\leqslant 11$ **b** $11 \leqslant$ longer edge $\leqslant 20$

Chapter 6 Test

1 $(-1, 1), (5, 3)$
2 $\left(2, \frac{9}{4}\right), \left(3, \frac{3}{2}\right)$
3 $(2, -2), \left(9, \frac{3}{2}\right)$
4 $0 < m < \frac{4}{3}$
5 $-18 < m < -6$
6 $(-1, -2), (5, 2)$
7 $(2, -5), (5, -3)$
8 $(-6, 2), (3, -4)$

Examination Questions

1 ± 3
2 $0 \leqslant k$
3 **(i)** $x < 2$ or $x > 6$ **(ii)** $0 < x < 8$; $0 < x < 2$ or $6 < x < 8$
4 $k < -6$
5 $0 \leqslant k \leqslant 4$
6 $x < 4$ or $x > 9$
7 $c \in \{-12, 12\}$
8 **a** $m = -1.25$ **b** $c = 2, d = 15$
9 $\frac{1}{2} < x < 6$
10 **(i)** $k \in \{-16, 16\}$ **(ii)** $(-2, 10), (2, -10)$
11 $(-1, -5), (3, 3)$
12 $x < -1$ or $x > 2$
13 $-13 < m < 3$
14 $-1, \frac{1}{4}$

Chapter 7

Exercise 7.1

1 $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$
2 $a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$
3 $a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$
4 $a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b^7$

Exercise 7.3

1 $a^8 + 8a^7b + 28a^6b^2 + 56a^5b^3 + 70a^4b^4 + 56a^3b^5 + 28a^2b^6 + 8ab^7 + b^8$
2 $1 + 4x + 6x^2 + 4x^3 + x^4$
3 $243 + 405x + 270x^2 + 90x^3 + 15x^4 + x^5$
4 $64 + 192x + 240x^2 + 160x^3 + 60x^4 + 12x^5 + x^6$

Exercise 7.5

1 a $1 - 4x + 6x^2 - 4x^3 + x^4$ b $1 - 10x + 40x^2 - 80x^3 + 80x^4 - 32x^5$
c $8 - 12x + 6x^2 - x^3$ d $729 - 2916x + 4860x^2 - 4320x^3 + 2160x^4 - 576x^5 + 64x^6$
e $1 + 2x + \frac{3}{2}x^2 + \frac{1}{2}x^3 + \frac{1}{16}x^4$ f $32 - 20x + 5x^2 - \frac{5}{8}x^3 + \frac{5}{128}x^4 - \frac{1}{1024}x^5$
g $8 + 9x + \frac{27}{8}x^2 + \frac{27}{64}x^3$ h $1 - \frac{3}{5}x + \frac{3}{25}x^2 - \frac{1}{125}x^3$
i $81 + 108x + 54x^2 + 12x^3 + x^4$ j $32 - 240x + 720x^2 - 1080x^3 + 810x^4 - 243x^5$
k $1 + 2x + \frac{4}{3}x^2 + \frac{8}{27}x^3$ l $4096 - 6144x + 3840x^2 - 1280x^3 + 240x^4 - 24x^5 + x^6$
m $\frac{1}{16} - x + 6x^2 - 16x^3 + 16x^4$ n $\frac{1}{243} + \frac{5}{27}x + \frac{10}{3}x^2 + 30x^3 + 135x^4 + 243x^5$
o $\frac{1}{64} - \frac{3}{8}x + 3x^2 - 8x^3$ p $64 + 96x + 60x^2 + 20x^3 + \frac{15}{4}x^4 + \frac{3}{8}x^5 + \frac{1}{64}x^6$

2 a $1 - \frac{4}{x} + \frac{6}{x^2} - \frac{4}{x^3} + \frac{1}{x^4}$ b $x^{10} + 10x^8 + 40x^6 + 80x^4 + 80x^2 + 32$
c $x^6 - 3x^3 + 3 - \frac{1}{x^3}$ d $x^6 - 6x^4 + 15x^2 - 20 + \frac{15}{x^2} - \frac{6}{x^4} + \frac{1}{x^6}$

3 a −326 592 b −960 c 59 136 d 455
e 77 520 f 438 480 g 1 180 980 h −7 741 440

4 a $1 - 30x + 420x^2 - 3640x^3 \ldots$
b $1\,048\,576 + 10\,485\,760x + 49\,807\,360x^2 + 149\,422\,080x^3 \ldots$
c $1 - 36x + 594x^2 - \ldots$
d $1 + 36x + 612x^2 + 6528x^3 + 48\,960x^4 + \cdots$
e $1 + 40x + 760x^2 + 9120x^3 + 77\,520x^4 + \cdots$
f $\left(\frac{1}{2}\right)^{12} - 12\left(\frac{1}{2}\right)^{11}x + 66\left(\frac{1}{2}\right)^{10}x^2 - \cdots$
g $65\,536 - 262\,144x + 491\,520x^2 - 573\,440x^3 \ldots$
h $1 - 5x + \frac{35}{3}x^2 - \frac{455}{27}x^3 \ldots$

5 a 1.22 b 0.785 c 1.60 d 0.970

Exercise 7.6

1 a $1 + 4u + 6u^2 + 4u^3 + u^4$ b $1 + 4x + 10x^2 + 16x^3 + 19x^4 + 16x^5 + 10x^6 + 4x^7 + x^8$
2 252
3 a $1 - 6x + 15x^2 - 20x^3 \ldots$ b −85
4 $\frac{495}{16}$
5 −512
6 a $1 - 5u + 10u^2 - 10u^3 + 5u^4 - u^5$
b $1 - 5x + 15x^2 - 30x^3 + 45x^4 - 51x^5 + 45x^6 - 30x^7 + 15x^8 - 5x^9 + x^{10}$
7 a $x^8 - 8x^6 + 28x^4 - 56x^2 + 70 - \frac{56}{x^2} + \frac{28}{x^4} - \frac{8}{x^6} + \frac{1}{x^8}$ b 2
8 $n = 10$; $a = 2$; $b = 960$

Chapter 7 Summative Exercise

1 1 10 45 120 210 252 210 120 45 10 1
2 a $a^3 + 3a^2b + 3ab^2 + b^3$
b $a^9 + 9a^8b + 36a^7b^2 + 84a^6b^3 + 126a^5b^4 + 126a^4b^5 + 84a^3b^6 + 36a^2b^7 + 9ab^8 + b^9$
c $a^{10} + 10a^9b + 45a^8b^2 + 120a^7b^3 + 210a^6b^4 + 252a^5b^5 + 210a^4b^6 + 120a^3b^7 + 45a^2b^8 + 10a^2b^9 + b^{10}$

3 a $1+4x+6x^2+4x^3+x^4$ b $1-4x+6x^2-4x^3+x^4$

c $16+32x+24x^2+8x^3+x^4$ d $x^4-8x^3+24x^2-32x+16$

4 a $\frac{7\times 6\times 5}{3\times 2\times 1}$ b $\frac{9\times 8\times 7}{3\times 2\times 1}$ c $\frac{10\times 9\times 8}{3\times 2\times 1}$ d $\frac{14\times 13}{2\times 1}$

5 a $1-3x+3x^2-x^3$ b $16x^4-32x^3+24x^2-8x+1$ c $27-27x+9x^2-x^3$

d $16-96x+216x^2-216x^3+81x^4$

6 a $32+\frac{80}{x}+\frac{80}{x^2}+\frac{40}{x^3}+\frac{10}{x^4}+\frac{1}{x^5}$ b $\frac{16}{x^4}-\frac{32}{x^3}+\frac{24}{x^2}+\frac{8}{x}+1$

c $x^{12}+6x^8+15x^4+20+\frac{15}{x^4}+\frac{6}{x^8}+\frac{1}{x^{12}}$ d $x^8+8x^6+28x^4+56x^2+70+\frac{56}{x^2}+\frac{28}{x^4}+\frac{8}{x^6}+\frac{1}{x^8}$

7 a 10 264 320 b 768 768 c 1 959 552 d 672 e 495 f −560

8 $1+16x+112x^2+448x^3+1120x^4$; 1.171659

9 $4096-\frac{24\,576}{x}+\frac{67\,584}{x^2}-\frac{112\,640}{x^3}+\frac{126\,720}{x^4}-\frac{101\,376}{x^5}$; 3856.88702

10 a $64-576x+2160x^2-4320x^3$ b 1008

11 a $1-16x+112x^2-448x^3$ b −1232

12 a $99+70\sqrt{2}$ b $99-70\sqrt{2}$

13 $a=3, n=12$

14 −24 15 5376 16 40 095

17 $n=2a+1$ 18 $a=3, b=2, c=224$

19 a $p=a^n,\ q=na^{n-1}b,\ r=\frac{n(n-1)}{2}a^{n-1}b$ b 5

20 $a=-2, n=7$

Chapter 7 Test

1 a $1+10x+40x^2$ b 57

2 495

3 a $32+48x^2+2x^4$ b −3, 3

4 a 1120 b 1125

5 a $243+810x+1080x^2$ b $a=2, b=-2$

6 a 15 120 b 7 560

7 a $1-\frac{n}{3}x+\frac{n(n-1)}{18}x^2$ b 15

8 13440

Examination Questions

1 $64+192x+240x^2$; 48 2 (i) $32-80x+80x^2$ (ii) $\frac{1}{2}$

3 (i) $243x^5-405x^4+270x^3\ldots$ (ii) 135

4 a −102 b $k=\frac{3}{n-2}$ 5 $n=7$; $a=3$; $b=238$

6 a (i) $32+80x+80x^2+40x^3+10x^4+x^5$ (ii) $a=362$; $b=209$

b −2240

7 $n=8$; $p=-\frac{3}{2}$; $q=-189$ 8 (i) $k=3$ (ii) −39

9 3 10 (i) $32+80x+80x^2\ldots$ (ii) −80

11 (i) $1+5x+10x^2+10x^3+5x^4+x^5$ (ii) $41+29\sqrt{2}$ (iii) 82

12 (i) 1512 (ii) 504

13 (i) $32-240x+720x^2\ldots$ (ii) $a=2$; $b=9$; $c=-720$

14 (i) 240 (ii) 200

15 (i) $64-96x+60x^2-20x^3\ldots$ (ii) 4

Chapter 8

Exercise 8.1

1	$2x^2 - 3x + 13$;	-37	**2**	$3x^2 - 4x + 12$;	-27	**3**	$2x^2 + x - 5$;	3
4	$x^2 - 7x + 26$;	-105	**5**	$x^2 + x - 2$;	4	**6**	$2x^2 - 7x + 18$;	-31
7	$2x^2 - 9x + 31$;	-95	**8**	$3x^2 - 14x + 52$;	-205	**9**	$2x^2 + 5x + 9$;	11
10	$3x^2 + 14x + 60$;	237	**11**	$2x^2 + 9x + 23$;	67	**12**	$x^2 - x - 4$;	-9
13	$x^2 + 5x + 14$;	44	**14**	$2x^2 + 5x + 24$;	101	**15**	$2x^2 - x + 3$;	1
16	$3x^2 + 4x + 4$;	11	**17**	$2x - 5$;	$11x - 7$	**18**	$3x + 1$;	4
19	$x + 3$;	$-x + 1$	**20**	$2x - 5$;	$4x + 7$	**21**	$2x - 1$;	$10x$
22	$3x + 7$;	$20x + 10$	**23**	$x + 4$;	$4x$	**24**	$2x - 5$;	8

Exercise 8.2

1 -37	**2** -27	**3** 3	**4** -105	**5** 4	**6** -31
7 -95	**8** -205	**9** 11	**10** 237	**11** 67	**12** -9
13 44	**14** 101	**15** 1	**16** 11	**17** $-\frac{1}{2}$	**18** $\frac{1}{3}$
19 $\frac{37}{8}$	**20** $\frac{88}{27}$				

Exercise 8.3

1 $a = 1$	**2** $a = 5$	**3** $a = 9$	**4** $a = 4$; $b = -9$	**5** $a = -3$; $b = -2$
6 26	**7** 31	**8** $a = 0$	**9** $a = -5$; $b = 4$	**10** $a = 2$; $b = 2$

Exercise 8.4

1 **b** $f(x) = (x + 2)(x - 1)(x - 2)$ **c** $x \in \{-2, 1, 2\}$

2 **b** $f(x) = (x + 6)(x + 2)(2x - 5)$ **c** $x \in \left\{-6, -2, \frac{5}{2}\right\}$

3 **a** **(i)** $f(x) = (x + 2)(x - 1)(x - 3)$ **(ii)** $x \in \{-2, 1, 3\}$

b **(i)** $f(x) = (x + 4)(x + 1)(x - 1)$ **(ii)** $x \in \{-4, -1, 1\}$

c **(i)** $f(x) = (x + 5)(x + 2)(x - 1)$ **(ii)** $x \in \{-5, -2, 1\}$

d **(i)** $f(x) = (x + 3)(x - 2)(x - 2)$ **(ii)** $x \in \{-3, 2\}$ (double root at $x = 2$)

e **(i)** $f(x) = (x + 3)(x - 1)(x - 4)$ **(ii)** $x \in \{-3, 1, 4\}$

f **(i)** $f(x) = (x + 2)(x + 2)(x - 2)$ **(ii)** $x \in \{-2, 2\}$ (double root at $x = -2$)

g **(i)** $f(x) = (x + 2)(x - 2)(x - 5)$ **(ii)** $x \in \{-2, 2, 5\}$

h **(i)** $f(x) = (x + 5)(x + 2)(x + 1)$ **(ii)** $x \in \{-5, -2, -1\}$

i **(i)** $f(x) = (x + 3)(x - 2)(x - 4)$ **(ii)** $x \in \{-3, 2, 4\}$

j **(i)** $f(x) = (x + 3)(x - 2)(x - 3)$ **(ii)** $x \in \{-3, 2, 3\}$

k **(i)** $f(x) = (x + 2)(x - 3)(x - 3)$ **(ii)** $x \in \{-2, 3\}$ (double root at $x = 3$)

l **(i)** $f(x) = (x + 3)(x - 2)(x - 4)$ **(ii)** $x \in \{-3, 2, 4\}$

4 **a** **(i)** $f(x) = (x + 2)(2x + 1)(x - 1)$ **(ii)** $x \in \left\{-2, -\frac{1}{2}, 1\right\}$

b **(i)** $f(x) = (x + 4)(3x + 1)(x - 1)$ **(ii)** $x \in \left\{-4, -\frac{1}{3}, 1\right\}$

c **(i)** $f(x) = (x + 2)(x + 1)(2x - 3)$ **(ii)** $x \in \left\{-2, -1, \frac{3}{2}\right\}$

d (i) $f(x) = (x+3)(x+1)(2x+1)$ (ii) $x \in \left\{-3, -1, -\frac{1}{2}\right\}$

e (i) $f(x) = (x+3)(3x+1)(x-1)$ (ii) $x \in \left\{-3, -\frac{1}{3}, 1\right\}$

f (i) $f(x) = (x-1)(2x-3)(x-2)$ (ii) $x \in \left\{1, \frac{3}{2}, 2\right\}$

g (i) $f(x) = (x+4)(2x+1)(x-1)$ (ii) $x \in \left\{-4, -\frac{1}{2}, 1\right\}$

h (i) $f(x) = (x+2)(3x+1)(x-1)$ (ii) $x \in \left\{-2, -\frac{1}{3}, 1\right\}$

i (i) $f(x) = (x+3)(x-1)(2x-3)$ (ii) $x \in \left\{-3, 1, \frac{3}{2}\right\}$

j (i) $f(x) = (x+2)(2x+1)(x-2)$ (ii) $x \in \left\{-2, -\frac{1}{2}, 2\right\}$

k (i) $f(x) = (3x+1)(x-1)(x-2)$ (ii) $x \in \left\{-\frac{1}{3}, 1, 2\right\}$

l (i) $f(x) = (2x+3)(x-1)(x-2)$ (ii) $x \in \left\{-\frac{3}{2}, 1, 2\right\}$

Chapter 8 Summative Exercise

1 a $x^2 - 7x + 26; -84$ b $2x^2 + 9x + 29; 125$ c $3x^2 - 20x + 102; -514$
d $4x^2 + 10x + 15; 33$ e $x^3 - 6x^2 + 9x - 11; 4$ f $3x^3 + 11x^2 + 29x + 86; 261$
g $3x + 7; -17$ h $4x^2 - 7x + 23; -90x + 50$

2 a -4 b 6 c -136 d $\frac{27}{2}$ e 65 f 831 g 3 h 4

3 -2 **4** -6 **5** 2 **6** $3, -1$

7 b $(x+2)(x+3)(x-2)$ c $-3, -2, 2$

8 b $(x+2)(2x-1)(x-3)$ c $-2, \frac{1}{2}, 3$

9 b $(x-3)(2x+3)(2x-3)$ c $-\frac{3}{2}, \frac{3}{2}, 3$

10 a 3 b $(x-1)(2x-3)(x+3)$ c $-3, 1, \frac{3}{2}$

11 $a = 1, b = -1$

12 $a = -3, b = -1, c = 3$; D is $(0, 3)$

13 $(1, -2), (3, 4), (5, 10)$

14 b $2, 1-\sqrt{2}, 1+\sqrt{2}$ c 4

15 a $(x-\alpha)(x-\beta)(x-\gamma)$
b $x^3 - (\alpha+\beta+\gamma)x^2 + (\alpha\beta + \beta\gamma + \gamma\alpha)x - \alpha\beta\gamma$

16 $-3 < x < -1$ or $x > 2$

Chapter 8 Test

1 b $(3x-1)(x^2+1)$ (x^2+1) has no real roots.

2 a 12 b -42

3 $a = -2, b = 4$

4 $-6, 4, 1$

5 a $2a + b = 4$ b (i) $a + b - 2$ (ii) $3a + b$

6 a $a = -1, b = 12$ b 18 c $-3, 2, 2$

7 a $a = -1, b = -22$ b $(x+5)(x-2)(x-4)$ c $-5, 2, 4$

Examination Questions

1 **(i)** $k = 7$ **(ii)** 200 **2** **(ii)** $b = -2, -1, \frac{2}{3}$

3 $a = 1; b = -11$ **4** $x \in \left\{-2, \frac{1}{2}, \frac{2}{3}\right\}$

5 **(ii)** $a \in \left\{-3, \frac{1}{2}, 2\right\}$ **6** **(i)** $a = 10; b = -3$ **(ii)** $x \in \left\{\frac{3-\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}, 3\right\}$

7 **(i)** $f(x) = x^3 - 6x - 4$ **(ii)** 5 **(iii)** $x \in \{-1-\sqrt{3}, -1+\sqrt{3}, 2\}$

8 **(ii)** 3

9 **a** $k = 32$ **b** $x \in \{-2, 3-\sqrt{5}, 3+\sqrt{5}\}$

10 $x \in \left\{-4, 1, \frac{3}{2}\right\}$

11 $a = -12; b = 8$ **12** $x \in \left\{-5, \frac{1}{2}, 3\right\}$

Term test 2A (Chapters 5 to 8)

1 **a** $a = 3, b = 2, c = 4$ **b** $f(x) \geqslant 4$

2 $a = 4, b = -3$

3 **a** $1 + 12x + 60x^2 + 160x$ **b** 460

4 **a** $a = 19, b = -19$ **b** $(x + 4)(3x - 1)(2x - 1)$ **c** $-4, \frac{1}{3}, \frac{1}{2}$

5 2, 3 **6** $(-4, -2), (2, 4)$

7 **a** $a - 2$ **b** $n = 4$

Chapter 9

Exercise 9.1

1 **a** 5 **b** 13 **c** 10 **d** 17 **e** 13 **f** $2\sqrt{61}$ **g** $2\sqrt{13}$
h $5\sqrt{2}$ **i** $\sqrt{41}$ **j** $8\sqrt{5}$

2 **a** (0, 6) **b** (−5, 6) **c** (1, −3) **d** (1, −1.5) **e** (−3, −5) **f** (0, −6) **g** (2, 4)
h (0, 0) **i** (−2, −3.5) **j** (0, −3)

3 **a** $-\frac{3}{5}$ **b** 3 **c** −6 **d** −2 **e** 1 **f** 2 **g** 4
h −3 **i** $\frac{4}{3}$ **j** $\frac{3}{13}$ **k** $\frac{6}{5}$ **l** −3 **m** $-\frac{2}{5}$ **n** $-\frac{5}{7}$
o $\frac{3}{2}$

4 **a** *GV* **b** *HP* **c** *DR* **d** *OG* **e** *SK* **f** *AH* **g** *NI*
h *MB* **i** *FU* **j** *MQ* **k** *RJ* **l** *HQ* **m** *QT*

5 **a** *MB* **b** *OC* **c** *DR* **d** *HQ* **e** *NI* **f** *SX* **g** *RJ*
h *SK* **i** *AH* or *PS* **j** *GV* **k** *MQ* **l** *FU* **m** *OG* **n** *QT*

6 25 **7** (−3, 3.5) **8** **b** (2, −4) **c** $9\sqrt{2}$, 54

9 **b** (−4.5, −8.5) **c** $\sqrt{20.5}$, 20.5 **10** **b** (1, 3) **c** $5\sqrt{2}$, 40

11 **a** (−5, 1) **c** 106 **12** (−3, −10) **13** (8, −10) (−4, 2)

Exercise 9.2

1 **a** $y = x - 4$ **b** $y = 4x + 12$ **c** $y = \frac{1}{9}x - 8$ **d** $y = \frac{1}{3}x - 6$ **e** $y = x + 2$
f $y = -\frac{1}{2}x + 5$ **g** $y = -\frac{3}{2}x - 6$ **h** $y = \frac{1}{5}x - 2$ **i** $y = -\frac{1}{4}x + 8$ **j** $y = 3x - 2$

2 **a** $4x + 3y = 24$ **b** $2x - y = -6$ **c** $4x + 3y = -24$ **d** $2x - 3y = 12$ **e** $2x - y = 8$
f $x - y = -6$ **g** $8x + 3y = -24$ **h** $2x + y = 8$ **i** $x + y = 6$ **j** $4x - 3y = -24$

3 a $y+2=3(x-2)$ b $y+6=5(x+4)$ c $y-3=-6(x+6)$ d $y-4=-4(x+2)$

e $y+3=\frac{1}{4}(x+2)$ f $y-9=-4(x+8)$ g $y-4=3(x-2)$ h $y+2=\frac{9}{4}(x-2)$

i $y+1=\frac{1}{5}(x-5)$ j $y-3=-\frac{3}{10}(x+6)$

4 a $y-2=-2(x+2)$ b $y-8=7(x+1)$ c $y+2=\frac{3}{4}\left(x+\frac{3}{2}\right)$ d $y+2=-5(x-0)$

e $y+5=-5(x-1)$ f $y-2=\frac{3}{2}(x-1)$ g $y-3=\frac{2}{3}(x-2)$ h $y+5=\frac{7}{2}(x-2)$

i $y+1=\frac{4}{3}(x-1)$ j $y-2=-\frac{4}{5}(x-2)$

Chapter 9 Summative Exercise

1 a 10 b 13 c 17 2 a $(-2,-1)$ b $(1, 1)$ c $(-3,-2)$

3 a 2 b -3 c $-\frac{1}{3}$ 4 a $AG \parallel BD$ b $FE \perp BD$ c $(8, 3)$

5 a $y=\frac{5}{3}x-3$ b $y=2x+6$ c $y=-2x-8$ d $y=-2x+10$

6 a $(y-4)=-\frac{5}{7}(x+6)$ b $(y-8)=-2(x+8)$ c $(y-3)=\frac{9}{8}(x-2)$ d $(y-3)=-5(x-2)$

