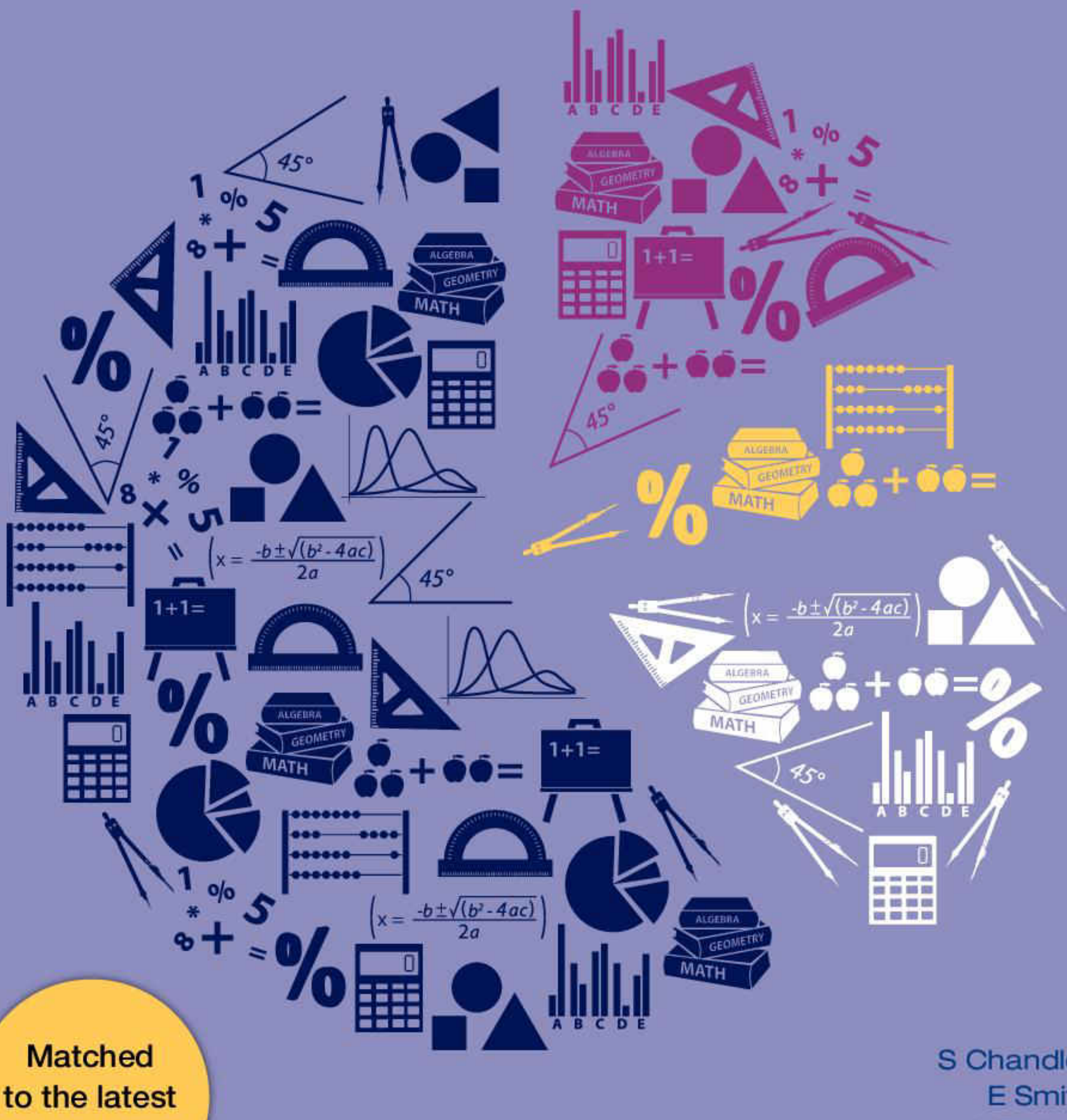
Mathematics for CSEC®

2nd Edition



Matched to the latest syllabus

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INTRODUCTION

This book has been designed for those of you who are going to enter the CSEC General Proficiency examination in Mathematics. The book contains complete coverage of the syllabus with plenty of straightforward questions.

We have also included comprehensive review tests after groups of chapters. These enable you to revise the work covered in those chapters. The multiple choice questions are intended to be worked without the use of a calculator.

The book concludes with three multiple choice exercises and three Model Examination papers that test the whole of the CSEC syllabus. We have also included a sample paper with worked solutions to show the type of answer that is expected.

Most chapters end with 'mixed exercises'. These will help you revise what you have done, either when you have finished the chapter or at a later date.

Guidance on the SBA has been made available on a support website.

Finally a word of advice: when you are doing numerical work, whether with the help of a calculator or not, remember always ask yourself 'Is my answer a reasonable one for the question that was asked?'

Access your support website for additional guidance on the School Based Assessment here:
www.oxfordsecondary.com/9780198414568

PREFACE

To the teacher

The general aims of the book are:

1 to help students to
   - attain solid mathematical skills
   - connect mathematics to their everyday lives and understand its role in the development of our contemporary society
   - see the importance of thinking skills in solving everyday problems
   - discover the fun of doing mathematics and reinforce their positive attitudes to it.

2 to encourage teachers to include historical information about mathematics in their programme.

In writing this book the authors attempted to present topics in such a way that students will understand the connections in Mathematics, be encouraged to see and use mathematics as a means to help make sense in the real world.

Topics from the history of mathematics have been incorporated to ensure that mathematics is not dissociated from its past. This should lead to an increase in the levels of enthusiasm, interest and fascination for mathematics, and should also enrich the teaching of it.

Careful grading of exercises makes the books approachable.

Some suggestions:

1 Before each lesson give a brief outline of the topic to be covered in the lesson. As examples are given refer back to the outline to show how the example fits into it.

2 List terms on the chalkboard that you consider new to the students. Solicit additional words from the class and encourage students to read from the text and make their own vocabulary.
   Remember that mathematics is a foreign language. The ability to communicate mathematically must involve the careful use of the correct terminology.

3 When possible have students construct alternative ways to phrase questions. This ties in with seeing mathematics as a language. Students tend to concentrate on the numerical or 'maths' part of the question and pay little attention to the instructions which give information which is required to solve the problem.

4 When solving problems have students identify their own problem-solving strategies and listen to others. This practice should create an atmosphere of discussion in the class centred around different approaches to the same problem.

   As the students try to solve problems on their own they will make mistakes. This is healthy, as this was the experience of the inventors of mathematics: they tried, guessed, made many mistakes and worked for hours, days and sometimes years before reaching a solution. There are enough problems in the exercises to allow the students to try and try again. The excitement, disappointment and struggle with a problem until a solution is found provide a healthy classroom atmosphere.

To the student

This book is written for you. As you study:

Try to break up the material in a chapter into manageable bits.
Always have paper and pencil when you study mathematics.
When you meet a new word write it down together with its meaning.
Read your questions carefully and rephrase them in your own words.
The information which you need to solve your problem is given in the wording of the problem, not the number part only.
Your success in mathematics may be achieved through practice.
You are therefore advised to try to solve as many problems as you can.
Always try more problems than those set by your teacher for homework.
Remember that the greatest cricketer or netball player became great by practising for many hours.

We have provided enough problems in the books to allow you to practise. Above all don't be afraid to make mistakes as you are learning. The greatest mathematicians all made many mistakes as they tried to solve problems.

You are now on your way to success in mathematics – **GOOD LUCK!**

Acknowledgements

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Picture credits

Nautilus shell, page 361, Corel 712 (NT); Halley's Comet, page 457, Digital Vision 9 (NT); Pyramids, Giza, Egypt, page 478, Corel 281 (NT).

1 NUMBER THEORY AND COMPUTATION

AT THE END OF THIS CHAPTER YOU SHOULD BE ABLE TO...

1 Know the difference between a prime number and a composite number.

2 List the factors and multiples of numbers, and find the HCF and LCM of a set of numbers.

3 Recognise the place value of a digit of a number in any base.

4 Know the number properties.

5 Approximate a given number to a given place value or significant figure.

6 Add, subtract, multiply and divide fractions and decimals.

7 Use scientific notation.

8 Convert between fractions, decimals and percentages.

9 Calculate a fraction or percentage of a given quantity.

10 Find one quantity as a fraction or percentage of another quantity.

11 Understand and use ratios.

12 Distinguish between sets of numbers.

MATHS IS OUT THERE

Euclid showed that there are an infinite number of prime numbers.

Time for a library search: Find the proof that this is true and rewrite it using your own words.

The largest prime number discovered so far has 9 808 358 digits.

BEFORE YOU START you need to know:

✓ pairs of numbers that add up to ten and that add up to 100
✓ the sum of any two numbers less than 10
✓ the product of any two numbers up to 12 × 12
✓ how to add, subtract, multiply and divide with whole numbers.

KEY WORDS

associative, binary number, closure, commutative, composite number, counting number, cube, cube root, denary number, digit, distributive, equivalent fraction, exponent, factor, fractional index, highest common factor, identity, index, integer, inverse, irrational, least common multiple, multiple, natural number, negative index, negative number, percentage, place value, positive number, power, prime number, product, ratio, rational number, real number, reciprocal, significant figure, square, square root, scientific notation, whole number

Whole numbers

- A **digit** is any one of the symbols 0, 1, 2, 3, 4, 5, 6, 7, 8 or 9.
- The **natural numbers**, also called **counting numbers**, are 1, 2, 3, 4, 5, ... The set of natural numbers is denoted by $\mathbb{N}$.

- The **whole numbers** are 0, 1, 2, 3, 4, 5, ... The set of whole numbers is denoted by $\mathbb{W}$. Notice that $\mathbb{N}$ is included in $\mathbb{W}$.
- A **factor** of a whole number will divide into the number exactly without leaving a remainder.
- A **prime number** is a whole number that has only two different factors: itself and 1.
- All other whole numbers are called **composite numbers**. A composite number can be expressed as the product of two factors other than 1 and itself.
- Any composite number can be expressed as a product of prime numbers.
- A **multiple** of a number is the product of the number and any other whole number.
- The **highest common factor** (HCF) of a set of numbers is the largest number that is a factor of every number in the set.
- The **least common multiple** (LCM) of a set of numbers is the smallest number that is a multiple of every number in the set.
- An **index** that is a whole number is a shorthand way of writing a number multiplied by itself, possibly several times. In the number 4^3, the superscript 3 is the index. It means the **product** of 3 fours, i.e. $4 \times 4 \times 4$. An index is also called a **power** or an **exponent**.

> The factors of 6 are 1, 2, 3 and 6.

> 1 is not a prime number because it has only one factor. 2 is the smallest prime number. 5 is a prime number because 1 and 5 are its only factors. 15 is a composite number as $15 = 3 \times 5$.

> The multiples of 3 are 3, 6, 9, 12, ...

> The plural of index is indices. $5^1 = 5$ and $5^0 = 1$, and for any number, a, $a^0 = 1$

EXERCISE 1a

1 Write down all the factors of 24.

2 Write down three two-digit prime numbers starting with 1.

3 Write down two multiples of 25 beginning with the digit 2.

4 Write down two two-digit prime numbers beginning with 5.

5 Write down two multiples of 13 bigger than 100.

6 Write down all the factors of 36.

7 Write as a single number in index form

a $2 \times 2 \times 2 \times 2$ **b** $3 \times 3 \times 5 \times 5$

c $3 \times 3 \times 7 \times 7 \times 7 \times 7$ **d** $2 \times 2 \times 2 \times 2 \times 3 \times 3 \times 13$

e $5 \times 5 \times 5 \times 7 \times 7 \times 13$ **f** $3 \times 2 \times 5 \times 5 \times 3 \times 5 \times 2 \times 3$

> These facts about divisibility are useful when you are looking for factors:
> - an even number is divisible by 2
> - a number ending in 0 or 5 is divisible by 5
> - a number whose digits add up to a multiple of 3 is itself divisible by 3.

Example:

Find the value of **a** $(3^2)^3$ **b** $4^2 + 12^0$

a $(3^2)^3 = (9)^3 = 9 \times 9 \times 9 = 81 \times 9 = 729$

b $4^2 + 12^0 = 16 + 1 = 17$

> First find the value of the number inside the bracket.

8 Find the value of

a 2^3 **b** 3^2 **c** 5^4 **d** 7^3 **e** $2^5 \times 3^4$

Example:

Express 116 as a product of its prime factors.

$$\begin{array}{r|l} 2 & 116 \\ 2 & 58 \\ 29 & 29 \\ & 1 \end{array}$$

so $116 = 2 \times 2 \times 29 = 2^2 \times 29$

You can usually see the prime factors of small numbers, e.g. $6 = 2 \times 3$. For larger numbers, you need an organised approach: start by dividing by the smallest prime factor and repeat until you can no longer divide by it. Then divide by the next smallest prime factor and so on until you get to 1.

9 Express each number as a product of its prime factors.

a 35 **b** 28 **c** 108 **d** 1144 **e** 1936 **f** 30 030

Example:

Find the HCF of 28 and 36.

$28 = 4 \times 7 = 2 \times 2 \times 7$
$36 = 4 \times 9 = 2 \times 2 \times 3 \times 3$
The HCF $= 2 \times 2 = 4$

First write each number as a product of prime factors. Then pick out the factors that are common to each number.

This shows an alternative method for finding the prime factors: start by writing the number as a product of any two factors, then express each of those which are not prime as the product of any two factors and continue this until all the factors are prime.

Find the HCF of the following sets of numbers.

10 **a** 45, 60 **b** 64, 72

11 **a** 32, 52, 56 **b** 18, 54, 72

12 **a** 351, 648 **b** 432, 768

13 **a** 105, 147, 196 **b** 273, 975, 1638

Example:

Find the LCM of 15 and 20.

$15 = 3 \times 5$
$20 = 4 \times 5 = 2 \times 2 \times 5$

The LCM $= 3 \times 5 \times 2 \times 2 = 60$

First write each number as a product of prime factors. Then choose all the factors of the smaller number and add in those factors of the larger number that are not already included.

Find the LCM of

14 **a** 45, 60 **b** 64, 72

15 **a** 56, 84 **b** 104, 169

16 **a** 44, 121, 66 **b** 36, 48, 108

17 For these three numbers: 20, 30, 35, find
a the HCF **b** the LCM.

Example:

The lights on three buoys A, B and C flash at intervals.
A flashes every 2 seconds, B flashes every 3 seconds and C flashes every 5 seconds.
A is started, B is started 1 second later then C is started 1 second after B.

a How many seconds are there between the times at which the three lights flash together?

b How long is it before they first flash together?

a The LCM of 2, 3 and 5 is 30.
The lights flash together every 30 seconds.

The number of seconds between the times at which the lights flash together has to be a multiple of 2, 3 and 5.

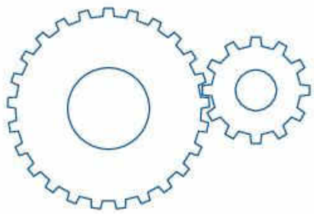
b If they first flash together n seconds after A is started, then n is a multiple of 2, $n - 1$ is a multiple of 3 and $n - 2$ is a multiple of 5.
Try $n - 2 = 10$: $n - 1 = 11$ and 11 is not a multiple of 3.
Try $n - 2 = 20$: $n - 1 = 21$ which is a multiple of 3 so $n = 22$.
They first flash together after 22 seconds.

As n is a multiple of 2, n is even so $n - 2$ is an even multiple of 5. Try $n - 2 = 10$

18 Find the shortest length that can be divided exactly into equal sections of lengths either 5 m, 6 m or 12 m.

19 What is the least sum of money into which \$35, \$50 and \$75 will divide exactly?

20 A gear wheel with 25 teeth drives another wheel with 60 teeth. Two particular teeth are in contact when the wheels start. How many times must each wheel turn before the same two teeth are in contact again?



21 Two cyclists ride around a track. One takes 5 minutes and the other $6\frac{1}{2}$ minutes to complete a circuit. They start together.
How long will it be before they are side by side again?

22 A room measures 400 cm by 550 cm. Find the side of the largest square tile that can be used to tile the floor without any cutting.

23 At a Vacation School the visitors can be divided into groups of 15, 20 and 24 with no one left out. Find the smallest number of visitors that makes this possible.

24 Find the largest number of students who can share 42 pens and 70 pencils equally.

25 Three bells ring at intervals of 3, 4 and 5 seconds. All three are rung together. How long will it be until all three are rung together again?

26 Anna does a big shop at the supermarket every 5 days, Maria every 6 days and Mo every 8 days. They all shop together on 2 January in a year that is not a leap year.
a How many days will it be before they shop together again?
b How many times will they shop together during the year?

27 A No. 3 bus leaves the Bus Station every 6 minutes, a No. 5 every 8 minutes and a No. 8 every 9 minutes. All three leave the Bus Station at 8 a.m.
How long will it be before all three bus services leave together again?

INVESTIGATION

h is the HCF of the two numbers a and b.
Prove that the LCM of a and b is $\frac{ab}{h}$.

Number bases and place value

The position of a digit in a number is called its **place value**. From this we can work out the value of a digit.

The numbers we use every day are based on powers of 10. The first digit at the right-hand end gives the number of units, the next digit to the left gives the number of tens, the next digit gives the number of hundreds, and so on. Any place value is ten times the place value of its right-hand neighbour.

These numbers are called **denary numbers**, and the number base is 10.

When the base of a number system is 4, the first digit at the right-hand end gives the number of units, the next digit to the left gives the number of fours, the next digit to the left gives the number of 4^2 and so on. Any place value is four times the place value of its right-hand neighbour.

So the number $2102_4 = 2\times4^3 + 1\times4^2 + 0\times4^1 + 2\times4^0$

This shows the base of the number system. If it is missing, assume the base is 10.

Base four numbers need just the digits 0, 1, 2 and 3.

Binary numbers have a base 2. They need only the digits 1 and 0.

$10011_2 = 1\times2^4 + 0\times2^3 + 0\times2^2 + 1\times2^1 + 1\times2^0 = (16 + 2 + 1)_{10} = 18_{10}$

The number 5946 means
5 thousands + 9 hundreds + 4 tens + 6 units
$= 5\times1000 + 9\times100 + 4\times10 + 6$
$= 5\times10^3 + 9\times10^2 + 4\times10^1 + 6\times10^0$

In the number 42760, the digit 7 has place value 10^2 so it represents $7\times10^2 = 700$.
Note that base ten numbers use the digits 0, 1, 2, 3, 4, 5, 6, 7, 8 and 9. There is no digit called ten.

When the base is 5, the place values are powers of 5
e.g. $4341_5 = 4\times5^3 + 3\times5^2 + 4\times5^1 + 1\times5^0$.
Base 5 numbers need the digits 0, 1, 2, 3 and 4.

EXERCISE 1b

Example:

a What is the value of the digit 2 in 3203_4?

b Write the number 35_{10} in base 4.

a $2 \times 4^2 = 32$

The place value of the digit 2 is 4^2, so it represents $2 \times 4^2 = 32$

b
4)35 = 8(4s) remainder 3 (units)
4) 8 = 2(4^2s) remainder 0(4s)
4) 2 = 2 (4^3s) remainder 2 (4^2s)
0

So $35_{10} = 203_4$

Notice that the digits are in reverse order of the remainders in the working.

To write 35 in base 4 you need to find how many 16s, 4s and units the number contains. You can do this by dividing by 4 repeatedly.

1 What is the value of the green 1 in each of the following binary numbers?
a 1101 **b** 1011 **c** 11101 **d** 101010

2 What is the value of each digit?
a the 3 in 2301_{four} **b** the 2 in 1021_{eight} **c** the 2 in 3210_{four}

3 What is the value of each digit?
a the 3 in 2311_{four} **b** the 2 in 3521_{10} **c** the 2 in 7234_{eight}

4 What is the value of each digit?
a the 3 in 2230_4 **b** the 2 in 1120_8 **c** the 7 in 3744_8

5 Express as a denary number
a 201_8 **b** 201_4 **c** 101_2

6 Express in the base 4
a 20_{10} b 30_{10} c 50_{10}

7 Express as a denary number
a 1001_2 b 11001_2 c 111000_2

8 Convert 345_8 into a number to base 4.

9 Convert 505_8 into a number to base 4.

10 Convert 2321_4 into a number to base 8.

11 Convert 135_8 into a number to base 10.

12 Convert 257_8 into a number to base 4.

13 Convert 345_8 into a number to base 2.

14 Convert 233_4 into a number to base 10.

MATHS IS OUT THERE

All digital devices use binary numbers to interpret information. Hexadecimal numbers (base 16) are used for machine code programming: A is used for 10, B for 11, C for 12, D for 13, E for 14 and F for 15.

Write the hexadecimal number 1E2A92 as a denary number.

Example:

Determine

a $1223_4 + 1310_4$ b $170_8 - 25_8$

a

4^3	4^2	4^1	units
1	2	2	3
1 (+1)	3	1	0
3	1 (5 = 4 + 1)	3	3

5 = 4 + 1 so 1 goes in this column and one 4 carries into the next column.

$1223_4 + 1310_4 = 3133_4$

b

8^2	8	units
1	7 (−1)	0 (8)
	2	5
1	4	3

Take one 8 from the second column.

$170_8 - 25_8 = 143_8$

Write the numbers in headed columns.

15 Calculate
a $101_2 + 11_2$ b $1011_2 - 110_2$

16 Calculate
a $37_8 + 25_8$ b $32_8 - 25_8$

17 Calculate
a $33_4 + 22_4$ b $123_4 - 32_4$

18 Calculate
a $32_4 + 13_4$ b $32_4 - 13_4$

19 Calculate
a $34_8 + 23_8$ b $43_8 - 23_8$

20 Calculate
a $26_8 + 55_8$ b $161_8 - 55_8$

21 Calculate
a $11_2 + 10_2$ b $101_2 - 11_2$

22 Calculate
a $1101_2 + 1010_2$ b $1001_2 - 101_2$

Negative numbers

The whole numbers can be shown as equally-spaced points on a number line:

When we extend the number line to the left, the numbers less than 0 are called **negative numbers** and numbers greater than 0 are called **positive numbers**.

The set of positive and negative numbers shown as points on the number line above are called **integers**. The symbol $\mathbb{Z}$ is used to denote the set of integers. The integers include the whole numbers.

There is a difference between a positive or negative number and the operations 'add' or 'subtract' although the same symbols are used. $-2 + 4$ means start with negative 2 and add 4: $-2 + 4 = 2$. $3 - 6$ means start with positive 3 and subtract 6: $3 - 6 = -3$.

The rules for working with positive and negative numbers are:

- when you add or subtract a number from another then $b + (+a) = b - (-a) = b + a$ and $b + (-a) = b - (+a) = b - a$
- when you multiply or divide two numbers
  $(+a) \times (+b) = (-a) \times (-b) = +ab$ and
  $(+a) \times (-b) = (-a) \times (+b) = -ab$
  $(+a) \div (+b) = (-a) \div (-b) = +\frac{a}{b}$ and
  $(+a) \div (-b) = (-a) \div (+b) = -\frac{a}{b}$

Both rules can be remembered as 'if the signs are the same the answer is positive, if the signs are different the answer is negative'.

Order of operations

When a calculation involves a mixture of operations, work out any calculation inside the brackets first, then do the multiplications and divisions before the additions and subtractions.

You can remember this from the mnemonic Bless My Dear Aunt Sally.

Properties of operations

We combine numbers using the operations of addition, subtraction, multiplication and division.

- An operation is **commutative** when the order of the numbers does not matter. Addition and multiplication are commutative because, for example, $2 + 5 = 5 + 2\ (= 7)$ and $2 \times 5 = 5 \times 2\ (= 10)$. Subtraction and division are not commutative.

$6 - 3 \neq 3 - 6$ and $6 \div 3 \neq 3 \div 6$

- An operation is **associative** when brackets can be disregarded. Addition and multiplication are associative because, for example, $(2 + 3) + 4 = 2 + (3 + 4)$ and $(2 \times 3) \times 4 = 2 \times (3 \times 4)$. Subtraction and division are not associative.

$(12 \div 3) \div 2 \neq 12 \div (3 \div 2)$ and
$(12 - 3) - 2 \neq 12 - (3 - 2)$

- An operation is **distributive** over another operation when it can be applied separately to each number in the second operation. Multiplication is distributive over addition, for example, $2 \times (3 + 4) = (2 \times 3) + (2 \times 4)$, but addition is not distributive over multiplication.

$2 + (3 \times 4) \neq (2 + 3) \times (2 + 4)$

- A set has **closure** under an operation when any two members of the set combine to give another member of the set.

  The sum or product of any two whole numbers gives another whole number so $\mathbb{W}$ is closed under addition and multiplication. $\mathbb{W}$ is not closed under subtraction and division.

  $\mathbb{Z}$ is closed under addition, subtraction and multiplication but not under division.

$3 - 4$ is not a whole number, nor is $2 \div 3$.
$3 - 4$ is an integer but $2 \div 3$ is not.

- An **identity** is a number that does not change the value of another number when combined with it. For integers, 0 is the identity under addition or subtraction, 1 is the identity for multiplication and division.

$a + 0 = a - 0 = a$
and $a \times 1 = a \div 1 = a$

- The **inverse** of a number is such that when it is combined with the number it gives the identity. The inverse of a under addition is $-a$ because $a + (-a) = 0$. The inverse of a under multiplication is $\frac{1}{a}$ because $a \times \frac{1}{a} = 1$. Note that $\frac{1}{a}$ is called the **reciprocal** of a.

Note that $\frac{1}{a}$ means $1 \div a$.

EXERCISE 1c

Example:

Calculate $5 - 4 \times 3 - (5 - 6)$

$$5 - 4 \times 3 - (5 - 6) = 5 - 12 - (-1)$$
$$= 5 - 12 + 1 = -7 + 1 = -6$$

Bracket first: $5 - 6 = -1$,
multiplication next: $4 \times 3 = 12$

Find the value of

1 $8 + (-1) - (-5)$

2 $(-3) + (-3) + (-3)$

3 $5 - (-4) + (-2)$

4 $4 \times 3 + 7(6 - 3)$

5 $8 \times 4 \times (-3)$

6 $7 \times 4 - 2(3 - 7)$

7 $(-2)^2 \times 3 - 14 \div (-2)$

8 $(10 - 7) \div (2 - 5)$

9 $(4 - 12) \div (-4 \times 2)$

10 $12 \div (-4) \times (-5)$

11 $(-2)^3 \times (-5)$

12 $7 - 2 \div (-1)$

13 $12 \div 3 \div (-4)$

14 $3 \times (-7 - 3) \div 5$

15 Find the difference between $12 \times (-3) \div 4$ and $16 \div (-4) - 3$.

Remember: the difference between two quantities is always positive.

Fractions

A common fraction, e.g. $\frac{3}{4}$, describes part of a unit. The denominator (bottom number) describes the number of equal parts the unit is divided into. The numerator (top number) describes the number of those parts we have.

Fractions are called **rational numbers** and they are denoted by $\mathbb{Q}$.

An integer can be written as a fraction, e.g. $3 = \frac{3}{1}$ and $-4 = -\frac{4}{1}$ so the integers are included in the rational numbers.

Equivalent fractions have the same value.
We can find equivalent fractions by multiplying (or dividing) the numerator and the denominator of a fraction by the same number.

$\frac{3}{4}$ of an orange means 3 out of 4 equal parts of the orange.

When the numerator is greater than the denominator, the fraction is **improper**. $\frac{5}{4}$ is an improper fraction. It means 5 quarters which is 1 whole and a quarter and can be written as $1\frac{1}{4}$. This Is a **mixed number** and it means $1 + \frac{1}{4}$.

Simplifying fractions means finding an equivalent fraction with a smaller numerator and denominator. When the numerator and denominator are as small as possible, the fraction is in its lowest possible terms or its simplest form.

$\frac{18}{24} = \frac{18 \div 6}{24 \div 6} = \frac{3}{4}$
The process of dividing top and bottom by the same number is called cancelling.

Operations on fractions

- Fractions can be added or subtracted by first changing them into equivalent fractions with the same denominator, then adding or subtracting the numerators.

$\frac{2}{3} + 1\frac{1}{6} = \frac{2}{3} + \frac{7}{6} = \frac{4 + 7}{6}$
$= \frac{11}{6} = 1\frac{5}{6}$

- Fractions can be multiplied together by multiplying the denominators and multiplying the numerators. Before multiplying, cancel where possible to reduce the size of the numbers.
- To divide by a fraction, turn the fraction upside down (i.e. find its reciprocal) and multiply.

$\frac{3}{4} \times 1\frac{1}{2} = \frac{3}{4} \times \frac{3}{2} = \frac{3 \times 3}{4 \times 2} = \frac{9}{8} = 1\frac{1}{8}$

$\frac{5}{6} \div \frac{2}{7} = \frac{5}{6} \times \frac{7}{2} = \frac{35}{12} = 2\frac{11}{12}$

EXERCISE 1d

Example:

Calculate $\frac{4}{3} + 1\frac{2}{9} \div \frac{1}{6}$

$$\frac{3}{4} + 1\frac{2}{9} \div \frac{1}{6} = \frac{3}{4} + \frac{11}{9} \div \frac{1}{6}$$
$$= \frac{3}{4} + \frac{11}{\cancel{9}_3} \times \frac{\cancel{6}^2}{1} = \frac{3}{4} + \frac{22}{3}$$
$$= \frac{9}{12} + \frac{88}{12} = \frac{97}{12} = 8\frac{1}{12}$$

First change $1\frac{2}{9}$ into an improper fraction then do the division: turn $\frac{1}{6}$ upside down and multiply.

Next do the addition: turn the fractions into equivalent fractions with a common denominator. Choose the LCM of 4 and 3 for the common denominator.

Evaluate using a calculator or otherwise.

1 $12\frac{1}{2} + 5\frac{3}{7}$

2 $3\frac{1}{2} + 2\frac{1}{4} + 1\frac{1}{8}$

3 $\frac{7}{8} + \frac{4}{3} + \frac{7}{9}$

4 $\frac{5}{6} - \frac{11}{18}$

5 $\frac{19}{20} - \frac{3}{5} + \frac{3}{4}$

6 $6\frac{1}{4} - 1\frac{5}{8}$

7 $4\frac{7}{12} - 3\frac{5}{8} + 1\frac{2}{3}$

8 $7\frac{3}{7} - 2\frac{3}{14} - 3\frac{1}{2}$

9 $\frac{8}{13} \times \frac{5}{12}$

10 $7\frac{3}{5} \times \frac{5}{19}$

11 $4\frac{1}{4} \times \frac{13}{34}$

12 $3\frac{1}{6} \times 1\frac{5}{7} \times 5\frac{1}{4}$

13 $1 \div \frac{4}{3}$

14 $\frac{18}{11} \div \frac{27}{22}$

15 $1 \div 2\frac{1}{4}$

16 $1 \div 4\frac{4}{7}$

17 $4\frac{2}{5} \div 5\frac{1}{2}$

18 $6\frac{1}{2} \div 2\frac{3}{5}$

19 $8\frac{1}{8} \div 7\frac{3}{7}$

20 $\left(2\frac{1}{4} - 1\frac{1}{3}\right) \div \frac{11}{24}$

21 $1\frac{1}{3} \times 1\frac{8}{9} \div 3\frac{7}{9}$

22 $\left(4\frac{1}{3} - 2\frac{2}{7}\right) \times \frac{49}{86}$

23 $\left(1\frac{3}{8} + 2\frac{3}{4}\right) - \left(\frac{1}{3} + \frac{5}{6}\right)$

24 $\frac{4}{3} \times \left(1\frac{1}{4} - \frac{1}{5}\right)$

25 $\dfrac{\frac{3}{7} \times \frac{5}{9}}{6\frac{1}{4} - 5\frac{4}{15}}$

26 $\dfrac{1\frac{3}{8} + 1\frac{5}{7}}{2\frac{6}{7} + 1\frac{3}{4}}$

27 $1\frac{5}{7}$ of $\frac{1}{2} \div 3\frac{3}{7}$

28 $\dfrac{12\frac{1}{6} + 5\frac{1}{4}}{8\frac{3}{4} - 3\frac{1}{2}}$

29 $\dfrac{2\frac{1}{4} - \frac{2}{3} \times 1\frac{5}{6}}{\frac{1}{5} \times 3\frac{1}{3} + \frac{13}{16}}$

30 $5\frac{1}{8} - 2\frac{7}{24} + 1\frac{1}{2}$

31 $\left(3\frac{3}{8} - 1\frac{1}{2}\right) \times 2\frac{2}{3}$

32 $\dfrac{3\frac{1}{3} \times 1\frac{1}{2}}{1\frac{1}{4}}$

33 $\dfrac{\frac{3}{4} - \frac{2}{5}}{3\frac{1}{2}}$

34 $\dfrac{5\frac{1}{5} \times 4\frac{1}{4}}{5\frac{1}{5} - 4\frac{1}{4}}$

35 $\dfrac{\frac{5}{7} - \frac{2}{3}}{\frac{1}{6} \div \frac{3}{11}}$

36 $\dfrac{2\frac{1}{2}}{3\frac{1}{4} - \frac{5}{8}}$

37 In a mathematics textbook $\frac{2}{5}$ deals with arithmetic, $\frac{3}{7}$ algebra and the remainder geometry. If the book has 210 pages, how many pages of geometry are there?

38 During a 2-hour bus journey the rest periods amount to $\frac{1}{5}$ of the time. If the average speed of the bus when it is moving is 50 mph, what distance is the journey?

39 A petrol storage tank is three-quarters full. After 75 litres have been drawn off it is three-fifths full. What is the capacity of the tank?

40 After spending $\frac{5}{6}$ of my money I have $1072 remaining. How much money did I have to start with?

41 Terry poured $\frac{5}{8}$ of a can of oil into his car engine. He had 3 litres left over. How much did he use?

42 Andy and John went into business together. Andy provided $\frac{7}{12}$ of the capital and John the remainder of $10 400. How much did Andy contribute?

43 Anne, Betty and Cheryl decide to open a hairdressing salon. To do this they require $10 720. Anne contributes $\frac{7}{20}$ of it, Betty $\frac{3}{10}$ and Cheryl the remainder. How much does each contribute?

44 In a book containing four short stories, the first is $\frac{1}{6}$ of the whole, the second $\frac{1}{8}$, the third 126 pages and the fourth is $\frac{1}{3}$ of the whole. How many pages are in the book?

45 The product of two numbers is 8. If one of the numbers is $3\frac{1}{3}$ find the other number.

46 The product of two numbers is 21. If one of the numbers is $2\frac{4}{7}$, find the other number.

47 How many jars, each of which holds $\frac{3}{8}$ kg, can be filled from a tin containing 21 kg?

48 The area of a blackboard is $8\frac{3}{4}$ square metres. If the board is $1\frac{2}{3}$ m wide, how long is it?

49 If it takes $3\frac{1}{3}$ minutes to fill $\frac{3}{8}$ of a water storage tank, how long will it take to fill it completely?

50 When the larger of two fractions is divided by the smaller, the result is $1\frac{7}{18}$. If the smaller fraction is $2\frac{2}{5}$, find the larger.

Powers and roots

- The **square** of a number is the number multiplied by itself. The square of 5 is called 5 squared and is written as 5^2.
- When a number can be expressed as the product of two equal factors, that factor is the **square root** of the number.

  A positive number has two square roots, one positive and one negative, e.g. the square roots of 4 are 2 and −2.

  A negative number has no square roots because there are no equal numbers whose product is negative.

$16 = 4 \times 4$, so 4 is a square root of 16.
$16 = -4 \times -4$, so -4 is also a square root of 16.
By convention, however, $\sqrt{16}$ means the positive square root of 16.

- The **cube** of a number is the product of three of the number. The cube of 2 is $2 \times 2 \times 2$ and is called 2 cubed. It is written as 2^3.
- When a number can be expressed as the product of three equal factors, that factor is called the **cube root** of the number.

  Any number has one cube root; a positive number has a positive cube root and a negative number has a negative cube root.

> $27 = 3 \times 3 \times 3$, so 3 is the cube root of 27. The symbol $\sqrt[3]{3}$ means 'the cube root of'.
> $-27 = -3 \times -3 \times -3$ so $\sqrt[3]{-27} = -3$

Irrational numbers

The square root of two, i.e. $\sqrt{2}$, cannot be written exactly as either a decimal or a fraction whose numerator and denominator are integers. All square roots that are not exact are irrational, for example $\sqrt{3}$, $\sqrt{5}$, $\sqrt{\frac{7}{8}}$. There are many more numbers like this, such as π. They are called **irrational numbers.**

Indices

Other words for indices are **powers** and **exponents**.
These are the laws for working with indices.

- We can multiply the same number to different powers by adding the indices, for example $5^2 \times 5^4 = 5^{2+4} = 5^6$.

> $5^2 \times 5^4 = 5 \times 5 \times 5 \times 5 \times 5 \times 5 = 5^6$

- We can divide the same number to different powers by subtracting the indices, for example $4^7 \div 4^2 = 4^{7-2} = 4^5$.

> $4^7 \div 4^2 = \frac{4 \times 4 \times 4 \times 4 \times 4 \times \cancel{4} \times \cancel{4}}{\cancel{4} \times \cancel{4}} = 4^5$

- A **negative index** means the reciprocal, for example, 3^{-2} means the reciprocal of 3^2, i.e. $\frac{1}{3^2}$.

> $3^1 \div 3^3 = \frac{3}{3 \times 3 \times 3} = \frac{1}{3^2}$
> and $3^1 \div 3^3 = 3^{1-3} = 3^{-2}$

- We can simplify a power of a power by multiplying the indices, for example, $(2^4)^3 = 2^{12}$.

> $(2^4)^3 = 2^4 \times 2^4 \times 2^4 = 2^{3 \times 4}$

To determine $\frac{5^3}{5^3}$ we can subtract the indices. This gives 5^0.

But $\frac{5^3}{5^3} = 1$. So we deduce that $5^0 = 1$

Now consider $\frac{a^b}{a^b}$ where a and b are any two numbers.

Subtracting the indices gives $\frac{a^b}{a^b} = a^{b-b} = a^0$. But $\frac{a^b}{a^b} = 1$ so we deduce that $a^0 = 1$

Therefore **any number to the power zero is equal to 1**.

EXERCISE 1e

Example:

Write $3^{-5} \times 3^2$ as a single number in index form.

$3^{-5} \times 3^2 = 3^{-3}$

> Add the powers.

Write as a single number in index form

1 **a** $2^4 \times 2^2$ **b** $3^3 \times 3^{-3}$ **c** $5^0 \times 5^2$

2 **a** $2^{\frac{1}{2}} \times 2^3$ **b** $3^{-2} \times 3^{\frac{1}{2}}$ **c** $(3^2)^3$

Example:

Find the value of $(-3)^2 + \left(\frac{1}{2}\right)^{-3}$

$(-3)^2 + \left(\frac{1}{2}\right)^{-3} = 9 + 2^3 = 9 + 8 = 17$

$-3 \times -3 = 9$ and $\left(\frac{1}{2}\right)^{-3} = 1 \div \left(\frac{1}{2}\right)^3 = 2^3$

Find the value of

3 a 4^{-1}	b 100^{-2}	c 47^{-2}
4 $\left(\frac{4}{9}\right)^{-2}$	5 $9^0 \times 10^2$	6 $12^2 \times 3^{-2}$
7 $(2^4)^{-3}$	8 $(-4)^2 \times \left(\frac{1}{2}\right)^{-3}$	9 $\frac{(10-3)^3}{9-2}$
10 $\frac{(8-5)^2}{3^2+6}$	11 $\frac{2^3+3^2}{3+2}$	12 $\frac{5^3-10^2}{10-5}$

Decimals

A decimal is another way of describing part of a unit.

A point (called the decimal point) is placed after the units and the place values to the right of this are called decimal places.

The first decimal place after the point represents tenths, the second decimal place represents hundredths, the third decimal place represents thousandths, and so on.

5.27 means $5 + \frac{2}{10} + \frac{7}{100}$

Operations on decimals

- Decimals are added and subtracted in the same way as whole numbers. Make sure that you add tenths to tenths, hundredths to hundredths, and so on. Add zeros if needed so that both numbers have the same number of decimal places.

The easiest way to make sure that you add the same place values together is to write them in a column with the decimal points lined up.
e.g. 2.56 −0.7:

```
  2.56
 −0.70
  1.86
```

- To multiply decimals, ignore the decimal point and multiply the numbers as you would with whole numbers. Then use the total number of decimal places in the original numbers as the number of decimal places in the result.

$3.24 \times 0.5 = 1.620 = 1.62$ $(324 \times 5 = 1620)$
2 d.p. 1 d.p. 3 d.p.

- A decimal can be divided by a whole number. Proceed as you would with whole numbers. Add zeros after the point if necessary to continue the division.

$3.16 \div 8 = 0.395$

```
  0.395
8)3.160
```

- Division by a decimal can be converted to division by a whole number. Use the fact that a division can be written as a fraction. Then multiply top and bottom of the fraction by a number that makes the denominator a whole number.

$2.15 \div 0.5 = \frac{2.15}{0.5} = \frac{21.5}{5} = 4.3$
(multiplying top and bottom by 10)

```
  4.3
5)21.5
```

Approximations

Many decimal calculations result in more decimal places than we need and some of them are infinite. We deal with this by rounding the number to a given place value or significant figure. This is called correcting the number.

The first **significant figure** in a number is the first non-zero digit, reading from left to right. The second significant figure is the next digit whether or not it is zero.

The first significant figure in each of these numbers is 3:
3.79, 0.334, 3670, 0.000 315

To correct a number to a given place value, look at the digit on its right:

if this is 5 or larger, round up; if it is less than 5, leave it alone.

Note that any number that has been rounded is not exact. It is an approximation to the number.

2.5|7203 = 2.6 correct to 2 significant figures
2.57|203 = 2.57 correct to 2 decimal places

Scientific notation

A number in **scientific notation** is a number between 1 and 10 multiplied by a power of 10.

4.27×10^2 is in scientific notation but 0.75×10^3 is not.

EXERCISE **1f**

Example:

Calculate $\frac{0.5508}{3.4}$

a exactly **b** correct to 2 decimal places
c correct to 1 significant figure **d** in scientific notation.

$$\frac{0.5508}{3.4} = \frac{5.508}{34} = 0.162 \qquad 34\overline{)5.506}\ \ 0.162$$

a 0.162

b 0.162 = 0.16 correct to 2 d.p.

c 0.162 = 0.2 correct to 1 s.f.

d $0.162 = 1.62 \times 10^{-1}$

$0.162 = 1.62 \div 10 = 1.62 \times 10^{-1}$

1 Give each number correct to 1 significant figure.
a 6.350 b 473.5 c 45.98 d 0.0709

2 Give each number correct to 2 significant figures.
a 386.5 b 0.0807 c 3.225 d 53.92

3 Give each number correct to 3 significant figures.
a 0.067 78 b 0.999 99 c 45.73 d 765.4

4 Give each number correct to 2 decimal places.
a 1.546 b 0.7070 c 3.1426 d 65.888

5 Calculate exactly.
a $\frac{0.7191}{4.7}$ b $\frac{92.61}{73.5}$ c $\frac{20.223}{5.4}$ d $\frac{571.2}{8.5}$

6 Give each number in scientific notation.
a 54 000 b 23.5 c 0.000 72 d 0.010 05

7 Calculate $17.697 \div 10.2$
a exactly
b correct to 2 significant figures
c correct to 2 decimal places.

8 Find, correct to two decimal places
a $0.41 \div 2.6$ b $0.937 \div 26.4$ c $0.643 \div 0.55$ d $89.2 \div 137$

9 Find, correct to 3 significant figures
a $40.1 \div 0.076$ b $3.21 \div 0.00434$ c $16.3 \div 0.0876$

10 Calculate the exact value of
a $5.2^2 + 3.7^2$ b $0.8^3 + 0.8$ c $\frac{0.21}{0.7} - 0.007$

11 Find the exact value of
a $18.7 - 3.4^2$ b $\frac{5.3 + 7.39}{1.35}$

12 Find the exact value of
a $6.4^2 - 3.9^2$ b $\frac{26 + 50.32}{5.3 \times 0.6}$

13 Find the exact value of
a $\frac{3.848}{0.52}$ b $\frac{0.72 \times 0.4}{36}$

14 Find, correct to 3 significant figures
a $\sqrt{(4.63^2 + 6.49^2)}$ b $\sqrt{(0.894^2 + 1.465^2)}$ c $1 \div 0.767^2$

15 Find, correct to 2 decimal places
a $\sqrt{(2.65^2 + 3.06^2)}$ b $1 \div \sqrt{345.7}$ c 0.945^2

16 Find, correct to 3 significant figures
a $\frac{14.7 - 9.54}{3.77 + 7.23}$ b $\frac{3.45 \times 6.25}{67.3 - 45.9}$ c $\frac{5.38^2}{\sqrt{33.7}}$

17 Find the value of $(13.2^2) - (0.675 \div 3)$
a exactly b to 2 s.f. c in scientific notation.

18 If $a = 5 \times 10^4$ and $b = 25 \times 10^3$ find, in scientific notation
a $a + b$ b ab c $a \div b$ d $b \div a$

19 If $a = 3.6 \times 10^4$ and $b = 4 \times 10^{-3}$ find, in scientific notation
a ab b $a \div b$

> Remember that you can multiply a string of numbers in any order. e.g. $2 \times 10^3 \times 3 \times 10^5 = 2 \times 3 \times 10^3 \times 10^5 = 6 \times 10^8$.
> But when a calculation includes a mixture of operations, do $\times$ and $\div$ before $+$ and $-$.
> e.g. $2 \times 10^3 + 3 \times 10^5 = 2 \times 1000 + 3 \times 10000000 = 2000 + 30000000 = 30002000$

20 If $a = 4.2 \times 10^{-3}$ and $b = 6 \times 10^{-5}$ find, in scientific notation

a $a + b$ **b** $a - b$ **c** ab **d** $a \div b$

21 The thickness of a 250-page book (i.e. 125 sheets or leaves) is 11.25 mm.

a Find the thickness of **i** 1 sheet **ii** 80 sheets **iii** a ream (500 sheets).

b How thick is a book with **i** 200 pages **ii** 560 pages?

c How many sheets have a total thickness of **i** 28.8 mm **ii** 3.42 mm?

d How many pages are there in a book whose thickness, excluding the cover, is **i** 19.8 mm **ii** 7.2 mm?