7 a $x-2y=6$ b $4x-7y=-52$ c $x+7y=8$ d $3x+8y=-24$

8 $y=-x-3$ 9 $y=4x-2$ 10 $2y=x+10$ 11 $2x+4y=9$

12 a $3x+y=11$ b $y-2=\frac{1}{3}(x-3)$ c $5, -6, -1$ d 60

13 a $(1, 3)$ b $y-3=-2(x-1)$ c 5 d $q=4, r=-3$ e 20

14 a $y=-\frac{4}{3}(x-3)$ b M is $(0, 4)$; C is $(2, 5.5)$; D is $(-2, 2.5)$ c 37.5

Chapter 9 Test

1 a $y=7x-9$ b $C(-2, 2), D(5, 1)$ c 62.5 units

2 a $5\sqrt{2}$ b $x+y+2=0$

3 a $M(6, 2), N(2, 3.5), P(4, 6.5)$ b AN: $x+y=8$; BM: $y=\frac{1}{4}x+3$ c $(4, 4)$

4 a $(-4,-3), (2, 6)$ b $3\sqrt{13}$ c $4x+6y=5$

Examination Questions

1 $y+\frac{3}{2}=-\frac{2}{3}(x-1)$ 2 10.3 3 (i) $C(6, 3)$ $B(8, 4)$ $D(4, 2)$ (ii) 26.1

4 $A(5,15), B\left(0,16\frac{2}{3}\right), C\left(-3\frac{1}{3},6\frac{2}{3}\right)$ 5 (i) 2 6 16.7

7 (i) $AD: y-4=\frac{3}{5}(x+2)$ $CD: y-2=4(x-6)$ (ii) $(8, 10)$

8 25 9 $(6, 4)$; $(16, -6)$; $(11, -1)$ 10 $4y=x+41$

12 (ii) $(10, 0)$ (iii) $1:3$ 13 $(4, -3)$ 14 (i) $y=2x$ (ii) $(2, 4)$ (iii) $(-2, 1)$ (iv) 40 unit^2

15 (i) $C(4, 6)$; $Q(8.5, 0)$ (ii) 37.5

Chapter 10

Exercise 10.1

1 The ratio is always = 1

2 The ratio is always = 2

3 The ratio is always = 3

4 The ratio is always = 4

5 The ratio is not constant. It varies according to the lower bound and the object set size.

6 For a linear function, the ratio will be constant and will be equal to the coefficient of x in the function.

Exercise 10.2

1 $f'(x) = 4x^3$ **2** $f'(x) = 5x^4$ **3** $f'(x) = 6x^5$ **4** $f'(x) = 10x^9$ **5** $f'(x) = 100x^{99}$

Exercise 10.3

1 **a** 0 **b** 0

2 **a** $e'(x) = 4x^3$ **b** $f'(x) = 6x^5$ **c** $g'(x) = 9x^8$

d $h'(x) = 12x^{11}$ **e** $e'(x) = -\frac{1}{x^2}$ **f** $f'(x) = \frac{2}{x^3}$

g $g'(x) = \frac{3}{x^4}$ **h** $h'(x) = \frac{9}{x^{10}}$ **i** $e'(x) = \frac{1}{2\sqrt{x}}$

j $f'(x) = \frac{1}{3\sqrt[3]{x^2}}$ **k** $g'(x) = \frac{1}{4\sqrt[4]{x^3}}$ **l** $h'(x) = \frac{1}{5\sqrt[5]{x^4}}$

m $e'(x) = \frac{3}{2}\sqrt{x}$ **n** $f'(x) = \frac{2}{3\sqrt[3]{x}}$ **o** $g'(x) = \frac{3}{4\sqrt[4]{x}}$

p $h'(x) = \frac{5}{2}\sqrt{x^3}$ **q** $e'(x) = -\frac{1}{2\sqrt{x^3}}$ **r** $f'(x) = -\frac{1}{3\sqrt[3]{x^4}}$

s $g'(x) = -\frac{3}{4\sqrt[4]{x^3}}$ **t** $h'(x) = -\frac{2}{5\sqrt[5]{x^7}}$

3 **a** $f'(x) = 2x + 4x^3$ **b** $g'(x) = 3x^2 - \frac{1}{2\sqrt{x}}$ **c** $h'(x) = \frac{1}{2\sqrt{x}} + \frac{1}{3\sqrt[3]{x^2}}$

d $f'(x) = 8x - 12x^3$ **e** $g'(x) = \frac{3}{2\sqrt{x}} + 4$ **f** $h'(x) = 5 - \frac{9}{2}\sqrt{x}$

g $f'(x) = -\frac{1}{x^2} - \frac{2}{x^3}$ **h** $g'(x) = -\frac{6}{x^3} + \frac{12}{x^4}$ **i** $h'(x) = -\frac{35}{x^8} - \frac{20}{x^6}$

j $f'(x) = \frac{2}{\sqrt{x^3}} + \frac{1}{\sqrt[3]{x^4}}$ **k** $g'(x) = \frac{-9}{4\sqrt[4]{x^7}} - \frac{2}{3\sqrt[3]{x^4}}$ **l** $h'(x) = \frac{-4}{\sqrt[5]{x^7}} + \frac{6}{\sqrt[4]{x^7}}$

Exercise 10.4

1 **a** $3x^2 - 2x$ **b** $f'(x) = 12x^3 + 12x^2$ **c** $\frac{dy}{dx} = 10x^4 + 16x^3 - 18x^2$

d $6x - \frac{4}{x^2}$ **e** $g'(x) = -\frac{4}{x^3} + \frac{9}{x^4}$ **f** $\frac{dy}{dx} = 8x + \frac{2}{\sqrt{x^3}}$

g $6x^2 + \frac{12}{x^3}$ **h** $h'(x) = -\frac{3}{x^2} + \frac{12}{x^3}$ **i** $\frac{dy}{dx} = \frac{3}{\sqrt{x}} - \frac{4}{\sqrt{x^3}}$

j $\frac{3}{\sqrt[3]{x^2}} - \frac{2}{\sqrt{x}}$ **k** $k'(x) = \frac{9}{\sqrt[4]{x}} + 6\sqrt{x}$ **l** $\frac{dy}{dy} = 6x - \frac{2}{x^4}$

2 $a = 23$ **3** $x = 3$ **4** 7

5 $(-2,-8),(2,8)$ 6 -4 7 $a=4$ A is $(2,-12)$

8 $(-2, 36)$ and $(3, -69)$ 9 $a=1, b=-3; (-4, 10)$ 10 $a=4, b=-3; (-1,-1), (-0.5, 5.5), (0.5, 5.5)$

Exercise 10.5

1 a $12x^2+6x$ b $f''(x)=40x^3+36x^2-8$ c $\frac{d^2y}{dx^2}=126x^5-60x^4+100x^3$

d $4-\frac{2}{x^3}$ e $g''(x)=\frac{18}{x^4}+\frac{24}{x^5}$ f $\frac{d^2y}{dx^2}=6-\frac{6}{\sqrt{x^5}}$

g $24x+\frac{12}{x^4}$ h $h''(x)=\frac{4}{x^3}+\frac{6}{x^4}$ i $\frac{d^2y}{dx^2}=-\frac{3}{\sqrt{x^3}}+\frac{6}{\sqrt{x^5}}$

j $-\frac{4}{\sqrt[3]{x^5}}+\frac{2}{\sqrt{x^3}}$ k $k''(x)=-\frac{3}{\sqrt[4]{x^5}}-\frac{6}{\sqrt{x}}$ l $\frac{d^2y}{dx^2}=8x+\frac{12}{x^5}$

Exercise 10.6

1 a $y=x$ b $y=-5x+3$

2 a $y=-4x+9$ b $y=2x+12$

3 a (i) $y=3x+7$ (ii) $(-1, 4)$

b (i) $y=-x+3$ (ii) $(3, 0)$

4 a $y=-3x+12$ b $3y=x+16$ c $\left(-18,-\frac{2}{3}\right)$

5 a $y=x+1$ b $(-1, 0)$

6 a $y=-6x+22$ b $(-2, 34)$

7 a $8y=-3x+36$ b $(-2, 3)$ c $3y=8x-30$

8 a $y=3(x-2)$ b $y=\frac{1}{3}(x-2)$ c $\left(-3,-3,\frac{5}{3}\right)$

9 a $3y=-x+8$ b $\left(-\frac{6}{7},\frac{62}{21}\right)$

Chapter 10 Summative Test

1 a $f'(x)=5x^4$ b $f'(x)=8x^7$ c $f'(x)=11x^{10}$ d $f'(x)=15x^{14}$

e $\frac{dy}{dx}=-\frac{4}{x^5}$ f $\frac{dy}{dx}=-\frac{5}{x^6}$ g $\frac{dy}{dx}=-\frac{6}{x^7}$ h $\frac{dy}{dx}=-\frac{10}{x^{11}}$

2 a $g'(x)=6x^5-3x^2$ b $g'(x)=7x^6-\frac{3\sqrt{x}}{2}$ c $g'(x)=\frac{1}{2\sqrt{x}}+\frac{1}{3\sqrt[3]{x^2}}$

d $\frac{dy}{dx}=\frac{-3}{x^2}-\frac{10}{x^6}$ e $\frac{dy}{dx}=\frac{-28}{x^5}+\frac{2}{\sqrt[3]{x^4}}$ f $\frac{dy}{dx}=\frac{-2}{\sqrt[4]{x^5}}+\frac{2}{\sqrt[5]{x^6}}$

3 64 4 3.75 5 $1-2\sqrt{2}, 1+2\sqrt{2}$

6 $(-2. 6), (2, -2)$ 7 a $S=8\pi r^2$ b $\frac{dS}{dr}=16\pi r$

8 $\frac{dV}{dr}=4\pi r^2$ 9 $\frac{dV}{dx}=10\sqrt{3}x$

10 a $y = 10x - 24$ b $y + 4 = -\frac{1}{10}(x - 2)$

11 a (4, 4) b $y = x$; $y = -x + 8$

12 a $a = 10$, $b = -8$ b $\left(-\frac{10}{9}, \frac{244}{243}\right)$

13 a (−0.5, 22) b (11, 8.5)

14 a $y = -4x + 2$ b (−1, 6) c $4y = x - 26$ d 84.5 square units

Chapter 10 Test

1 $\frac{3}{16}$

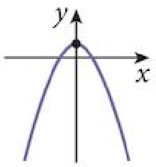
2 b (i) $\frac{dy}{dx} = 2 + \frac{6}{x^3}$ (ii) $\frac{d^2y}{dx^2} = -\frac{18}{x^4}$

3 a −7 b (−3, 0), (1, 0), (2, 0) c 20, −4, 5

4 a $y = -5x + 1$ b B (0, 1) c $5y = x + 18$

5 a $a = 5$, $b = 3$ b 72

6 a Cuts axes at: b $4y = -68x + 51$



x-axis $\left(-\frac{2}{3}, 0\right)\left(\frac{3}{4}, 0\right)$

y-axis (0, 6)

Chapter 11

Problem 11.1

Angle	x	y
0	1.00	0.00
10	0.98	0.17
20	0.94	0.34
30	0.87	0.50
40	0.77	0.64
50	0.64	0.77
60	0.50	0.87
70	0.34	0.94
80	0.17	0.98
90	0.00	1.00

Problem 11.2

Point	Angle	$\cos\theta$	$\sin\theta$
P	θ	$\cos\theta$	$\sin\theta$
Q	$90 - \theta$	$\sin\theta$	$\cos\theta$
R	$90 + \theta$	$-\sin\theta$	$\cos\theta$
S	$180 - \theta$	$-\cos\theta$	$\sin\theta$
T	$180 + \theta$	$-\cos\theta$	$-\sin\theta$
U	$270 - \theta$	$-\sin\theta$	$-\cos\theta$
V	$270 + \theta$	$\sin\theta$	$-\cos\theta$
W	$360 - \theta$	$\cos\theta$	$-\sin\theta$

Exercise 11.1

1 There are many other answers to these questions. These are the simplest ones.

a 390°, 750°, −330°, −690°
b 440°, 800°, −280°, −640°
c 480°, 840°, −240°, −600°
d 510°, 870°, −210°, −570°
e 540°, 900°, −180°, −540°
f 570°, 930°, −150°, −510°
g 615°, 975°, −105°, −465°
h 630°, 990°, −90°, −450°
i 670°, 1030°, −50°, −410°
j 700°, 1060°, −20°, −380°
k 60°, 780°, −300°, −660°
l 170°, 890°, −190°, −550°
m 270°, 990°, −90°, −450°
n 350°, 1070°, −10°, −370°
o 330°, 690°, −30°, −390°

2 a $-\cos 60°$ b $\sin 70°$ c $-\tan 45°$ d $-\cos 20°$ e $\sin 30°$ f $\tan 30°$
g $-\cos 45°$ h $-\sin 60°$ i $\tan 75°$ j $-\cos 80°$ k $-\sin 80°$ l $-\tan 60°$
m $\cos 50°$ n $-\sin 35°$ o $-\tan 30°$

3 a $\cos 15°$ b $-\sin 40°$ c $-\tan 65°$ d $\cos 70°$ e $-\sin 85°$ f $\tan 65°$
g $-\cos 50°$ h $-\sin 20°$ i $\tan 15°$ j $-\cos 10°$ k $\sin 30°$ l $-\tan 60°$
m $\cos 60°$ n $\sin 40°$ o $\tan 10°$

4 a $-\sin 30°$ b $-\sin 45°$ c $-\sin 60°$ d $-\sin 70°$ e $-\sin 60°$ f $-\sin 45°$
g $-\sin 30°$ h $-\sin 15°$ i $-\sin 10°$ j $\sin 10°$ k $\sin 30°$ l $\sin 45°$
m $\sin 80°$ n $\sin 20°$ o $\sin 20°$

5 a cos: $0° \leqslant \theta \leqslant 180°$ sin: $-90° \leqslant \theta \leqslant 90°$ tan: $-90° < \theta < 90°$
b no

Exercise 11.2

1 a {−75.5°, 75.5°} b {23.6°, 156.4°} c {−145.0°, 35.0°} d {−25.8°, 25.8°}
e {58.2°, 121.8°} f {−168.7°, 11.3°} g {−107.5°, 107.5°} h {−150°, −30°}
i {−63.4°, 116.6°} j {−138.6°, 138.6°} k {−64.2°, −115.8°} l {−77.5°, 102.5°}
m {−90°, 90°} n {−90°} o {−99.5°, 80.5°}

2 a {33.2°, 146.8°, 213.2°, 326.8°}
b {14.8°, 45.2°, 134.8°, 165.2°, 254.8°, 285.2°}
c {18.5°, 63.5°, 108.5°, 153.5°, 198.5°, 243.5°, 288.5°, 333.5°}
d {25.2°, 94.8°, 145.2°, 214.8°, 265.2°, 334.8°}
e {6.7°, 38.3°, 96.7°, 128.3°, 186.7°, 218.3°, 276.7°, 308.3°}
f {5.7°, 95.7°, 185.7°, 275.7°}
g {31.7°, 58.3°, 121.7°, 148.3°, 211.7°, 238.3, 301.7°, 328.3°}
h {116.6°, 153.4°, 296.6°, 333.4°}
i {36.1°, 96.1°, 156.1°, 216.1°, 276.1°, 336.1°}
j {82.8°} k {133.3°} l {306.9°}
m {78.5°, 281.5°} n {11.5°, 168.5°} o {108.4°, 288.4°}

3 a {22.8°, 127.2°, 202.8°, 307.2°}
b {17.2°, 69.5°, 137.2°, 189.5°, 257.2°, 309.5°}
c {26.6°, 71.6°, 116.6°, 161.6°, 206.6°, 251.6°, 296.6°, 341.6°}
d {60°, 100°, 180°, 220°, 300°, 340°}
e {25.9°, 59.1°, 115.9°, 149.1°, 205.9°, 239.1°, 295.9°, 329.1°}
f {51°, 141°, 251°, 341°}
g {46.3°, 153.7°, 231°, 321°}
h {37.5°, 97.5°, 217.5°, 277.5°}
i {190.0°}

4 a {−120°, −90°, 90°, 120°}
b {−180°, 0°, 19.5°, 160.5°, 180°}
c {−180°, −116.6°, 0°, 63.4°, 180°}
d {−120°, −48.2°, 48.2°, 120°}
e {−90°, 30°, 150°}
f {−146.3°, −123.7°, 33.7°, 56.3°}

Chapter 11 Summative Exercise

1 There are many possible answers. These are the simplest.
a 50°, 410° b −230°, −310° c 230°, 490°
d −130°, −230° e 310°, 490° f −50°, −230°

2 a $-\cos 20°$ b $\sin 70°$ c $-\tan 40°$
d $-\cos 70°$ e $-\sin 40°$ f $\tan 80°$
g $\cos 40°$ h $-\sin 50°$ i $-\tan 20°$

3 a 72.5°, 287.5° b 23.6°, 156.4° c 56.3°, 236.3°
d 101.5°, 258.5° e 197.4°, 342.5° f 138.0°, 318.0°

4 a −26.6°, 26.6° b −102.3°, −77.7°, 17.7°, 92.3°, 137.7°, 162.3°
c −159.0°, −69.0°, 21.0°, 110.0° d −167.7°, −72.3°, −47.7°, 47.7°, 72.3°, 167.7°
e −57.9°, −32.1°, 122.1°, 147.9° f −142.7°, −82.7°, −22.7°, 37.3°, 97.3°, 157.3°

5 a $-270° \leqslant x \leqslant -90°$
b $-540° \leqslant x \leqslant -360°$

6 a 27°, 63°, 207°, 243°
b 105°, 165°, 285°, 345°
c 15°, 135°, 195°, 315°
d 10.2°, 190.2°

7 a −101.3°, 78.7°
b −153.4°, 26.6°
c −18.4°, 161.6°
d −45°, 135°

8 a 45°, 108.4°, 225°, 288.4°
b 41.8°, 138.2°, 225.6°, 311.4°
c 53.1°, 104.5°, 255.5°, 306.9°
d 76.0°, 153.4°, 256.0°, 333.4°

9 a −90°, 30°, 90°, 150°
b −109.5°, 0°, 109.5°, 180°
c −75.5°, 0°, 75.5°, 180°
d −45°, 19.5°, 135°, 160.5°
e −120°, −116.6°, 63.4°, 120°

10 a $270° < A + B < 630°$
b $-90° < A - B < 270°$

11 a $\frac{3}{5}$
b $\frac{3}{4}$

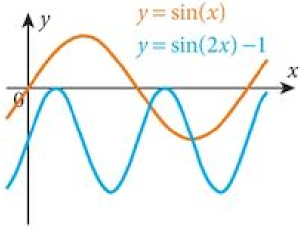
12 a $\frac{12}{13}$
b $-\frac{15}{17}$

13 a $-\frac{24}{25}$
b $\frac{40}{41}$

Chapter 11 Test

1 a 95.8°, 174.2°, 275.8°, 354.2°
b 71.6°, 251.6°
c 60°, 120°, 240°, 300°

2 a



b 2

3 a 10°, 130°
b 63.4°, 243.4°
c 60°, 109.4°, 250.5°, 300°

4 a $\frac{2\sqrt{5}}{5}$
b $\frac{\sqrt{5}}{2}$
c 26.6°

5 a $-1, -\frac{2}{3}, \frac{3}{2}$
b 221.8°, 270°, 318.2°

Term test 3A (Chapters 9 to 11)

1 $9\sqrt{26}$

2 b $\frac{dy}{dx} = 4x + \frac{6}{x^2}$

3 a 126.9°, 306.9°
b 130°, 250°

4 a $3x + 2y = 15$
b $k = -1$
c $(9, -6)$
d 65

5 a $-6x(2 - x^2)^2$
b $\frac{dy}{dx} = -\frac{2}{x^2}$
c $f'(x) = -\frac{1}{\sqrt{x^3}} + \frac{1}{2\sqrt{x}}$

6 a 18.43°, 108.4°, 198.4°, 288.4°
b 166.0°, 346.0°
c 81.8°, 178.2°

Chapter 12

Problem 12.1

H	Degrees	Radians
0.1	0.017453284	0.998334166
0.01	0.017453292	0.999983333
0.001	0.017453293	0.999999833
0.0001	0.017453293	0.999999998
0.00001	0.017453293	1
0.000001	0.017453293	1
0.0000001	0.017453293	1
0.00000001	0.017453293	1

Exercise 12.1

1 **a** $\frac{\pi}{18}$ **b** $\frac{\pi}{9}$ **c** $\frac{\pi}{8}$ **d** $\frac{\pi}{6}$ **e** $\frac{5\pi}{12}$ **f** $\frac{7\pi}{12}$ **g** $\frac{2\pi}{3}$
h $\frac{3\pi}{4}$ **i** $\frac{5\pi}{6}$ **j** $\frac{7\pi}{6}$ **k** $\frac{5\pi}{4}$ **l** $\frac{4\pi}{3}$ **m** $\frac{3\pi}{2}$ **n** $\frac{5\pi}{3}$
o $\frac{7\pi}{4}$ **p** $\frac{11\pi}{6}$ **q** $\frac{5\pi}{2}$ **r** 3π **s** $\frac{7\pi}{2}$ **t** 2π **u** 3π

2 **a** 1.0° **b** 5.0° **c** 15.0° **d** 25.0° **e** 50.0° **f** 70.0° **g** 100.0°
h 110.0° **i** 165.0° **j** 195.0° **k** 255.0° **l** 285.0° **m** 320.0° **n** 345.0°
o 372.4° **p** 401.1° **q** 429.7° **r** 458.4° **s** 487.0° **t** 515.7° **u** 573.0°

3 **a** **(i)** $l = \frac{5\pi}{6}$ cm **(ii)** $a = \frac{25\pi}{12}$ cm² **b** **(i)** $l = \frac{8\pi}{3}$ cm **(ii)** $a = \frac{32\pi}{3}$ cm²
c **(i)** $l = \frac{5\pi}{6}$ cm **(ii)** $a = \frac{25\pi}{6}$ cm² **d** **(i)** $l = 10\pi$ cm **(ii)** $a = 100\pi$ cm²
e **(i)** $l = 10\pi$ cm **(ii)** $a = 150\pi$ cm² **f** **(i)** $l = \frac{25\pi}{63}$ cm **(ii)** $a = \frac{625\pi}{3}$ cm²

4 **a** **(i)** $\theta = 2$ rad **(ii)** $a = 25$ cm² **b** **(i)** $\theta = \frac{15}{8}$ rad **(ii)** $a = 60$ cm²
c **(i)** $\theta = 5$ rad **(ii)** $a = 40$ cm² **d** **(i)** $\theta = 2.25$ rad **(ii)** $a = 450$ cm²
e **(i)** $\theta = 1$ rad **(ii)** $a = 450$ cm² **f** **(i)** $\theta = 2.4$ rad **(ii)** $a = 3000$ cm²

5 **a** **(i)** $\theta = 0.8$ rad **(ii)** $l = 4$ cm **b** **(i)** $\theta = \frac{15}{32}$ rad **(ii)** $l = 3.75$ cm
c **(i)** $\theta = 2.5$ rad **(ii)** $l = 10$ cm **d** **(i)** $\theta = \frac{9}{40}$ rad **(ii)** $l = 4.5$ cm
e **(i)** $\theta = \frac{3}{20}$ rad **(ii)** $l = 2$ cm **f** **(i)** $\theta = \frac{12}{125}$ rad **(ii)** $l = 4.8$ cm