Take care that you understand the difference between a page and a sheet.

Converting fractions to decimals

The fraction $\frac{a}{b}$ means $a \div b$, so a fraction can be converted to a decimal by dividing the numerator by the denominator.

Most fractions do not convert to a decimal that terminates.

For example, $\frac{2}{11}$: $11\overline{)2.000000...}$ = 0.181818... The digits 18 recur infinitely.

We write this as $0.\dot{1}\dot{8}$. The dots show the pattern of digits that recur.

All fractions convert to a decimal that either terminates or recurs. Conversely, any recurring decimal can be expressed as an exact fraction.

The set of irrational numbers and rational numbers are called **real numbers**. The set of real numbers is denoted by $\mathbb{R}$.

The fraction $\frac{2}{5}$ means $2 \div 5$: place a decimal point after the 2 and add zeros as needed. So $\frac{2}{5} = 0.4$ $5\overline{)2.0}$ = 0.4

$0.1\dot{2}5\dot{6}$ means 0.125 625 625 6...

There are some numbers, such as $\sqrt{2}$ and π that cannot be expressed as either a fraction or as an exact decimal. These are called **irrational** numbers.

Converting decimals to fractions

Any terminating decimal can be expressed as a fraction by ignoring the decimal point and writing it over the power of 10 equal to the number of decimal places. For example, $0.125 = \frac{125}{10^3} = \frac{125}{1000}$ which cancels to $\frac{1}{8}$.

Percentages

A **percentage** is a fraction with a denominator of 100.

$27\% = \frac{27}{100}$ and $180\% = \frac{180}{100} = 1\frac{80}{100} = 1\frac{4}{5}$

So to convert a percentage to a fraction, remove the % sign and write it over 100, then cancel the fraction when possible.

To convert a percentage to a decimal remove the % sign and divide it by 100.

Conversely, to convert a fraction or a decimal to a percentage, multiply by 100 and add a % sign.

If you remember this, it makes converting percentages to fractions or decimals, and vice versa, easy to remember.

$24\% = \frac{24}{100} = \frac{6}{25}$ and
$24\% = 24 \div 100 = 0.24$
$\frac{3}{8} = \frac{3}{8} \times 100\% = \frac{300}{8}\% = \frac{75}{2}\% = 37\frac{1}{2}\%$ and
$1.765 = 1.765 \times 100\% = 176.5\%$

One quantity as a fraction or percentage of another

To find one quantity as a fraction or percentage of another, first make sure both quantities are measured in the same unit. Then place the first quantity over the second. Cancel this to give a fraction in its lowest possible terms or multiply it by 100 and add a percentage sign to give a percentage.

25 cm as a fraction of 2 m $= \frac{25}{200} = \frac{1}{8}$
(converting 2 m to cm)
25 cm as a percentage of 2 m
$= \frac{25}{200} \times 100\% = 12.5\%$

A fraction or percentage of a quantity

To find a fraction of a quantity, divide the quantity by the denominator and multiply by the numerator.

To find a percentage of a quantity, divide the quantity by 100 and multiply by the percentage (without the percentage sign).

$\frac{2}{5}$ of \$5400: $\frac{1}{5}$ of \$5400 = \$5400 ÷ 5 = \$1080,
so $\frac{2}{5}$ of \$5400 = \$1080 × 2 = \$2160
30% of 250 m: 1% of 250 m = 250 m ÷ 100 = 2.5 m
so 30% of 250 m = 30 × 2.5 m = 75 m

EXERCISE **1g**

1 Copy and complete the following table.

Fraction	Decimal	Percentage
$\frac{3}{4}$		
	0.8	
		56%
		$37\frac{1}{2}$%
$\frac{7}{8}$		
	0.625	

Example:

Which of these numbers is nearest in value to a half?
$\frac{5}{9}$, 0.604, 59%, $\frac{7}{15}$

$\frac{5}{9} = 0.555...$, 0.604, 59% = 0.59, $\frac{7}{15} = 0.466...$

0.467, 0.556, 0.59, 0.604

$0.5 - 0.467 = 0.033$ and $0.556 - 0.5 = 0.056$

$0.033 < 0.056$ so $\frac{7}{15}$ is nearest in value to $\frac{1}{2}$.

The sizes of numbers are easier to compare when they are written as decimals. We can then write the numbers in order of size. $\frac{1}{2} = 0.5$, so 0.466... and 0.555... are the numbers nearest to $\frac{1}{2}$. It is clear that 0.466... is nearer to $\frac{1}{2}$ than 0.555.... To confirm this we can find the difference between these numbers and $\frac{1}{2}$.

2 Express each decimal as a fraction in its lowest terms.
a 0.45 **b** 1.25 **c** 0.875 **d** 0.6875

3 Express each fraction as **i** an exact decimal **ii** a percentage.
a $\frac{3}{50}$ **b** $\frac{7}{25}$ **c** $\frac{19}{8}$ **d** $3\frac{7}{8}$

4 Express the first quantity as
i a fraction of the second quantity
ii a percentage of the second quantity.
a \$1.75, \$5 **b** 2.5 kg, 4 kg **c** 33 cm^2, 60 cm^2
d 15 cm, 1 m **e** 560 m, 2 km **f** 0.75 cm^3, 250 mm^3

Be careful with the units.

5 Find
a $\frac{3}{8}$ of 10.4 kg **b** $\frac{7}{12}$ of \$6.72 **c** $\frac{4}{9}$ of 351 cm **d** $\frac{7}{15}$ of 9.45 m.

6 Arrange $\frac{5}{6}$, 0.82, $\frac{6}{7}$, 76% in ascending order.

7 Change
a 0.57 to a percentage **b** 28% to a decimal
c $\frac{39}{50}$ to a percentage **d** 0.48 to a fraction in its lowest terms.

8 Arrange $\frac{7}{11}$, 63%, $\frac{11}{16}$, 0.625 in descending order.

9 Arrange these fractions in ascending order: $\frac{9}{19}, \frac{14}{31}, \frac{6}{13}, \frac{18}{37}$.

10 Express each fraction as a decimal correct to 3 decimal places and hence arrange these numbers in descending order:
$\frac{5}{3}$, 1.55, $\frac{15}{11}$, 1.47, $1\frac{2}{7}$

11 Find as a decimal, correct to 3 decimal places, the difference between the largest fraction and the smallest fraction: $\frac{9}{11}, \frac{11}{14}, \frac{14}{17}, \frac{17}{20}$

12 Arrange these decimals in ascending order:
0.7071, 0.7701, 0.0771, 0.7107

13 Write the following numbers in ascending order.

a $\frac{7}{3}$, 2.7, $\sqrt{5}$, $\frac{12}{7}$, 3.4
b 5.9, $\frac{23}{4}$, $\sqrt{40}$, $\frac{27}{5}$, 5.2
c 4.05, 4.4, $\sqrt{20}$, $\frac{40}{9}$, 4.5
d $\sqrt{24}$, 6.3, $\frac{17}{4}$, 3.41, π

14 Write the following numbers in descending order.

a $\sqrt{5}$, $\frac{9}{4}$, 3.81, $\sqrt{8}$, $\frac{11}{4}$
b 2.8, $\frac{7}{2}$, $\sqrt{12}$, 1.46, $\frac{13}{3}$
c $\frac{21}{5}$, 3.5, π, $\frac{25}{8}$, $\sqrt{10}$
d $\sqrt{30}$, $2\sqrt{8}$, 5.6, $\frac{23}{4}$, 7.3

15 Write the following numbers in order with the largest first.

a $\frac{17}{3}$, $\frac{33}{5}$, 6.9, $\sqrt{50}$, π
c $\sqrt{12}$, 3.72, $\frac{10}{3}$, $\frac{3}{10}$, 3.27
c 9.23, $\sqrt{10}$, 2.93, 3.29, $\frac{13}{6}$

16 Write the following numbers in order with the smallest first.

a 4.52, $\frac{18}{5}$, 6.43, $\frac{9}{2}$, 2.64
c 8.9, $2\sqrt{12}$, $\frac{100}{3}$, $3\sqrt{8}$, 9.8
c 0.59, $\frac{1}{\sqrt{3}}$, $\frac{1}{\sqrt{2}}$, 7.07, $\frac{15}{2}$

17 Find the value of

a 30% of 8 m
b 65% of 5 cm
c 85% of 2.5 km
d $66\frac{2}{3}$% of 368 kg

Example:

A man buys a table for \$3500 and sells it on e-bay for \$4900. Find the profit that the man makes as a percentage of the price he paid for the table.

Profit = \$4900 − \$3500 = \$1400

\$1400 as a percentage of \$3500 $= \frac{\cancel{1400}^{2}}{\cancel{3500}_{5}} \times 100\% = \frac{2}{5} \times 100\%$

$= 40\%$

18 In a science test, Maria got 26 marks out of 40. What percentage is this?

19 The cost of running a car is 28% for diesel, 35% for tax, insurance and repairs, and the remainder depreciation. Joe's costs for a year were \$9250. By how much did his car depreciate?

20 In a form, 75% of the students are boys. There are 9 girls. How many students are there in the form?

21 A retailer buys 100 articles for \$360 and sells them at \$4.80 each. Find her percentage profit.

22 In a cricket match the local team's score was standing at 180 for 9 wickets when the last pair came together. These two increased the total by 35%. Find

a how many runs they put on for the last wicket

b the innings score.

23 A boy owed his father \$72. He paid back \$12. What percentage of the original amount did he still owe?

24 A tank contains 2150 litres of petrol. If 774 are drawn off what percentage of the original amount remains?

Ratio

A **ratio** is a way of comparing related quantities.

If a batch of compost is made by mixing 2 litres of sand with 6 litres of soil, we say that the ratio of sand to soil is 2 to 6 by volume. The ratio 2 to 6 is written as 2 : 6.

> To compare the lengths 2 m and 75 cm, we can write 2 m : 75 cm.
> To leave out the units, both quantities must be measured in the same unit. Converting 2 m to cm, we can write the ratio of the lengths = 200 : 75

A ratio involving two quantities can be written as a fraction. 2 to 6 can be written as $\frac{2}{6}$.

A ratio can involve three or more quantities.

If three sisters are aged 12 years, 14 years and 18 years, the ratio of their ages is 12 : 14 : 18.

A ratio can be simplified in the same way as a fraction: divide each part of the ratio by the same number.

> 200 : 75 = 8 : 3 (dividing both parts by 25)
> 12 : 14 : 18 = 6 : 7 : 9 (dividing all parts by 2)

Division in a given ratio

Any quantity can be divided in a given ratio: divide the quantity by the sum of the numbers in the ratio, then multiply the result by each number in the ratio in turn to give the parts.

> To divide 250 kg in ratio 3 : 2,
> start by dividing 250 kg by 5 (3 + 2).
> 250 kg ÷ 5 = 50 kg.
> So one part = 3 × 50 kg = 150 kg,
> the other part = 2 × 50 kg = 100 kg.
>  = 250 kg

EXERCISE 1h

Example:

An oil was made by mixing 5 litres of palm oil and 550 ml of sunflower oil.
In what ratio are the oils mixed?

5 litres : 550 ml = 5000 : 550 = 100 : 11

The ratio of palm oil to sunflower oil is 100 : 11 by volume.

> The units can be left out provided both are measured in the same unit. But when describing the ratio, it is important to state what type of quantity is being compared.

1 Express the following ratios in their lowest terms.

a 40 cm : 130 cm b \$8.40 : 42 c

c 30 minutes : 1 hour d 1500 mm : 2 m

2 Express the following ratios in their lowest terms.

a 49 g : 108 g **b** 1.5 kg : 500 g **c** 650 cm^3 : $1\frac{1}{2}$ litres

3 Which ratio is the smaller?

a 3 : 4 or 5 : 7 **b** 9 : 5 or 26 : 17 **c** 9 : 2 or 7 : 3

4 In a year group at Kimpton school, there are 216 boys and 288 girls. Find the ratio of

a boys to girls **b** girls to pupils **c** girls to boys

5 Lance earns $800 per week and Imran earns $4000 per calendar month. Find the ratio of their yearly earnings.

Assume that there are 52 weeks in a year.

6 A salesman spends $3\frac{1}{2}$ hours of his working day of $10\frac{1}{2}$ hours travelling. Find the ratio of

a the time he is travelling to the time he is not travelling

b the time he is not travelling to the length of his working day.

7 The population of a village rose from 153 to 238. In what ratio did the population increase?

8 The air fare between two American cities is $1635 at high season and $1308 at low season. Find the ratio of the high fare to the low fare.

9 The area of Barbados is 450 km^2 and the area of Jamaica is 10 850 km^2. Find, in its simplest form, the ratio of the area of Barbados to the area of Jamaica.

10 The population of Dominica is 74 000 and the population of Trinidad and Tobago is 1.3 million. Find, in its simplest form, the ratio of the population of Trinidad and Tobago to the population of Dominica.

11 Last year the estimated number of visitors to Grenada was 110 000. The population of Grenada is 90 000. Find, in its simplest form, the ratio of the number of visitors to the size of the population.

Example:

A bonus of $140 000 is to be divided between three employees in the ratio of their salaries. A's salary is $250000, B's salary is $300 000 and C's salary is $150 000. How much does each employee get?

Ratio of salaries = 250 000 : 300 000 : 150 000
= 25 : 30 : 15 = 5 : 6 : 3

First find the ratio of the salaries and simplify it.

5 + 6 + 3 = 14 and 140 000 ÷ 14 = 10 000

A gets 5 × $10 000 = $50 000

B gets 6 × $10 000 = $60 000

C gets 3 × $10 000 = $30 000

12 Divide

a $42 in the ratio 4 : 3 **b** 54 cm in the ratio 7 : 11

13 Divide

a 55 c in the ratio 2 : 3 **b** 3.6 m in the ratio 4 : 5

14 Find two numbers whose sum is 144 and whose ratio is 9 : 7.

15 Find two numbers whose difference is 15 and whose ratio is 4 : 3.

16 Divide 156 mm in the ratio 5 : 4 : 3.

17 Divide 450 g in the ratio 5 : 7 : 13.

18 Divide $98 between Alison, Beryl and Chris in the ratio 2 : 5 : 7.

19 Divide an 80 cm rod into three sections whose lengths are in the ratio 4 : 7 : 9.

20 Divide $88 in the ratio $\frac{1}{2} : \frac{1}{4} : \frac{1}{6}$.

21 The sides of a triangle are in the ratio 5 : 6 : 7. The perimeter of the triangle is 63 cm. Find the lengths of the sides.

22 Three people A, B and C enter into partnership in a business. They agree to find capital in the ratio 4 : 5 : 6.
If C puts in $43 200, calculate
a the total capital required
b how much B invests in the business.

23 Two cricket clubs receive a grant from a sponsor in proportion to their respective memberships. If one has 338 members and the other 494, how would a grant of $11 520 be divided between them?

24 In three consecutive innings an opening batsman scored 185, 111 and 148 runs. Find the ratio of
a the lowest score to the highest score
b the lowest score to the total score
c the total score to the highest score.

25 The ratio of the length to the width of a rectangular carpet is 7 : 4. The perimeter of the carpet is 1232 cm. Find its dimensions.

26 Divide $165 between A, B and C so that A's share is twice B's share and three times C's share.

27 Profits in a business amounting to $2871 are to be divided between the partners John, Ria and Roland in the ratio of the capital they invested. If John invested $5280, Ria $4224 and Roland $2112, how much does Ria receive?

28 An alloy consists of zinc, copper and tin in the ratio of 2 : 7 : 4. Find the amount of each metal in 65 g of alloy.

ABCD MIXED EXERCISE 1

Several answers are given for these questions.

Write down the letter that corresponds to the correct answer.

1 A shop charges VAT at 6% on all sales. What is the total cost of a dress marked $125 + VAT?
A $7.50 **B** $117.50 **C** $132.50 **D** $200

2 Imram bought a table marked $280. He received a discount of 10% by paying cash. How much does he pay for the table?
A $252 **B** $277.20 **C** $280 **D** $308

3 Which of the following numbers is prime?
A 237 **B** 243 **C** 261 **D** 271

4 If n is an even number which of the following must be odd?
A $n + 4$ **B** $n - 4$ **C** $n + 3$ **D** $2n$

5 The HCF of 126 and 231 is
A 14 **B** 7 **C** 21 **D** 42

6 The LCM of 168 and 294 is
A 672 **B** 840 **C** 1008 **D** 1176

7 If $146_n = 83_{10}$ then n is
A 5 **B** 6 **C** 7 **D** 8

8 In the number 234_5 the digit 3 represents the denary number
A 3 **B** 15 **C** 25 **D** 75

9 324_5 written in base 10 is
A 45 **B** 64 **C** 89 **D** 110

10 The binary equivalent of 14_{10} is
A 111_2 **B** 110_2 **C** 1110_2 **D** 1010_2

11 Using the distributive law, $21 \times 4 + 7 \times 21$ is
A 28×25 **B** $28 + 25$ **C** 21×11 **D** $21 + 11$

12 $9 + (2 + 7) = (9 + 2) + 7$ is an example of
A the distributive law **B** the associative law
C the commutative law **D** an identity

13 $5 \times (7 - 3) = 5 \times 7 + 5 \times (-3)$ is an example of
A the distributive law **B** the associative law
C the commutative law **D** an identity

14 $7 + 5 + 3 = 7 + 3 + 5 = 10 + 5 = 15$ is an example of using
A the distributive law **B** the associative law
C the commutative law **D** an identity

15 The fractions $\frac{2}{3}, \frac{5}{8}, \frac{4}{7}$ written in descending order of magnitude are
A $\frac{2}{3}, \frac{5}{8}, \frac{4}{7}$ **B** $\frac{5}{8}, \frac{2}{3}, \frac{4}{7}$ **C** $\frac{2}{3}, \frac{4}{7}, \frac{5}{8}$ **D** $\frac{4}{7}, \frac{2}{3}, \frac{5}{8}$

16 The value of $\left(\frac{4}{9}\right)^{-\frac{3}{2}}$ is
A $\frac{8}{27}$ **B** $\frac{27}{8}$ **C** $-\frac{27}{8}$ **D** $-\frac{8}{27}$

17 The value of $\left(\frac{1}{2}\right)^{-3} \times 3^0$ is
A 24 **B** $\frac{1}{8}$ **C** $\frac{1}{24}$ **D** 8

18 The value of $\frac{4 + 3^3}{4^2 - 3^2}$ is
A 49 **B** 343 **C** 7 **D** -49

19 If $a = 5 \times 10^3$ and $b = 3 \times 10^4$ then $ab =$
A 1.5×10^7 **B** 8×10^7 **C** 1.5×10^8 **D** 15×10^8

20 0.55 written as a fraction in its simplest form is
A $\frac{1}{55}$ **B** $\frac{5}{10}$ **C** $\frac{11}{100}$ **D** $\frac{11}{20}$

21 The number 3 776 890 written as a number correct to 4 significant figures is
A 3777 **B** 3 777 000 **C** 3 776 900 **D** 37 776 890

22 \$4.50 as a percentage of \$9 is
A 5% **B** 45% **C** 50% **D** 90%

23 When \$45 is divided in the ratio 4 : 5, the smaller part is
A \$4 **B** \$20 **C** \$25 **D** \$30

24 $27 \times 38.9 = 1050.3$ so $2.7 \times 0.389 =$
A 0.10503 **B** 1.0503 **C** 10.503 **D** 105.03

INVESTIGATION

1 A perfect number is defined as a number whose factors (apart from the number itself) add up to the number.

For example $6 = 1 + 2 + 3$

and $496 = 1 + 2 + 4 + 8 + 16 + 31 + 62 + 124 + 248$. There is another perfect number between 6 and 496. Can you find it?

2 Given that n is a composite number, investigate whether $2^n - 1$ is also a composite number.

MATHS IS OUT THERE

Euclid, 300 BC, is best known for his work on geometry but he also was responsible for the first recorded written study of perfect numbers.

IN THIS CHAPTER YOU HAVE SEEN THAT...

- the factors of a number include 1 and the number itself
- a prime number is a whole number that has only two different factors: itself and 1. All other numbers, except 1, are called composite numbers
- we can find the highest common factor (HCF) of two or more numbers by finding the highest product common to the products of prime factors of all of the numbers
- the multiples of a number include the number itself
- we can find the least common multiple (LCM) of two or more numbers by starting with the prime factors of the smallest number then multiplying them by the factors of the other numbers that are not already included
- the position of a digit in a number tells you its value. In the number 437_{10} the 7 stands for 7×10^0, the 3 for 3×10^1 and the 4 for 4×10^2. For the number 321_4 the 1 stands for 1×4^0, the 2 for 2×4^1, the 3 for 3×4^2
- addition and multiplication of numbers are commutative and associative, e.g. $3 + 7 = 7 + 3$, $4 \times 8 = 8 \times 4$ and $1 + 2 + 3 = 3 + 3 = 1 + 5$, $2 \times 3 \times 4 = 6 \times 4 = 2 \times 12$. Subtraction and division are not commutative or associative
- to multiply powers of the same number, add the indices. To divide powers of the same number subtract the indices, e.g. $4^2 \times 4^3 = 4^5$ and $5^5 \div 5^2 = 5^3$. Note that $3^{-1} = \frac{1}{3}$, $4^0 = 1$, and $(5^2)^3 = 5^{2 \times 3}$
- we can convert a fraction to a decimal by dividing the numerator by the denominator. The decimal will be either exact or recurring
- you can convert a decimal to a fraction by writing it as a number of tenths, hundredths, thousandths, ... $0.215 = \frac{215}{1000}$
- the first significant figure in a number is the first non-zero digit reading from left to right, e.g. for 75640 the first significant figure is 7, the second significant figure is 5, and so on

- a number in scientific notation is a number between 1 and 10 multiplied by a power of 10, e.g. 3.1×10^2 is in scientific notation but 32×10^3 is not
- we can convert percentages to fractions or decimals by dividing the percentage by 100 and removing the % sign. We do the opposite to convert a fraction or a decimal to a percentage
- a ratio compares two or more related quantities. We can multiply or divide both sides of a ratio by another number without affecting its value, and we can omit the units provided they are both the same, e.g. 0.5 m to 30 cm = 50 cm : 30 cm = 5 : 3
- to divide a quantity in a given ratio, divide the quantity by the sum of the numbers in the ratio, then multiply the result by each number in the ratio to give the parts.

2 SETS

AT THE END OF THIS CHAPTER YOU SHOULD BE ABLE TO...

1 Describe a set in words and in set builder notation.
2 List the members of a set from a given description.
3 Know the meaning of equivalent sets, equal sets and the empty set.
4 Find the cardinal number of a set.
5 Distinguish between finite and infinite sets.
6 Find subsets and calculate the number of subsets of a given set.
7 Find the complement of a given set.
8 Find the union and intersection of two sets.
9 Use Venn diagrams.

MATHS IS OUT THERE

Most discoveries in mathematics are part of a long process where ideas evolve until inspiration gives an important theory.

Set theory is different. It is the creation of one person, Georg Cantor, 1845–1918.

BEFORE YOU START

you need to know:

✓ the meaning of natural numbers, whole numbers, integers and irrational numbers
✓ the meaning of even, odd and prime numbers
✓ the meaning of inequalities
✓ how to form and solve simple linear equations
✓ how to substitute values into an algebraic equation.

KEY WORDS

cardinal number, complement, disjoint, element, intersection, subset, union, universal set, Venn diagram

Describing a set

A set is a clearly defined collection of different things having something in common.

The things in the set are called members or **elements**.

We can describe a set in words.

We can also describe a set by listing the elements enclosed in curly brackets.

For example, the set of even numbers between 1 and 9 can also be described as {2, 4, 6, 8}. The members of a set can be written in any order so {2, 6, 8, 4} is the same set.

Another way of describing a set is to write $\{x: 1 \leqslant x < 5, x \in \mathbb{N}\}$.

This means the values of x such that x is greater than or equal to 1 and less than 5 and is a member of the set of natural numbers.

This is called set builder notation. The set it describes is {1, 2, 3, 4}.

Definitions and notation

- The symbol $\in$ means 'is a member of' and the symbol $\notin$ means 'is not a member of'.

> $2 \in \{1, 2, 3, 4\}$ but
> $5 \notin \{1, 2, 3, 4\}$

- The number of members in a set is written $n(A)$. n is called the **cardinal number** of the set.

> When $A = \{2, 4, 6, 8)$, $n(A) = 4$.

- Two sets are equal if they contain exactly the same elements.
- Two sets are equivalent if they contain the same number of elements, i.e. if their cardinal numbers are the same.

> The sets {1, 2, 3, 4} and {a, b, c, d} are equivalent but not equal.

- A set with no members is called the empty set. It is denoted by Ø or {}.
- A set is finite when all the members can be written down.
- A set is infinite when all the members cannot be written down.

> {prime numbers} is infinite, whereas {days of the week} is finite.

- A set that contains all the elements of the sets under consideration is called the **universal set** and is written as U or $\mathscr{E}$.

> If A = {knife, teaspoon, fork}, U could be {cutlery}

- The **complement** of a set A is all the members in the universal set that are not members of A. The complement of A is written as A'.
- A set B is a **subset** of a set A if all the members of B are also members of A. B is a proper subset of A when B does not contain all the members of A, in which case we write $B \subset A$. If B contains all the members of A, we write $B \subseteq A$.

> $C \supset D$ means D is a subset of C.

- The set of natural numbers, i.e. {1, 2, 3, ...} is denoted by $\mathbb{N}$.
- The set of whole numbers, i.e. {0, 1, 2, 3, ...} is denoted by $\mathbb{W}$.
- The set of integers, i.e. {..., −1, 0, 1, ...} is denoted by $\mathbb{Z}$.
- The set of **rational numbers** includes the natural numbers, the whole numbers, the integers and fractions $\frac{a}{b}$ where a and b are integers. The rational numbers are denoted by $\mathbb{Q}$.
- The set of **real numbers** includes the real and irrational numbers and is denoted by $\mathbb{R}$.
  $\mathbb{R} \subset \mathbb{Q} \subset \mathbb{Z} \subset \mathbb{W} \subset \mathbb{N}$

EXERCISE 2a

1 U = {integers bigger than 10 but smaller than 40}
A = {prime numbers}
B = {integers exactly divisible by 3 and 4}
C = {factors of 24}
List the sets A, B and C.

2 **a** List the members of the following sets.
$A = \{x : x$ such that x is a factor of 48}
$B = \{x : x$ is a prime number between 10 and 30}
$C = \{x : x$ is a vowel in the English alphabet}
b Find the number of elements in each set.

3 List the members of the following sets.
$P = \{x : -3 < x \leqslant 10$, where x is an integer$\}$
$Q = \{x : -3 < x < 2, x \in \mathbb{Z}\}$
$R = \{x : x \leqslant 12, x \in \mathbb{N}\}$

Remember that
$\mathbb{N} = \{1, 2, 3, ...\}$ and
$\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$

4 **a** List any six members of the set $\{x : x = 2n, n \in \mathbb{N}\}$
b Is the set finite or infinite?

5 List the members of the set
$A = \{(x, y) : y = 2x, -2 \leqslant x < 2, \in \mathbb{Z}\}$

Example:

$A = \{1, 2, 3, 4\}$

a Write down all the subsets of A.

b How many subsets does A have?

c If U = {the natural numbers} describe the complement of A.

d Is A' finite or infinite? Give a reason for your answer.

a $\{\emptyset\}$

$\{1\}, \{2\}, \{3\}, \{4\},$

$\{1, 2\}, \{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}, \{3, 4\}$

$\{1, 2, 3\}, \{1, 2, 4\}, \{1, 3, 4\}, \{2, 3, 4\}$

$\{1, 2, 3, 4\}$

b 16

c A' = {all the natural numbers except 1, 2, 3 and 4}

d Infinite because there are an infinite number of natural numbers and removing four of them still leaves an infinite number.

Do this in an organised way to make sure you do not miss any. The empty set is a subset of any set, then start with the sets with one member, then the sets with two members and so on. Remember that the order does not matter, so {1, 2} is the same as {2, 1}.

6 Find $n(A)$ where $A = \{x : x = \sqrt{n},\ 0 < n < 30, x \in \mathbb{Z}\}$

7 $U = \{x : 1 < x < 16, x \in \mathbb{N}\}$
A = {even numbers}, B = {multiples of 3},
C = {prime numbers} and D = {irrational numbers}
a List the sets A, B, C and D.
b Find the number of elements in each set.

8 Are the following sets finite or infinite?
a A = {odd numbers}
b B = {cricketers in Tobago}
c C = {prime numbers}
d D = {consonants}

9 Find the number of elements in each set.
a {cricketers on the field when an innings is about to start}
b {players in a quartet}

10 P = {prime numbers bigger than 2 but less than 20}
a Find $n(P)$.
b Write down the subset of P whose elements are all odd numbers. Is this a proper subset of P?
c Write down the subset of P whose elements are even numbers. What special name is given to this set?

11 Give the complement of each of the following sets.

a $A = \{2, 4, 6, 8\}$ if $U = \{\text{whole numbers less than } 10\}$

b $B = \{\text{Friday, Saturday, Sunday}\}$ if $U = \{\text{days of the week}\}$

c $P = \{\text{pupils in my maths class}\}$ if $U = \{\text{pupils in my school}\}$

d $Q = \{\text{odd numbers}\}$ if $U = \{\}$

12 $U = \{\text{whole numbers less than } 10\}$
Give the complement of each of the following sets.

a $P = \{1, 3, 5, 7\}$

b $Q = \{\text{numbers smaller than } 6\}$

c $R = \{\text{numbers of } U \text{ which give a whole number when divided by } 3\}$

d $S = \{\text{numbers of } U \text{ greater than } 10\}$

13 The universal set, U, is given as $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.
The sets A and B are subsets of U such that $A = \{\text{factors of } 10\}$ and $B = \{\text{multiples of } 3\}$.

a List the members of set A.

b List the members of the complement of B.

c Find $n(A')$.

d Find a subset of A with a least two members.

14 Give a subset with at least three members for each of these sets.

a {Caribbean countries} **b** {cricketing countries}

c {insects} **d** {pupils in your class}

15 a i If $A = \{2, 4, 6, 8, 10\}$ and $A' = \{1, 3, 5, 7, 9\}$ what is U?

ii How many subsets does A have with 4 members?

b i If $B = \{\text{homes with a computer}\}$ and
$B' = \{\text{homes without a computer}\}$ what is U?

ii Is B' a finite or infinite set?

c If $C = \{r, s, t, u, v\}$ and $C' = \{w, x, y, z\}$ what is U?

16 $U = \{1, 2, 3, 4, ..., 9, 10\}$, $A = \{1, 2, 3, 4, 5, 6\}$ and $B = \{\text{factors of } 6\}$
Which of the statements are true?

a $B \subset A$ **b** $B \subseteq A$ **c** $B' \subset A$

d $7 \in B'$ **e** $6 \notin A'$

17 List all the subsets of $P = \{2, 4, 9, 16\}$.

18 $U = \{0 < x < 10, x \in \mathbb{R}\}$
Which of the following sets are in U?
$A = \{2, \sqrt{3}, \pi\}$, $B = \{4, 6.5, 10\}$, $C = \{1.5, 2.8, \sqrt{11}\}$

19 $U = \{0 < x < 10, x \in \mathbb{Q}\}$
Which of the following sets are in U?
$A = \mathbb{Z}$, $B = \mathbb{N}$, $C = \mathbb{R}$

Why is the empty set a subset of every set?
HINT: either it is or it isn't.

$\{1, 2, 3, 4\} \cup \{2, 4, 6, 8\} = \{1, 2, 3, 4, 6, 8\}$
Notice that 2 and 4 are in both sets but we only include one 2 and one 4 in the union.

$\{1, 2, 3, 4\} \cap \{2, 4, 6, 8\} = \{2, 4\}$

Union and intersection

The **union** of two sets A and B is the set that contains the members that are either in A or in B.

We write the union of A and B as $A \cup B$.

The intersection of two sets A and B is the set that contains the members that are in both A and B.

We write the intersection of A and B as $A \cap B$.

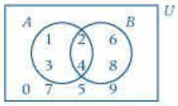
Venn diagrams

A **Venn diagram** is a way of illustrating sets.

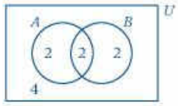
The **universal set** is represented by a rectangle and the subsets of the universal sets are represented by closed curves.

A Venn diagram can show either the actual members or the number of members in each region.

These Venn diagrams illustrate the sets
U = {whole numbers less than 10}
and $A = \{1, 2, 3, 4\}$
and $B = \{2, 4, 6, 8\}$.

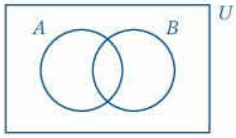


This shows the actual members.

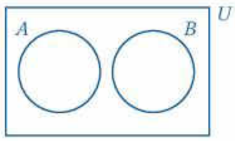


This shows the number of members.

When sets A and B have at least one member in common, the Venn diagram looks like this:

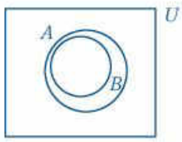


When two sets have no members in common, the Venn diagram looks like this:
These sets are called **disjoint**.



There are no members common to A and B, so $A \cap B = \varnothing$.

When one set is a proper subset of the other, the Venn diagram looks like this:



All the members of B are also in A, so $B \subset A$.

EXERCISE 2b

1 $A = \{2, 3, 4\}$ and $B = \{3, 4, 5\}$
List the members in $A \cup B$ and $A \cap B$.

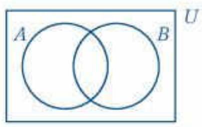
2 If P = {letters in the word ALGEBRA} and Q = {letters in the word GEOMETRY}, list the members in $P \cup Q$ and $P \cap Q$.

TIP List the members first.

3 $L = \{p, q, r\}$ and $M = \{q, r, s, t\}$
Find the value of $n(L \cup M)$ and $n(L \cap M)$.

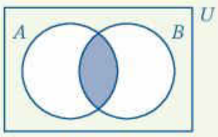
Example:

Use copies of the Venn diagram on the right to shade in the regions representing

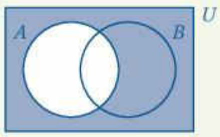


a $A \cap B$ **b** A' **c** $A \cup B$ **d** $A \cup B'$ **e** $A' \cap B'$ **f** $(A \cup B)'$

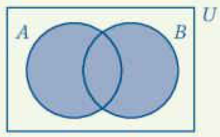
a



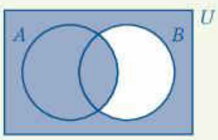
b



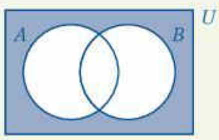
c



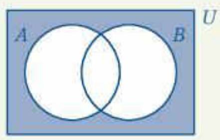
d



e



f



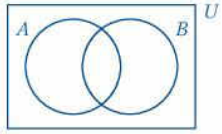
Parts **e** and **f** show that $A' \cap B' = (A \cup B)'$

$A \cup B'$ means the members in A together with those not in B.

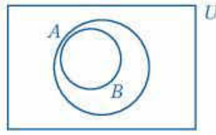
$A' \cup B'$ means the members that are not in A and also not in B.

$(A \cup B)'$ means the members that are not $A \cup B$

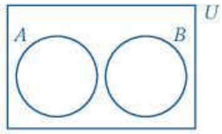
4 Use copies of this Venn diagram to shade in the regions representing
a B' **b** $A' \cap B$ **c** $A \cap B'$



5 Use copies of this Venn diagram to shade in the regions representing
a $A \cap B$ **b** $A \cup B$ **c** B'

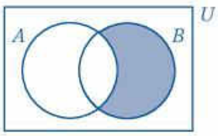


6 Use copies of this Venn diagram to shade the regions representing
a $A \cup B$ **b** A' **c** $(A \cap B)'$

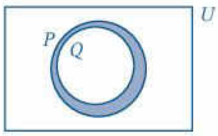


7 Describe the shaded region in each diagram in terms of the sets shown.

a



b



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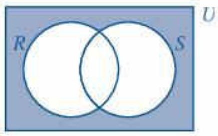


Example:

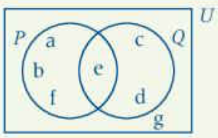
$U = \{a, b, c, d, e, f, g\}$, $P = \{a, b, e, f\}$ and $Q = \{c, d, e\}$

a Draw a Venn diagram showing the sets and their elements.

b List the members of the set $P \cap Q$.

c Write down the value of $n(P \cup Q)$.

a



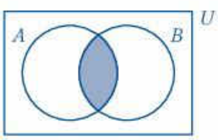
e is in P and Q so the circles overlap and e is in the overlapping region.

b $P \cap Q = \{e\}$ **c** $n(P \cup Q) = 6$

$n(P \cup Q)$ is the number of elements in union of P and Q.

8 $U = \{x: 1 \leqslant x \leqslant 10, x \in \mathbb{N}\}$,
$A = \{\text{multiples of } 2\}$ and $B = \{\text{multiples of } 3\}$
Write down the elements of the sets represented by each shaded region.

a



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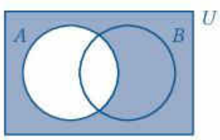

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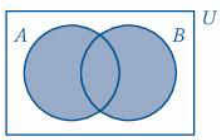
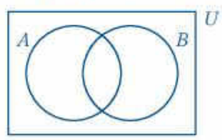


Start by entering the members of the sets in



9 $U = \{a, b, c, d, e, f, g\}$, $P = \{c, e, f, g\}$ and $Q = \{a, c, f\}$
Show U, P and Q on a Venn diagram entering all the members.
Hence list the sets
a P' **b** Q' **c** $P \cup Q$
d $P \cap Q'$ **e** $P' \cap Q'$ **f** $(P \cup Q)'$

10 $U = \{1, 2, 3, 4, \ldots, 15\}$, $A = \{\text{factors of } 12\}$ and $B = \{\text{multiples of } 3\}$
a Show U, A and B on a Venn diagram marking all the elements.
b List the elements of the set **i** $A \cap B$ **ii** $(A \cup B)'$.
c Find **i** $n(A \cap B)$ **ii** $n(A \cup B)'$

11 $U = \{1, 2, 3, 4, 5, 6, 7\}$, $A = \{\text{even numbers}\}$, $B = \{\text{prime numbers}\}$
  a Illustrate these sets and their elements on a Venn diagram.
  b List the members of **i** $A \cap B$ **ii** $A \cup B$.

12 $U = \{\text{students in my class}\}$, $A = \{\text{those who play cricket}\}$ and $B = \{\text{those who play football}\}$
  Describe the following sets.
  a A' **b** B' **c** $(A \cup B)'$ **d** $A' \cap B'$

13 $U = \{\text{different letters in the word GEOGRAPHY}\}$
  $A = \{\text{different letters in the word GREY}\}$
  $B = \{\text{different letters in the word GRAPH}\}$
  Show U, A and B on a Venn diagram entering all the elements.
  Hence list the sets
  a A' **b** B' **c** $A \cup B$
  d $(A \cup B)'$ **e** $A' \cup B'$

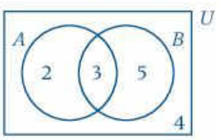
14 $U = \{\text{different letters in the word GOVERNMENT}\}$
  $P = \{\text{different letters in the word TORN}\}$
  $Q = \{\text{different letters in the word OVERT}\}$
  Show U, P and Q on a Venn diagram entering all the elements.
  Hence list the sets
  a P' **b** Q' **c** $P \cup Q$
  d $(P \cup Q)'$ **e** $P' \cup Q'$

15 $U = \{x: 0 \leqslant x < 7, x \in \mathbb{Z}\}$, $P = \{\text{prime numbers}\}$ and $Q = (\text{odd numbers})$
  a Find P', Q', $P' \cup Q'$, $P' \cap Q'$ and $(P \cup Q)'$
  b Is it true that $P \subset Q$? Give a reason for your answer.

16 Draw a Venn diagram to illustrate each of the following.
  a $A \subset B$ **b** $C \supset D$ **c** $P \cap Q = \varnothing$ **d** $R' \cap S = \varnothing$

Use the information given in the following Venn diagrams to find $n(A)$, $n(B)$, $n(A')$, $n(B')$, $n(A \cup B)$, $n(A \cap B)$, $n(A' \cup B')$ and $n(A \cap B)'$ for each of the given pairs of sets.

17



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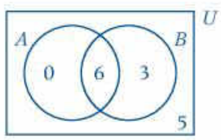

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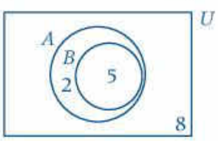

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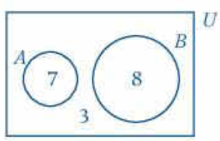
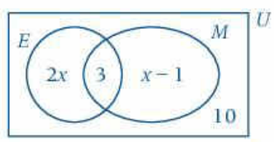


Example:

The Venn diagram shows the number of pupils studying mathematics (M) and economics (E) in a class of 30.

a Use the information to form an equation in terms of x.

b Find the number of pupils studying economics.



a $2x + 3 + x - 1 + 10 = 30$

$3x + 12 = 30$

The numbers in each region add up to give $n(U)$, i.e. the number in the class.
This gives the equation. Simplify the left-hand side.

b $x = 6$

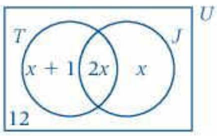
The number of students studying economics is $2x + 3$.

When $x = 6$, $2x + 3 = 15$, so 15 students are studying economics.

Solve the equation to find x. From the Venn diagram, the number studying economics is $2x + 3$.

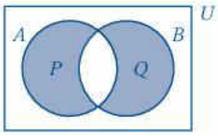
21 The Venn diagram shows the number of students able to tango (T) and able to jive (J) in a group of 25.

a Form an equation in x and solve it.
b How many students can jive?
c How many students cannot tango?



22 In the Venn diagram $P \subset A$ and $Q \subset B$.

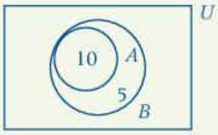
a $n(A) = 42$, $n(B) = 31$, $n(A \cap B) = x$
Find, in terms of x, the values of $n(P)$ and $n(Q)$.
b If $n(A \cup B) = 60$, find the value of x.



Example:

A and B are two sets and $A \subset B$. $n(A) = 10$ and $n(B) = 15$.

Find the value of **a** $n(A \cap B)$ **b** $n(A \cup B)$.



Start by drawing a Venn diagram. A is a proper subset of B, so the circle showing A is inside the circle showing B. There are 10 members in A and 15 in B, so there are 5 members in B that are not in A.

a $n(A \cap B) = 10$

$A \cap B$ is the set of members common to A and B. This is A.

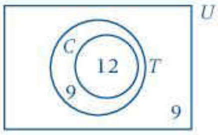
b $n(A \cup B) = 15$

$A \cup B$ is the set of elements in either A or B. This is B.

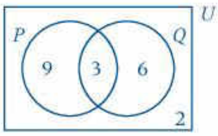
23 P and Q are two sets such that $P \subset Q$, $n(P) = 7$ and $n(Q) = 18$.
Find the value of **a** $n(P \cap Q)$ **b** $n(P \cup Q)$.

24 This Venn diagram shows that in a group of 30 students 12 live in a home with a car and a telephone and 9 live in a home with a telephone but no car.
Find

a $n(C \cup T)$ **b** $n(C')$ **c** $n(T')$ **d** $n(C \cap T)$.



25 Use the information in the Venn diagram to find $n(P)$, $n(Q)$, $n(P \cup Q)$, $n(U)$, $n(P' \cap Q)$, $n(P' \cup Q)$ and $n(P' \cup Q')$.