6 **a** **(i)** $r = 9.55$ cm **(ii)** $a = 23.9$ cm² **b** **(i)** $r = 11.5$ cm **(ii)** $a = 68.8$ cm²
c **(i)** $r = 15.3$ cm **(ii)** $a = 153$ cm² **d** **(i)** $r = 6.37$ cm **(ii)** $a = 95.5$ cm²
e **(i)** $r = 11.5$ cm **(ii)** $a = 138$ cm² **f** **(i)** $r = 8.68$ cm **(ii)** $a = 217$ cm²

Exercise 12.2

1 $50(2\sqrt{3} - \pi)$ **2** $20(\pi + 3)$ **3** **a** $40\sqrt{3} + \frac{140\pi}{3}$ **b** $800\sqrt{3} - \frac{1100\pi}{3}$ **4** **a** $\frac{40\pi}{3}$ **b** $\frac{200\pi}{3} - 50\sqrt{3}$

5 **a** $\frac{40\sqrt{3}}{3} + \frac{20\pi}{3}$ **b** $\frac{400\sqrt{3}}{3} - \frac{200\pi}{3}$

Chapter 12 Summative Exercise

1 a $\frac{\pi}{12}$ b $\frac{\pi}{6}$ c $\frac{2\pi}{9}$ d $\frac{5\pi}{12}$ e $\frac{\pi}{2}$

f $\frac{2\pi}{3}$ g $\frac{3\pi}{4}$ h $\frac{7\pi}{6}$ i $\frac{5\pi}{4}$ j $\frac{5\pi}{3}$

2 a 15° b 30° c 45° d 60° e 75°

f 240° g 255° h 54° i 130° j 300°

3

	a	b	c	d	e	f
(i)	$\frac{3\pi}{2}$	$\frac{25\pi}{6}$	15π	18π	12π	24π
(ii)	$\frac{9\pi}{2}$	$\frac{125\pi}{6}$	135π	216π	192π	180π

4

	a	b	c	d	e	f
(i)	$\frac{3\pi}{8}$	$\frac{2\pi}{5}$	$\frac{\pi}{3}$	$\frac{5\pi}{6}$	$\frac{3\pi}{4}$	$\frac{4\pi}{3}$
(ii)	54π	$\frac{16\pi}{5}$	$\frac{27\pi}{2}$	135π	96π	24π

5

	a	b	c	d	e	f
(i)	3	5	9	10	8	6
(ii)	2π	$\frac{15\pi}{4}$	5π	6π	$\frac{10\pi}{3}$	5π

6 a 0.6 b 6.574 c 0.852

7 a 1.3 b 12.87 c 16.35

8 a $\theta = \frac{20}{r} - 2$ b $A = 10r - r^2$

9 a $\theta = 2$ b $A = 35$

10 a $\frac{10\pi}{13}$ b 65π

Chapter 12 Test

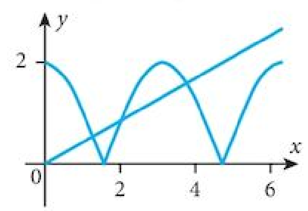
1 a 2.07, 3.927 b 0.777, 2.348, 3.919, 5.489 c $\frac{\pi}{6}, \frac{5\pi}{6}$, 3.481, 5.943

2 a 0.644 b 14.435 c 8.175

3 a 12.28 b 3.26

4 a $10\left(\frac{\pi}{3} + \sqrt{3}\right)$ b $25\left(\sqrt{3} - \frac{\pi}{3}\right)$

5



a 3

b 1

Examination Questions

1 **(i)** 1.2 rad **(ii)** 7.456 cm **(iv)** 9.275 cm^2 **2** **(i)** 57.1 cm^2 **(ii)** 44.6 cm
3 32 cm **4** **(ii)** 64.76 cm^2 **(iii)** 41.53 cm **5** **(ii)** 13.6 cm **(iii)** 7.50 cm^2
6 **(ii)** 24.91 cm **(iii)** 39.64 cm^2 **7** **(i)** 1.25 rad **(ii)** 25.7 cm^2
8 **(i)** 0.8 rad **(ii)** 90 cm^2
9 2.68, or 7.6, or 15.2

Chapter 13

Exercise 13.1

1 **a** (2.5, −2.25) **b** (−2, −1) **c** (0, 4) **d** (1.5, −0.25) **e** (−2, 9)
f (−2, 32) **g** (−1, −2), (1, 2) **h** (0, 0), (4, −32) **i** (−1, 12), (3, −52)
j (−2, 16), (0, 0), (2, 16) **k** (2, 48) **l** (−2, −152), (1, 37), (3, −27)

2 **a** maximum (−1, 9) **b** minimum (3, 0) **c** maximum (2, 11) **d** minimum (−5, −20)
e minimum $\left(\frac{1}{12}, -\frac{25}{24}\right)$ **f** minimum $\left(-\frac{3}{20}, -\frac{49}{40}\right)$ **g** minimum $\left(-\frac{2}{3}, -\frac{16}{9}\right)$, maximum $\left(\frac{2}{3}, \frac{16}{9}\right)$
h maximum (0, 0), minimum (2, −4) **i** maximum (−2, 18), minimum (2, −14)
j maximum (−2, 16), minimum (0, 0), maximum (2, 16) **k** maximum $\left(\frac{3}{2}, \frac{243}{8}\right)$
l minimum (−1, 0), maximum (0, 5), minimum (2, −27)

3 **a** maximum $\left(-\frac{1}{2}, -4\right)$, minimum $\left(\frac{1}{2}, 4\right)$ **b** minimum (−2, 12) **c** minimum (0.6, 3.21)
d minimum (2, 3) **e** minimum $\left(\frac{4}{9}, 27\right)$ **f** maximum (−3, −27)

Exercise 13.2

1 **a** 10.05 **b** 9.056 **c** 25.02
2 **a** 10.0033 **b** 5.0133 **c** 14.9926
3 **a** −2.09 **b** −18.024 **c** 47.5
4 **a** −17.2 **b** 10.2 **c** 13.075
5 12 cm^2

Exercise 13.3

2 500 cm^2 s^{-1} **3** 1 cm s^{-1} **4** 40π cm^2 s^{-1} **5** $\frac{1}{4\pi}$ cm s^{-1} **6** 0.1 cm s^{-1}

7 600π cm^2 s^{-1} **8** 1 cm^2 s^{-1} **9** $\frac{1}{5\sqrt{2\pi}}$ cm s^{-1} **10** $\frac{1}{2}$ cm s^{-1} **11** 2 cm s^{-1}

12 **a** $-\frac{2}{25\pi}$ cm s^{-1} **b** $-\frac{\sqrt{3}}{50\pi}$ cm s^{-1} **13** $\frac{1}{20}$ cm s^{-1} **14** $\frac{3}{400}$ cm s^{-1}

Exercise 13.4

1 $x = 25$ m **3** **a** $(25 - x)$ m **b** 12.5 m
4 $20 \times 20 \times 5$ cm (note: $x = 15$ is not a valid solution)
5 $10.4 \times 20.8 \times 13.87$ cm (note: better make the longest edge the height)
6 **a** $r = \sqrt[3]{\frac{100}{\pi}}$ cm, $h = 2\sqrt[3]{\frac{100}{\pi}}$ cm **b** $h = 2r$ **c** Multiply lengths by $\sqrt[3]{2}$
d h is usually doubled. This makes manufacturing cheaper. Also, usually, $h \neq 2r$!
7 $2x = 12.6$ cm, $l = 7.27$ cm **8** $x = 5\sqrt{2}$ **9** **b** $x = \frac{10}{3}$, $V = \frac{500\pi}{9}$

Chapter 13 Summative Exercise

1 a (1, 12) Max., (2, −15) Min.
b (−1.5, 109.625) Min., (1, 31) Max, (2, 19), Min.
c (−2, −2) Max., (2, 6) Min.
d (2.25, 36), Min.

2 b $A = 24x^2 + \frac{750}{x}$ c $\frac{dA}{dx} = 48x - \frac{750}{x^2}, x = \frac{5}{2}$ d 450 Min.

3 1000.3

4 1.014

5 a $\frac{2}{\pi}$ cm s^{-1} b 40 cm^2

6 a 0.006 b 34.25

7 a (i) $V = 4\pi r^3$ (ii) $S = 10\pi r^2$
b (i) $192p\pi$ (ii) $80\,p\pi$

8 a $\theta = \frac{16}{r} - 2$ b −2 rad min^{-1} c 2 cm^2 min^{-1}

9 a (i) $V = 4x^3$ (ii) $S = 18x^2$ b 6 m^3 min^{-1}

10 a $L = 4x + 4y = 240$ b $x = 40$ cm, $V = 32\,000$ cm^3, Max.

11 $x = 7.5$ cm, Max $A = 30$ cm^2.

Chapter 13 Test

1 b $\frac{75}{2}$, Maximum

2 a $\frac{dy}{dx} = 4 - \frac{1}{(x-4)^2}, \frac{d^2y}{dx^2} = \frac{2}{(x-4)^3}$ b/c $\left(\frac{7}{2}, 12\right)$ maximum, $\left(\frac{9}{2}, 20\right)$ minimum

3 a $\frac{dy}{dx} = 3\sqrt{2x+1}$ b $9p$

4 b $\frac{dA}{dt} = \frac{1600}{3r}$

5 a 15.1 cm^2 s^{-1} b 22.6 cm^3 s^{-1}

6 a (3, 27) b minimum

Examination Questions

1 (i) $\frac{dy}{dx} = \frac{-16}{(2x-1)^2}$ (ii) −0.05 units s^{-1} 2 (i) $\frac{60-2x^2}{3x}$

3 (i) $\frac{dy}{dx} = \frac{1}{2\sqrt{x}} - \frac{9}{2\sqrt{x^3}}, \frac{d^2y}{dx^2} = \frac{27}{4\sqrt{x^5}} - \frac{1}{4\sqrt{x^3}}$ (iii) Minimum

4 (i) $\frac{dy}{dx} = 12x - \frac{96}{x^4}$ (ii) 0.72 5 (i) $x^2 + \frac{32}{x^4}$ (ii) $\frac{dS}{dx} = 2x - \frac{128}{x^5}$ (iii) $x = 2, OP = \sqrt{6}$

6 (i) $PQ = 27 - t^2$ (ii) $t = 3$ (iii) A = 108, Maximum

7 (i) −6 (ii) $-6p$

Chapter 14

Exercise 14.1

1 **A**(2 by 2), **B**(1 by 3), **C**(3 by 4), **D**(2 by 2), **E**(3 by 3), **F**(3 by 4)

2 No

3 **a** $\begin{pmatrix} -2 & 6 \\ 4 & -2 \end{pmatrix}$ **b** $(16 \quad -8 \quad -12)$ **c** $\begin{pmatrix} 3 & -12 & 6 & 9 \\ -6 & 9 & 3 & 6 \\ -3 & 6 & 15 & 0 \end{pmatrix}$ **d** $\begin{pmatrix} 1 & 7 \\ 1 & -4 \end{pmatrix}$

e $\begin{pmatrix} 3 & -9 & 5 & 4 \\ -2 & 5 & -3 & -1 \\ 0 & 2 & 5 & -4 \end{pmatrix}$ **f** $\begin{pmatrix} 5 & 5 \\ -4 & -5 \end{pmatrix}$ **g** $\begin{pmatrix} -1 & -2 & 0 & 7 \\ -6 & 5 & 11 & 12 \\ -5 & 6 & 15 & 8 \end{pmatrix}$ **h** $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

4 $a = 2, b = 5, c = 6, d = -4$

5 $a = 4, b = 7, c = -1, d = -1$

Exercise 14.2

1 **a** $(5 \; 4 \; 1 \; 3 \; 2)\begin{pmatrix} 3 \\ 4 \\ 6 \\ 2 \\ 5 \end{pmatrix}$ **b** \$ 53 **c** $(5 \; 4 \; 1 \; 3 \; 2)\begin{pmatrix} 100 \\ 150 \\ 400 \\ 80 \\ 200 \end{pmatrix}$ **d** 2140 g

2 **a** $\begin{pmatrix} 100 & 50 & 30 & 10 & 5 \\ 200 & 150 & 100 & 100 & 50 \\ 500 & 400 & 400 & 200 & 100 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}$ **b** $\begin{pmatrix} 195 \\ 600 \\ 1600 \end{pmatrix}$

c $\begin{pmatrix} 1 & 0.5 & 0.3 & 0.1 & 0.05 \\ 2 & 1.5 & 1 & 1 & 0.5 \\ 5 & 4 & 4 & 2 & 1 \end{pmatrix}\begin{pmatrix} 4 \\ 5 \\ 6 \\ 8 \\ 10 \end{pmatrix}$ **d** $\begin{pmatrix} 9.60 \\ 34.50 \\ 90 \end{pmatrix}$

3 **a** $\begin{pmatrix} 10 & 5 & 1 & 6 & 3 \\ 8 & 10 & 2 & 12 & 1 \\ 5 & 6 & 1 & 4 & 2 \\ 4 & 5 & 3 & 8 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \\ 8 \\ 2 \\ 6 \end{pmatrix}$ **b** $\begin{pmatrix} 88 \\ 110 \\ 67 \\ 78 \end{pmatrix}$

c $(1 \; 1 \; 1 \; 1)\begin{pmatrix} 10 & 5 & 1 & 6 & 3 \\ 8 & 10 & 2 & 12 & 1 \\ 5 & 6 & 1 & 4 & 2 \\ 4 & 5 & 3 & 8 & 1 \end{pmatrix}$ **d** $\begin{pmatrix} 27 \\ 26 \\ 7 \\ 30 \\ 7 \end{pmatrix}$

e $(1 \; 1 \; 1 \; 1)\begin{pmatrix} 10 & 5 & 1 & 6 & 3 \\ 8 & 10 & 2 & 12 & 1 \\ 5 & 6 & 1 & 4 & 2 \\ 4 & 5 & 3 & 8 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ 4 \\ 8 \\ 2 \\ 6 \end{pmatrix}$ **f** \$ 343

4 a $\begin{pmatrix} 20 & 10 & 5 & 1 \\ 30 & 40 & 4 & 1 \\ 10 & 15 & 1 & 1 \\ 15 & 20 & 3 & 1 \\ 25 & 30 & 4 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 5 & 0 \\ 20 & 5 \\ 30 & 10 \end{pmatrix}$ b $\begin{pmatrix} 220 & 35 \\ 370 & 30 \\ 145 & 15 \\ 220 & 25 \\ 310 & 30 \end{pmatrix}$ c $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ d $\begin{pmatrix} 255 \\ 400 \\ 160 \\ 245 \\ 340 \end{pmatrix}$

e $\begin{pmatrix} 20 & 10 & 5 & 1 \\ 30 & 40 & 4 & 1 \\ 10 & 15 & 1 & 1 \\ 15 & 20 & 3 & 1 \\ 25 & 30 & 4 & 1 \end{pmatrix} \begin{pmatrix} 2 & 0 \\ 5 & 0 \\ 20 & 5 \\ 30 & 10 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ f $\begin{pmatrix} 255 \\ 400 \\ 160 \\ 245 \\ 340 \end{pmatrix}$

5 a The total ticket money collected.

b $ 22,200 c None

d (i) The total number of seats of each type.

(ii) The income from each type of full bus.

e Nothing useful.

6 a $\begin{pmatrix} 410 \\ 456 \\ 612 \\ 665 \end{pmatrix}$ b $\begin{pmatrix} 230 \\ 265 \\ 225 \\ 260 \end{pmatrix}$

Problem 14.1

b $\mathbf{A}^{-1} = \frac{1}{17}\begin{pmatrix} 4 & 3 \\ 3 & -2 \end{pmatrix}$ c $x = 5, y = 2$

Exercise 14.3

1 a $\mathbf{P} = \begin{pmatrix} -9 & -2 \\ 7 & -10 \end{pmatrix}$ b $\begin{pmatrix} \frac{3}{5} & -\frac{1}{5} \\ 1 & 1 \end{pmatrix}$

2 a $p = 5$ b $q = 9$ 4 $m = \frac{3}{2}, k = 5$

5 a $\begin{pmatrix} 5 & 1 \\ 6 & 8 \end{pmatrix}$ b $\begin{pmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{3}{4} & \frac{1}{4} \end{pmatrix}$ c $\begin{pmatrix} -\frac{5}{4} & \frac{1}{2} \\ -\frac{3}{2} & 1 \end{pmatrix}$

6 $\frac{1}{3}\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$

7 a $x = 2, y = -4$ b $x = 2, y = -5$ c $x = 4, y = -3$ d $x = 1, y = -4$

e $x = 4, y = -1$ f $x = -2, y = -2$ g $x = -1, y = -7$ h $x = 3, y = -1$

i $x = 2, y = -5$ j $x = 4, y = 1$ k $x = 3, y = -1$ l $x = 4, y = 3$

m $x = 1, y = -1$ n $x = 1, y = -2$ o $x = 10, y = -1$

Chapter 14 Summative Exercise

1 **a** (3 by 3) **b** (2 by 3) **c** (2 by 2) **d** (2 by 2) **e** (2 by 3) **f** (3 by 3)

2 **a** $\begin{pmatrix} -5 & 2 & -2 \\ 9 & -13 & 8 \\ 2 & -1 & -6 \end{pmatrix}$ **b** $\begin{pmatrix} 5 & 5 & 10 \\ 12 & 9 & 1 \end{pmatrix}$ **c** $\begin{pmatrix} -12 & 17 \\ 22 & -35 \end{pmatrix}$

3 **a** $\begin{pmatrix} 10 & 9 & -7 \\ 1 & -11 & 18 \end{pmatrix}$ **b** $\begin{pmatrix} -8 & 11 & -12 \\ 16 & -4 & 12 \end{pmatrix}$ **c** $\begin{pmatrix} -14 & 19 \\ 36 & -49 \end{pmatrix}$

d $\begin{pmatrix} 0 & 17 & -17 \\ -1 & -45 & 44 \end{pmatrix}$ **e** $\begin{pmatrix} 2 & 5 & -4 \\ 7 & -5 & 10 \\ -3 & 0 & 6 \end{pmatrix}$ **f** $\begin{pmatrix} -23 & 31 & -29 \\ 13 & -17 & 17 \end{pmatrix}$

4 **a** $\begin{pmatrix} 0.2 & 0.4 \\ 0.4 & 0.4 \\ 0.4 & 0.2 \end{pmatrix}\begin{pmatrix} 50 & 35 & 35 \\ 40 & 60 & 55 \end{pmatrix} = \begin{pmatrix} 26 & 31 & 29 \\ 36 & 38 & 36 \\ 28 & 26 & 25 \end{pmatrix}$ **b** $\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

5 **a** $L = \begin{pmatrix} 20 \\ 50 \\ 25 \end{pmatrix}$ **b** $V = \begin{pmatrix} 60 & 100 & 30 \\ 80 & 150 & 40 \end{pmatrix}$ **c** $\begin{pmatrix} 6950 \\ 10100 \end{pmatrix}$

d $\begin{pmatrix} 1 & 1 \end{pmatrix}\begin{pmatrix} 6950 \\ 10100 \end{pmatrix} = 17050$ **e** The total number of each category of visitor.

6 **a** The determinant is the area scale factor of the transformation.
b When the determinant is zero.

7 **a** 1 **b** 2

8 **a** $\begin{pmatrix} -8 & -3 \\ -5 & -2 \end{pmatrix}$ **b** $\frac{1}{2}\begin{pmatrix} 3 & 5 \\ 2 & 4 \end{pmatrix}$

9 **a** **(i)** $\begin{pmatrix} -26 & 31 \\ -16 & 19 \end{pmatrix}$ **(ii)** $\begin{pmatrix} -7 & -2 \\ 1 & 0 \end{pmatrix}$

b **(i)** $\mathbf{DC}^{-1}$ **(ii)** $\mathbf{C}^{-1}\mathbf{D}$

10 **(i)** $\frac{1}{2}\begin{pmatrix} 19 & -31 \\ 16 & -26 \end{pmatrix}$ **(ii)** $\frac{1}{2}\begin{pmatrix} 0 & 2 \\ -1 & -7 \end{pmatrix}$

11 **a** $x = 3, y = -2$ **b** $x = -1, y = 4$
12 **a** $x = -4, y = 7$ **b** $x = 2, y = -7$

Chapter 14 test

1 **a** $\begin{pmatrix} 6 & 7 \\ 32 & -1 \end{pmatrix}$ **b** $\frac{1}{2}\begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$ **c** $\begin{pmatrix} 2 & 0 \\ 6 & 4 \end{pmatrix}$ **d** $\begin{pmatrix} 13 & 5 \\ 23 & 9 \end{pmatrix}$

2 $\frac{1}{2}\begin{pmatrix} 4 & -5 \\ 2 & -3 \end{pmatrix}$

3 **a** $\begin{pmatrix} 7 & 6 \\ 8 & 7 \end{pmatrix}$ **b** $\begin{pmatrix} 11 & 9 \\ 14 & 12 \end{pmatrix}$ **c** $\begin{pmatrix} 47 & -21 \\ 54 & -24 \end{pmatrix}$

4 **a** $\begin{pmatrix} 7 & -1 \\ 2 & 14 \end{pmatrix}$ **b** $\begin{pmatrix} 7 & 3 \\ -6 & -14 \end{pmatrix}$ **c** $\frac{1}{10}\begin{pmatrix} 4 & 1 \\ -2 & -3 \end{pmatrix}$ **d** $x = 4, y = -3$

5 **a** $\begin{pmatrix} -\frac{5}{a} & -\frac{3}{a} \\ \frac{3}{b} & \frac{2}{b} \end{pmatrix}$ **b** $\begin{pmatrix} -17 & -\frac{9b}{a} \\ \frac{11a}{b} & 5 \end{pmatrix}$

Examination Questions

1 $ 27,600 **2** $\mathbf{A}^{-1} = \begin{pmatrix} -\frac{1}{2} & 1 \\ \frac{3}{2} & -2 \end{pmatrix}, \mathbf{B}^{-1} = \begin{pmatrix} \frac{3}{8} & \frac{-1}{8} \\ \frac{1}{4} & \frac{1}{4} \end{pmatrix}$ **(i)** $\begin{pmatrix} 3 & -1 \\ -5 & 7 \end{pmatrix}$ **(ii)** $\begin{pmatrix} \frac{9}{8} & \frac{5}{8} \\ \frac{7}{4} & \frac{3}{4} \end{pmatrix}$

3 **(i)** $(12 \;\; 5)\begin{pmatrix} 300 \\ 40 \end{pmatrix} = 3800$ **(ii)** $(10 \;\; 4)\begin{pmatrix} 180 & 400 \\ 40 & 150 \end{pmatrix} = (1960 \quad 4600)$ **(iii)** $ 10,360

4 **(i)** $\begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}$ **(ii)** $\begin{pmatrix} -3 & 1 \\ 5 & -5 \end{pmatrix}$ **5** $\begin{pmatrix} 5 & 3 & 0 \\ 4 & 1 & 2 \\ 4 & 0 & 4 \\ 2 & 1 & 4 \\ 1 & 1 & 6 \end{pmatrix}\begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 18 \\ 13 \\ 12 \\ 7 \\ 4 \end{pmatrix}$ **6** $p = 2, k = 5$

7 **(i)** $(5 \;\; 8 \;\; 4 \;\; 10)\begin{pmatrix} 300 & 60 & 40 \\ 150 & 50 & 20 \\ 120 & 40 & 0 \\ 100 & 0 & 0 \end{pmatrix}$ **(ii)** $(4180 \quad 860 \quad 360)$ **(iii)** $\begin{pmatrix} 0.05 \\ 0.1 \\ 0.2 \end{pmatrix}$ **(iv)** 367