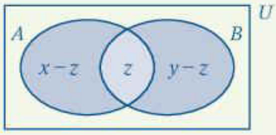


Example:

Prove that, for any two sets A and B, $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

Let $n(A) = x$, $n(B) = y$ and $n(A \cap B) = z$.

The Venn diagram illustrating this is



From the diagram, $n(A \cup B) = x - z + z + y - z$

$= x + y - z$

$= n(A) + n(B) - n(A \cap B)$

The numbers of elements in A and B can be any numbers, so choose letters to represent these numbers. When you give a proof you must explain your reasoning.

You can use the result $n(A \cup B) = n(A) + n(B) - n(A \cap B)$, without proof, to solve problems.

Given $n(A) = 5$, $n(B) = 4$ and $n(A \cup B) = 6$, find $n(A \cap B)$:

Substituting in
$n(A \cup B) = n(A) + n(B) - n(A \cap B)$
gives $6 = 5 + 4 - n(A \cap B)$, so $n(A \cap B) = 3$

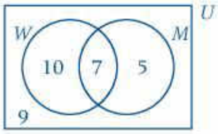
26 A and B are two sets such that $n(A) = 10$, $n(B) = 3$ and $n(A \cup B) = 1$. Find $n(A \cup B)$.

27 A and B are two sets such that $n(A) = 15$, $n(B) = 6$ and $n(A \cap B) = 2$. Find $n(A \cup B)$.

28 A and B are two sets such that $n(A) = 3x$, $n(B) = 2x$, $n(A \cap B) = 2$ and $n(A \cup B) = 8$. Find the value of $n(A)$.

29 The students in a class were asked if they did woodwork (W) or metalwork (M) in school. Their answers are shown in the Venn diagram.

- a How many students are there in the class?
- b How many students did woodwork?
- c How many students did one but not both of these subjects?
- d How many students did not do metalwork?



30 A group of television viewers were asked if they watched documentary and comedy programmes. The replies showed that 27 watched both kinds of programmes but 7 watched comedy programmes only. All 40 watched either documentary or comedy programmes. Show this information on a Venn diagram and use it to find the numbers of viewers who

- a watched documentaries
- b did not watch comedy programmes
- c watched either documentaries or comedy programmes but not both.

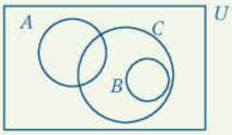
Harder problems

You can use the ideas in this chapter for problems involving three sets.

EXERCISE 2c

Example:

A, B and C are three sets such that $B \subset C$, $A \cap C \neq \varnothing$ and $A \cap B = \varnothing$.
Draw a Venn diagram to show the relationships between A, B and C.

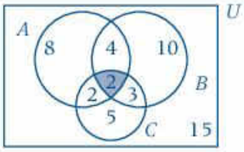


$B \subset C$ tells you that the circle for B is inside C.
$A \cap B = \varnothing$ tells you that A and B have no members in common so their circles do not overlap.
$A \cap C \neq \varnothing$ tells you that A and C do have members in common so their circles do overlap.

1. A, B and C are three sets such that $A \cap C = \varnothing$, $A \cap B \neq \varnothing$ and $B \cap C \neq \varnothing$. Draw a Venn diagram to show the relationships between A, B and C.
2. A, B and C are three sets such that $C \subset B$ and $A \cap B = \varnothing$. Draw a Venn diagram to show the relationships between A, B and C.
3. A, B and C are three sets such that $A \cap B = \varnothing$, $A \cap C = \varnothing$ and $B \cap C = \varnothing$. Draw a Venn diagram to show the relationships between A, B and C.
4. A, B and C are three sets such that $A \cap B \neq \varnothing$, and $(A \cap B) \cap C \neq \varnothing$. Draw a Venn diagram to show the relationships between A, B and C.
5. A, B and C are three sets such that $B \subset A$ and $C \subset B$. Draw a Venn diagram to show the relationships between A, B and C.

Example:

The Venn diagram shows the sets A, B and C.



a State the relationship shown by the shaded region.

b Find the value of **i** $n(A)$ **ii** $n(A \cup B)$ **iii** $n((A \cup B) \cap C')$.

a The shaded region shows the members common to A and B and C so it shows $(A \cap B) \cap C$.

b i $n(A) = 8 + 4 + 2 + 2 = 16$

$n(A)$ = the number of elements inside the circle showing A.

ii $n(A \cup B) =$
$8 + 4 + 10 + 2 + 2 + 3 = 29$

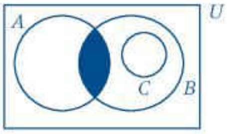
$n(A \cup B)$ = the number of elements in A and B combined.

iii $n((A \cup B) \cap C') = 8 + 4 + 10 = 22$

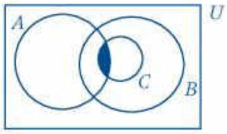
$n(A \cup B) \cap C')$ = number of elements common to (the union of A and B) and those not in C.

6 Each of these Venn diagrams show the relationships between three sets. State the relationship shown by the shaded region.

a



b



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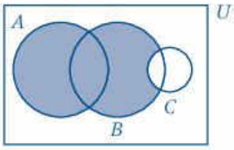

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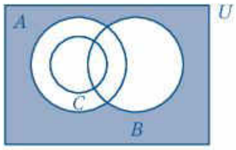


7 Given that $U = \{a, b, c, d, e, f, g\}$, $A = \{a, b, c\}$, $B = \{e, f, g\}$ and $C = \{b, c\}$

a draw a Venn diagram showing the sets U, A, B and C and their members

b list the members of the set A'

c list the members of the set $(A \cup C) \cup B$.

8 Given that $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$, $A = \{3, 5, 7\}$, $B = \{\text{even numbers}\}$ and $C = \{\text{prime numbers}\}$

a draw a Venn diagram showing the sets U, A, B and C and their members

b write down the value of $n((A \cup B)' \cap C)$

c list the members of $(A \cap B') \cup C$.

9 A, B and C are subsets of the universal set $U = \{1, 2, 3, 4, \ldots, 20\}$.

$A = \{x : x \text{ is a factor of } 30\}$

$B = \{x : x \text{ is a factor of } 48\}$

$C = \{x : x \text{ is a factor of } 72\}$

a Illustrate U, A, B and C on a Venn diagram.

b List the elements of the sets $A \cap B \cap C$.

10 $U = \{A, B, C, D, E, F, L, M, N, O, R, S, T\}$

A = {different letters in the word REFLECT}

B = {different letters in the word ROTATE}

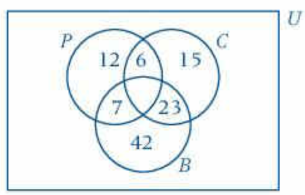
C = {different letters in the word TRANSLATE}

a Show these sets on a Venn diagram entering every element.

b List the sets **i** $B \cup C$ **ii** $B \cap C$ **iii** $(B \cup C) \cap A$.

c Find **i** $n(B \cup C)'$ **ii** $n((B \cup C)' \cup A)$.

11 The Venn diagram shows the number of students in a year group who study physics (P), chemistry (C), biology (B) or a combination of these. There are 120 students altogether and every student studies at least one of these subjects.



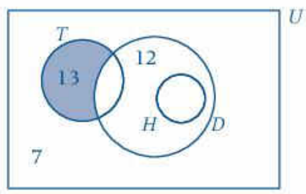
How many students

a study all three subjects?

b study exactly two subjects?

c do not study biology?

12 This Venn diagram shows the numbers of members of an athletics club that participate in long distance running (D), hurdling (H) and track events (T). The total membership is 44.



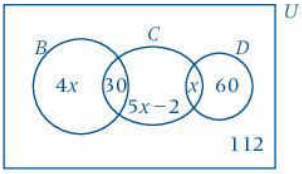
a State the relationship shown by the shaded region in the diagram.

b State the relationship between T and H.

c The same number, x, participate in hurdling as participate in both track and long distance running. Find the number who participate in hurdling.

TIP Enter x in the correct places in the diagram and then form an equation

13 The Venn diagram shows the numbers of pupils in the basketball club (B), the cricket club (C) and the diving club (D) out of a school of 500 pupils.



a Find an expression in terms of x for the number of pupils who are members of at least one of these clubs.

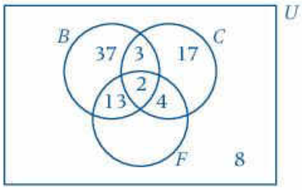
b Find the number of pupils who are only in the cricket club.

14 In a sports club with 200 members, $4x$ members only swim, $3x$ members only use the fitness equipment and $5x$ members both swim and use the fitness equipment. There are 20 members who do not use the equipment or swim.

a Draw a Venn diagram showing this information.

b Find the number of members who only swim.

15 100 customers in a supermarket were asked how they had got there. Was it by car (C), by bus (B), on foot (F), was it in some other way or was it a combination of these? The results of the survey are shown in the Venn diagram.



a How many customers got there on foot only?

b How many customers used a bus?

c How many customers did not use a car?

d How many customers used exactly two modes of transport?

e Suggest a way in which some customers could have come which was different from the three ways named.

16 At a vegetable show 61 people entered exhibits. 11 showed carrots only, 9 showed potatoes only and 5 showed parsnips only. 6 showed carrots and potatoes only, 11 showed potatoes and parsnips only and 7 showed carrots and parsnips only. If the number who showed all three vegetables is half the number who didn't show any of these, how many showed potatoes?

17 The businesses on an industrial estate own an estate car, a van, a lorry or a combination of these.
13 have an estate car only, 12 have a van only and 8 have a lorry only, 4 have an estate car and a van only, 2 have a van and a lorry only and 1 has an estate car and a lorry only.
There are 49 businesses on the estate and the number who do not have a vehicle is twice the number who have all three.
How many businesses

a have one of each type of vehicle?

b have more than one vehicle?

c do not have a lorry or a van?

18 A works canteen provides ham, eggs and baked beans for lunch.
80 workers chose at least one of these, 65 chose ham,
43 chose eggs, 31 chose ham and eggs, 38 baked beans and ham and 22 eggs and baked beans.
15 chose all three.
How many chose

a ham only? **b** eggs only? **c** baked beans only?

19 In a survey of 60 boys each one was found to read one or more of a car magazine (C), a bodybuilding magazine (B) or a sports magazine (S). The following facts were revealed by the survey.
32 boys read a car magazine, 28 read a bodybuilding magazine and 34 read a sports magazine.
The same number of boys read a car magazine and a sports magazine as read a car magazine and a bodybuilding magazine.
Half as many read both a bodybuilding magazine and a sports magazine as read both a car magazine and a sports magazine.
Ten read a car magazine and a bodybuilding magazine but not a sports magazine.
Using all this information and a suitable Venn diagram find how many boys read

a a car magazine only
b a bodybuilding magazine only
c a sports magazine only.

ABCD MIXED EXERCISE 2

Several answers are given for these questions.

Write down the letter that corresponds to the correct answer.

1 If $U = \{1, 2, 3, ..., 10\}$ and $P = \{1, 2, 3, 5, 7\}$ then P' is
A $\{8, 9, 10\}$ **B** $\{4, 6\}$ **C** $\{4, 6, 8, 9, 10\}$ **D** $\{4, 5, 6, 7, 8, 9, 10\}$

2 The set of numbers greater than -1 but less than 4 may be written
A $\{x: -1 > x > 4\}$ **B** $\{x: -1 < x < 4\}$
C $\{x: -1 \leqslant x < 4\}$ **D** $\{x: -1 \leqslant x \leqslant 4\}$

3 If $P = \{2, 5, 6, 7\}$, $Q = \{2, 5, 9, 10\}$ and $S = \{4, 5, 6, 10\}$ then $P \cap Q \cap S$ is
A $\{\ \}$ **B** $\{5\}$ **C** $\{2, 5\}$ **D** $\{5, 10\}$

4 The set of numbers greater than -2 but less than or equal to 6 may be written
A $\{x: -2 > x > 6\}$ **B** $\{x: -2 < x < 6\}$
C $\{x: -2 < x \leqslant 6\}$ **D** $\{x: -2 \leqslant x \leqslant 6\}$

5 The set P defined by $P = \{x: 3 < x \leqslant 7\}$ means that the elements of P are
A greater than or equal to 3 but less than 7
B greater than 3 but less than or equal to 7
C greater than 3 but less than 7
D less than 3 and less than 7

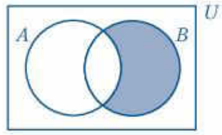
6 If U = {whole numbers less than 15}, A = {prime numbers} and B = {odd numbers} then $A \cap B$ is
A $\{1, 3, 5, 15\}$ **B** $\{3, 5, 7, 13\}$
C $\{3, 5, 7, 13, 15\}$ **D** $\{3, 5, 7, 11, 13\}$

7 If U = {whole numbers less than 13}, A = {multiples of 3} and B = {multiples of 4} then $(A \cap B)'$ is
A $\{12\}$ **B** $\{4, 8, 12\}$
C $\{3, 6, 9, 12\}$ **D** $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$

8 If U = {whole numbers less than 15}, A = {factors of 12} and B = {even numbers} then $(A \cup B)'$ is

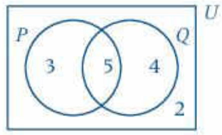
A { 5, 7, 9, 11, 13} **B** {1, 2, 3, 4, 6, 12}
C {2, 4, 6, 8, 10, 12, 14} **D** {1, 2, 3, 4, 5, 6, 7, 9, 11, 12, 13}

9 The region shaded on this Venn diagram represents



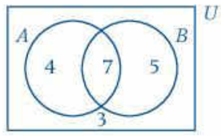
A $A' \cap B$ **B** $A \cap B'$ **C** $(A \cup B)'$ **D** $A' \cup B'$

10 On this Venn diagram the value of $n(P \cup Q)'$ is



A 2 **B** 3 **C** 4 **D** 5

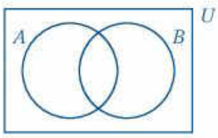
11 On this Venn diagram the value of $n(A' \cap B)$ is



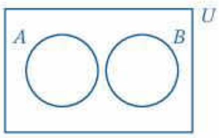
A 8 **B** 7 **C** 5 **D** 4

12 A and B are two sets such that $A \subset B$.
Which of these Venn diagrams shows these sets?

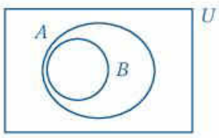
A



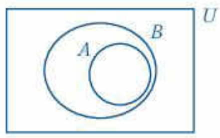
B



C



D



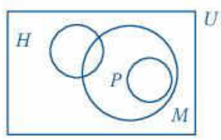
13 If $P = \{1, 2, 3, 4\}$ and $Q = \{2, 3, 8\}$, then $P \cap Q =$

A { } **B** {2}
C {2, 3} **D** {1, 2, 3, 4, 7, 8}

14 A and B are two sets such that B is a subset of A. $n(A) = 6$ and $n(B) = 4$. $n(A \cup B) =$

A 2 **B** 4 **C** 6 **D** 10

15 The Venn diagram shows the set of students studying mathematics (M), the set of students studying history (H) and the set of students studying physics (P).
Which of the following deductions can be made?



1. No student studies physics and history.
2. Some students study both mathematics and history.
3. Some students study physics and chemistry.

A none **B** 1 **C** 1 and 2 **D** 1, 2, and 3

16 A and B are two sets such that $n(A) = 10$, $n(B) = 3$ and $(A \cap B) = \varnothing$. $n(A \cup B) =$

A 3 **B** 7 **C** 10 **D** 13

17 A is the set $\{x : x < 5, x \in \mathbb{N}\}$. $A =$

A {2, 3, 4} **B** {1, 2, 3, 4} **C** {0, 1, 2, 3, 4} **D** {1, 2, 3, 4, 5}

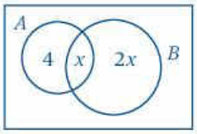
18 Which of the following sets is a proper subset of {w, x, y, z}?

A {w, z} **B** {a, b, c} **C** {u, w, x} **D** {w, x, y, z}

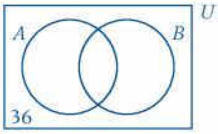
19 If $A = \{0, 1\}$, the number of proper subsets of A is

A 1 **B** 2 **C** 3 **D** 4

20 A and B are two sets such that $n(A) = 10$, $n(B) = 3$ and $n(A \cap B) = 2$.
$n(A \cup B) =$
A 5 **B** 7 **C** 11 **D** 15

21 

$n(A) = n(B)$.
$n(A \cap B) =$
A 1 **B** 2 **C** 3 **D** 4

22 

$n(U) = 50$.
$n(A \cup B) =$
A 14 **B** 24 **C** 36 **D** 86

23 If A = {members of the school cricket team}, B = {members of the school football team} and $(A \cap B) = \varnothing$, which of the following deductions can be made?
1. Some members of the cricket team are also members of the football team.
2. No members of the cricket team are members of the football team.
3. Some members of the cricket team play football.

A 1 only **B** 2 only **C** 1 and 2 **D** 1 and 3

IN THIS CHAPTER YOU HAVE SEEN THAT...

- the cardinal number of a set is the number of members in a set
- two sets are equal if they contain exactly the same elements
- two sets are equivalent if they contain the same number of elements, i.e. if their cardinal numbers are the same
- a set is finite when the elements can be counted. For example, the number of grains of sand on a beach is finite and can in theory, but not in reality, be counted
- a set is infinite when the elements cannot, even in theory, be counted. For example, the set of even numbers is infinite
- the complement of a set A is all the members in the universal set that are not members of A. The complement of A is written as A'
- a set B is a subset of a set A if all the members of B are also members of A. B is a proper subset of A when B does not contain all the members of A, in which case we write $B \subset A$. If B contains all the members of A, we write $B \subseteq A$
- the union of two sets is the set that contains the members that are in either set
- the intersection of two sets is the set that contains the members that are in both A and B
- $A' \cap B' = (A \cup B)'$
- $n(A \cup B) = n(A) + n(B) - n(A \cap B)$.

MATHS IS OUT THERE

Fuzzy set theory is a fairly new development of sets. Membership of a fuzzy set is based on probability. Its main applications are in fuzzy logic such as decision making situations.

Use an internet search engine to find out more about fuzzy logic.

3 ALGEBRA 1

AT THE END OF THIS CHAPTER YOU SHOULD BE ABLE TO...

1 Understand the basic rules of algebra.
2 Simplify algebraic expressions.
3 Substitute numbers into algebraic expressions.
4 Solve linear equations in one unknown.
5 Solve two linear simultaneous equations in two unknowns.
6 Solve a linear inequality in one unknown.
7 Solve problems using algebra.

BEFORE YOU START

you need to know:
✓ how to work with positive and negative numbers
✓ how to simplify numeric fractions
✓ the laws of arithmetic
✓ the laws of indices
✓ the meaning of the set of real numbers
✓ the meaning of commutative, associative and distributive.

KEY WORDS

coefficient, equation, expression, identity, inequality, like terms, simultaneous equations, term, unlike terms, variable

MATHS IS OUT THERE

J.B.S. Haldane (1892–1964), a British geneticist, and populariser of science wrote

'If you are faced by a difficulty or a controversy in science, an ounce of algebra is worth a ton of verbal argument.'

Algebra

Algebra is generalised arithmetic so the rules of arithmetic apply.
It uses letters to represent unknown numbers or quantities or **variables**.

For example, if we do not know the price of a particular computer, we can say the price is $\$x$.
In this case, x represents an unknown, but fixed, number of dollars.
The depth of water in a harbour varies depending on the state of the tide. We can say that the depth is h metres; in this case h is a variable.

Definitions and conventions

- When letters, or numbers and letters, are multiplied together, we leave out the multiplication signs. So $a \times b$ is written as ab and $3 \times x \times y \times z$ is written as $3xyz$. Also, $5pq$ means $5 \times p \times q$. We use indices when a letter is multiplied by itself. So $a \times a$ is written as a^2 and $t \times t \times t$ as t^3.
- An **expression** is any collection of numbers and letters connected by plus and minus signs, for example, $2a + 5b$, $3 - 2xy + x^2$.
- A **term** in an expression is any collection of letters and/or numbers that are not separated by plus or minus signs.

3, $2xy$ and x^2 are the terms in the expression $3 - 2xy + x^2$.

- **Like terms** contain exactly the same combination of letters where the powers of the letters are the same but the numbers can be different. For example, $3xy^2$ and $5xy^2$ are like terms but $3x^2y$ and $5xy^2$ are not. They are **unlike terms**.

- An **equation** is two expressions connected by an equals sign, for example, $2x + 3 = 5xy$.
- An **identity** is an equation where the two expressions are alternative ways of writing the same thing, for example, $3 + x = x + 3$ because the addition of numbers is commutative.
- An **inequality** is two expressions related such that one is greater than, or greater than or equal to, the other.

Simplifying expressions

Algebraic expressions can be simplified by:

- adding or subtracting like terms
- using the distributive law to multiply out a bracket
- cancelling common factors in fractions
- using the laws of indices.

Adding or subtracting like terms is called collecting like terms.

EXERCISE 3a

Example:

Simplify **a** $4x - 2y + 6x - 7$ **b** $a^2(a^2 + 2ab)$

c Show that $2(5x + 4) - 3(x - 7) = 7x + 29$

a $4x - 2y + 6x - 7 = 4x + 6x - 2y - 7$

$= 10x - 2y - 7$

4x and 6x are like terms; they can be added. This is called collecting like terms. No further simplification is possible as there are no other like terms.

b $a^2(a^2 + 2ab) = a^2 \times a^2 + a^2 \times 2ab$

$= a^4 + 2a^3b$

Multiply out the bracket by using the distributive property of multiplication across addition and subtraction. This is called expanding the expression.
Then use the commutative property of multiplication to simplify the terms.
($a^2 \times 2ab = 2 \times a \times a^2 \times b$)

c $2(5x + 4) - 3(x - 7) = 10x + 8 - 3x + 21$

$= 7x + 29$

Start with the left-hand side.
Remember that $-3 \times (-7) = +21$.