8 **(i)** $x = 3, y = 4$ **(ii)** $\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$ **9** **(i)** $\begin{pmatrix} 14 & 23 \\ -10 & -1 \end{pmatrix}$ **(ii)** $\begin{pmatrix} 3 & -1 \\ 2 & 4 \end{pmatrix}$

10 $m = -3, n = -5$ **11** **(i)** $(0.3 \quad 0.3 \quad 0.2 \quad 0.2)\begin{pmatrix} 8 & 12 & 14 \\ 7 & 10 & 2 \\ 10 & 12 & 0 \\ 6 & 8 & 4 \end{pmatrix}\begin{pmatrix} 300 \\ 500 \\ 800 \end{pmatrix}$ **(ii)** $ 11,770

12 **(i)** $\begin{pmatrix} 12 & -18 \\ 6 & -4 \end{pmatrix}$ **(ii)** $\begin{pmatrix} 7 \\ 2 \end{pmatrix}$ **(iii)** $\begin{pmatrix} 0.9 & -0.1 \\ -0.6 & 1.8 \end{pmatrix}$ **13** $\mathbf{A}^{-1} = \begin{pmatrix} 0.4 & -0.6 \\ -0.7 & 1.3 \end{pmatrix} x = 2, y = 2.5$

14 $\mathbf{A}^{-1} = \begin{pmatrix} 0.4 & -0.6 \\ -0.3 & 0.7 \end{pmatrix} x = 5, y = -3$ **15** **(i)** $\begin{pmatrix} 6 & 3 & 1 & 2 \\ 3 & 2 & 4 & 3 \\ 2 & 5 & 5 & 0 \\ 1 & 2 & 2 & 7 \end{pmatrix}\begin{pmatrix} 5 \\ 3 \\ 2 \\ 1 \end{pmatrix}$ **(ii)** $\begin{pmatrix} 43 \\ 32 \\ 35 \\ 22 \end{pmatrix}$

16 **(i)** $\begin{pmatrix} 0 & -6 \\ 2 & -12 \end{pmatrix}$ **(ii)** $\begin{pmatrix} 11 \\ 10 \end{pmatrix}$ **(iii)** $\begin{pmatrix} 0.3 & 0.1 \\ -0.4 & 0.2 \end{pmatrix}$ $\begin{pmatrix} 0.5 & -0.9 \\ 0 & 1.2 \end{pmatrix}$

Chapter 15

Problem 15.1

b $\frac{1}{n+1}x^{n+1}$

Exercise 15.1

1 **a** $f(x)=\frac{1}{3}x^3+c$ **b** $f(x)=\frac{1}{4}x^4+c$ **c** $f(x)=\frac{1}{5}x^5+c$ **d** $f(x)=2x^3+c$

e $f(x)=2x^4+c$ **f** $f(x)=3x^5+c$ **g** $f(x)=-\frac{1}{x}+c$ **h** $f(x)=-\frac{1}{2x^2}+c$

i $f(x)=-\frac{1}{3x^3}+c$ **j** $f(x)=\frac{5}{x}+c$ **k** $f(x)=\frac{3}{x^2}+c$ **l** $f(x)=\frac{3}{x^3}+c$

2 **a** $y=\frac{1}{3}x^3-\frac{1}{4}x^4+c$ **b** $y=2x^4-x^2+c$ **c** $y=2x^5-\frac{3}{4}x^4+c$ **d** $y=3x^3-\frac{1}{x}+c$

e $y=\frac{1}{4}x^4+\frac{1}{x^2}+c$ **f** $y=\frac{1}{5}x^5-\frac{1}{x^3}+c$ **g** $y=\frac{8}{3}\sqrt{x^3}+4\sqrt{x}+c$

h $y=\frac{2}{5}\sqrt{x^5}-3\sqrt{x^3}+c$ **i** $y=\frac{3}{4}\sqrt[3]{x^4}+9\sqrt[3]{x}+c$

3 **a** $y=2x^2-2x^3+3$ **b** $y=x^5-2x^4-2$ **c** $y=2x^5-x^4+4$ **d** $y=3x^3-\frac{1}{x}-6$

e $y=2x^4+\frac{3}{x}+1$ **f** $y=3x^2-\frac{2}{x^2}+\frac{7}{2}$ **g** $y=4\sqrt{x^3}+4\sqrt{x}+2$

h $y=6\sqrt{x^5}-3\sqrt{x^3}+4$ **i** $y=9\sqrt[3]{x^4}+\sqrt[3]{x}+4$ **j** $y=4\sqrt{x^3}-4\sqrt{x^5}-4$

4 **a** $2x^4+2x^3+c$ **b** $2y^4+2y^3+c$ **c** $4\sqrt{t^3}+6\sqrt{t}+c$ **d** $v^5+\frac{2}{v}+c$

e $\frac{3}{2}\sqrt[3]{y^4}+12\sqrt{y^3}+c$ **f** $\frac{12}{5}\sqrt{t^5}-\frac{6}{\sqrt{t}}+c$

5 $y=x^3+3x^2+4x+2$

6 $y=-\frac{1}{x}+2x+3$

7 $y=x^3+3x^2-11x+6$

8 $y=x^3-\frac{1}{x}+\frac{1}{x^3}-x$

9 $y=x^3-6x^2+9x+1$; (1, 5) maximum; (3, 1) minimum

10 **a** $y=\frac{1}{6}(2x+1)^3+c$ **b** $\frac{4}{3}x^3+2x^2+x+k$ **c** $y=\frac{4}{3}x^3+2x^2+x+\frac{1+c}{6}$ $6k=1+c$

Exercise 15.2

1 **a** 4 **b** 18 **c** $\frac{64}{5}$ **d** 21 **e** 40 **f** $\frac{45}{4}$

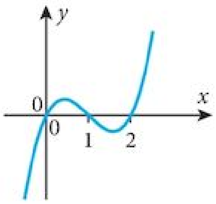
g $-\frac{14}{3}$ **h** $\frac{14}{3}$ **i** 10 **j** $\frac{3}{2}$ **k** 1 **l** $\frac{3}{4}$

m -8 **n** $\frac{16}{3}$ **o** $\frac{1}{6}$

2 **a** 2 **b** $\frac{20}{3}$ **c** $-\frac{1}{32}$ **d** -6 **e** $\frac{10}{3}$ **f** -4

3 **a** $\frac{2}{3}$ **b** $\frac{16}{3}$ **c** $\frac{38}{3}$ **d** $\frac{125}{6}$ **e** 15 **f** $\frac{25}{4}$

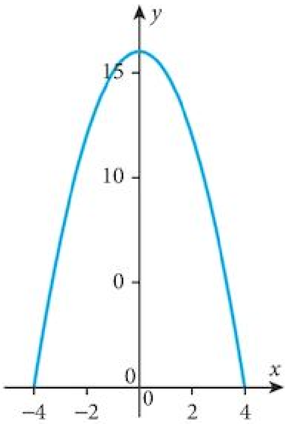
4 **a**



b **(i)** $\frac{1}{4}$ **(ii)** $-\frac{1}{4}$ **(iii)** 0

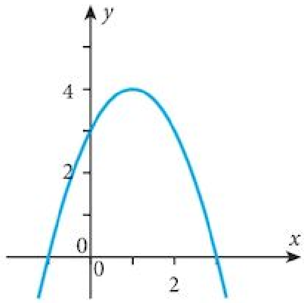
c some is above the axis some is below; $\frac{1}{2}$

5 **a**



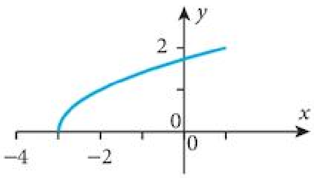
b $\frac{256}{3}$

6 **a**



b $\frac{32}{3}$

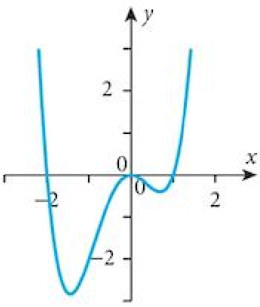
7 **a**



b $\frac{9\sqrt[3]{3}}{4}$

8 **a** (−1.443, −2.833), Minimum (0, 0), Maximum; (0.693, −0.397), Minimum

b



c $\frac{63}{20}$

Chapter 15 Summative Exercise

1 **a** $f(x) = 2x^5 + c$ **b** $f(x) = x^8 + c$ **c** $f(x) = x^{15} + c$

d $f(x) = -\frac{2}{x^4} + c$ **e** $f(x) = -\frac{5}{x^2} + c$ **f** $f(x) = -\frac{3}{x^3} + c$

2 **a** $y = 4\sqrt{x^3} + c$ **b** $y = 4\sqrt{x^5} + c$ **c** $y = 2\sqrt{x^7} + c$

d $y = 6\sqrt[3]{x^4} + c$ **e** $y = 8\sqrt[4]{x^5} + c$ **f** $y = 5\sqrt[5]{x^6} + c$

3 **a** $y = 4\sqrt{x} + c$ **b** $y = -\frac{8}{\sqrt{x}} + c$ **c** $y = -\frac{4}{\sqrt{x^3}} + c$

d $y = 3\sqrt[3]{x^2} + c$ **e** $y = \frac{8}{\sqrt[4]{x^3}} + c$ **f** $\frac{10}{\sqrt[4]{x^5}} + c$

4 **a** $y = x^2 - 3x + 5$ **b** $y = 2x^2 + \frac{6}{x} - 5$ **c** $y = 2x^3 - 2x^2 + 3x + 1$

d $y = x^3 + \frac{10}{2x - 1} + 4$

5 $y = -2x^3 + 2x^2 + 30x - 20$

6 $y = x^4 + 2x^3 - 6x^2 + x - 3$

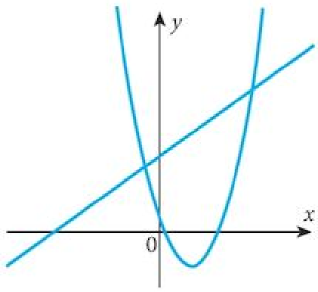
7 a $f'(x) = \dfrac{-3}{(3x+4)^2}$ b $y = \dfrac{-4}{3x+4} - 2$

8 a $\dfrac{45}{4}$ b $\dfrac{14}{3}$ c $\dfrac{69}{2}$ d $\dfrac{32}{15}$ e $\dfrac{788}{5}$ f $\dfrac{125}{12}$

9 b $\dfrac{132}{7}$

10 8

11 a $(-1, 6), (6, 13)$ b



c $\dfrac{343}{6}$

12 $\dfrac{64}{3}$

13 $\dfrac{253}{12}$

14 $\dfrac{3}{2}$

Chapter 15 Test

1 $\dfrac{4}{3}$

2 a $y - 21 = 2(x - 1)$ b $(5, 29)$ c $21\dfrac{1}{3}$

3 a $\dfrac{x}{(2x+1)^{\frac{3}{2}}}$ b $\dfrac{2}{3}$

4 a $x - \dfrac{6}{x^2} + c$ b $-1, 1.5, 3$

5 a $\dfrac{2}{9}(3x+1)^{\frac{3}{2}} + c$ b $\dfrac{1}{9}\left(250 - 26\sqrt{13}\right)$

6 a $2x^2 + \dfrac{1}{x-4} + c$ b $\dfrac{19}{2}$

7 a $\dfrac{2}{15}(5x-4)^{\frac{3}{2}} + c$ b 304

Examination Questions

1 $\dfrac{7}{4}$

2 (i) $y = \dfrac{10}{x^2} + 1$ (ii) $\sqrt{10}$

3 (i) $y = x^3 - x^2 - 5x - 3$

4 (i) $y = 3\sqrt{4x+1} + 5$ (ii) $(0, 15), (30, 0)$

5 (i) $X\ (16, 0);\ M\ (4, 4)$ (ii) $\dfrac{128}{3}$

6 (i) $\left(\dfrac{4}{3}, \dfrac{256}{27}\right)$ (ii) $\dfrac{64}{3}$

Term test 4A (Chapters 12 to 15)

1 a $3\sqrt{2}\left(\sqrt{3}+1\right)$ b $\dfrac{5\pi}{6}$ c $3\sqrt{2}\left(\sqrt{3}+1\right) + 5\pi$ d $3(5\pi - 3)$

2 a $f(x) = 3 - x^2$ b $\dfrac{2\sqrt{3}}{3}$

3 a $y = 5 + 4x - 3x^2$ b $(2, 4)$

4 a $\begin{pmatrix} 2 & -14 & 9 \\ 1 & -10 & 7 \end{pmatrix}$ b $\begin{pmatrix} -22 \\ -2 \end{pmatrix}$ c $\begin{pmatrix} -2 & 3 \\ 3 & -4 \end{pmatrix}$

5 a $y = 4\sqrt{2x+1} - 3$ b 12

6 a $\frac{1}{2}\begin{pmatrix} 3 & -2 \\ 5 & -4 \end{pmatrix}$ b $a = 3, b = -2, c = 4, d = -12$

Chapter 16

Problem 16.1

a $OC = \cos\theta$; $CB = \sin\theta$; $DP = \tan\theta$
b $OP = \sec\theta$; $OQ = \operatorname{cosec}\theta$; $AQ = \cot\theta$

Exercise 16.1

1 a $\theta \in \{60°, 300°\}$ b $\theta \in \{40.0°, 220.0°\}$ c $\theta \in \{60°, 120°\}$
d $\theta \in \{145.0°, 215.0°\}$ e $\theta \in \{146.3°, 326.3°\}$ f $\theta \in \{233.1°, 306.9°\}$

2 a $\theta \in \{-50.0°, 50.0°\}$ b $\theta \in \{60.0°, -120.0°\}$ c $\theta \in \{75.0°, 105.0°\}$
d $\theta \in \{-126.9°, 126.9°\}$ e $\theta \in \{-21.8°, 158.2°\}$ f $\theta \in \{-168.5°, -11.5°\}$

3 a $\theta \in \{-1.16, 1.16\}$ b $\theta \in \{-2.82, 0.322\}$ c $\theta \in \{0.253, 2.89\}$
d $\theta \in \{-1.91, 1.91\}$ e $\theta \in \{-0.464, 2.68\}$ f $\theta \in \{-3.04, -0.100\}$

4 a $\theta \in \{1.28, 5.00\}$ b $\theta \in \{1.19, 4.33\}$ c $\theta \in \{0.340, 2.80\}$
d $\theta \in \{\pi\}$ e $\theta \in \{1.77, 4.91\}$ f $\theta \in \{3.31, 6.12\}$

5 a $\theta \in \{33.2°, 146.8°, 213.2°, 326.8°\}$ b $\theta \in \{6.1°, 66.1°, 126.1°, 186.1°, 246.1°, 306.1°\}$
c $\theta \in \{7.2°, 82.8°, 187.2°, 262.8°\}$ d $\theta \in \{33.8°, 86.2°, 153.8°, 206.2°, 273.8°, 326.2°\}$
e $\theta \in \{32.2°, 77.2°, 122.2°, 167.2°, 212.2°, 257.2°, 302.2°, 347.2°\}$
f $\theta \in \{46.8°, 88.2°, 136.8°, 178.2°, 226.8°, 268.2°, 316.8°, 358.2°\}$
g $\theta \in \{18.2°, 131.8°, 198.2°, 311.8°\}$ h $\theta \in \{13.9°, 73.9°, 133.9°, 193.9°, 253.9°, 313.9°\}$
i $\theta \in \{2.2°, 77.8°, 182.2°, 257.8°\}$

6 a $\theta \in \{0.421, 2.72, 3.56, 5.86\}$ b $\theta \in \{0.155, 1.20, 2.25, 3.30, 4.34, 5.39\}$
c $\theta \in \{0.206, 1.37, 3.35, 4.51\}$ d $\theta \in \{0.620, 1.47, 2.71, 3.57, 4.81, 5.66\}$
e $\theta \in \{0.488, 1.27, 2.06, 2.84, 3.63, 4.41, 5.20, 5.99\}$
f $\theta \in \{0.849, 1.51, 2.42, 3.08, 3.99, 4.65, 5.56, 6.22\}$
g $\theta \in \{0.650, 1.49, 3.79, 4.63\}$ h $\theta \in \{1.04, 2.08, 3.13, 4.18, 5.22, 6.27\}$
i $\theta \in \{0.333, 1.38, 3.47, 4.52\}$

Exercise 16.2

1 a $\theta \in \{63.4°, 243.4°\}$ b $\theta \in \{71.6°, 251.6°\}$
c $\theta \in \{59.0°, 239.0°\}$ d $\theta \in \{30°, 150°, 210°, 330°\}$
e $\theta \in \{60°, 120°, 240°, 300°\}$ (Note that $\tan\theta$, $\sec\theta$ are not defined for $\theta = 90°$)
f $\theta \in \{45°, 135°, 225°, 315°\}$ g $\theta \in \{38.2°, 141.8°\}$
h $\theta \in \{56.3°, 135°, 236.3°, 315°\}$ i $\theta \in \{60°, 120°, 240°, 300°\}$
j $\theta \in \{71.6°, 116.6°, 251.6°, 296.6°\}$ k $\theta \in \{221.8°, 270°, 318.2°\}$
l $\theta \in \{144.3°, 155.7°, 294.3°, 335.7°\}$

2 a $\theta \in \{-1.25, 1.89\}$ b $\theta \in \{-0.464, 2.68\}$
c $\theta \in \left\{-\frac{\pi}{4}, \frac{3\pi}{4}\right\}$ d $\theta \in \left\{-\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}\right\}$
e $\theta \in \left\{-\frac{5\pi}{6}, \frac{\pi}{6}, \frac{\pi}{6}, \frac{5\pi}{6}\right\}$ f $\theta \in \{-2.53, -0.615, 0.615, 2.53\}$
g $\theta \in \left\{-\frac{\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}\right\}$ h $\theta \in \{-1.23, 3.14, -3.14, 1.23\}$

i $\theta \in \{-2.09, -1.91, 1.91, 2.09\}$ **j** $\theta \in \left\{-\frac{3\pi}{4}, -1.11, \frac{\pi}{4}, 2.03\right\}$

k $\theta \in \left\{-\frac{5\pi}{6}, -\frac{\pi}{6}, 0.848, -5.44\right\}$ **l** $\theta \in \{-2.59, -1.02, 0.554, 2.12\}$

Problem 16.3

c The basic graph is dilated in the x-direction by scale factor $\frac{1}{k}$.

Problem 16.4

c The basic graph is translated in the x-direction by $-\theta°$.

Problem 16.5

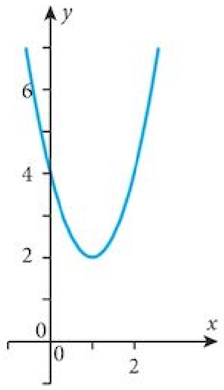
c The basic graph is dilated in the y-direction by scale factor $\frac{1}{k}$.

Exercise 16.3

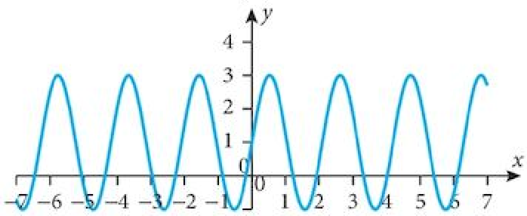
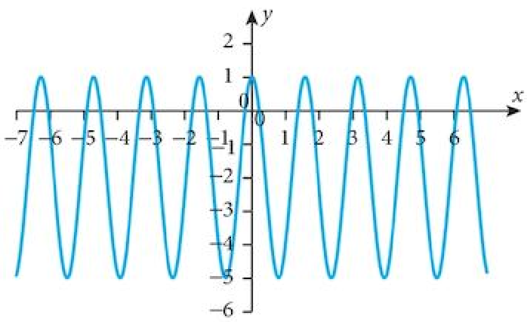
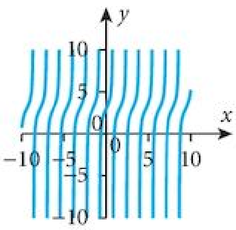
1 **a** $y = 2(x-1)^2 + 2$

c Dilation in the y-direction by scale factor 2 followed by a translation of (1, 2) units.

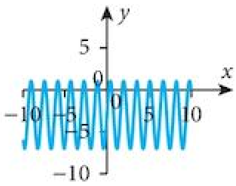
d



2

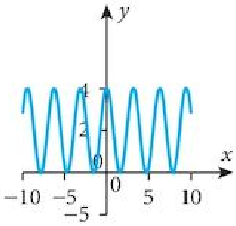
	(i)	(ii)	(iii)	(iv)
a	Dilation by factors $\left(\frac{1}{3}, 2\right)$ translation by (0, 1)		$120°, \frac{2\pi}{3}$	2
b	Dilation by factors $\left(\frac{1}{4}, 3\right)$ translation by (0, −2)		$90°, \frac{\pi}{2}$	3
c	Dilation by factors $\left(\frac{1}{2}, 1\right)$ translation by (0, 3)		$180°, \pi$	–

d Dilation by factors $\left(\frac{1}{4}, 4\right)$ translation by $(0, -3)$



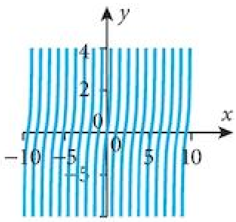
$90°, \frac{\pi}{2}$ –

e Dilation by factors $\left(\frac{1}{2}, 2\right)$ translation by $(0, 2)$



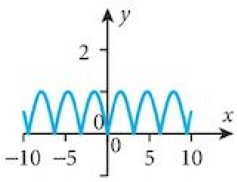
$180°, \pi$ 2

f Dilation by factors $\left(\frac{1}{3}, 2\right)$ translation by $(0, 0)$



$120°, \frac{2\pi}{3}$ –

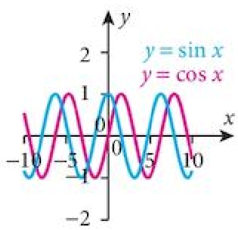
3



4

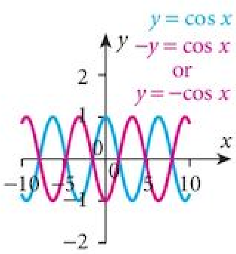
	Maximum	@	Minimum	@
a	2	45°	−4	135°
b	6	120°	−2	60°
c	4	30°	2	90°
d	6	15°	2	45°
e	1	60°	−5	30°
f	2	72°	−2	36°

5



translation $(-90°, 0)$; $\cos x = \sin(x + 90°)$

6 **a**



translation $(-90°, 0)$; $\cos x = \sin(x + 90°)$

c The transformation $x \mapsto -x$ does not change the graph. $\cos(-x) = \cos x$.

Chapter 16 Summative Exercise

1 **a** 70°, 290° **b** 50°, 130° **c** 140°, 320°
d 70°, 110°, 250°, 290° **e** 110°, 160°, 200°, 340° **f** 10°, 100°, 190°, 280°

2 **a** $-\frac{\pi}{5}, \frac{\pi}{5}$ **b** $\frac{\pi}{3}, \frac{2\pi}{3}$ **c** $-\frac{3\pi}{8}, \frac{5\pi}{8}$

d $-\frac{31\pi}{24}, -\frac{7\pi}{24}, \frac{7\pi}{24}, \frac{31\pi}{24}$ **e** $-\frac{3\pi}{8}, -\frac{\pi}{8}, \frac{5\pi}{8}, \frac{7\pi}{8}$ **f** $-1.256, 1.884$

3 **a** $5°, 35°$ **b** $-170°, -70°$ **c** $40°, -140°$ **d** $-156.7°, -90°, 30°, 36.7°, 83.3°, 150°$

e $-120°, -90°, 60°, 90°$ **f** $-160°, -100°, -40°, 20°, 80°, 140°$

4 **a** $-2.80, -\frac{\pi}{2}, 0.380, \frac{\pi}{2}$ **b** $-2.19, -0.955, 0.955, 2.19$ **c** $-\pi, -\frac{\pi}{3}, \frac{\pi}{3}, \pi$

d $-\frac{\pi}{3}, \frac{\pi}{3}$ **e** $-2.03, -1.11, 1.11, 2.03$ **f** $-\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{3\pi}{4}$

5 **a** 1.95, 4.34 **b** 1.11, 1.89, 4.25, 5.03 **c** 1.23, 5.05

d 0.339, 2.80

6 **a** 1, 5 **b** 1, 4, 5, 7, 10

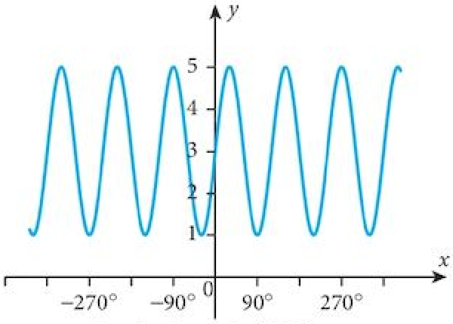
7 **b** 2 **c** $\frac{1+\tan x}{1-\tan x}$

9 There are many alternative answers to this question.

a $y = \sin 2x$ **b** $y = -\sin\frac{1}{2}x$ **c** $y = \frac{1}{2}\cos 3x$

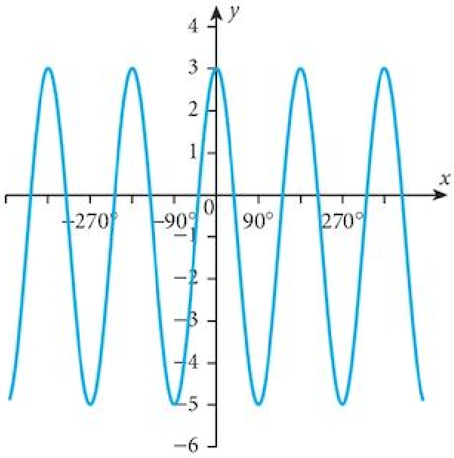
d $y = \frac{1}{2}\cos(x + 90°)$ **e** $y = -\cos\left(\frac{1}{2}x\right)$ **f** $y = -\tan x$

10 **a**



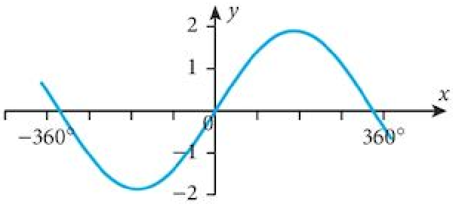
max 5, min 1, period 120°

b



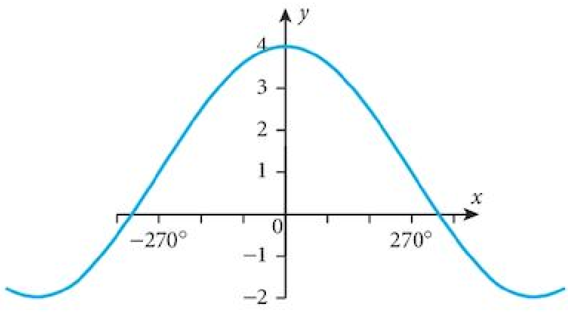
max 3, min −5, period 180°

c



max 2, min −2, period 720°

d

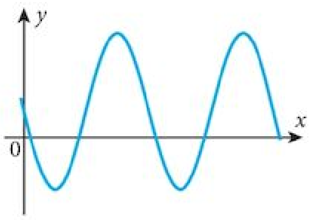


max 4, min −2, period 1080°

Chapter 16 Test

2 **a** $0, \pi, 3\pi$ **b** $\frac{3\pi}{2}, 3\pi$

3 **a**



b 3 **c** π

4 **a** 3.52, 0.38 **b** 0.33, 3.48, 1.908, 5.04

5 **a** $p\cos\theta$ **b** $\frac{p}{\sin\theta}$

6 1.107, 4.248

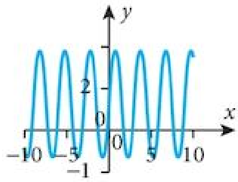
Examination Questions

1 $\theta \in \{143.1°, 323.1°\}$

2 **a** 113.6° or 246.4° **b** $y = 3.28$ radians

3 $k = 2$

4 **(i)** $a = 2; b = 3$ **(ii)** $c = 1$ **(iii)**



5 $x = 14.1$

7 **(i)** Amplitude = 3; period = $\frac{\pi}{2}$ **(ii)** Max $\left(\frac{\pi}{2}, 8\right)$; Min $\left(\frac{\pi}{4}, 2\right), \left(\frac{3\pi}{4}, 2\right)$

8 **(i)** 78.7° (1.37 rad) **(ii)** $x^2 + y^2 = 13$

9 $1 + \sqrt{2}$

10 **(i)** $x \in \{30°, 60°, 120°, 150°\}$ **(ii)** $y \in \{90°, 221.8°, 318.2°\}$ **(iii)** $z \in \left\{\frac{\pi}{6}, \frac{5\pi}{6}\right\}$

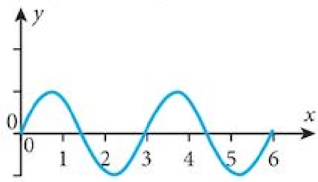
11 **(i)** 5 **(ii)** π **(iii)** Max = 8; min = −2

13 **a** **(i)** $x \in \{56.3°, 236.3°\}$ **(ii)** $y \in \{60°, 300°\}$ (Note: $\cos y = -2$ is not a valid solution)

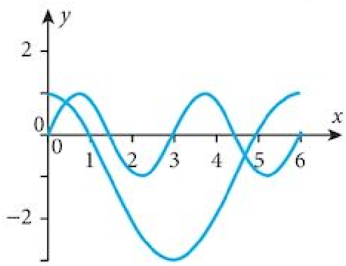
b $z \in \{0.06, 0.51\}$

14 **b** **(i)** $x \in \{70.5°, 180°, 289.5°\}$ **(ii)** $y \in \{3.66, 5.76\}$

15 **(i)**



(ii) $y = 2\cos x - 1$



(iii) 2

Chapter 17

Problem 17.1

a $2x$ **b** $3x^2$ **c** x^5 **d** $5x^4$ **e** $6x^3$, No

Exercise 17.1

1 **a** $(x + 3)(1) + (1)(x - 4)$ **b** $(2x + 3)(-4) + (2)(5 - 4x)$

c $(2 - 3x)(-2) + (-3)(6 - 2x)$ **d** $f'(x) = (x + 6)(2x - 4) + (1)(x^2 - 4x)$

e $f'(x) = (2x^3 - 3x^2)(-2) + (6x^2 - 6x)(1 - 2x)$ **f** $f'(x) = (2 - 3x)(-2 + 2x) + (-3)(6 - 2x + x^2)$

g $\frac{dy}{dx} = (x^2 - 6x)(4x + 3) + (2x - 6)(2x^2 + 3x)$ **h** $\frac{dy}{dx} = (2x - x^3)(4 - 4x) + (2 - 3x^2)(4x - 2x^2)$

i $\frac{dy}{dx} = (3x - 2x^2)(2) + (3 - 4x)(4 + 2x)$

2 **a** $(x^2)(1) + (2x)(x-4)$ **b** $(x^3)[2(5-4x)(-4)] + (3x^2)\,(5-4x)^2$
c $(x^4)[3(6-2x)^2(-2)] + [4x^3](6-2x)^3$ **d** $f'(x) = (3x^2)[2(2x+5)(2)] + [6x]\,(2x+5)^2$
e $f'(x) = (4x^3)[4(1-2x)^3(-2)] + [12x^2](1-2x)^4$ **f** $f'(x) = (5x^3)[3(6-2x)^2(-2)] + (15x^2)(6-2x)^3$

g $\dfrac{dy}{dx} = (4x^3)(6x-4) + (12x^2)(3x^2-4x+2)$

h $\dfrac{dy}{dx} = (2x-1)^3[2(4-2x)(-2)] + [3(2x-1)^2(2)](4-2x)^2$

i $\dfrac{dy}{dx} = (3x-5)^2[4(2x-1)^3(2)] + [2(3x-5)(3)]\,(2x-1)^4$

3 **a** $(x^2-4)^2[2(2x^3-3x)(6x^2-3)] + [2(x^2-4)(2x)](2x^3-3x)^2$
b $(2x-x^3)^2[3(4x-2x^2)^2(4-4x)] + [2(2x-x^3)(2-3x^2)]\,(4x-2x^2)^3$
c $(x^2-6x)^3[4(2x^2+3x)^3(4x+3)] + [3(x^2-6x)^2(2x-6)](2x^2+3x)^4$
d $f'(x) = (3x^2-2x)^3[2(2x+5)(2)] + 3(3x^2-2x)^2(6x-2)(2x+5)^2$
e $f'(x) = (4x^3+2x)^2[4(x^2-2x^3)^3(2x-6x^2)] + [2(4x^3+2x)\,(12x^2+2)](x^2-2x^3)^4$
f $f'(x) = (2x^2-5x)^4[3(6-2x)^2(-2)] + [4(2x^2-5x)^3(4x-5)](6-2x)^3$

g $\dfrac{dy}{dx} = (x^2-4)^3[5(3x^2-4x+2)^4(6x-4)] + [3(x^2-4)^2(2x)](3x^2-4x+2)^5$

h $\dfrac{dy}{dx} = (x^2-3x+1)^5[3(3x-4x^2)^2(3-8x)] + [5(x^2-3x+1)^4(2x-3)](3x-4x^2)^3$

i $\dfrac{dy}{dx} = (4x^2-6x)^6[5(3x^3-4x^2)^4(9x^2-8x)] + [6(4x^2-6x)^5(8x-6)]\,(3x^3-4x^2)^5$

4 **a** $(x)\left[\dfrac{1}{2\sqrt{x-4}}\right]+[1]\sqrt{(x-4)}$ **b** $(2x)\left[\dfrac{-1}{2\sqrt{3-x}}\right]+[2]\sqrt{3-x}$ **c** $(2x^2)\left[\dfrac{3}{2\sqrt{3x-1}}\right]+[4x]\sqrt{3x-1}$

d $f'(x) = \sqrt{x-4}\left[\dfrac{2}{2\sqrt{2x+3}}\right]+\left[\dfrac{1}{2\sqrt{x-4}}\right]\sqrt{2x+3}$

e $f'(x) = \sqrt{x^2-4}\left[\dfrac{6x^2+3}{2\sqrt{2x^3+3x}}\right]+\left[\dfrac{2x}{2\sqrt{x^2-4}}\right]\sqrt{2x^2+3x}$

f $f'(x) = (x^2-5x)^3\left[\dfrac{12x-2}{2\sqrt{6x^2-2x}}\right]+[3(x^2-5x)^2(2x-5)]\sqrt{6x^2-2x}$

g $\dfrac{dy}{dx} = (x^2-4)\left[\dfrac{4x+4}{3\sqrt[3]{(2x^2+4x-2)^2}}\right] + [2x]\sqrt[3]{2x^2+4x-2}$

h $\dfrac{dy}{dx} = (x^2-3x)^2\left[\dfrac{3-8x}{4\sqrt[4]{(3x-4x^2)^3}}\right] + [2(x^2-3x)(2x-3)]\sqrt[4]{3x-4x^2}$

i $\dfrac{dy}{dx} = 5[(2x^2-3x)\,(4x^3-2x^2)]^4[(2x^2-3x)\,(12x^2-4x) + (4x-3)\,(4x^3-2x^2)]$

5 tgt: $y-8 = 28(x-2)$; nml: $y-8 = -\dfrac{1}{28}(x-2)$

6 $6p$ **7** $\dfrac{361}{9}$

8 **a** **(i)** $V = (x+4)(x-1)^2$ **(ii)** $S = 4(x+4)\,(x-1) + 2(x-1)^2$ **b** $\dfrac{8}{11}$

Problem 17.3

a $f'(x) = u(x) \times s'(x) + u'(x) \times s(x)$

b $s'(x) = v(x) \times w'(x) + v'(x) \times w(x)$

c $f'(x) = u(x) \times [v(x) \times w'(x) + v'(x) \times w(x)] + u'(x) \times [v(x) \times w(x)]$

$f'(x) = u(x) \times v(x) \times w'(x) + u(x) \times v'(x) \times w(x) + u'(x) \times v(x) \times w(x)$

Problem 17.4

a $-\dfrac{v'(x)}{[v(x)]^2}$

b $u(x) \times \left[-\dfrac{v'(x)}{[v(x)]^2}\right] + u'(x) \times \left[\dfrac{1}{v(x)}\right]$

Exercise 17.2

1 a $\dfrac{(x-4)[1]-[1](x+3)}{(x-4)^2}$

b $\dfrac{(5-4x)[2]-[-4](2x-3)}{(5-4x)^2}$

c $\dfrac{(6-2x)[-3]-[-2](2-3x)}{(6-2x)^2}$

d $f'(x) = \dfrac{(x^2-4x)[1]-[2x-4](x+6)}{(x^2-4x)^2}$

e $f'(x) = \dfrac{(1-2x)[6x^2-6x]-[-2](2x^3-3x^2)}{(1-2x)^2}$

f $f'(x) = \dfrac{(6-2x+x^2)[-3]-[-2+2x](2-3x)}{(6-2x+x^2)^2}$

g $\dfrac{dy}{dx} = \dfrac{(2x^2+3x)[2x-6]-[4x+3](x^2-6x)}{(2x^2+3x)^2}$

h $\dfrac{dy}{dx} = \dfrac{(4x-2x^2)[2-3x^2]-[4-4x](2x-x^3)}{(4x-2x^2)^2}$

i $\dfrac{dy}{dx} = \dfrac{(4+2x)[3-4x]-[2](3x-2x^2)}{(4+2x)^2}$

2 a $\dfrac{(x-4)[2x]-[1](x^2)}{(x-4)^2}$

b $\dfrac{(5-4x)^2[3x^2]-[2(5-4x)(-4)](x^3)}{(5-4x)^4}$

c $\dfrac{(6-2x)^3[4x^3]-[3(6-2x)^2(-2)](x^4)}{(6-2x)^6}$

d $f'(x) = \dfrac{(2x+5)^2[6x]-[2(2x+5)(2)](3x^2)}{(2x+5)^4}$

e $f'(x) = \dfrac{(1-2x)^4[12x^2]-[4(1-2x)^3(-2)](4x^3)}{(1-2x)^8}$

f $f'(x) = \dfrac{(6-2x)^3[15x^2]-[3(6-2x)^2(-2)](5x^3)}{(6-2x)^6}$

g $\dfrac{dy}{dx} = \dfrac{(3x^2-4x+2)[12x^2]-[6x-4](4x^3)}{(3x^2-4x+2)^2}$

h $\dfrac{dy}{dx} = \dfrac{(x^3+4x)[2+2x]-[3x^2+4](2x+x^2)}{(x^3+4x)^2}$

i $\dfrac{dy}{dx} = \dfrac{(x^3-4)[2x]-[3x^2](x^2-6)}{(x^3-4)^2}$

3 a $\dfrac{(2x^3-3x)^2[2(x^2-4)(2x)]-[2(2x^3-3x)(6x^2-3)](x^2-4)^2}{(2x^3-3x)^4}$

b $\dfrac{(4x-2x^2)^3[2(2x-x^3)(2-3x^2)]-[3(4x-2x^2)^2(4-4x)](2x-x^3)^2}{(4x-2x^2)^6}$

c $\dfrac{(2x^2+3x)^4[3(x^2-6x)^2(2x-6)]-[4(2x^2+3x)^3(4x+3)](x^2-6x)^3}{(2x^2+3x)^8}$

d $f'(x) = \dfrac{(2x+5)^2[3(3x^2-2x)^2(6x-2)]-[2(2x+5)(2)](3x^2-2x)^3}{(2x+5)^4}$

e $f'(x) = \dfrac{\left(x^2-2x^3\right)^4\left[2\left(4x^3+2x\right)\left(12x^2+2\right)\right]-\left[4\left(x^2-2x^3\right)^3\left(2x-6x^2\right)\right]\left(4x^3+2x\right)^2}{\left(x^2-2x^3\right)^8}$

f $f'(x) = \dfrac{(6-2x)^3\left[4\left(2x^2-5x\right)^3(4x-5)\right]-\left[3(6-2x)^2(-2)\right]\left(2x^2-5x\right)^4}{(6-2x)^6}$

g $\dfrac{dy}{dx} = \dfrac{\left(3x^2-4x+2\right)^5\left[3\left(x^2-4\right)^2(2x)\right]-\left[5\left(3x^2-4x+2\right)^4(6x-4)\right]\left(x^2-4\right)^3}{\left(3x^2-4x+2\right)^{10}}$

h $\dfrac{dy}{dx} = \dfrac{\left(3x-4x^2\right)^3\left[5\left(x^2-3x+1\right)^4(2x-3)\right]-\left[3\left(3x-4x^2\right)^2(3-8x)\right]\left(x^2-3x+1\right)^5}{\left(3x-4x^2\right)^6}$

i $\dfrac{dy}{dx} = \dfrac{\left(3x^3-4x^2\right)^5\left[6\left(4x^2-6x\right)^5(8x-6)\right]-\left[5\left(3x^3-4x^2\right)^4\left(9x^2-8x\right)\right]\left(4x^2-6x\right)^6}{\left(3x^3-4x^2\right)^{10}}$

4 There are alternative approaches to some of the questions which lead to equivalent answer that look a little different from those here.

a $\dfrac{\sqrt{x-4}[1]-\left[\dfrac{1}{2\sqrt{x-4}}\right]x}{x-4}$ **b** $\dfrac{\sqrt{3-x}[2]-\left[\dfrac{-1}{2\sqrt{3-x}}\right]2x}{3-x}$ **c** $\dfrac{\sqrt{3x-1}[4x]-\left[\dfrac{3}{2\sqrt{3x-1}}\right]2x^2}{3x-1}$

d $f'(x) = \dfrac{\sqrt{2x+3}\times\left[\dfrac{1}{2\sqrt{x-4}}\right]-\left[\dfrac{2}{2\sqrt{2x+3}}\right]\sqrt{x-4}}{2x+3}$

e $f'(x) = \dfrac{\sqrt{2x^3+3x}\times\left[\dfrac{x}{\sqrt{x^2-4}}\right]-\left[\dfrac{6x^2+3}{2\sqrt{2x^3+3x}}\right]\sqrt{x^2-4}}{2x^3+3x}$

f $f'(x) = \dfrac{\sqrt{6x^2-2x}\times\left[3\left(x^2-5x\right)^2(2x-5)\right]-\left[\dfrac{12x-2}{2\sqrt{6x^2-2x}}\right]\left(x^2-5x\right)^3}{6x^2-2x}$

g $\dfrac{dy}{dx} = \dfrac{\sqrt[3]{2x^2+4x-2}\times[2x]-\left[\dfrac{4x+4}{\sqrt[3]{\left(2x^2+4x-2\right)^2}}\right]\left(x^2-4\right)}{\sqrt[3]{\left(2x^2+4x-2\right)^2}}$

h $\dfrac{dy}{dx} = \dfrac{\sqrt[4]{3x-4x^2}\times\left[2\left(x^2-3x\right)(2x-3)\right]-\left[\dfrac{3-8x}{4\sqrt[4]{\left(3x-4x^2\right)^3}}\right]\left(x^2-3x\right)^2}{\sqrt{3x-4x^2}}$

i $\dfrac{dy}{dx} = \dfrac{\left[(2x-3)\left(x^2+x\right)\right]^2[2x]-\left[2(2x-3)\left(x^2+x\right)[2\left(x^2+x\right)+(2x+1)(2x-3)\right]x^2}{\left[(2x-3)\left(x^2+x\right)\right]^4}$

5 $\left(-3,\dfrac{3}{2}\right);\left(1,\dfrac{3}{2}\right)$ **6** tgt: $y-4=3(x-2)$, nml: $y-4=-\dfrac{1}{3}(x-2)$

7 **a** (0.293, 2.17) Maximum; (1.707, 7.83) Minimum.
b (–1.414, 0.25); (1.414, 0,25) both Maximum.