Simplify

1 $3x + 4x + 5$	**2** $5x - 2x - 9$
3 $4a + b - 3a + 6b$	**4** $2x - 3y + 7y - x$
5 $6p - 2q + 5 - 2q$	**6** $3a^2 - 2b + 6b - 4$
7 $5p + 3q + 8 - 4p - 3q$	**8** $7x^2 + 9x - 4x + 3$
9 $x^2 + 6x + 36 - 6x$	**10** $x^2(x - 3y)$
11 $a(3a^2b + b^2)$	**12** $a^3(a + b)$
13 $x^2(3x^2 + y^2)$	**14** $a^2b(a^3 + b^3)$
15 $3a^2(2ab - 3b^2)$	**16** $5p^2q(2q^2 - 3p)$
17 $4(x - 3) + 3(x + 1)$	**18** $3(x + 5) + 3(x - 7)$
19 $7(2a - 1) + 2(3a + 5)$	**20** $5(a + 4) - 2(a + 2)$
21 $3(x - 2) - 2(x - 7)$	**22** $3(4x + 1) - 2\,(3x + 2)$
23 $4(5x + 1) - 3(2x - 4)$	**24** $5(2a - 3) - 2(2b - 3)$

25 $8(2p - 1) + 4(3 - 4p)$

26 $3(2x + y) - 2(y + z)$

27 $9(5a - 2) + 7(3 - 4a)$

28 $5(2 - y) - 3(5 - 2y)$

29 $6(5b - 2c) - 3(7b + c)$

30 $4(2m - 3n) - (n - 3m)$

31 $3x(x + y) - 4y(x - y)$

32 $5p(2p - 3q) + 3q(p - q)$

33 $x(x + 3) + x(x - 2)$

34 $2a(a + 4) - a(a + 6)$

Show that

35 $5p(p - 3) - 2p(p - 3) = 3p^2 - 9p$

36 $4a(a - 3) - 3a(a - 4) = a^2$

37 $3a(a - 2) - 5(a + 6) = 3a^2 - 11a - 30$

38 $5x(x + 3) - 4(x - 2) = 5x^2 + 11x + 8$

39 $2p(3 - p) + 3p(1 - 3p) = 9p - 11p^2$

40 $3xy^2(x^2 + y) - 2x^2y(x - y^2) = 3x^3y^2 + 3xy^3 - 2x^3y + 2x^2y^3$

Factorising

Factorising an expression means expressing it as a product of factors.

The distributive property of multiplication across addition or subtraction gives $a(b + c) = ab + ac$. We can reverse this for a sum or difference of terms when the terms have a common factor, i.e. $ab + ac = a(b + c)$.

EXERCISE 3b

Example:

Factorise **a** $3x - 6y$ **b** $2x^2 + 4xy$ **c** $2pq - 4p^2q + 6pq^2$

a $3x - 6y = 3(x - 2y)$

$3x = 3 \times x$ and $6y = 2 \times 3 \times y$ so 3 is a common factor of $3x$ and $6y$.

b $2x^2 + 4xy = 2x(x + 2y)$

$2x^2 = 2 \times x \times x$ and $4xy = 2 \times 2 \times x \times y$ so $2x^2$ and $4xy$ have a common factor of $2x$.

c $2pq - 4p^2q + 6pq^2 = 2pq(1 - 2p + 3q)$

$2pq = 2 \times p \times q$, $4p^2q = 2 \times 2 \times p \times p \times q$, $6pq^2 = 2 \times 3 \times p \times q \times q$

Factorise

1 $5x + 5y$

2 $9a - 12b$

3 $7t + 14$

4 $10 - 5a$

5 $3x - 6y - 3z$

6 $12a - 8b + 16c$

7 $x^2 + 6x$

8 $3a^2 + 6$

9 $4p^2 - 2p$

10 $5a^2 - a$

11 $12x^2 - 6x$

12 $10a - 2a^2$

13 $4x^2 + 8x - 4$

14 $x^2 - 2xy$

15 $4a^2 - 2ab$

16 $a^3 + 2a^2$

17 $12x^3 - 9x^2$

18 $10p^2 - 15p^4$

19 $8xy + 16xz + 8x$

20 $9a - 12ab + 3ab^2$

21 $5x^2y - 10xy^2 + 15xy$

22 $3p - 6p^2 + 9qp$

23 $4xy - 6x^2y + 4xy^2$

24 $x + x^2 - 2x^3$

25 $4xy - 2xz + 3x$

26 $3a^3 - 6a^2b + 9ab^2$

27 $\pi r^2 - \pi r + r^3$

28 $\pi a^2b - \pi ab^2$

29 $\pi r^2 - 2\pi rh$

30 $\frac{1}{2}ah - \frac{1}{2}bh$

Simplifying fractions

Algebraic fractions can be simplified by using the ordinary rules of arithmetic by factorising and cancelling common factors, adding, subtracting, multiplying or dividing them.

EXERCISE 3c

Example:

Simplify $\frac{x^2 - 3x}{x^2}$

$\frac{x^2 - 3x}{x^2} = \frac{x^1(x - 3)}{{}_x x^2} = \frac{x - 3}{x}$

Factorise the numerator. Now you can see that x is a common factor of the numerator and the denominator so it can be cancelled.

Simplify

1 $\frac{a^2 + 3a}{a}$ **2** $\frac{4x - 2x^2}{2x}$ **3** $\frac{a^4b + a^3b^2}{a^2b}$

4 $\frac{3a}{ab - ab^2}$ **5** $\frac{6a^2 - 3a + 6ab}{3a}$ **6** $\frac{15x^2y}{5xy - 10xy^2}$

7 $\frac{ab^2 + a^2b}{ab}$ **8** $\frac{3x^2 + 6x}{6x}$ **9** $\frac{pq + p^2q^2}{3pq}$

10 $\frac{4a^2b - 3ab^2}{5ab}$ **11** $\frac{6ab - 4a^2b + 2ab^2}{2ab}$ **12** $\frac{15x^2y + 20xy - 10xy^2}{5xy}$

13 $\frac{3a^3b^2 - 6a^2b^3}{3ab^2}$ **14** Show that $\frac{8x^3y^3 + 2x^2y^2}{4xy} = \frac{4x^2y^2 + xy}{2}$

Example:

Simplify **a** $\frac{3x}{4} \times \frac{5x}{9}$ **b** $\frac{1}{3a - 1} \div \frac{2}{a - 3}$ **c** $\frac{x^2}{y} \times \frac{y^3}{x^4}$

a $\frac{3x}{4} \times \frac{5x}{9} = \frac{3^1 x \times 5x}{4 \times 9_3} = \frac{5x^2}{12}$

b $\frac{1}{3a - 1} \div \frac{2}{a - 3} = \frac{1}{(3a - 1)} \times \frac{(a - 3)}{2} = \frac{a - 3}{2(3a - 1)}$

c $\frac{x^2}{y} \times \frac{y^3}{x^4} = \frac{x^{2\,1} \times y^{3\,y^2}}{y_1 \times x^4{}_{x^2}} = \frac{y^2}{x^2}$

We multiply fractions by multiplying the numerators and multiplying the denominators. Cancel common factors first.

To divide by a fraction, multiply by its reciprocal. Brackets placed round denominators and numerators help us avoid mistakes.

x^2 and x^4 have a common factor of x^2.
y^3 and y have a common factor of y.

15 $\frac{5a}{7} \times \frac{3}{4}$ **16** $\frac{4x^2}{3} \times \frac{3}{8}$ **17** $\frac{2x}{3} \times \frac{4x}{7}$

18 $\frac{5a}{4} \times \frac{a^2}{10}$ **19** $\frac{3p^2}{2} \times \frac{8p}{9}$ **20** $\frac{5a}{b} \times \frac{3ab^2}{2}$

21 $\frac{9xy}{2} \times \frac{4xy^2}{5}$ **22** $\frac{a}{b} \div \frac{c}{d}$ **23** $\frac{ab}{c} \div \frac{bc}{a}$

24 $\frac{5a^2b}{3} \div 3ab^2$ **25** $\frac{12pq^2r}{5} \div 2qr^2$ **26** $\frac{5ab^2}{2} \div \frac{4a^2b}{5}$

27 $\frac{x^2y^2}{3} \div \frac{4xy^3}{7}$ **28** $\frac{10pq^3}{3} \div \frac{2p^3q}{5}$ **29** $\frac{2}{x + 3} \div \frac{3}{x + 2}$

30 $\frac{1}{2a+3} \div \frac{4}{3a-1}$ **31** $\frac{x}{3} \div \frac{(x-3)}{4}$ **32** $\frac{x}{4} \div \frac{(x-1)}{2}$

33 $\frac{4}{2x+1} \div \frac{2}{x-1}$ **34** $\frac{3}{4x-1} \div \frac{2}{4x+1}$ **35** $\frac{p^2}{q^4} \times \frac{p^3}{q^3}$

36 $\frac{3ab^2}{5c} \times \frac{a^2b}{6c^2}$ **37** $\frac{p}{q} \times \frac{q^2}{p^2}$ **38** $\frac{a^2}{b} \times \frac{b^2}{a^3}$

39 $\frac{ab^2}{3c} \times \frac{bc^3}{5ab^4}$ **40** $\frac{2x^2}{y^3} \times \frac{3y^2}{x^3}$ **41** Show that $\frac{6x^2}{y^3} \times \frac{2y^4}{x} = 12xy$

42 $\frac{5a^3}{b^2} \times \frac{ab^4}{3}$ **43** $\frac{5a^2}{2bc} \times \frac{6bc^2}{20a^2b}$ **44** $\frac{15x^2}{4yz} \times \frac{8xy^2z}{3} \times \frac{2z^3}{5x^2y}$

Example:

Simplify **a** $\frac{3a}{2b} - \frac{2a}{5b}$ **b** $\frac{2x-3}{9} - \frac{4x-3}{6}$

c $\frac{p}{2} - \frac{3p-2}{4p}$ **d** $\frac{1}{1-x} + \frac{2}{x+2}$

a $\frac{3a}{2b} + \frac{2a}{5b} = \frac{{}^{5}\cancel{10b}}{10b} \times \frac{3a}{\cancel{2b}_1} + \frac{{}^{2}\cancel{10b}}{10b} \times \frac{2a}{\cancel{5b}_1}$

$= \frac{15a + 4a}{10b} = \frac{19a}{10b}$

To add fractions we need to find equivalent fractions with a common denominator. The LCM of $2b$ and $5b$ is $10b$.

b $\frac{2x-3}{9} - \frac{4x-3}{6} = \frac{{}^{2}\cancel{18}}{18} \times \frac{(2x-3)}{\cancel{9}_1} - \frac{{}^{3}\cancel{18}}{18} \times \frac{(4x-3)}{\cancel{6}_1}$

$= \frac{2(2x-3) - 3(4x-3)}{18} = \frac{4x - 6 - 12x + 9}{18} = \frac{3-8x}{18}$

Place brackets round the numerators before multiplying by the common denominator. Then expand the brackets and collect like terms.

c $\frac{p}{2} - \frac{3p-2}{4p} = \frac{{}^{2}\cancel{4p}}{4p} \times \frac{p}{\cancel{2}_1} - \frac{{}^{1}\cancel{4p}}{4p} \times \frac{(3p-2)}{\cancel{4p}_1}$

$= \frac{2p^2 - (3p-2)}{4p} = \frac{2p^2 - 3p + 2}{4p}$

d $\frac{1}{1-x} + \frac{2}{x+2} = \frac{(x+2)}{(x+2)} \times \frac{1}{(1-x)} + \frac{(1-x)}{(1-x)} \times \frac{2}{(x+2)}$

$= \frac{x+2+2-2x}{(x+2)(1-x)} = \frac{4-x}{(x+2)(1-x)}$

45 $\frac{x}{2} + \frac{x}{3}$ **46** $\frac{2x}{3} + \frac{3x}{4}$ **47** $\frac{3a}{5b} + \frac{4a}{15b}$

48 $\frac{5p}{6} - \frac{7p}{12}$ **49** $\frac{3x}{4y} - \frac{2x}{3y}$ **50** $\frac{x}{2} - \frac{x}{3} + \frac{x}{6}$

51 $\frac{5a}{3} + \frac{a}{6} - \frac{5a}{12}$ **52** $\frac{x+3}{4} + \frac{x+1}{2}$ **53** $\frac{a+6}{3} + \frac{3a+1}{2}$

54 $\frac{5a+2}{6} - \frac{a+4}{5}$ **55** $\frac{4b-3}{2} - \frac{2b+1}{3}$ **56** $\frac{5y-3}{7} - \frac{4y+9}{3}$

57 $\frac{3x+2}{4} - \frac{x+5}{8}$ **58** $\frac{5x-2}{12} - \frac{4x-5}{8}$ **59** $\frac{6p-5}{10} - \frac{9-2p}{5}$

60 $\frac{5a+2b}{7} + \frac{a+5b}{3}$ **61** $\frac{4x-3y}{5} + \frac{2x+y}{10}$ **62** $\frac{3a+b}{6} - \frac{2a-3b}{5}$

63 $\frac{4p+q}{9} - \frac{3p-5q}{5}$ **64** $\frac{a}{4} + \frac{5a-3}{2}$ **65** $\frac{2a}{3} - \frac{3a+2}{2a}$

66 $\frac{5p-2}{3p} - \frac{7p}{2}$ 67 $\frac{7a-3}{4b} + \frac{3b-4}{2a}$ 68 $\frac{2}{x+1} + \frac{1}{x+2}$

69 $\frac{5}{x+3} + \frac{1}{x+4}$ 70 $\frac{4}{a+2} - \frac{3}{a+3}$ 71 $\frac{9}{x+1} - \frac{5}{x+4}$

72 $\frac{5}{p-1} + \frac{3}{p+2}$ 73 $\frac{4}{x-2} + \frac{5}{x-3}$ 74 $\frac{4}{b-2} + \frac{5}{b-3}$

Substituting values into algebraic expressions

When we know the numeric values of the letters in an expression we can replace those letters with their values and hence find the value of the expression.

EXERCISE 3d

Example:

If a = 4, b = −2 and c = 2, find the value of

a $\frac{2a^2 - ab^2}{a+c}$ **b** b^3c **c** $\sqrt{2a + b^2 + c^2}$

a $\frac{2a^2 - ab^2}{a+c} = \frac{2(4)^2 - (4)(-2)^2}{(4)+(2)} = \frac{32 - (4)(4)}{6}$

$= \frac{16}{6} = \frac{8}{3} = 2\frac{2}{3}$

Place the numbers in brackets; this helps us avoid mistakes with signs.

b $b^3c = (-2)^3 \times 2 = -8 \times 2 = -16$

c $\sqrt{2a + b^2 + c^2} = \sqrt{2(4) + (-2)^2 + (2)^2}$

$= \sqrt{8+4+4} = \sqrt{16} = 4$

$(-2)^3 = [(-2) \times (-2)] \times (-2) = (4) \times (-2) = -8$
In general, $(-\text{ve number})^{\text{odd power}}$ = negative numbers
$(-\text{ve number})^{\text{even power}}$ = positive number

1 If $x = 5y - 2$, find the value of x when
a $y = 2$ **b** $y = -4$ **c** $y = -\frac{1}{5}$

2 If $x = 4$, $y = 3$, $z = 2$, find the value of
a $5x - 2y + 3z$ **b** $4y + 5z - 2x$ **c** $\frac{3x + y - 2z}{x + y + z}$ **d** $\frac{4x - 3y + z}{3x - 2y + z}$

3 If $x = 2$, $y = 3$, $z = -4$, find the value of
a $3xy - z^2$ **b** $4xz^2 + 3yz^2$ **c** $5xyz + 2xy^2$ **d** $3x^2y - 2xz^2$

4 Find the value of $a(b - 2c)$ if
a $a = -2$, $b = 6$ and $c = 3$ **b** $a = 1$, $b = 2$ and $c = 3$
c $a = 4$, $b = -3$ and $c = -2$

5 If $a = 3$, $b = -2$ and $c = 1$, find the value of
a $4abc$ **b** $a^2 + b^2 - c$ **c** $a^2 - b^2 + 3c$

6 If $p = -4$, $q = 7$ and $r = 3$, find the value of
a $\frac{2p^2 + q^2}{9r}$ **b** $\sqrt{2p^2 + q^2}$

7 Find the value of $\frac{a^2 + b^2}{c}$ when
a $a = 3$, $b = 4$ and $c = 5$ **b** $a = 2$, $b = -1$ and $c = -5$
c $a = \frac{1}{2}$, $b = \frac{3}{4}$ and $c = \frac{1}{8}$

8 If $a = 0$, $b = -4$ and $c = -2$, find the value of

a $\sqrt{a^2 + 2bc}$ **b** $\sqrt{a^2 + 2b^2 + c^2}$

9 If $p = 6$ and $q = 5$, find the value of

a $2p^2 - 3q^2$ **b** $\sqrt{q^2 - \frac{p^2}{4}}$

10 If $a = -2$, $b = 3$ and $c = -1$, find the value of

a $a^2 + b^2 - c^2$ **b** $a^2 + b^2 - 3abc$ **c** $\sqrt{a^2 + 4b^2 + 4c}$

11 Find the value of $\frac{1}{x} + \frac{1}{y}$ if

a $x = 2, y = 3$ **b** $x = 2, y = -3$

c $x = \frac{1}{4}, y = \frac{1}{9}$ **d** $x = -\frac{1}{4}, y = \frac{1}{9}$

12 Given that $\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$, find the value of f when

a $u = 3, v = 4$ **b** $u = 2, v = -3$ **c** $u = \frac{1}{4}, v = -\frac{1}{3}$

13 If $a = \frac{1}{3}$, $b = -2$ and $c = 5$, find the value of

a $6a - 3b$ **b** $9a^2 - 2b^2 + c^2$ **c** $\sqrt{\frac{9a^2 - 2b^2 + c^2}{6a - b + 5}}$

14 If $p = 3$, $q = \frac{1}{3}$ and $r = -4$, find the value of

a $pq - 6qr$ **b** $p^2 - 9q^2 + r^2$ **c** $\sqrt{(2p^2 - 6q + 3r^2)}$

15 Given that $a = 4$, $b = -4$ and $c = 5$, find the value of

a $ab(a - b)$ **b** $a^2 + b^2 + c^2$ **c** $\frac{a^2 - bc}{c - b}$

16 If $p = \frac{1}{2}$, $q = \frac{2}{3}$ and $r = 9$, find the value of

a $p^2 + q^2$ **b** pq^2r **c** $8p^2 - 3qr$

17 If $a = 5$, $b = -4$ and $c = -2$, find the value of

a $\frac{ab}{2c}$ **b** $\frac{2ac}{b}$ **c** $\sqrt{(a^2 - b^2)}$

18 If $p = -2$ and $q = 5$, find the value of

a $p^2 - 2pq + q^2$ **b** $p(3p + 2q)$ **c** $(5p)^2 - 4q^2$

Linear equations in one unknown

A linear equation in one unknown contains one letter to the power 1. The letter may appear more than once.

Solving an equation means finding the value of the unknown that makes the equality true. We can rearrange an equation provided we keep the equality true. We can do this by

- adding or subtracting any term from both sides of an equation
- multiplying or dividing the whole expression on each side of the equation by the same number or term.

To solve a linear equation in one unknown

- eliminate any fractions
- expand any brackets
- simplify the expressions on each side of the equals sign
- collect the terms containing the letter on the side of the equals sign with the greater number of them and the numbers on the other side by adding to or subtracting terms from both sides
- divide both sides by the coefficient of the letter.

The **coefficient** of a letter is the number multiplied by it.

EXERCISE 3e

Example:

Solve the equations **a** $4 - x = 7 - 3x$ **b** $2 - x - 3(x - 4) = 6$

a $4 - x = 7 - 3x$ Add 3x to both sides.

$4 - x + 3x = 7$

$4 + 2x = 7$ Subtract 4 from both sides.

$2x = 3$ so $x = 1\frac{1}{2}$. Divide both sides by 2.

This gives a positive x term.

b $2 - x - 3(x - 4) = 6$ Expand the bracket.

$2 - x - 3x + 12 = 6$ Collect like terms.

$14 - 4x = 6$ Add 4x to both sides.

$14 = 6 + 4x$

$8 = 4x$ so $x = 2$

Solve

1 $4x - 13 = 3$

2 $3x - 9 = 0$

3 $7x = 0$

4 $12x = -36$

5 $5x + 10 = 5$

6 $7 - 3x = 1$

7 $19 - 5x = 6$

8 $7x - 3 = -31$

9 $-3p - 5 = 4$

10 $2x + 5 = x + 9$

11 $5x + 4 = 3x + 12$

12 $7y - 3 = 4y + 9$

13 $11(x - 3) - 7 = 37$

14 $3 - x + 2(x - 3) = 1$

15 $5 + 2x - 3(x - 4) = 13$

16 $4 + x - 3(x - 1) = 13$

17 $3(2x + 1) - x + 11 = 4$

18 $x - 4 - 2(3x - 1) = 8$

19 $5(1 - 2x) - 3 + 11x = 3$

20 $11 - 4(3 - 5x) + x = 83$

Example:

Solve the equations **a** $x + \frac{3x}{2} = 2$ **b** $\frac{3x - 4}{3} - \frac{2x + 1}{6} = \frac{1}{2}$

a $\frac{x}{1} + \frac{3x}{2} + \frac{2}{1}$

$2x + {}^{1}\cancel{2} \times \frac{3x}{\cancel{2}_{1}} = 4$

$2x + 3x = 4$

$5x = 4$ so $x = \frac{4}{5}$

We can eliminate fractions by multiplying every term by the LCM of the denominators. The LCM of 1, 2 and 1 is 2 so we multiply every term by 2.

b $\frac{3x - 4}{3} - \frac{2x + 1}{6} = \frac{1}{2}$

$^{2}\cancel{6} \times \frac{(3x - 4)}{\cancel{3}_{1}} - {}^{1}\cancel{6} \times \frac{(2x + 1)}{\cancel{6}_{1}} = {}^{3}\cancel{6} \times \frac{1}{\cancel{2}_{1}}$

$2(3x - 4) - (2x + 1) = 3$

$6x - 8 - 2x - 1 = 3$

$4x - 9 = 3$

$4x = 12$ so $x = 3$

6 is the LCM of 3, 6 and 2 so multiply every term by 6.