8 $\frac{4p}{9}$

9 $\frac{361}{162}$

10 a $\frac{4}{9}$ b $-\frac{1}{9}$

Chapter 17 Summative Exercise

1 a $3(2x-7)+2(3x+4)$

b $f'(x)=-2(5-3x)-3(6-2x)$

c $\frac{dy}{dx}=(2-2x)(5x+x^3)+(5+3x^2)(2x-x^2)$

d $(2x-2)(x^3+3x)+(3x^2+3)(x^2-2x)$

e $f'(x)=\left(1+\frac{1}{2\sqrt{x}}\right)\left(x^2-\sqrt{x}\right)+\left(2x-\frac{1}{2\sqrt{x}}\right)\left(x+\sqrt{x}\right)$

f $\frac{dy}{dx}=\left(\frac{1}{4\sqrt[4]{x^3}}\right)\left(\sqrt[3]{x}+4\right)+\left(\frac{1}{3\sqrt[3]{x^2}}\right)\left(\sqrt[4]{x}-3\right)$

2 a $3(x^2-4x)^2(2x-4)(3x-2)^2+6(3x-2)(x^2-4x)^3$

b $f'(x)=2[(x^2+2x)(x-3)][(2x+2)(x-3)+(x^2+2x)]$

c $\frac{dy}{dx}=3[(3x-8)^3(x^2-5x)]^2[9(3x-8)^2(x^2-5x)+(2x-5)(3x-8)^3]$

d $\frac{1}{\sqrt{x}}\left(\sqrt{x}+4\right)\left(3-\sqrt{x}\right)^3-\frac{3}{2\sqrt{x}}\left(3-\sqrt{x}\right)^2\left(\sqrt{x}+4\right)^2$

e $f'(x)=2(x^3+\sqrt{x})\left(3x^2+\frac{1}{2\sqrt{x}}\right)\left(\sqrt{x}+4\right)^3+\frac{3}{2\sqrt{x}}\left(\sqrt{x}+4\right)^2\left(x^3+\sqrt{x}\right)^2$

f $\frac{dy}{dx}=\left(\frac{1}{2\sqrt[4]{x^3}}\right)\left(\sqrt[4]{x}+3\right)\left(\sqrt[3]{x}-4\right)^3+\left(\frac{1}{\sqrt[3]{x^2}}\right)\left(\sqrt[3]{x}-4\right)^2\left(\sqrt[4]{x}+3\right)^2$

3 a $\frac{4(x-1)(2x-4)-(2x-4)^2}{(x-1)^2}$

b $f'(x)=\frac{(x-3)(2x+2)-(x^2+2x)}{(x-3)^2}$

c $\frac{dy}{dx}=\frac{9\sqrt{x^2-5x}\,(3x-8)^2-\frac{1}{2\sqrt{x^2-5x}}(2x-5)(3x-8)^3}{x^2-5x}$

d $\frac{3\left(\sqrt{x}+2\right)-\frac{1}{2\sqrt{x}}(3x-2)}{\left(\sqrt{x}+2\right)^2}$

e $f'(x)=\frac{2\left(\sqrt{x}+4\right)\left(x^3+\sqrt{x}\right)\left(3x^2+\frac{1}{2\sqrt{x}}\right)-\frac{\left(x^3+\sqrt{x}\right)^2}{2\sqrt{x}}}{\left(\sqrt{x}+4\right)^2}$

f $\frac{dy}{dx}=\frac{\frac{\left(\sqrt[3]{x}-4\right)^2}{4\sqrt[4]{x^3}}-\frac{2\left(\sqrt[4]{(x)}+3\right)\left(\sqrt[3]{x}-4\right)}{3\sqrt[3]{x^2}}}{\left(\sqrt[3]{x}-4\right)^4}$

4 a $(4, 0), (1, 27)$

b $(0, 0), \left(\frac{2}{3}, \frac{4}{9}\right)$

c $(1, 3)$

d $(-1, 32), (3, 0)$

e $(0, 0)$ $(2, 12)$

f $(3, 1)$

5 a $-\frac{1}{3}$

b $-\frac{1}{3}p$

6 $\frac{1}{4}p$

7 $37p$

8 a -3

b 4

9 a $10\text{ cm}^3\text{ s}^{-1}$

b $8\text{ cm}^2\text{ s}^{-1}$

10 $\frac{3}{22}$ cm^3 s^{-1}

11 **a** $y-2=-\frac{1}{4}(x-3)$ **b** $y-2=4(x-3)$

12 **a** $y+\frac{5}{4}=\frac{11}{16}(x+2)$ **b** $y+\frac{5}{4}=-\frac{16}{11}(x+2)$

13 **a** $y-28=-17(x-4)$ **b** $y-28=\frac{1}{17}(x-4)$

14 $a=8, b=-21$

Chapter 17 Test

1 10

2 **a** $y-11=\frac{1}{9}(x-6)$ **b** $\left(-76,-\frac{17}{9}\right)$

3 $a=-3, b=4$

4 **a** $\frac{dy}{dx}=\frac{-x^2-4x-3}{(2+x)^2}$ $\frac{d^2y}{dx^2}=\frac{-2}{(2+x)^3}$ **b/c** $(-3, 6)$ maximum; $(-1, 2)$ minimum

5 0.4 units s^{-1}

6 $(0, 0)$; $\left(\frac{2}{3},\frac{4}{9}\right)$ 7 $-\frac{7p}{4}$

Examination Questions

1 **(i)** $k=-8$ **(ii)** $y=2(x+2)$ **(iii)** -0.1

2 **(i)** $k=12$ **(ii)** $\frac{20}{3}$

3 **(i)** $\frac{dy}{dx}=\frac{(x^2+5)[3]-[2x](3x-2)}{(x^2+5)^2}$ **(iii)** $-\frac{5}{3}, 3$

4 **(i)** $k=5$ **(ii)** $189+90p$ **(iii)** $\frac{186}{5}$

5 **(i)** $k=\frac{3}{2}$ **(ii)** $\frac{20}{3}$

6 17

7 **(i)** $\frac{dy}{dx}=\frac{10}{(x+3)^2}$ This is never zero. **(ii)** $\frac{4}{5}$

8 $y-6=\frac{1}{2}(x-4)$

9 **a** **(i)** $\frac{-2}{(2x-1)^3}+c$ **(ii)** $\frac{1}{4}x^4-\frac{2}{3}x^3+\frac{x^2}{2}+c$ **b** **(ii)** $\frac{2}{3}(x-5)\sqrt{(x+4)}+c$

10 **(i)** ± 3 **(ii)** 0.32 units per second

11 **(i)** $k=6$ **(ii)** 20

Chapter 18

Problem 18.1

a

x	0	1	2	3	4	5	6	7	8	9	10
2^x	1	2	4	8	16	32	64	128	256	512	1024

b x is an integer between 0 and 10

c The graph consists of dots at the points plotted. It is not a continuous curve.

d The graph would then be a continuous curve.

e 1 : 1

f For $x > 0$, the curves get steeper more quickly and rise above the previous ones.
For $x < 0$, the curves lie below the previous ones and eventually get squashed between them and the x-axis.

g (0, 1)   h They are all 1 : 1

Exercise 18.1

1 a 3   b 8   c 10   d −2   e −4   f −6
g 4   h 3   i 4   j 6   k 4   l 5

2 a 5.64   b 3.91   c 4.32

Exercise 18.2

1 a 512   b 1024   c 2048   d 64   e 64   f 0.25
g 512   h 1024   i 256   j 128   k 128   l 16

2 a 3   b $\frac{1}{25}$ c 1

3 a 2.22   b 395   c 3.55

4 a log 15   b log 27   c log 16   d log 200   e log 2   f log 36
g $\log x^2y$ h $\log x^3y$ i $\log x^{\frac{1}{2}}y^2$

Exercise 18.3

1 a 2.73   b 2.89   c 3.29   d 3.48   e 3.12   f 3.06

2 a 6.13   b 1.38   c 31.0   d 2.82   e −3.22   f 1.41

3 a $243, \frac{1}{243}$ b $4096, \frac{1}{4096}$ c $512, \frac{1}{512}$ d 23.8, 0.0421   e 71.9, 0.0139   f 29.4, 0.0341

4 a {1, 2}   b {0, 2}   c {0, 2}   d {1.26, 1.46}   e {1, 2}   f {1, 3}
g {0, 0.631}   h {1.58, 2.32}   i {0.683, 0.861}

5 a 1.58   b 0.631   c No solution   d 2.02   e 1.16   f 0.631

6 a {−1}   b {−0.387, −0.226}   c {−0.792, −0.208}   d {−2, 1}   e {−0.252}   f {−0.368}

7 b $(y-1)(y-2)(y-3)$ c {0, 0.5, 0.792}

8 $x = 1000, y = 100$

Exercise 18.4

1 $A = 4; B = 3$ 2 $A = 1.33; B = 5$ 3 $A = 12; B = 2$

4 a $xy = A + Bx^2$; plot xy against x^2 b $A = 120; B = 3$ c 39   d 6.32

5 a $\frac{h}{t} = u + \frac{1}{2}at$; plot $\frac{h}{t}$ against t b $u = 25, a = -10$

6 plot lg A against t; $P = 4980, r = 1.1$

7 a plot lg N against x; $A = 500, b = 1.5$ b 3797

8 a plot lg G against lg t; $A = 3.20, n = 5$ b 10 000

9 a plot lg q against p; $A = 7.045, b = 1.26$ b 17.76

10 a plot lg N against t; $A = 3.195, b = 2.30$ b 473

Chapter 18 Summative Exercise

1 a 4   b 3   c 4   d 3   e 2.5   f 1.5

2 a log 6   b log 16   c log 3   d log 72   e log 128   f log xy

3 a 1.26   b 8.84   c −5.128   d 1.66   e −1.18   f 1.26

4 a 2.36   b 10   c 4   d $\frac{1}{4}, 2$ e $\frac{1}{3}$ f 0.388
g $\frac{1}{8}, 1$ h 7.61

5 a (5, 7)   b $p = 1, q = 1$ c $x = 2, y = 3$
d $x = 5, y = 8$ e $x = -1, y = -4$ f $x = 9, y = 8$

6 a 0.792   b 1.58   c 0, 2.81   d 1.26   e 0.585   f 0.285, 0.511

7 a $\frac{y}{x^2} = ax + b$ b $a = 1.74, b = -0.96$ c 139

8 $y = 10^{\frac{x}{2}}$

9 $A = 1.46, b = 0.525$
10 $A = 5, b = 3$
11 $A = 50, b = 1.1$

12 $y = \dfrac{33}{4x^2} - \dfrac{13}{8x}$

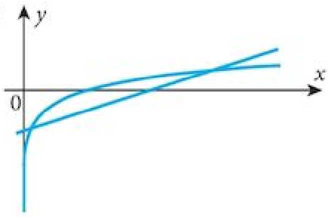
Chapter 18 Test

1 **a** $3p - \dfrac{1}{2}q$ **b** $\dfrac{2}{p} - 4q$ **c** $\dfrac{2}{p+q}$

2 **a** 9 **b** 3, 6

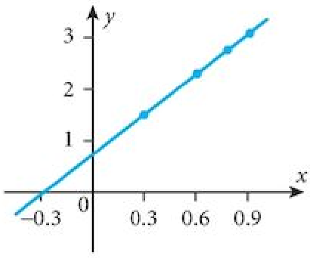
3 **a** 2 **b** 5 **c** $\dfrac{7}{10}$

4 **a/c**



b $\lg x = \dfrac{1}{2}x - 1, a = \dfrac{1}{2}, b = -1$ **c** $y = \dfrac{1}{2}x - 1$

5 **a** $\log N = n \log m + \log A$
b



c $A = 6.4, n = 2.97$ **d** 762

6 $\log_a 8$ $(= 3 \log_a 2)$
7 $x = 2, y = 5$

Examination Questions

1 **(i)** $y = 1.6x + \dfrac{12}{x}$ **(ii)** 4.54
2 8

3 $\dfrac{2}{3}$

4

	Y	X	m	c
(i)	$\lg y$	x	$\lg b$	$\lg a$
(ii)	$\lg y$	$\lg x$	k	$\lg a$
(iii)	$\dfrac{1}{y}$	$\dfrac{1}{x}$	$-\dfrac{q}{p}$	$\dfrac{1}{p}$

5 **a** 1.0287 **b** 6.3077 6 **(ii)** $k = 15020; n = -0.8936$

7 **a** 2 **b** $\dfrac{3}{4}$

8 **a** **(i)** 4^u **(ii)** $2 - u$ **(iii)** $\dfrac{3}{2u}$ **b** $\left\{\dfrac{1}{81}, 9\right\}$

9 **a** $\frac{8}{11}$ **b** $c = a - 3b$ **10** **(ii)** $y = \frac{x}{2.2 - 2x}$ **(iii)** 0.2538

11 **(i)** 2.32 **(ii)** 4 **12** **(ii)** $y = 1.2x + \frac{24}{x}$ **(iii)** 11.84

Chapter 19

Exercise 19.1

1 **a** $2\cos 2x$ **b** $-3\sin 3x$ **c** $4\sec^2 4x$
d $15\cos 5x$ **e** $-14\sin 7x$ **f** $12\sec^2 3x$
g $2\cos\frac{1}{2}x$ **h** $-2\sin\frac{1}{3}x$ **i** $3\sec^2\frac{1}{4}x$
j $2\cos\left(x + \frac{\pi}{3}\right)$ **k** $6\sin(3x - 2)$ **l** $\frac{1}{2}\sec^2\left(\frac{x+\pi}{4}\right)$

2 **a** $\frac{dy}{dx} = 2x\cos(x^2)$ **b** $\frac{dy}{dx} = 2\sin x\cos x$ **c** $\frac{dy}{dx} = -3x^2\sin(x^3)$
d $\frac{dy}{dx} = -6x\sin(x^2 - 1)$ **e** $\frac{dy}{dx} = 3\tan^2 x\sec^2 x$ **f** $\frac{dy}{dx} = 3(3x^2 + 2x)\sec^2(x^3 + x^2)$
g $\frac{dy}{dx} = 4(2x - 2)\cos(x^2 - 2x)$ **h** $\frac{dA}{dx} = -12\cos(2x - 1)\sin(2x - 1)$
i $\frac{dA}{dx} = 2\sec^2\left(x - \frac{\pi}{3}\right)$

3 **a** $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}\cos\sqrt{x}$ **b** $\frac{dy}{dx} = \frac{\cos x}{2\sqrt{\sin x}}$ **c** $\frac{dy}{dx} = -\frac{3}{2}\sqrt{x}\sin\left(\sqrt{x^3}\right)$
d $\frac{dy}{dx} = \frac{-x}{\sqrt{x^2 - 1}}\sin\sqrt{x^2 - 1}$ **e** $\frac{dy}{dx} = 24\sin^3(6x)\cos(6x)$ **f** $\frac{dy}{dx} = \frac{\cos 2x}{\sqrt{1 + \sin 2x}}$
g $\frac{dy}{dx} = \frac{-2\cos x}{\sin^3 x}$ **h** $\frac{dy}{dx} = \frac{2\sin 4x}{\sqrt{\cos^3 4x}}$ **i** $\frac{dy}{dx} = \frac{-6x^2\sec^2\left(x^3\right)}{\tan^3\left(x^3\right)}$

4 **a** $\frac{dy}{dx} = x\cos x + \sin x$ **b** $\frac{dy}{dx} = 2x\cos(x^2) - 2x^3\sin(x^2)$ **c** $\frac{dy}{dx} = x\sec^2 x + \tan x$
d $\frac{dy}{dx} = (2x + 1)^2\cos(x^2 + x) + 2\sin(x^2 + x)$ **e** $\frac{dy}{dx} = \cos^2 x - \sin^2 x$
f $\frac{dy}{dx} = 2\sin x\cos^3 x - 2\sin^3 x\cos x$ **g** $\frac{dy}{dx} = \frac{\sin x - x\cos x}{\sin^2 x}$
h $\frac{dy}{dx} = \frac{4\sin 2x}{[1 + \cos 2x]^2}$ **i** $\frac{dy}{dx} = \frac{(1 + \cos x)\cos x + (1 + \sin x)\sin x}{[1 + \cos x]^2}$
j $\frac{dy}{dx} = \frac{3(x + 2)\sin 3x + \cos 3x}{[x + 2]^2}$ **k** $\frac{dy}{dx} = \frac{2\sin^2 2x + 2(1 + \cos 2x)\cos 2x}{\sin^2 2x}$
l $\frac{dy}{dx} = 2x\sin x\cos x + x^2\cos^2 x - x^2\sin^2 x$

5 **a** $-4\cos 2x + c$ **b** $2\sin 3x + c$ **c** $-3\cos(2x + 1) + c$
d $3\sin(3x + \pi) + c$ **e** $2\tan 2x + c$ **f** $2\cos(1 - 2x) + c$

6 **a** $\sin(x^2) + c$ **b** $\sin(x^3) + c$ **c** $-2\cos(x^4 + 1) + c$
d $-2\cos(x^2 + x) + c$ **e** $2\sin(\sqrt{x}) + c$ **f** $3\tan(2x + 1) + c$

7 a $\frac{dy}{dx} = 3\sin^2 x\cos x$ b $2\sin^3 x + c$

8 a $\frac{dy}{dx} = -4\cos^3 x\sin x$ b $-3\cos^4 x + c$

9 a $\frac{dy}{dx} = 2x\cos(x^2)$ b $3\sin(x^2) + c$

10 a $\frac{dy}{dx} = 1 - \cos x$ b $(x - \sin x)^3 + c$

11 a $\frac{dy}{dx} = \sin x$ b $4\sqrt{4 - \cos x} + c$

12 b $x\sin x + \cos x + c$

Exercise 19.2

1 tgt: $y - \frac{\pi}{12} = \left(\frac{6 + \pi\sqrt{3}}{12}\right)\left(x - \frac{\pi}{6}\right)$ nml: $y - \frac{\pi}{12} = \left(\frac{-12}{6 + \pi\sqrt{3}}\right)\left(x - \frac{\pi}{6}\right)$

2 tgt: $y - \frac{\sqrt{3}+1}{2} = \left(\frac{1 - \sqrt{3}}{2}\right)\left(x - \frac{\pi}{3}\right)$ nml: $y - \frac{\sqrt{3}+1}{2} = \left(\frac{2}{\sqrt{3}-1}\right)\left(x - \frac{\pi}{3}\right)$

3 tgt: $y - \frac{\pi}{6} = \left(\frac{3 - \pi\sqrt{3}}{6}\right)\left(x - \frac{\pi}{3}\right)$ nml: $y - \frac{\pi}{6} = \left(\frac{6}{\pi\sqrt{3} - 3}\right)\left(x - \frac{\pi}{3}\right)$

4 tgt: $y - 2\left(2 - \sqrt{3}\right) = \left(8\sqrt{3} - 14\right)\left(x - \frac{\pi}{3}\right)$ nml: $y - 2\left(2 - \sqrt{3}\right) = \frac{\left(7 + 4\sqrt{3}\right)}{2}\left(x - \frac{\pi}{3}\right)$

5 tgt: $y - \frac{1}{\sqrt{3}} = -\frac{2}{3}\left(x - \frac{\pi}{6}\right)$ nml: $y - \frac{1}{\sqrt{3}} = \frac{3}{2}\left(x - \frac{\pi}{6}\right)$

6 $\frac{\pi^2}{144} + \frac{\pi\sqrt{3}}{12} + \frac{3}{4}$ 7 2.55, 5.70

8 $0, \frac{\pi}{2}, \frac{3\pi}{2}$ 9 a $\frac{\sin x}{\cos^2 x}$ b $0, \pi, 2\pi$

10 b/c $\left(\frac{\pi}{2}, 0\right)$; $\left(\frac{7\pi}{6}, -\frac{3\sqrt{3}}{4}\right)$, minimum; $\left(\frac{11\pi}{6}, \frac{3\sqrt{3}}{4}\right)$, maximum;

11 b $0, \pi$ 12 $\frac{\sqrt{2}}{10}$ 13 $\frac{3\pi}{10}$ 14 $\frac{5}{8}$ 15 $\frac{\sqrt{3}}{60}$

16 b $\frac{\pi}{6}$ 17 $5\sqrt{2} - \frac{5\sqrt{3}}{3} \approx 4.18$ 18 b 100 19 b $48\sqrt{3}$ 20 π

21 1 22 1 23 a $\frac{dy}{dx} = 2\sin x\cos x$ b $\frac{1}{4}$

24 a $3\sin^2 x\cos x$ b $\frac{1}{3}$

25 a $\left(\frac{\pi}{3}, \frac{\sqrt{3}}{2}\right)$ b $\frac{1}{4}$ 26 a $\left(\frac{\pi}{4}, \frac{\sqrt{2}}{2}\right)$ b $\sqrt{2} - 1$

Chapter 19 Summative Exercise

1 a $3\cos(3x + 1)$ b $-4\sin(4x - 2)$ c $2\sec^2(2x + 1)$

d $\frac{dy}{dx} = (2x + 2)\cos(x^2 + 2x)$ e $\frac{dy}{dx} = \frac{-1}{2\sqrt{x}}\sin(\sqrt{x})$ f $\frac{dy}{dx} = 4\tan(2x - 1)\sec^2(2x - 1)$

g $f'(x) = \frac{-x\cos(x^2)}{\sqrt{1 - \sin(x^2)}}$ h $f'(x) = -6x\cos^2(x^2)\sin(x^2)$ i $f'(x) = \frac{-3\sec^2(3x)}{\tan^2(3x)}$

2 a $\cos^2 x - \sin^2 x$ b $3\cos 3x \cos 2x - 2\sin 3x \sin 2x$

c $2\sin x \cos^2 x - \sin^3 x$ d $\frac{dy}{dx} = \cos^3 x - 2\sin^2 x \cos x$

e $\frac{dA}{dx} = 2\sin x \cos^3 x - 2\cos x \sin^3 x$ f $\frac{dy}{dx} = \cos^2 x \tan x - \sin^2 x \tan x + \sin x \sec x$

3 a $2\cos 3x + c$ b $2\sin 4x + c$ c $2\tan 3x + c$

d $2\sin x^2 + c$ e $\cos(x^2 - 2x) + c$ f $\tan(x^2) + c$

4 a $f'(x) = -\operatorname{cosec} x \cot x$ b $f'(x) = \sec x \tan x$ c $f'(x) = -\operatorname{cosec}^2 x$

6 b $\sin x - x\cos x$

8 $\frac{1}{2}$ 9 $\left(\frac{\pi}{3}, \frac{\sqrt{3}}{3}\right)$

10 a $\frac{dy}{dx} = 2\sec^2(2x)$ b $\frac{2}{\sqrt{3}}$

11 a $\frac{dy}{dx} = 1 + \cos(2x)$ b $\frac{2\pi}{3} + \frac{\sqrt{3}}{2}$

12 a $\frac{dy}{dx} = 4\sin(2x)\cos(2x)$ b $y - \frac{3}{4} = -\sqrt{3}\left(x - \frac{\pi}{3}\right)$ c $-\sqrt{3}p$

13 b $\frac{dA}{dx} = -16\sin x + 16(\cos^2 x - \sin^2 x); x = \frac{\pi}{6}$ c $12\sqrt{3}$ Maximum

Chapter 19 Test

1 a $2x\sin x + x^2\cos x$ b $\cos^2 x - \sin^2 x$ c $\frac{(x-1)\sec^2 x - \tan x}{(x-1)^2}$

2 $y - \frac{3\sqrt{3}}{2\pi} = \left(\frac{3}{2\pi} - \frac{9\sqrt{3}}{2\pi^2}\right)\left(x - \frac{\pi}{3}\right)$

3 $\frac{6\sqrt{2}}{5}$

4 a $y + 1 = -\frac{1}{4}\left(x - \frac{\pi}{4}\right)$ b $-4 + \frac{\pi}{4}$

5 0.365

6 a $-3\sin 3x$ b $-3p$

7 a $2\sec^2 2x$ b $\frac{\pi}{2} + 1$

Examination Questions

1 (i) $x\cos x + \sin x$ (ii) $\frac{\pi}{2} - 1$ 2 $2 + \sqrt{2}$ 3 (ii) 4

4 $\frac{1+\sqrt{3}}{6}$

Term test 5A (Chapters 16 to 19)

1 a $\frac{\pi}{2}, \frac{7\pi}{6}, \frac{5\pi}{2}, \frac{19\pi}{6}$ b 0.464, 2.678, 3.605, 5.819, 6.747, 8.961, 9.888, 12.103

c 1.389, 5.036, 7.672, 11.319

2 a $\frac{9x-4}{2\sqrt{(3x-2)}}$ b $y - 4 = \frac{7}{2}(x-2)$ c $96 - 16\sqrt{10}$

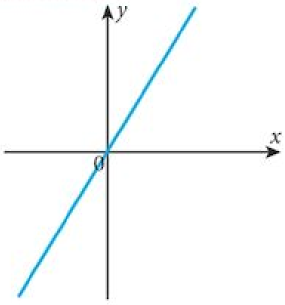
3 a $a = 2, b = x, c = y$ b $\frac{1}{\log_2 x}$ c 8, 32

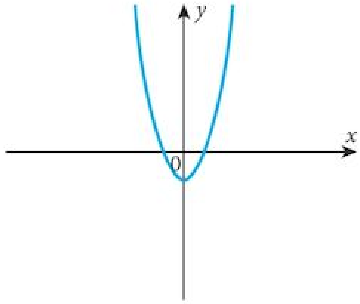
4 a $y-\sqrt{3}=-\frac{1}{3}(x-2\pi)$ b $(2\pi+3\sqrt{3},0)$ c $\frac{3}{2}$