Solve

21 $5 - \frac{2x}{3} = 9$ **22** $7 + \frac{3x}{4} = 19$ **23** $2x + \frac{x}{3} = 2$

24 $y - \frac{2y}{5} = 3$ **25** $4x + \frac{3x}{4} = 3$ **26** $\frac{3x - 2}{4} = 7$

27 $\frac{3 - x}{2} + 1 = 0$ **28** $\frac{x - 2}{3} + \frac{x - 4}{2} = 1\frac{1}{2}$ **29** $\frac{4x + 5}{5} + \frac{2x + 1}{2} = 6$

30 $\frac{2x - 1}{3} + \frac{3x + 1}{5} = 10$ **31** $\frac{3x - 2}{4} - \frac{x - 3}{3} = 3$ **32** $\frac{2x + 1}{3} - \frac{5x - 1}{6} = \frac{3}{2}$

33 $\frac{2x - 3}{5} + \frac{x + 14}{2} = 1$ **34** $\frac{6x + 1}{3} - \frac{5x - 2}{4} = \frac{5}{4}$ **35** $\frac{x - 2}{4} - \frac{x + 5}{8} = 2$

36 $\frac{5x + 3}{6} - \frac{x + 9}{4} = 0$ **37** $\frac{3x + 1}{5} - \frac{7x - 3}{7} = \frac{6}{7}$ **38** $\frac{2x + 1}{3} - \frac{x}{2} = 5 - \frac{3x + 4}{4}$

39 $\frac{7(2x + 3)}{4} - 4x = 2\frac{3}{4}$ **40** $\frac{x - 3}{12} + \frac{3(x - 1)}{8} = 1\frac{2}{3}$

Forming expressions and equations

We can often translate a problem in words into an algebraic form. For example, if 20 oranges are bought, each at the same price, we can form an expression for the total cost by allocating a letter for the unknown cost of one orange.

If one orange costs $\$x$, then 20 oranges costs $\$20x$. If we also know that the total cost is \$30, then we can equate $20x$ and 30 to form an equation, i.e. $20x = 30$.

EXERCISE 3f

Example:

Jason bought some pens and Sharon bought 5 fewer pens than Jason.

a Write down an expression for the total number of pens that Jason and Sharon had between them.

b Jason lost 8 of his pens. He then had half as many pens as Sharon. Form an equation and find the number of pens that Jason bought.

a If Jason bought x pens then Sharon bought $(x - 5)$ pens. The total number of pens they had between them is $x + x - 5$, i.e. $2x - 5$.

We do not know how many pens Jason bought so allocate a letter for this number.

b Jason now has $x - 8$ pens.

So $x - 8 = \frac{1}{2}(x - 5)$

$2x - 16 = x - 5$

$x = 11$

Jason bought 11 pens.

We use the fact that the number of pens Jason has now is equal to half the number of pens that Sharon has to form an equation.

1 I think of a number x, add 9 and divide the sum by 3. The result is 8. Find the number.

2 When we add 43 to a number x and divide the total by 5 the result is 10. Find x.

3 When we add 55 to a number x and divide the total by 4 the result is three times the original number. Find x.

4 When we subtract 5 from a number x and divide the answer by 7 the result is 23 less than x. Find x.

5 Molly buys 2 dresses and 3 pairs of pants for \$675. The cost of a pair of pants is \$25 less than the cost of a dress.
  a If a dress costs \$$x$ write down, in terms of x, the cost of a pair of pants.
  b Form an equation in x and solve it.
  How much is the cost of **i** a dress? **ii** a pair of pants?

6 One parcel weighs x kg and another is three times as heavy. If 2 kg is added to the heavier parcel and 8 kg to the lighter parcel they will be of equal weight. Find the weight of each.

7 A boy is 12 kg heavier than his sister. Together they weigh 74 kg. Find the weight of each.

8 A man is 25 years old when his daughter is born. How many years later will the daughter's age be half the father's age?

9 Tim had 30 days off work. He spent x days at home, twice as long with his parents and three times as long abroad. How long did he spend at home?

10 When 2 is subtracted from two-thirds of a number the result is half the original number. Find the number.

11 The price of a fork is $\frac{4}{7}$ that of a spoon. 10 forks and 6 spoons cost \$41. Find the price of **a** a spoon **b** a fork.

12 The sum of three consecutive integers is 48. Find them.

Let the middle number be x.

13 The lengths of the three sides of a triangle are x cm, $(x + 1)$ cm and $(x + 2)$ cm. Find the lengths of the three sides if the perimeter of the triangle is 135 cm.

14 The middle number of three consecutive numbers is x. When 25 is subtracted from the sum of these three numbers the result is half the middle number. Find the numbers.

15 The mass of a fish is 1 kg plus half its mass. What is the mass of the fish?

16 Lance and Clarrie share \$19 such that $\frac{3}{5}$ of Lance's share is equal to $\frac{2}{3}$ of Clarrie's.
  a If Lance has \$$x$ how much, in terms of x, does Clarrie have?
  b Form an equation in x and solve it to find how much each has.

17 The perimeter of a rectangle is 72 cm. The rectangle is x cm wide and is twice as long as it is wide. Calculate its dimensions.

18 The middle number of three consecutive odd numbers is x. The sum of the numbers is 57. Find the numbers.

19 The perimeter of a rectangular sheet is 13 m and the length is $\frac{1}{2}$ m more than the width. Find the width of the sheet.

20 A mother is 5 times her daughter's age. In 6 years time the mother will be 3 times her daughter's age. How old is the daughter now?

21 Emily's age is half John's age. Four years ago Emily was one-quarter of John's age in 4 years time. Find the age of each now.

22 Find the number x such that when one-third of it is added to 10 the result is half as much again as x.

23 87 people are divided into three groups. One group has 6 more than the smallest group and another group has 12 more than the smallest group. How many people are there in the smallest group?

24 In a triangle the shortest side is $\frac{2}{3}$ the length of the longest side and the third side is $\frac{3}{4}$ the length of the longest side. If the perimeter is 29 cm find the lengths of the three sides.

So far in this exercise, instructions in words have been converted into expressions or equations. Sometimes we need to reverse this and express an algebraic expression or equation in words.

Example:

A number x is such that $3x - 2 = 13$.
Express this equation in words.
A number is multiplied by 3 and then 2 is subtracted. The result is 13.

25 A number x is such that $\frac{x}{2} - 5 = 7$.
Express this equation in words.

26 The area of a rectangle is found from the formula $A = xy$ where x cm is the length and y cm is the width.
Express this formula in words.

27 The area of a triangle is found from the formula $A = \frac{1}{2}xd$ where x cm is the length of the base of the triangle and d cm is the perpendicular height of the triangle.
Express this formula in words.

28 The cost of n oranges is given by the expression np where p is the price of one orange.
Express this expression in words.

29 Two numbers x and y are such that $x^2y = 40$.
Express this equation in words.

30 Two numbers x and y are such that $\sqrt{\frac{x}{y}} = 5$.
Express this equation in words.

31 The perimeter of a rectangle is given by the expression $2x + 2y$ where x cm is the length and y cm is the width.
Express this expression in words.

32 The area of a circle is given by the formula $A = \pi r^2$.
Express this formula in words.

33 A number x is such that $3x - 40 = \frac{1}{2}x$.
Express this equation in words.

34 Two numbers x and y are such that $\frac{x}{2} - y^2 = 3 - y$.
Express this equation in words.

INVESTIGATION

Buying his weekly purchase of fifty cents' worth of sweets, Sugar Ray was pleased to find that he was given five more sweets than usual.

When he looked at the price he saw it had gone down ten cents per dozen. What was the new price per dozen?

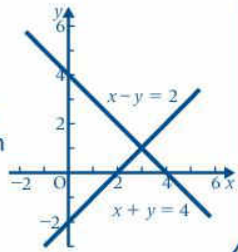
Simultaneous linear equations in two unknowns

The linear equation $x + y = 4$ has two unknowns and an infinite set of pairs of values that satisfy it.

e. g. ($x = 3, y = 1$), ($x = -6, y = 10$), ($x = \frac{1}{2}, y = 3\frac{1}{2}$) ...

If we add another condition on x and y, say $x - y = 2$, we find there is just one pair of values ($x = 3$ and $y = 1$) that satisfies the equations simultaneously. We say that $x = 3$ and $y = 1$ is the solution to the **simultaneous equations** $x + y = 4$ and $x - y = 2$.

If we plot values of y against values of x for both equations, we get two straight lines that intersect at just one point. The value of x (3) and the value of y (1) at this point satisfy, both equations.



There are several methods for solving a pair of linear simultaneous equations. We look at two of them here.

Solution by elimination

We can eliminate one unknown by adding or subtracting the equations when the number of xs or ys is the same in both equations. If it is not, we use a multiple of one or both equations to achieve the same numbers of xs or ys.

This gives a linear equation in one unknown which we can solve.

We add when the signs of the letter to be eliminated are different. We subtract when the signs of the letter to be eliminated are the same.

Solution by substitution

We start by rearranging one equation to give one letter equal to an expression containing the other letter. We then substitute this expression for the first letter in the other equation. This again gives a linear equation in one unknown.

EXERCISE 3g

Example:

Solve the equations **a** $2x - y = 5$, $3x + y = 10$ **b** $2x - y = 5$, $2x + 3y = 3$

a

$2x - y = 5$ [1]

$3x + y = 10$ [2]

eqn [1] + eqn[2] $5x = 15$ so $x = 3$

$x = 3$ in eqn[2] $9 + y = 10$ so $y = 1$

The solution is $x = 3$ and $y = 1$.

The coefficients of y are equal and the signs are different so adding the equations will eliminate y.

b

$2x - y = 5 \quad [1]$

$2x + 3y = 3 \quad [2]$

eqn[2] − eqn[1] $\quad (2x + 3y) - (2x - y) = 3 - 5$

$3y + y = -2$

$4y = -2$ so $y = -\frac{1}{2}$

$y = -\frac{1}{2}$ in eqn[1] $\quad 2x - (-\frac{1}{2}) = 5$

$2x + \frac{1}{2} = 5$ so $2x = 4\frac{1}{2}$ giving $x = 2\frac{1}{4}$

The solution is $x = 2\frac{1}{4}$ and $y = -\frac{1}{2}$.

The coefficients of x are equal and the signs are the same so subtracting the equatons wll eliminate x.

Not all these steps need to be written down but they do avoid mistakes with signs.

1 $2x + y = 5$, $x - y = 4$

2 $5x + y = 8$, $x - y = 4$

3 $3x + y = 11$, $x + y = 3$

4 $4x + y = 15$, $7x - y = 7$

5 $3x + 2y = 7$, $3x + y = 5$

6 $9x - 2y = 20$, $7x - 2y = 12$

7 $x + 3y = 13$, $x + 2y = 9$

8 $9x - 2y = 20$, $7x + 2y = 12$

9 $x - 3y = 7$, $x + y = 3$

10 $7a + 4b = 2$, $7a - 4b = 26$

11 $3x - 2y = 0$, $x + 2y = 8$

12 $x - 2y = 2$, $x + 2y = -2$

13 $3x - y + 2 = 0$, $x - y = 0$

14 $3x + y = 13$, $2x - y = 7$

15 $5x + 2y = 4$, $4x - 2y = \frac{1}{2}$

Example:

Solve the simultaneous equations **a** $4x - 3y = 8$, $3x - 2y = 10$ **b** $x = 2y + 5$, $2x - 3y = 6$

a

$4x - 3y = 8 \quad [1]$

$3x - 2y = 10 \quad [2]$

eqn [1] × 2 $\quad 8x - 6y = 16 \quad [3]$

eqn[2] × 3 $\quad 9x - 6y = 30 \quad [4]$

eqn[4] − [3] $\quad x = 14$

$x = 14$ in eqn[2] $\quad 42 - 2y = 10$

$42 = 10 + 2y$

$32 = 2y$ so $y = 16$

The solutions is $x = 14$ and $y = 16$.

We need the coefficients of either x or y to be the same in each equation. We can achieve this by multiplying [1] by 2 and [2] by 3 to give $6y$ in both equations. We could also multiply [1] by 3 and [2] by 4 to give $12x$ in both equations but this gives larger numbers so we choose the first option.

b

$x = 2y + 5 \quad [1]$

$2x - 3y = 6 \quad [2]$

$x = 2y + 5$ in eqn[2] $\quad 2(2y + 5) - 3y = 6$

$4y + 10 - 3y = 6$

$y + 10 = 6$ so $y = -4$

$y = -4$ in eqn[1] $\quad x = -8 + 5 = -3$

The solution is $x = -3$ and $y = -4$.

We can rearrange [1] to $x - 2y = 5$ and use the elimination method but, as [1] already gives x as an expression containing y, we can use the substitution method.

16 $5x + y = 15$
$7x - 2y = 4$

17 $4x - y = 14$
$3x + 2y = 5$

18 $2x + y = 7$
$3x + 2y = 12$

19 $5x - y = 17$
$2x + 3y = 0$

20 $4x - y = 10$
$3x + 5y = 19$

21 $5x - 3y = 11$
$4x + y = 19$

22 $5x + 7y = 4$
$x + 2y = 2$

23 $3x + 5y = -14$
$3x - 2y = 14$

24 $7x + 5y = 6$
$3x - 4y = 21$

25 $3x + 4y = 26$
$4x - 3y = 18$

26 $5p + q = 22$
$2p - 5q = 25$

27 $3m - 4n = 30$
$4m + 5n = 9$

28 $6x - 5y = 11$
$4x + 3y = 1$

29 $10x + 3y = -24$
$5x - 4y = -23$

30 $2x + 3y = 2$
$8x - 9y = 1$

31 $4a - 6b = 13$
$a + b = 2$

32 $12p - 3q = 1$
$4p + 6q = 5$

33 $7x - 5y = 16$
$4x + y = -14$

34 $x - 2y = 2$
$3x - y = -2$

35 $2x - 3y = 4$
$2x + 3y = -10$

36 $5p - 5q = 1$
$p + q = 3$

37 $2x + 3y = -1$
$2x - 2y = 5$

38 $2x + 3y = 4$
$x - y + 4 = 0$

39 $3x + 2y = 16$
$2x + 3y = 29$

40 $3x + y = 6$
$2x - y = 0$

41 $6a - 5b = -7$
$3a + 4b = 16$

42 $3x - 4y = 1$
$6x - 6y = 5$

43 $4x + 3y = 4$
$2x - 5y = 15$

44 $8p + 4q = 7$
$6p - 8q = 41$

45 $3x + 2y + 9 = 0$
$2x + y + 7 = 0$

46 $5x + 2y = 31$
$x = 3y + 13$

47 $x + 2y = 10$
$3x = 5y - 3$

48 $4x = -5y - 8$
$6x = 2y + 26$

Example:

The length of a rectangle is 5 cm more than its width and the perimeter is 30 cm. Find the width of the rectangle.

Let x cm be the length and y cm be the width.

Then $x - 5 = y$ [1]
and $2x + 2y = 30$ [2]

Allocate letters for the length and the width. There are two unknowns so we need to form two equations.

Substitute $x - 5$ for y in [2]: $2x + 2(x - 5) = 30$
$2x + 2x - 10 = 30$
$4x = 40$ so $x = 10$

$x = 10$ in [1] $10 - 5 = y$ so $y = 5$

The rectangle is 5 cm wide.

The perimeter of a shape is the distance round its sides.

49 A number x is 9 more than a number y.
Twice the smaller number is 1 less than the larger number.
Find the numbers.

50 A photographer charges x cents for a large print and y cents for a small one. Lance paid \$13.28 for 5 large prints and 12 small ones while Ashley paid \$14.24 for 7 large and 10 small. Find the cost of each size print.

51 A student bought x books at \$8.50 each and y books at \$12.50 each. The total cost of the seven books he bought was \$71.50. How many of each book did he buy?

52 Tea costs \$$x$ a packet and coffee \$$y$ a packet. Five packets of tea and seven of coffee cost \$39.40 while three packets of tea and five of coffee cost \$26.20.
Work out the cost of

a a packet of tea **b** a packet of coffee **c** two packets of each.

53 When Sheena bought x stamps at 25 cents each and y stamps at 50 cents each she paid \$5.25. Michael bought twice as many 25-cent stamps and half as many 50-cent stamps for which he paid \$4.50. How many of each denomination stamp did Sheena buy?

54 A bottle of orange squash costs \$$x$ and a bottle of lime squash costs \$$y$. Two bottles of orange squash and three bottles of lime squash cost \$7 while three bottles of orange and two bottles of lime cost \$6.75.
Find the cost of **a** a bottle of orange squash **b** a bottle of lime squash.

55 Joe is x years old and is 25 years older than his son Brian who is y years old. The sum of their ages is 49 years.
Find **a** Joe's age now **b** Brian's age in 5 years time.

56 Mrs Khan paid \$115 for the goods she bought in a shop. She used x five-dollar notes and y ten-dollar notes using 14 notes altogether. How many of each denomination note did she use?

57 A mother buys 7 snacks for her family. Snack A cost \$6.50 each and Snack B cost \$8.75 each. She spends \$54.50 altogether. How many of each snack does she buy?

58 A market stall sells mangoes and pears. Mangoes cost \$$x$ each and pears \$$y$ each. One woman pays \$17 for 5 mangoes and 4 pears while another woman pays \$24 for 3 mangoes and 7 pears.
Work out the cost of **a** a mango **b** a pear.

59 The cost of three roti and five pies is \$23.50 while the cost of two roti and six pies is \$21.
Find the cost of **a** a roti **b** a pie.

60 A number consists of two digits, x and y. x is bigger than y and their difference is 5. If this number is added to the number formed by reversing the digits, the answer is 121. Find the number.

INVESTIGATION

Try to solve the simultaneous equations

$$x = 5y + 8$$
$$2x - 10y = 3$$

Explain what happens and give a reason why it happens.

Linear inequalities

The symbols used are $>$ (greater than), $\geqslant$ (greater than or equal to), $<$ (less than), $\leqslant$ (less than or equal to).

Consider the statement $x > -2$.

This is an inequality and it means that x can be any real number greater than -2.

We can illustrate this on a number line:

and we can describe it as x: $x > -2$

The open circle shows that -2 is not included.

Solving an inequality means finding the values of the unknown for which the inequality is true. We can rearrange an inequality provided it remains true.

An inequality remains true when

- the same number is added to or subtracted from both sides
- both sides are multiplied or divided by the same positive number.

Multiplying or dividing both sides of an inequality by a negative number reverses the inequality.

For example, $-4 < 3$ and $-4 + 2 < 3 + 2$, $-4 - 2 < 3 - 2$ and $2(-4) < 2(3)$ but $(-2)(-4) > (-2)(3)$

EXERCISE 3h

Example:

Solve the inequalities and illustrate each solution on a number line.

a $3 - 2x \leqslant 5$ **b** $x - 3 < 2x + 1 < 5$

a $3 - 2x \leqslant 5$ Add $2x$ to both sides.

$3 \leqslant 5 + 2x$ Subtract 5 from both sides.

$-2 \leqslant 2x$ Divide both sides by 2.

$-1 \leqslant x$ so $x \geqslant -1$

The solid circle shows that -1 is included.

b $x - 3 < 2x + 1 < 5$

So $x - 3 < 2x + 1$ and $2x + 1 < 5$

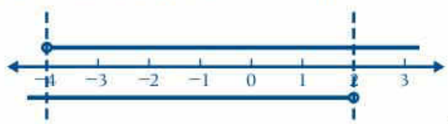
$-3 < x + 1$ $2x < 4$

$-4 < x$ so $x > -4$ $x < 2$

There are two inequalities here. The possible values of x have to satisfy them both.

therefore $-4 < x < 2$

Illustrating both on a number line shows that the two inequalities are satisfied for all values of x between -4 and 2.



In questions **1** to **18**, solve the inequalities, illustrating each solution on a number line.

1 **a** $5 - 2x \leqslant 3$ **b** $3x + 1 \leqslant 10$

2 **a** $4x - 3 \geqslant 5$ **b** $2x - 1 < 4$

3 **a** $5x + 2 > 6$ **b** $1 - 3x < 4$

4 **a** $\frac{x}{3} \leqslant 1.5$ **b** $\frac{3 + x}{2} \geqslant \frac{5}{6}$

5 **a** $7 < 2x + 1$ **b** $15 \geqslant 7 - 2x$

6 **a** $2x > 9 - 7x$ **b** $3x - 2 \leqslant 5x + 1$

7 **a** $\frac{4x+1}{2} < \frac{5x-2}{3}$ **b** $\frac{1-3x}{2} \geqslant \frac{2x+1}{5}$

8 **a** $\frac{x}{2} + 1 < \frac{1}{2}$ **b** $\frac{x}{2} > 1 - \frac{x}{3}$

9 $6 > x + 1 > 2$

10 $7 \leqslant 2 - x \leqslant 11$

11 $10 \geqslant 2x + 6 > x + 3$

12 $x + 1 > 3x + 4 \geqslant 7$

13 $2x - 3 \geqslant x + 2 \geqslant 9$

14 $7x + 1 \geqslant 3x - 4 \geqslant 2$

15 $4 - 2x > x + 5 \geqslant 4$

16 $-1 < 2x + 3 < 7$

17 $x - 3 \leqslant 2x < 4$

18 $3x + 2 < x + 5 < 3$

In questions **19** to **22**, solve the pair of inequalities and then find the range of values of x that satisfies both of them.

19 $x - 5 < 7$ and $x + 4 > 3$

20 $5x - 7 > 3$ and $3x - 1 < 8$

21 $2x + 1 > 3$ and $3x - 2 < 4$

22 $0 > 3 - 2x$ and $2x - 5 \leqslant 1$

23 Find **a** the smallest integer **b** the largest integer
that satisfies the inequalities $x + 2 < 3$ and $x - 1 > -5$

24 Find **a** the smallest integer **b** the largest integer
that satisfies the inequalities $x - 1 < 2x - 4 < 8$

25 Damon bought x boxes of envelopes and y boxes of paper.
He could carry a maximum of 20 boxes in his car.
Write down an inequality satisfied by x and y.

26 An internet bookseller uses a combination of large and small boxes to pack the books it sells. A small box holds at most 2 books and a large box holds at most 5 books. If the bookseller uses a large boxes and b small boxes to pack an order for 15 books or less, write down an inequality that must be satisfied by a and b.

INVESTIGATION

This is a sequence of equations:

$5x - 4 = 6,\quad 4x - 3 = 5,\quad 3x - 2 = 4,\quad 2x - 1 = 3, \ldots$

a Describe how one equation is derived from the equation before it.

b Write down the next two equations in the sequence.

c Investigate the solutions of these equations.

d Find the difference between successive equations.
Write down what you notice.

MIXED EXERCISE 3

Several answers are given for these questions.

Write down the letter that corresponds to the correct answer.