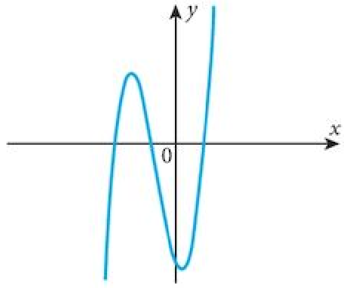
5 $y=2x+\frac{4}{x}$

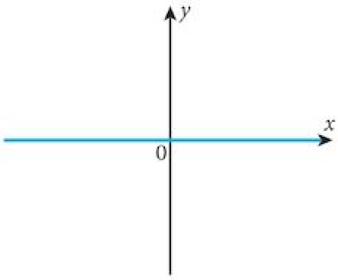
Chapter 20

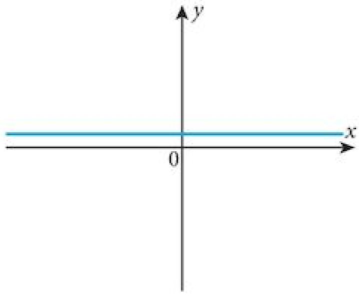
Problem 20.1

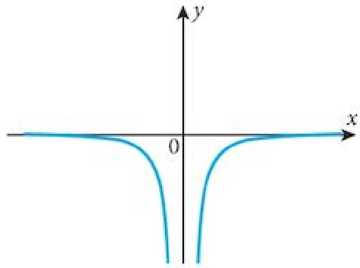
a


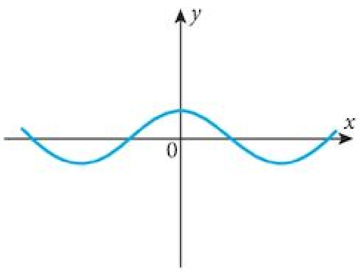
b


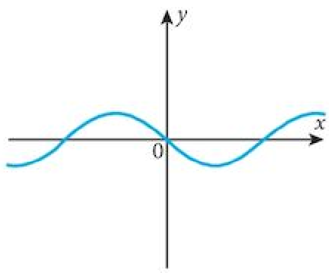
c


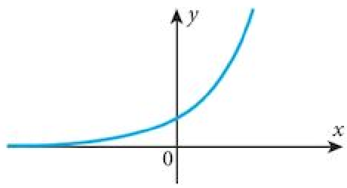
d


e


f


g


h


i


Problem 20.2

$y=0$

Problem 20.3

a 0.693

b

a	1	2	3	4
$\frac{a^h-1}{h}$	0	0.693	1.099	1.386

Exercise 20.1

1 a $\frac{dy}{dx}=4e^{4x}$ b $\frac{dy}{dx}=12\,e^{3x}$ c $\frac{dy}{dx}=2x\,e^{x^2}$ d $\frac{dy}{dx}=\frac{e^{\sqrt{x}}}{2\sqrt{x}}$

e $\frac{dy}{dx}=3e^{3x-1}$ f $\frac{dy}{dx}=-\frac{1}{x^2}e^{\frac{1}{x}}$ g $\frac{dy}{dx}=(2x+3)\,e^{x^2+3x}$ h $\frac{dy}{dx}=\frac{-2}{e^{2x}}$

2 **a** $\dfrac{dy}{dx} = (4x+1)e^{4x}$ **b** $\dfrac{dy}{dx} = e^{3x}(3\sin x + \cos x)$ **c** $\dfrac{dy}{dx} = e^{x^2}\left(2 - \dfrac{1}{x^2}\right)$

d $\dfrac{dy}{dx} = e^{\sqrt{x}}\left(\dfrac{\cos x}{2\sqrt{x}} - \sin x\right)$ **e** $\dfrac{dy}{dx} = e^{3x-1}(3x^2 + 8x + 2)$ **f** $\dfrac{dy}{dx} = e^{\frac{1}{x}}\left(\cos x - \dfrac{1}{x^2}\sin x\right)$

g $\dfrac{dy}{dx} = e^{x^2+3x}(\sec^2 x + (2x+3)\tan x)$ **h** $\dfrac{dy}{dx} = \dfrac{2\sin x\cos x - 2\sin^2 x}{e^{2x}}$

3 **a** $\dfrac{dy}{dx} = \dfrac{1}{x}$ **b** $\dfrac{dy}{dx} = \dfrac{1}{x}$ **c** $\dfrac{dy}{dx} = \dfrac{2}{x}$ **d** $\dfrac{dy}{dx} = \dfrac{2}{x}$

e $\dfrac{dy}{dx} = \dfrac{2}{x}\ln x$ **f** $\dfrac{dy}{dx} = 1 + \ln x$ **g** $\dfrac{dy}{dx} = 2x\ln x + x$

h $\dfrac{dy}{dx} = \dfrac{1}{x}\sin x + \cos x \ln x$

4 **a** ln 3 is a constant

b $\ln(x^2) = 2\ln x$. (One of the laws of logs.) They are different ways of writing the same thing.

5 **a** $\dfrac{1}{3}e^{3x} + c$ **b** $2e^{2x} + c$ **c** $-\dfrac{1}{4}e^{-4x} + c$ **d** $4e^{-3x} + c$

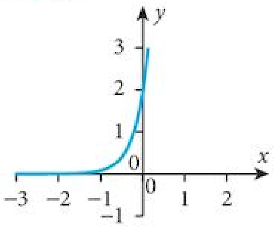
e $2e^{x^2} + c$ **f** $2(1+e^x)^3 + c$ **g** $4(1+e^x)^{\frac{3}{2}} + c$ **h** $2\sqrt{1+e^x} + c$

6 **b** $x\ln x - x + c$

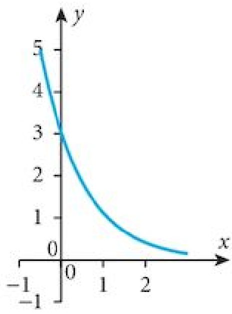
7 **a** $\dfrac{dy}{dx} = \dfrac{e^x}{e^x + 3}$ **b** $\ln(e^x + 3) + c$

Exercise 20.2

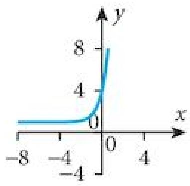
1 **a**



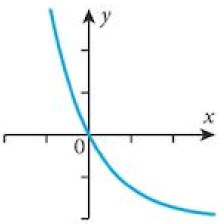
b



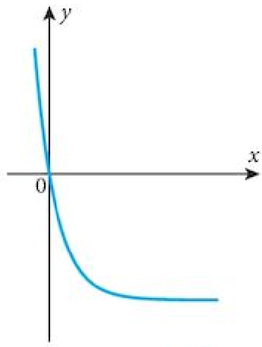
c



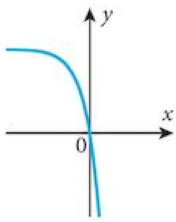
d



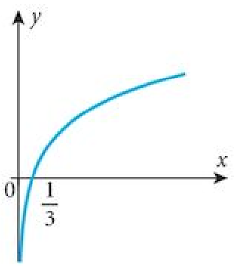
e



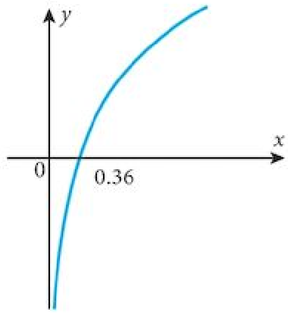
f



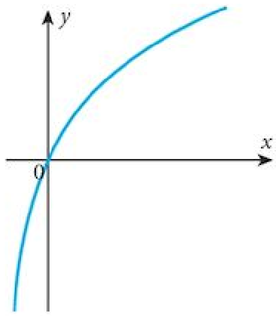
2 **a**



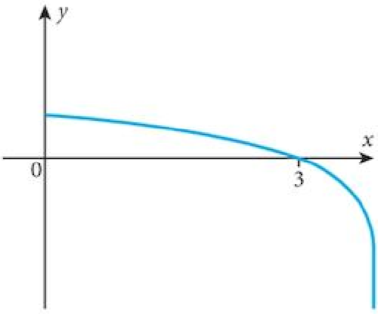
b



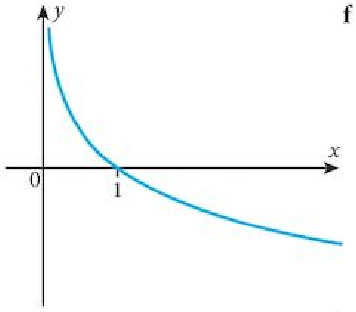
c



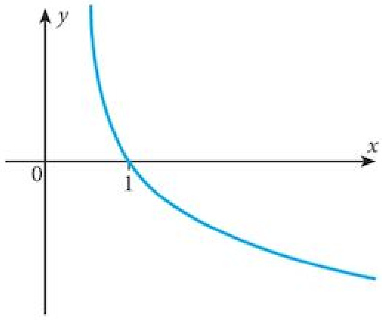
d



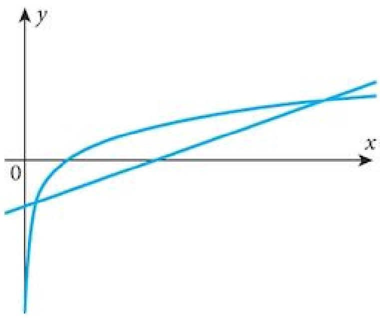
e



f



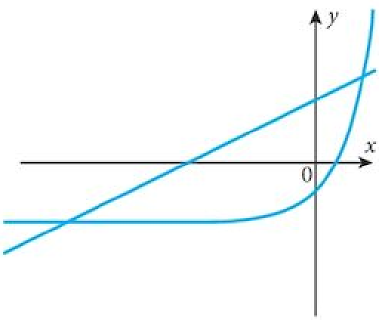
3 **a**



b $a = \frac{1}{2}\ b = -\frac{3}{2}$

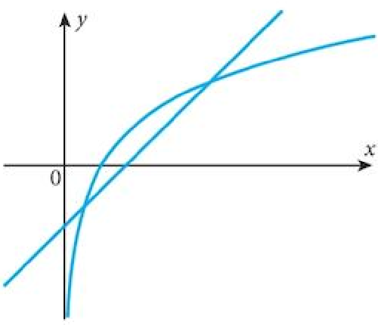
c 0.253, 6.85

4 **a**



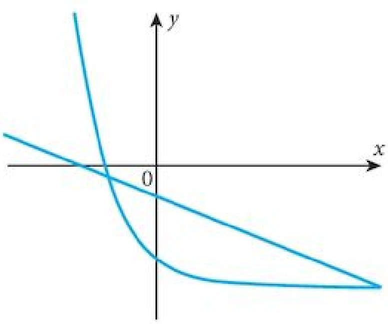
c –3.99, 0.782
$e^{2x} - 4 - x = 0$

5 **a**



c 0.302, 2.36

6 **a**



b –0.647, 2.997

Exercise 20.3

1 **a** 1.099 **b** 0.693, 1.386 **c** 0.288, 0.405

2 **a** 0.182 **b** 0.231 **c** −0.101, 0.101 **d** 0.347 **e** 1.099

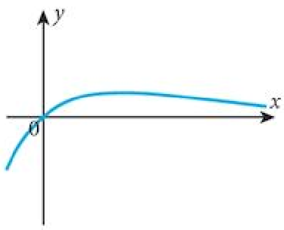
3 0.25, 2 **4** $x = 3, y = 4$

5 $x = -2, y = 5$ **6** $x = 2, y = 3$

7 **b**

x	0	0.5	1	1.5	2	3
y	0	0.91	1.10	1.00	0.81	0.45

c

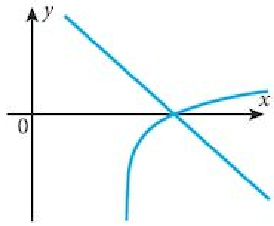


d A solution is 2.1 (1 d.p.)

8 **a**

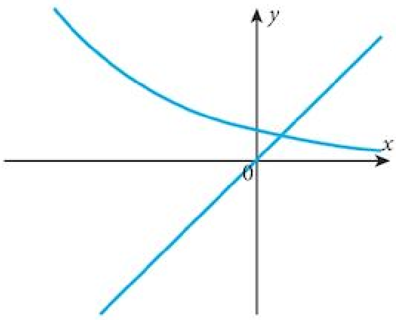
x	2.5	3	3.5	4	4.5	5
y	−0.69	0	0.41	0.69	0.92	1.10

b



c $y = 6 - 2x$; 3

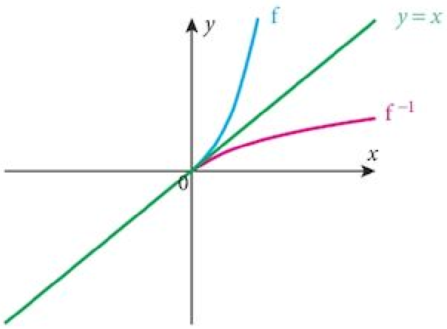
9 **a**



b 0.203

10 **a** 4.575 **b** $f^{-1}: x \mapsto \ln(x + 1)$ **c** Domain $x \geqslant 0$; Range $y \geqslant 0$

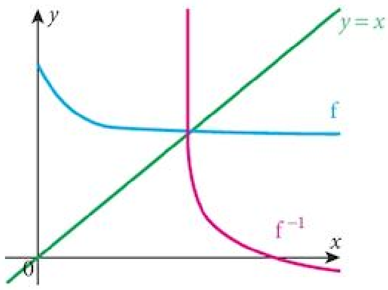
d

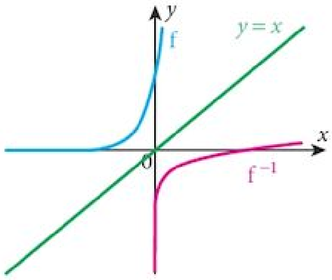


11 **a** $5 < f(x) \leqslant 8$ **b** 5.006 **c** $f^{-1}: x \mapsto \ln 3 - \ln(x - 5)$

d Domain : $5 < x \leqslant 8$;
Range : $y \geqslant 0$

e



12 a 

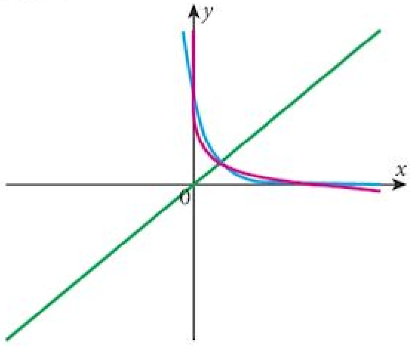
b $f(x) > 0$

c $f^{-1}: x \mapsto \frac{1}{2}\ln\left(\frac{x}{3}\right)$
Domain: $x > 0$
Range : $\mathbb{R}$

13 a 0.0549

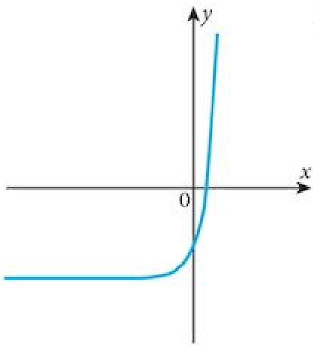
b 3.594

c $f^{-1}: x \mapsto -\frac{1}{2}\ln\left(\frac{x}{3}\right)$
Domain: $x > 0$
Range : $\mathbb{R}$

d 

14 1 **15 a** (2, e) **b** 2e, −e **16** $\left(\sqrt{2}, \ln\left(2+2\sqrt{2}\right)\right)$

17 a $\frac{dy}{dx} = \frac{3}{x}$ **b** $x = 4$ **c** 1

18 a 

b 9 **19** $y + 1 = -x$ **20** (0, 1)

21 b $x = -\frac{1}{k}\ln k$ **c** $y = -\frac{1}{k}\ln k$ **d** $k = \frac{1}{e}$

22 a $2x\,e^{-x} - x^2\,e^{-x}$ **b** (0, 0); minimum $(2, 4e^{-2})$; maximum

23 (0, 4); minimum **25** (1, 1); minimum **26** $(2, e^2)$; minimum

27 a $\frac{dy}{dx} = e^x(\cos x - \sin x)$ **b** $\frac{\pi}{4}$

28 (0, 0); minimum: $\left(-\frac{2}{3},\ 0.06\right)$; maximum

29 a $\frac{dy}{dx} = 10(x+3)e^x, \frac{d^2y}{dx^2} = 10(x+4)e^x$ **b** (−3, −0.498); minimum

30 (0.32, 2.49); minimum **31** (e, −e); minimum **32** $y = 6x^3$

33 60 **34 a** 41.6 **b** 4.16 **35 b** e^2 **36 a** $e^x(\cos x + \sin x)$ **b** $e^{\frac{\pi}{2}}$

37 a (0.792, 1.793) **b** 0.065

38 a $(e - 1, 3)$ **b** $1 + \ln(x+1)$ **c** $3e - 6$

Chapter 20 Summative Exercise

1 **a** 2.302 **b** 2.996 **c** 3.401 **d** 4.605
e 20.09 **f** 148.4 **g** 0.0498 **h** 0.000912

2 **a** $\frac{dy}{dx} = 2e^{2x}$ **b** $\frac{dy}{dx} = 8xe^{x^2}$ **c** $\frac{dy}{dx} = \frac{3}{2\sqrt{x}}e^{\sqrt{x}}$ **d** $\frac{dy}{dx} = (6x^2 - 2x)e^{2x^3 - x^2}$

e $\frac{dy}{dx} = (1 + x)e^x$ **f** $\frac{dy}{dx} = (2x + 3)e^x$ **g** $\frac{dy}{dx} = e^x(\sin x + \cos x)$ **h** $\frac{dy}{dx} = e^{-x}(\cos x - \sin x)$

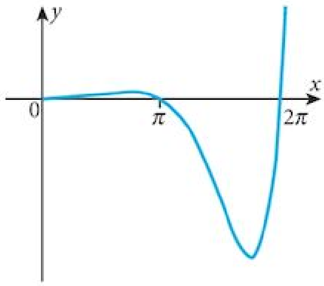
3 **a** $\frac{dy}{dx} = \frac{2}{x}$ **b** $\frac{dy}{dx} = 2x\ln x + x$ **c** $\frac{dy}{dx} = \frac{\cos x}{x} - \sin x \ln x$ **d** $\frac{dy}{dx} = \frac{3\ln(x)^2}{x}$

4 **a** $\frac{1}{2}e^{2x} + c$ **b** $-\frac{1}{4}e^{-4x} + c$ **c** $x^2 + e^{x^2} + c$ **d** $\frac{1}{3}\left(1 + e^x\right)^3 + c$

5 **b** $xe^x - e^x + c$

6 **b** $\frac{1}{2}x^2\ln x - \frac{x^2}{4} + c$

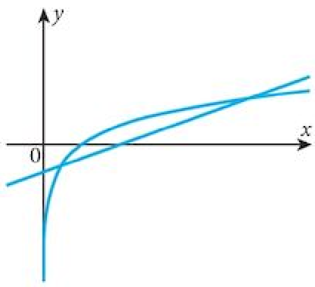
7



8 **a** 0.347 **b** 1.270 **c** 2.109

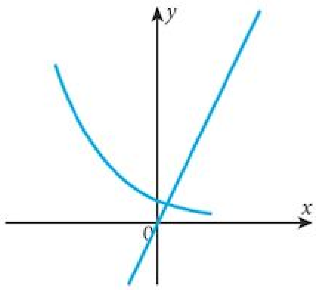
9 1.249 **10** $x = 7, y = -2$

11 **a** $\ln x = \frac{1}{2}x - 1$ **b, c**



d 0.464, 5.36

12



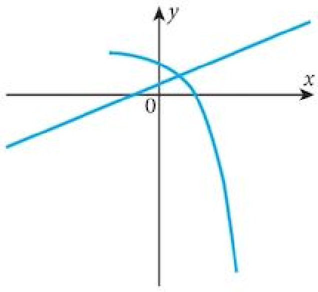
0.169

13 **b** 403.4 **c** $f^{-1} : x \mapsto \frac{1}{3}\ln x$ **d** 1.304

14 $\frac{dy}{dx} = (12x - 24)e^x$ (2, −88.7) Minimum

15 (4.174, −0.322)

16 **a** (0, 3), (1.386, 0) **b**



c $y = 1 + x$ **d** 0.792

17 (3, 1.344) Maximum

Chapter 20 Test

1 **a** $3x^2\,e^{4x} + 4x^3\,e^{4x}$ **b** $\dfrac{-\cos x}{3 - \sin x}$ **c** $2e^{2x}\tan x + e^{2x}\sec^2 x$

2 **a** $2 + 3e^{-x}$ **b** $5k$

3 **b** $\dfrac{2\ln x - 3}{x^3}$ **c** Maximum

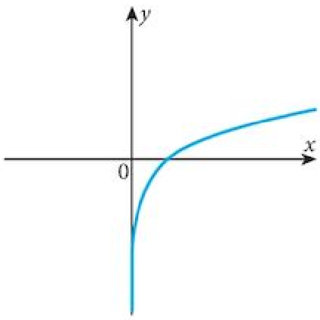
4 **a** $\dfrac{dy}{dx} = -2(x-2)\,e^{-(x-2)^2}$ **b** $y = 6\,e^{-(x-2)^2}$ **c** (2, 6) **d** $\dfrac{1}{2}$

5 **a** 80 **b** 90 **c** 85 days **d** 100

Examination Questions

1 **(i)** 0.916 **(ii)** $y = 7$

2 **(i)**



(ii) $y = \dfrac{1}{2}(2 - x)$

3 **(i)** $-\dfrac{1}{2}$ **(ii)** $k = 4$ **(iii)** minimum

4 **(i)** $\dfrac{(2x+3)\frac{1}{x} - 2\ln x}{(2x+3)^2}$ **(ii)** $\dfrac{p}{5}$ **(iii)** 0.6 units per sec.

5 **(i)** $x = 2.47$ **(ii)** $\ln x = 2e^x - 3$; $h(x) > -3$ **(iii)** $h^{-1}: x \mapsto \ln\left(\dfrac{x+3}{2}\right)$

6 **(i)** $\ln x$ **(ii)** 1.296 **7** **(i)** $\dfrac{dy}{dx} = \dfrac{e^{2x}(2\sin x - \cos x)}{\sin^2 x}$ **(ii)** 0.464

8 **(i)** $\dfrac{dy}{dx} = \dfrac{2}{2x-3}$ **(ii)** $1 + 2p$

9 **a** $\sec^2 x\,e^{\tan x}$ **b** $\dfrac{1}{2}(e - 1)$ **10** **(ii)** $\dfrac{1}{4}(x-4)\,e^{-\frac{x}{2}}$ **(iii)** $\left(2, \dfrac{2}{e}\right)$ **(iv)** Maximum

11 **(i)** $2x\ln x + x$ **12** $y = e^{-2x} + 5x - 10$

13 **(i)** $1 + \ln x$ **(ii)** $x\ln x - x + c$ **14** **(i)** $3x\,e^{3x}$ **(ii)** $\dfrac{1}{3}\left(xe^{3x} - \dfrac{e^{3x}}{3}\right) + c$

15 **(i)** $f(x) > e^{-1}$ **(ii)** $f^{-1}: x \mapsto 1 + \ln x$ **(iii)** $x > e^{-1}$

Chapter 21

Exercise 21.1

1 a $\begin{pmatrix} 6 \\ -3 \end{pmatrix}$ b $\begin{pmatrix} 2 \\ -6 \end{pmatrix}$ c $\begin{pmatrix} -8 \\ -20 \end{pmatrix}$ d $\begin{pmatrix} -9 \\ -6 \end{pmatrix}$ e $\begin{pmatrix} 8 \\ 14 \end{pmatrix}$

f $\begin{pmatrix} 3 \\ -4 \end{pmatrix}$ g $\begin{pmatrix} 0 \\ -6 \end{pmatrix}$ h $\begin{pmatrix} 2 \\ 5 \end{pmatrix}$ i $\begin{pmatrix} -1 \\ -5 \end{pmatrix}$ j $\begin{pmatrix} 6 \\ 12 \end{pmatrix}$

k $\begin{pmatrix} 1 \\ 7 \end{pmatrix}$ l $\begin{pmatrix} -12 \\ -18 \end{pmatrix}$ m $\begin{pmatrix} 3 \\ 13 \end{pmatrix}$ n $\begin{pmatrix} 0 \\ -13 \end{pmatrix}$ o $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$

2 a $\sqrt{5}$ b $\sqrt{10}$ c $\sqrt{29}$ d $\sqrt{13}$ e $\sqrt{65}$ f $4\sqrt{2}$

g $\sqrt{109}$ h $\sqrt{29}$ i $\sqrt{74}$ j $2\sqrt{17}$ k $\sqrt{65}$ l $\sqrt{317}$

m $\sqrt{193}$ n $\sqrt{65}$ o $3\sqrt{5}$

Exercise 21.2

1 a $\mathbf{a}-\mathbf{b}$ b $2\mathbf{a}$ c $-\mathbf{b}$ d $\mathbf{a}-\mathbf{b}$ e $\mathbf{a}+\mathbf{b}$ f $2\mathbf{a}-\mathbf{b}$

g $2(\mathbf{b}-\mathbf{a})$ f $\mathbf{a}-2\mathbf{b}$ i $-2\mathbf{b}$

2 a (i) $\mathbf{a}+\frac{1}{2}\mathbf{b}$ (ii) $\lambda\left(\mathbf{a}+\frac{1}{2}\mathbf{b}\right)$ (iii) $\left(\mathbf{b}-\frac{1}{2}\mathbf{a}\right)$ (iv) $\mu\left(\mathbf{b}-\frac{1}{2}\mathbf{a}\right)$ (v) $\mathbf{a}+\mu\left(\mathbf{b}-\frac{1}{2}\mathbf{a}\right)$

b $\lambda=\frac{4}{5}, \mu=\frac{2}{5}, \mathrm{OT}=\frac{2}{5}\mathbf{b}+\frac{4}{5}\mathbf{a}$

3 a (i) $\frac{1}{2}\mathbf{b}-\mathbf{a}$ (ii) $\lambda\left(\frac{1}{2}\mathbf{b}-\mathbf{a}\right)$ (iii) $\mathbf{a}+\lambda\left(\frac{1}{2}\mathbf{b}-\mathbf{a}\right)$

(iv) $\frac{1}{2}\mathbf{a}-\mathbf{b}$ (v) $\mu\left(\frac{1}{2}\mathbf{a}-\mathbf{b}\right)$ (vi) $\mathbf{b}+\mu\left(\frac{1}{2}\mathbf{a}-\mathbf{b}\right)$

b $\lambda=\frac{2}{3}, \mu=\frac{2}{3}, \mathrm{OT}=\frac{1}{3}(\mathbf{a}+\mathbf{b})$

4 a (i) $\mathbf{a}+\frac{1}{2}\mathbf{b}$ (ii) $\lambda\left(\mathbf{a}+\frac{1}{2}\mathbf{b}\right)$ (iii) $\mu\,\mathbf{a}$ (iv) $\mathbf{b}+\mu\,\mathbf{a}$

b $\lambda=2, \mu=2, \mathbf{OT}=2\mathbf{a}+\mathbf{b}$ c C is the mid-point of BT

5 a $\mathbf{b}-\mathbf{a}$ b $\frac{\lambda}{\lambda+\mu}(\mathbf{b}-\mathbf{a})$ c $\frac{1}{\lambda+\mu}(\mu\mathbf{a}+\lambda\mathbf{b})$

Exercise 21.3

1 $-\mathbf{i}-4\mathbf{j}$, speed $=\sqrt{17}$ ms^{-1} 2 47.2 km h^{-1} dirn 278° 3 412 km h^{-1} dirn 344°

4 a 25 m, 5 s b 65.4° to bank upstream c 5.5 s

5 7.82 km h^{-1} dirn 319° 6 65.1 ms^{-1}, 44.6°; or 89.4 ms^{-1}, 74.6°

7 a 21.2 m, 14.1 s b (i) 28.1° to bank (ii) 30 s

8 a $3\mathbf{i}-3\mathbf{j}$ b $-3\mathbf{i}+3\mathbf{j}$ c 4.24, 8.48 km, 12.7 km

9 a (i) $\sqrt{52}$ ms^{-1}, 56.3° to bank (ii) 50 s, 200 m downstream

b (i) 48.2° to bank, upstream (ii) 67 s

c (i) 32.3° to bank, upstream (ii) 93.5 s

10 a bearing 041.6° b 1 hr 20 m c 244.6° d 2 hr 20 m

Chapter 21 Summative Exercise

1 a 5 b 10 c 26 d 13

e 25 f 51 g 26 h 104

2 a $\frac{2\sqrt{5}}{5}\mathbf{i}+\frac{\sqrt{5}}{5}\mathbf{j}$ b $\frac{5\sqrt{34}}{34}\mathbf{i}+\frac{3\sqrt{34}}{34}\mathbf{j}$ c $\frac{-3\sqrt{13}}{13}\mathbf{i}-\frac{2\sqrt{13}}{13}\mathbf{j}$ d $\frac{-4}{5}\mathbf{i}-\frac{3}{5}\mathbf{j}$

3 $\mathbf{r} = 4.33\mathbf{i} + 2.5\mathbf{j}$ $\mathbf{s} = -3.76\mathbf{i} + 1.37\mathbf{j}$ $\mathbf{t} = -5\mathbf{i} - 8.66\mathbf{j}$ $\mathbf{u} = 1.03\mathbf{i} - 2.82\mathbf{j}$

4 $\mathbf{p} = -10\mathbf{i} + 24\mathbf{j}$ $\mathbf{q} = 12\mathbf{i} + 16\mathbf{j}$ 23.4

5 a $\frac{2}{3}\mathbf{a}+\frac{1}{2}\mathbf{b}$ b $\frac{1}{2}\mathbf{b}-\frac{1}{3}\mathbf{a}$ c $-\frac{2}{3}\mathbf{a}+\frac{(1+2\lambda)}{2}\mathbf{b}$ d $\lambda=\frac{1}{2}$

6 2, 2.4

7 19.06 km hr^{-1} from 150.5°

8 a 079.2°, 40 min b 280.8°, 20 min

9 47.8°, 36.3 m

10 a 039.5°, 22.22 km hr^{-1} b 2.5 km

11 066.4°, 55.9 min

12 45 ms^{-1}, 5.56 s

Chapter 21 Test

1 a $\frac{1}{13}\begin{pmatrix}-5\\12\end{pmatrix}$ b $\begin{pmatrix}-3\\21\end{pmatrix}$

2 a $150\sqrt{3}t\mathbf{i}+150t\mathbf{j}$ c 30 minutes

3 a $\frac{1}{2}(1-\lambda)\mathbf{a}+2\lambda\mathbf{b}$ b $(1-\mu)\mathbf{a}+\mu\mathbf{b}$ c $\lambda=\frac{1}{3}, \mu=\frac{2}{3}$

4 $-\frac{11}{2}, 6$

5 a $\mathbf{v} = 9\mathbf{i} - 12\mathbf{j}$ b $\mathbf{r} = 18\mathbf{i} - 24\mathbf{j}$ c $\mathbf{r} = (18\mathbf{i} - 24\mathbf{j}) + t(9\mathbf{i} - 12\mathbf{j})$
d $\mathbf{r} = 45\mathbf{i} - 60\mathbf{j}$ e $\mathbf{v} = -6\mathbf{i} - 20\mathbf{j}$

6 $\frac{4-\sqrt{3}}{2}, \frac{4+\sqrt{3}}{2}$

7 a $\mathbf{c} = 4\mathbf{i} + 8\mathbf{j}$ b $\frac{\sqrt{5}}{5}\mathbf{i}+\frac{2\sqrt{5}}{5}\mathbf{j}$

Examination Questions

1 (i) P: $50\mathbf{j} + t(20\mathbf{i} + 10\mathbf{j})$ Q: $80\mathbf{i} + 20\mathbf{j} + t(-10\mathbf{i} + 30\mathbf{j})$ (ii) $10\sqrt{5}$ km
(iii) Do not meet – equations are inconsistent.

2 $k = -3$ 3 speed = 42.9 km h^{-1}

4 (i) $\frac{3}{5}\mathbf{i}+\frac{4}{5}\mathbf{j}$ (ii) $12\mathbf{i} + 5\mathbf{j}$

5 Wind blowing on bearing 200.6° (from 020.6°)

6 (i) 108.4° (ii) 47 min 7 34 s

8 $-\frac{3}{5}\mathbf{i}+\frac{4}{5}\mathbf{j}$ 9 3 hr 15.5 mins 10 $\frac{8}{3}$

11 (i) $\frac{1}{3}m(2\mathbf{p}+\mathbf{q})$ (ii) $n\left(\frac{2}{5}\mathbf{q}-\mathbf{p}\right)$ (iii) $n=\frac{5}{9}, m=\frac{2}{3}$

12 (ii) $\frac{1}{10}(8\mathbf{i}+6\mathbf{j})$ (iii) $m = 3, n=\frac{2}{9}$

13 (i) $k = 26$ (ii) $k = 16$

14 (i) $\frac{1}{13}(5\mathbf{i}-12\mathbf{j})$ (ii) $p = 9, q = 2$ 15 (i) 5 ms^{-1} (ii) 73.7°

16 (i) $10\mathbf{i} + 10\mathbf{j}$ (ii) $16\mathbf{i} + 28\mathbf{j}$ (iii) $2\mathbf{i} + 4\mathbf{j}$ (iv) 13.30; $31\mathbf{i} + 43\mathbf{j}$

Chapter 22

Exercise 22.1

1 a $v = 24t - 6t^2$ b $s = 12t^2 - 2t^3$ c 64 m d $t = 6, v = -72$ ms^{-1}
e It will accelerate away in the negative direction.
2 a 8 m b $t = 1, 2, 4$ c $v = 3t^2 - 14t + 14$ d 3 ms^{-1}, -2 ms^{-1}, 6 ms^{-1}
e $a = 2.11$ m, 0.631 m
3 a $t = -3$ (not valid), 0, 3 b $v = 3t^2 - 9$ c $t = -\sqrt{3}$ (not valid), $\sqrt{3}$
d $x = -6\sqrt{3}$ m e $a = 6t$ f $6\sqrt{3}$ ms^{-2}
4 a $v = 20 - 2t$ b 100 m c 20 s
d v = Bird arrives back at the nest with a velocity of -20 ms^{-1} and an acceleration of -2 ms^{-2}. That is very dangerous !
5 a $v = 10\cos 2t$ b $a = -20\sin 2t$ c 10 ms^{-1}, at O d 5 m, π s
e $a = -4x$ f The acceleration is always directed towards O.
6 a $v = \sin t + t\cos t$ b $x = t\sin t$ c $a = 2\cos t - t\sin t$
d 2π s e It increases to infinity.

Exercise 22.2

1 a 80 ms^{-1} b 1600 m
2 a 180 m b 60 ms^{-1}
3 a 0.3 ms^{-2} b 540 m 4 a -3.125 ms^{-2} b 8 s
5 a 75 m b 5 s c 85 m 6 Distance 4650 m; time = 620 s
7 a 15 s after A left b 225 m c 30 ms^{-1}, 60 ms^{-1}
8 a 1.6 ms^{-1} b $6\frac{2}{3}$ s c $5\frac{1}{3}$ m 9 a 14 ms^{-1} b -1.5 ms^{-2} c $11\frac{2}{3}$ s
10 a 8 ms^{-1} (downwards) b 2 s c 7.2 m

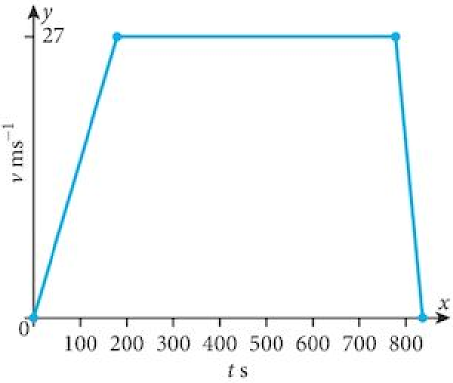
Exercise 22.3

1 a OA = 40 km/h AB = 130 km/h BC = 20 km/h CD = 50 km/h
b student's own description
2 a OA = 15 ms^{-2} AB = 2.5 ms^{-2} BC = 0 ms^{-2} CD = $-1\frac{2}{3}$ ms^{-2} DE = -20 ms^{-2}
b 190 m
3 a OA = 30 ms^{-1} AB = 0 ms^{-1} BC = 15 ms^{-1} CD = -40 ms^{-1} DE = -10 ms^{-1}
EF = 0 ms^{-1} FG = 22.5 ms^{-1} GH = -15 ms^{-1}
4 a AB = -20 ms^{-2} BC = -10 ms^{-2} CD = 40 ms^{-2} DE = -8 ms^{-2}
b 120 m
5 a $a = (v - u)/t$
b $s = \frac{1}{2}(u + v)t$
c student's other derivations

Chapter 22 Summative Exercise

1 a (i) 11 b (i) $\frac{-2\pi}{3}$ c (i) $0.7e^{-0.3}$
(ii) 6 (ii) $\frac{-\pi^2}{18}$ (i) $-0.18e^{-0.2}$
2 a 4 s b 8 s c 512 m d 288 ms^{-1}
3 a 5 s b 150 m
4 a $\frac{4\pi}{3}$ b -3.37 ms^{-2} c 64 m

5 **a**



b 27 ms^{-1} **c** 19 359 m

6 **a** 2 ms^{-1} **b** $-\frac{1}{6}$ ms^{-2} **c** $s = 6 - \frac{36}{2t+6}$

7 120 m $v = 9$ ms^{-1} 15 ms^{-1}

8 **a** 4 s **b** 40 m

c P has velocity 10 ms^{-1} downwards, Q has same velocity upwards

9 **a** 3 ms^{-2} **b** 333 m

Chapter 22 Test

1 **a** 9.43 ms^{-1} **b** $\frac{3\pi}{2}$ s

2 **a** **(i)** 5 s **(ii)** 0, 10 s **(iii)** 1000 m

(iv) –450 ms^{-1} **(v)** particle keeps moving away from O in the negative direction.

3 **a** 38 m **b** 5 s **c** –4 ms^{-1}

d 2 m **e** $a = \frac{72}{(t+1)^3}$ **f** $\frac{8}{3}$ ms^{-2}

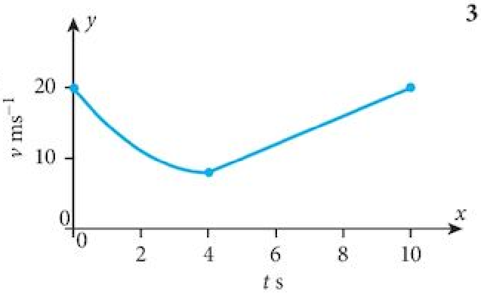
4 **a** $\frac{5}{3}$ s **b** 4.95 s **c** 120 m **d** 8 s **e** –88 ms^{-1}

Examination Questions

1 **(i)** 8 ms^{-1}

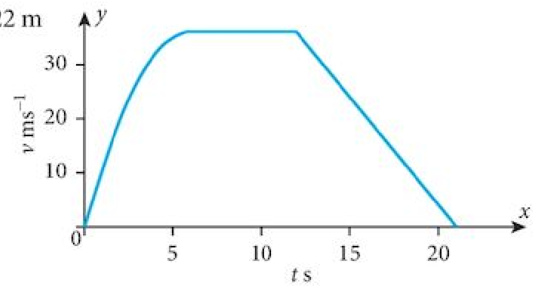
(ii) 48 m

(iii) 6 seconds



3 **(i)** 522 m

(ii)

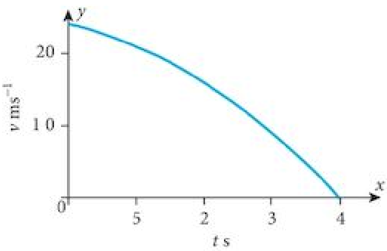


4 26.8 m

5 **(i)** 24 ms^{-1}

(ii) $58\frac{2}{3}$ m

(iii) $20\frac{5}{6}$ m



6 $p = \frac{3}{2}, q = 5$ **7** **(i)** A: –5 ms^{-2} B: 5 ms^{-2} **(ii)** $6\frac{1}{4}$ ms^{-1} (negative)

8 **(i)** –1.92 ms^{-2} **(ii)** 16 m **9** **(i)** $\frac{5}{4}$ ms^{-1} **(ii)** $-\frac{2}{25}$ ms^{-2} **(iii)** 2 m

Term test 6A (Chapters 20 to 22)

1 **a** $-2x\ln(2x-3)+\dfrac{2(3-x^2)}{2x-3}$ **b** $\dfrac{(1-e^{2x})\cos x-(-2e^{2x})\sin x}{(1-e^{2x})^2}$

2 **a** **(i)** $\sqrt{(85)}$ km h^{-1} **(i)** $\sqrt{(29)}$ km h^{-1} **b** $\mathbf{r}=(11+6u)\mathbf{i}+(3+7u)\mathbf{j}$
c **(i)** 1 hr **(ii)** $7\mathbf{i}+6\mathbf{j}$ **(iii)** 2 hrs

3 **a** $a=-48\sin 4t$ **b** $s=3\sin 4t$ **c** $\dfrac{\pi}{8}$ s **d** 3 m

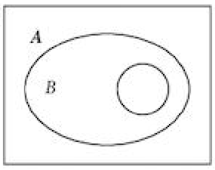
4 **b** $x\,e^x-e^x+c$

5 **a** 17 **b** $\lambda=-\dfrac{5}{22}, \mu=\dfrac{13}{22}$

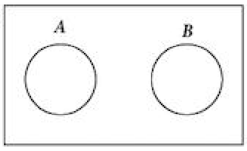
6 **a** 6 s **b** $\dfrac{4}{3}$ ms^{-2} **c** 216 m **d** 18 s

Practice paper 1

1 $p=2$ **2** **(i)** ℰ

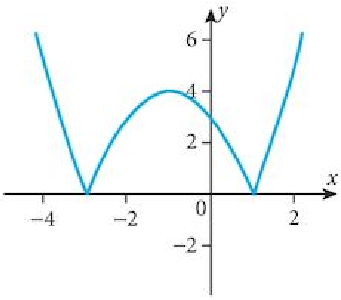


(ii) ℰ



b **(i)** 11
(ii) 20
(iii) 4

3 **a**



b $0<k\le 4$

4 $a=0.365$ **5** $x=-2, y=3$ **6** **a** AB, BC **b** $\begin{pmatrix}-2 & 3\\ 1 & -2\end{pmatrix}$

7 **b** 36.44 cm **c** 26.77 cm^2 **8** **a** **(i)** 360 **(ii)** 240 **(iii)** 48 **b** 714

9 **a** $S=2\pi r^2+\dfrac{10000}{r}$ **b** 1284.7 cm^2

10 **a** $\mathbf{v}=16\mathbf{i}+30\mathbf{j}$ **b** $\mathbf{l}=48\mathbf{i}+90\mathbf{j}$ **c** $\mathbf{r}=(t+3)(16\mathbf{i}+30\mathbf{j})$
d $128\mathbf{i}+130\mathbf{j}+t(-24\mathbf{i}+10\mathbf{j})$ **e** $t=2$ hr $\mathbf{r}=80\mathbf{i}+150\mathbf{j}$

11 **a** 90°, 131.8° **b** 24.01°, 335.98° **c** $\dfrac{\pi}{6}, \dfrac{\pi}{2}, \dfrac{5\pi}{6}, \dfrac{7\pi}{6}, \dfrac{3\pi}{2}, \dfrac{11\pi}{6}$

Index

Complete Additional Mathematics for Cambridge IGCSE® & O Level

Author
Tony Beadsworth

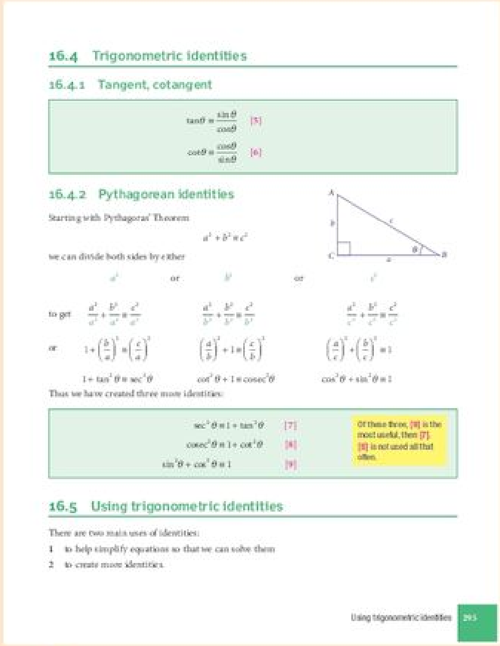
Complete Additional Mathematics for Cambridge IGCSE & O Level directly matches the latest Cambridge IGCSE and O Level Additional Mathematics syllabuses, with plenty of worked examples and practice exercises to cement understanding of key concepts. A stretching, skills-based approach progressively strengthens student ability, enabling confident exam performance.

- **Fully prepare for exams –** comprehensive coverage of the course
- **Develop advanced skills** – rigorous practice extends performance
- **Fully embed key concepts –** further revision tests and exam preparation online

Empowering every learner to succeed and progress

- ✓ Complete Cambridge syllabus match
- ✓ Comprehensive exam preparation
- ✓ Reviewed by subject specialists
- ✓ Embedded critical thinking skills
- ✓ Progression to the next educational stage

ASPIRE
SUCCEED
PROGRESS



16.4 Trigonometric identities

16.4.1 Tangent, cotangent

$$\tan\theta \equiv \frac{\sin\theta}{\cos\theta} \quad [5]$$

$$\cot\theta \equiv \frac{\cos\theta}{\sin\theta} \quad [6]$$

16.4.2 Pythagorean identities

Starting with Pythagoras' Theorem

$$a^2 + b^2 \equiv c^2$$

we can divide both sides by either

a^2 or b^2 or c^2

to get $\frac{a^2}{a^2} + \frac{b^2}{a^2} \equiv \frac{c^2}{a^2}$ $\quad \frac{a^2}{b^2} + \frac{b^2}{b^2} \equiv \frac{c^2}{b^2}$ $\quad \frac{a^2}{c^2} + \frac{b^2}{c^2} \equiv \frac{c^2}{c^2}$

or $1 + \left(\frac{b}{a}\right)^2 \equiv \left(\frac{c}{a}\right)^2$ $\quad \left(\frac{a}{b}\right)^2 + 1 \equiv \left(\frac{c}{b}\right)^2$ $\quad \left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 \equiv 1$

$1 + \tan^2\theta \equiv \sec^2\theta$ $\quad \cot^2\theta + 1 \equiv \operatorname{cosec}^2\theta$ $\quad \cos^2\theta + \sin^2\theta \equiv 1$

Thus we have created three more identities:

$$\sec^2\theta \equiv 1 + \tan^2\theta \quad [7]$$

$$\operatorname{cosec}^2\theta \equiv 1 + \cot^2\theta \quad [8]$$

$$\sin^2\theta + \cos^2\theta \equiv 1 \quad [9]$$

Of these three, [9] is the most useful, then [7]. [8] is not used all that often.

16.5 Using trigonometric identities

There are two main uses of identities:

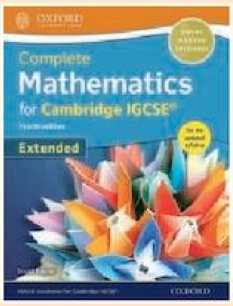
1 to help simplify equations so that we can solve them
2 to create more identities.

Using trigonometric identities 295

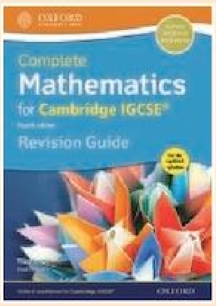
MyiMaths.com

Also available:

978 0 19 837835 8



978 0 19 837839 6



OXFORD
UNIVERSITY PRESS

How to get in contact:
web www.oxfordsecondary.com/cambridge
email schools.enquiries.uk@oup.com
tel +44 (0)1536 452620
fax +44 (0)1865 313472

ISBN 978-0-19-837670-5