1 $5a - (2a + 3)$ simplifies to
A $3a - 3$ **B** $3a + 3$ **C** $7a - 3$ **D** $7a + 3$

2 $8x - 3y - 4x + 7y$ simplifies to
A $12x - 4y$ **B** $4x - 4y$ **C** $4x + 4y$ **D** $12x + 4y$

3 $\frac{p^2 q^3}{p^3 q^2} =$
A pq^2 **B** $p^{-1}q^2$ **C** pq^{-1} **D** $p^{-1}q$

4 $\frac{2a}{3b} + \frac{3b}{2a}$ simplifies to
A $\frac{4a^2 + 9b^2}{5ab}$ **B** 1 **C** $\frac{4a + 9b}{6ab}$ **D** $\frac{4a^2 + 9b^2}{6ab}$

5 $\frac{4}{3x} - \frac{1}{x}$ simplifies to
A $\frac{3}{2x}$ **B** $\frac{1}{x}$ **C** $\frac{1}{3x}$ **D** $\frac{4}{3x}$

6 If $p = \frac{2}{3}$ and $q = \frac{4}{9}$, the value of $q \div p$ is
A $\frac{8}{27}$ **B** $\frac{27}{8}$ **C** $\frac{2}{3}$ **D** $\frac{3}{2}$

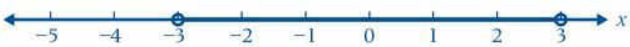
7 If $a = \frac{1}{2}$ and $b = -\frac{1}{3}$, the value of $a^2 + b^2$ is
A $\frac{1}{5}$ **B** $\frac{13}{36}$ **C** $\frac{5}{36}$ **D** $\frac{1}{36}$

8 If $\frac{3x - 7}{2} = 2$ then x is
A 3 **B** $3\frac{2}{3}$ **C** $1\frac{2}{3}$ **D** $1\frac{1}{2}$

9 The only value of p that does NOT satisfy $p - 4 < 10$ is
A -6 **B** 6 **C** 10 **D** 14

10 If $3x - 4 \leqslant 4x + 5$ then
A $x \geqslant -9$ **B** $x \leqslant 9$ **C** $x < -9$ **D** $x < 9$

11 If $7x - 2 < 5x - 8$ then
A $x > -3$ **B** $x > 3$ **C** $x < -3$ **D** $x < 3$

12 

The inequality in the diagram is defined by
A $-3 < x < 3$ **B** $-3 \leqslant x \leqslant 3$ **C** $-3 \leqslant x < 3$ **D** $-3 < x \leqslant 3$

13 If one orange costs \$3 and one banana costs \$2.50 then x oranges and 10 bananas cost
A \$(3 + 10x) **B** \$28 **C** \$(3x + 25) **D** \$(5.5x)

14 The statement '3 times a number subtracted from 35 gives 21' is represented by
A $3x - 35 = 21$ **B** $35 - 3x = 21$ **C** $3(x - 35) = 21$ **D** $35x - 21 = 3$

15 If $\frac{3x}{50} = 9$ then $x =$

A 1.5 **B** 15 **C** 150 **D** 1500

16 If $x + y = 6$ and $x - y = 8$ then $x =$

A −7 **B** −2 **C** 2 **D** 7

17 If $x + y = 5$ and $2x + y = 4$ then $x =$

A −3 **B** −1 **C** 1 **D** 3

18 If $3(x - 2) - 5x = 8$ then $x =$

A −7 **B** −1 **C** 1.75 **D** 7

19 $x(x - y) - y(x - y)$ simplifies to

A $x^2 - 2xy - y^2$ **B** $x^2 - y^2$ **C** $x^2 - 2xy + y^2$ **D** $x^2 + 2xy + y^2$

20 $\frac{x + 4}{2} - \frac{x + 3}{3} =$

A $\frac{x}{6}$ **B** x **C** $\frac{x + 6}{6}$ **D** $\frac{x + 18}{6}$

MATHS IS OUT THERE

This problem was written on the Rhind Papyrus in Egypt in about 1650 BC.

'Divide 100 loaves among 10 men including a boatman, a foreman and a doorkeeper, who receive double portions. What is the share of each?'

It can be solved using algebra. Solve it.

IN THIS CHAPTER YOU HAVE SEEN THAT...

- an expression with two or more terms can be factorised if there is a common factor, e.g. $3a^2 - 6ab = 3a(a - 2b)$ but $5ax + 4by$ cannot be factorised
- algebraic fractions can be added, subtracted, multiplied or divided using the rules of ordinary arithmetic, e.g. $\frac{2x}{3} + \frac{x}{2} = \frac{7x}{6}$, $\frac{2x}{3} - \frac{x}{2} = \frac{x}{6}$, $\frac{x}{4} \times \frac{y}{2} = \frac{xy}{8}$ and $\frac{x}{2} \div \frac{y}{3} = \frac{x}{2} \times \frac{3}{y} = \frac{3x}{2y}$
- letters in an expression can be replaced by numbers to give a numerical value for the expression
- an equation remains true when we add or subtract any term on both sides of the equals sign and when we multiply or divide both sides by the same number or term
- two simultaneous linear equations can be solved by elimination (by multiplying one or both equations by numbers if necessary to make the coefficients of one unknown the same in both equations, then adding or subtracting the equations) or by substitution (using one equation to give one unknown in terms of the other, then substituting into the other equation)
- inequalities can be solved in a similar way to solving equations. An inequality remains true when the same number is added to, or subtracted from, both sides and when both sides are multiplied or divided by the same positive number. Multiplication or division by a negative number reverses the inequality.

4 MEASUREMENT

AT THE END OF THIS CHAPTER YOU SHOULD BE ABLE TO...

1 Convert between different units of length, different units of mass, different units of time and different units of temperature.
2 Convert between metric and imperial units.
3 Convert between different currencies.
4 Understand that measurements are not exact.
5 Calculate the perimeters of polygons, circles and find the length of an arc of a circle.
6 Find the areas of polygons, circles and sectors of circles.
7 Find the volumes and surface area of prisms.

MATHS IS OUT THERE

The units described here for measuring length and mass are called metric units. These are used in most countries but are not in common use in the USA.

Find out what units are in common use in the USA and their equivalents in the metric system.

BEFORE YOU START

you need to know:
✓ how to work with decimals and fractions
✓ the laws of indices.

KEY WORDS

acre, arc, average speed, capacity, centimetre, circle, circumference, compound shape, convex polygon, cross-section, cube, cuboid, diameter, equilateral, exchange rate, foot, gallon, gram, hectare, hexagon, inch, kilogram, kilometre, kite, length, line segment, mass, metre, mile, milligram, millimetre, ounce, parallelogram, pentagon, perimeter, perpendicular, pint, polygon, pound, prism, quadrilateral, radius, rectangle, rhombus, sector, speed, square, surface area, time, ton, tonne, trapezium, triangle, volume, yard

Measurements

A measurement cannot be an exact value. The accuracy of a measurement depends on the accuracy of the measuring instrument.

We are not usually given the accuracy of a measurement so we assume it is corrected to the smallest place value given.

This means that when we calculate with given measurements, we should give an answer that is no more accurate than the measurements given.

For example, if we are told that a line is 2.6 cm long we assume this is correct to the nearest tenth of a centimetre. This means the line could be between 2.55 cm and 2.65 cm long. We say that the tolerance of 2.6 cm is 2.6 ± 0.5 cm and 0.5 is called the absolute error.

Units of measurement

The three basic measures are **length**, **mass** and **time**.

Units of length

The metric units of length in common use are the **kilometre** (km), the **metre** (m), the **centimetre** (cm) and the **millimetre** (mm) where

$$1\,\text{cm} = 10\,\text{mm}$$
$$1\,\text{m} = 100\,\text{cm}$$
$$1\,\text{km} = 1000\,\text{m}$$

Units of mass

The metric units of mass in common use are the **tonne** (t), the **kilogram** (kg), the **gram** (g) and the **milligram** (mg) where

$$1\,\text{t} = 1000\,\text{kg}$$
$$1\,\text{kg} = 1000\,\text{g}$$
$$1\,\text{g} = 1000\,\text{mg}$$

Mass is a measure of the matter in an object. In everyday language we use weight to mean mass but weight is a force and is measured in newtons.

Units of time

The basic units of time are the hour (h), the minute (min) and the second (s) where

$$1\,\text{h} = 60\,\text{min}$$
$$1\,\text{min} = 60\,\text{s}$$

Other units of time are the day, the week, the calendar month and the year where 1 day = 24 h and 1 week = 7 days. The number of days in a month varies depending on the month. 1 yr = 12 calendar months = 365 days, or 366 days in a leap year.

The time of day can be measured on a 12-hour clock as the number of hours and minutes after midnight (a.m. times) and the number of hours and minutes after midday (p.m. times) or on a 24-hour clock as the number of hours and minutes after midnight only.

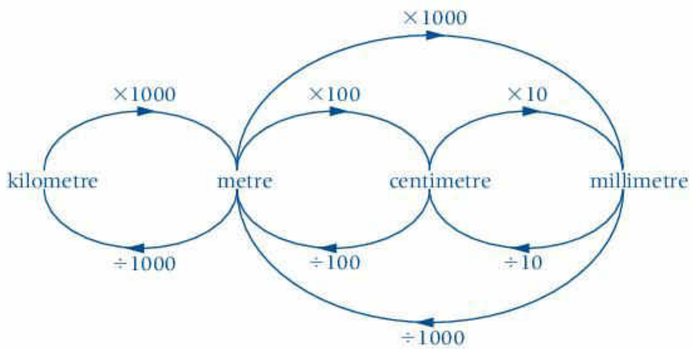
5.20 p.m. means 5 hours and 20 minutes after midday. Times in the 24-hour clock are written as a four-figure number, the first two give the hours and the second two give the minutes: 0750 hr means 7 h and 50 min after midnight. (0750 hr is also written as 07.50 hr and as 07:50 hr.) In the 24-hour clock, 1200 means midday and 0000 means midnight. In the 12-hour clock, 12.00 can mean midday or midnight so you need to use 12.00 noon for midday and 12.00 midnight for midnight.

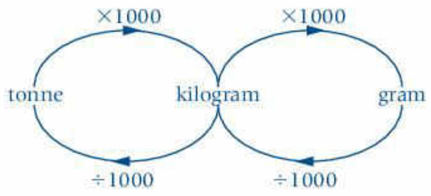
Converting between units of measure

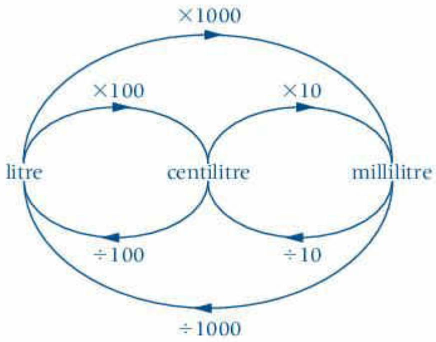
To convert a unit to a larger unit we divide.

To convert a unit to a smaller unit we multiply.

For example, to convert 500 g to kg, we divide 500 by 1000 and to convert 5 cm to mm, we multiply 5 by 10.







Temperature

There are two different everyday units used for measuring temperature. The most commonly used unit is degrees Celsius (°C). The other unit is degrees Fahrenheit (°F).

We can convert between them using the relationship

$$F = \frac{9 \times C}{5} + 32 \quad \text{or} \quad C = \frac{5}{9}(F - 32)$$

where F is the temperature in °F and C is the temperature in °C.

At standard atmospheric pressure, the freezing point of water is 0 °C = 32 °F, the boiling point of water is 100 °C = 212 °F.

Currencies

The units of currency vary. The dollar ($) is a popular unit used by several countries. Initials are used to identify the country, such as TT$ for Trinidad and Tobago. Some other units are the euro (€), the pound sterling (£) and the peso (p).

Currencies can be bought and sold like any other commodity such as sugar or coffee. The 'price' of a currency is given by the **exchange rate**. For example, the exchange rate JM$1.00 = €0.01113 tells us that 1 Jamaican dollar will 'buy' 0.01113 euros.

Exchange rates vary from day to day unless they have been fixed by governments.

Speed

Speed is the rate at which an object covers distance.

$$\textbf{Average speed} = \frac{\text{total distance covered}}{\text{time taken}}$$

The units of speed are kilometres per hour (km/h) and metres per second (m/s or ms^{-1}).

To calculate a speed in km/h, the distance must be in km and the time in hours. To calculate a speed in m/s, the distance must be in metres and the time in seconds.

Units of speed are compound units because they combine length and time.

EXERCISE 4a

Example:

Convert **a** 1.25 kg to grams **b** 35 °C to °F

a $1.25\,kg = 1.25 \times 1000\,g = 1250\,g$

b $F = \frac{9 \times 35^{7}}{5_{1}} + 32$

$= 63 + 32 = 95$

So $35\,°C = 95\,°F$

1 kg = 1000 g

Substitute 35 for C in $F = \frac{9 \times C}{5} + 32$

1 Which metric unit would it be most suitable to use to measure
- **a** the width of the nearest road
- **b** your height
- **c** the height of a street lamp
- **d** the distance between Miami, Florida and Bridgetown, Barbados
- **e** the thickness of a piece of window glass?

2 What units would you expect to use to measure the mass of
- **a** a car
- **b** a loaded petrol tanker
- **c** an aspirin tablet
- **d** one page from this book
- **e** a bag of mangoes?

3 What units would you expect to use to measure
- **a** the duration of a school lesson
- **b** the time taken to run a 100 m race
- **c** the time it takes for a student to qualify as a doctor of medicine
- **d** the time it takes for a banana plant grown from a cutting to produce fruit
- **e** the time it takes for sedimentary rocks to form on the Earth's crust?

4 Convert into metres

a 13 km	**b** 4.32 km	**c** 0.67 km	**d** 529 cm
e 4760 cm	**f** 26 cm	**g** 736 mm	**h** 38 mm

5 Convert into centimetres

a 34 m	**b** 0.47 m	**c** 0.029 m	**d** 44 mm
e 534 mm	**f** 0.05 mm	**g** 1.732 km	**h** 0.0274 km

6 Convert into millimetres

a 34 cm	**b** 8.12 cm	**c** 0.46 cm	**d** 26 m
e 0.77 m	**f** 0.034 m	**g** 1.243 km	**h** 0.092 km

7 Convert into kilometres

a 6000 m	**b** 10 000 m	**c** 750 m	**d** 7360 m
e 17 830 cm	**f** 49 350 cm	**g** 763 cm	**h** 183 400 mm

8 Convert into grams

a 250 mg	**b** 5700 mg	**c** 73 400 mg	**d** 4730 mg
e 5.42 kg	**f** 20.4 kg	**g** 0.73 kg	**h** 0.0493 kg

9 Convert into kilograms

a 7600 g **b** 491 g **c** 97.2 g **d** 6040 g
e 926 400 mg **f** 55 450 mg **g** 374 mg **h** 8497 mg

10 Convert into milligrams

a 54 g **b** 429 g **c** 1.24 g **d** 0.46 g
e 1.2 kg **f** 0.044 kg **g** 0.217 kg **h** 0.000 84 kg

11 Convert

a 3420 kg into tonnes **b** 1.74 t into kilograms
c 0.047 t into kilograms **d** 83 440 kg into tonnes
e 504 600 kg into tonnes **f** 25 000 m into kilometres
g 0.15 m into millimetres **h** 256 mm into centimetres

12 Copy and complete the following table.

24-hour clock time	a.m./p.m. time	Time in words
21.25		
	3.15 p.m.	
		Five to eleven in the morning
10.30		
	2.15 a.m.	
		Half past ten in the evening
18.40		
		A quarter to twelve at night

13 How many hours and minutes are there between

a 08.30 and 13.00 **b** 05.15 and 09.37
c 09.36 and 20.41 **d** 07.21 and 18.15
e 12.47 and 15.53 **f** 02.32 and 21.56?

14 What time is it

a 250 minutes after 10.45 a.m. **b** 123 minutes before midnight?

15 Convert the following temperatures from degrees Celsius to degrees Fahrenheit.

a 40 °C **b** 25 °C **c** 28 °C **d** −10 °C

16 Convert the following temperatures from degrees Fahrenheit to degrees Celsius

a 100 °F **b** 23 °F **c** 80 °F **d** 66 °F

17 Add the following lengths, giving each answer in metres.

a 3 m, 44 cm, 1 m 50 cm, 8 m 74 cm **b** 59 cm, 74 cm, 2430 mm
c 643 cm, 842 mm, 3914 mm **d** 2.1 km, 904 m, 8640 cm

18 Subtract the second length from the first, giving your answer in metres.

a 12 m, 200 cm **b** 60 m, 4930 cm
c 45.2 cm, 164 mm **d** 2934 mm, 1.36 m

19 Express in mixed quantities in terms of kilograms and grams.

a 9430 g **b** 1691 g **c** 3450 g **d** 13.74 kg **e** 28.4 kg

20 Express in mixed quantities in terms of hours, minutes and seconds.

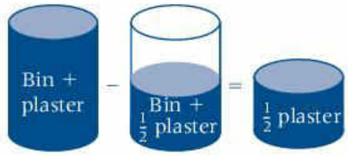
a 580 min **b** 1000 s **c** 5000 min **d** 2 500 000 s

Example:

A plastic bin is full of plaster. The mass of the bin and the plaster is 15.78 kg. The mass of the bin and half the plaster is 8.14 kg. Find the mass of the empty bin.

The mass of half the plaster is 15.78 kg − 8.14 kg = 7.64 kg

The mass of the empty bin is 8.14 kg − 7.64 kg = 0.5 kg



21 How many pencils of length 18 cm, placed end to end, are required to stretch 100 m?

22 How many concrete blocks, each 60 cm long, are required to build one course of a wall 33.6 m long?

23 A book is 6.6 mm thick. How many similar books may be stored on a shelf 1.65 m long?

24 A tin of tomatoes has a mass of 230 g. What is the mass, in kilograms, of 72 such tins?

25 The mass of a jar of marmalade is 454 g. What is the mass, in kilograms, of 48 such jars?

26 Into an empty carton of mass 80 g are placed six cakes, each of mass 195 g. Calculate the mass of the carton when full.

27 The mass of a tin of fruit cocktail is 326 g. Find the mass, in kilograms, of a box which contains 36 such tins, allowing 1 kg for the mass of the empty box.

28 Ham is sold in tins, each tin having a mass of 198 g. Find the total mass of a carton containing 72 such tins if the mass of the carton is 550 g.

29 The mass of the 10 c coins in a bag is 5.34 kg. If each coin has a mass of 3.56 g, what is the total value of these coins?

30 Fifty-four tins of tomato soup are placed in a box of mass 456 g. If the combined mass of the tins and box is 24 kg, find the mass, in grams, of one tin of soup.

31 A lorry is loaded with 64 boxes, each of mass 22 kg; 44 boxes, each of mass 34 kg; 152 boxes, each of mass 18 kg; and 98 boxes, each of mass 26 kg. Find the total mass of the load in tonnes.

32 Table salt is sold in boxes of mass 1.5 kg. If a salt cellar requires 35.5 g of salt to fill it, how many times could it be filled from one box of table salt? Give your answer correct to the nearest whole number.

33 A jug full of water has a mass of 1.8 kg. When the jug is only a quarter full of water the mass is 0.6 kg. Calculate the mass of the jug.

34 A plank is 5 m long. A length measuring x cm is cut off the plank.

Be careful with the units.

- **a** Write an expression in x for the length remaining.
- **b** The length cut off is a quarter of the length remaining.
  - **i** Form an equation in x.
  - **ii** Hence find the length of the piece cut off the plank.

35 1.5 kg of potting compost is made from 20% sand, 10% manure, 65% grit and the remainder is bonemeal. Find the mass of bonemeal in the compost. Give your answer in grams.

Example:

A banker exchanged JM$500 000 to TT$ at the exchange rate TT$1 = JM$10.83.
Calculate the amount that the banker received in TT dollars.

JM$500 000 = TT$500 000 ÷ 10.83

= TT$46 168.05 correct to 2 d.p.

One TT$ is a larger unit than one JM$ so to convert JM$ to TT$, divide by 10.83. The conversion rate is given to 2 d.p. so give the answer to 2 d.p.

Use this table of exchange rates for questions **36** to **41**

US$	UK£	Jamaican JM$	Trinidad & Tobago (TT) $	Eastern Caribbean $	Canadian (CAN)$	Barbadian (BD) $
1	0.52	63.5	5.97	2.70	1.16	1.98

36 Convert
- **a** US$50 into Jamaican dollars
- **b** US$500 into UK pounds
- **c** US$800 into Canadian dollars
- **d** US$100 into Barbadian dollars

37 Convert these amounts into US dollars giving each answer correct to the nearest dollar.
- **a** TT$700
- **b** £350
- **c** JM$1000
- **d** $1500 Eastern Caribbean

38 Convert
- **a** £65 into Jamaican dollars
- **b** BD$500 into TT dollars
- **c** Canadian $800 into Eastern Caribbean dollars
- **d** JM$1500 into Canadian dollars

39 A traveller converted US$500 into Jamaican dollars, spent 27 000 of them and converted what was left back into US dollars. How many US dollars did she get back?

40 A Canadian tourist changed CAN$120 into TT dollars, spent TT$500 and converted what remained back into Canadian dollars. How many dollars did he receive?

41 **a** A Barbadian visiting London changed BD$2400 into UK pounds. How much did he receive?
- **b** He spent £450 while visiting and converted what remained back into Barbadian dollars. How many dollars did he get?

42 A tourist exchanged BD$2000 into US dollars at the rate of BD$1 = US$0.50. The bank charged commission of 1.5%. How many US$ did the tourist receive?

43 This notice is in a bank in Jamaica:

Commission free exchange	
Bank sells	Bank buys
JM$1 = US$0.014	JM$1 = US$0.015

- **a** A tourist changes JM$10 000 into US dollars at this bank. How many US dollars does the tourist receive?
- **b** Another tourist changes US$100 into Jamaican dollars. How many Jamaican dollars does this tourist receive?

The bank is selling US$.

The bank is buying US$.

Example:

An aircraft leaves Bridge Town in Barbados at 11.45 a.m. local time and flies 3648 km to New Orleans where it arrives at 4.25 p.m. local time.

a Convert 4.25 p.m. to 24-hour time.

b The time in New Orleans is three hours behind the time in Barbados. Calculate the average speed of the aircraft.

a 4.25 p.m. = (1200 + 0425) hr = 1625 hr

b The aircraft arrives in New Orleans at 1625 hr + 3 h Barbados time = 1925 hr

This means that when the time in New Orleans is 0000 hr, the time in Barbados is 0300 hr.

Time taken is 19 h 25 min − 11 h 45 min = 7 h 40 min.

Average speed $= 3648 \div 7\frac{2}{3}$ km/h

$= 475.82...$ km/h

$= 476$ km/h to the nearest km/h.

To find an average speed in km/h, the distance must be in km and the time in hours so convert 40 min to a fraction of an hour.

44 A girl walked for 2 km and cycled for 5 km. The journey took her 45 minutes. Calculate her average speed.

45 A car travelled at an average speed of 30 km/h for 20 minutes and then at an average speed of 80 km/h for 30 minutes. Calculate
a the total distance travelled
b the average speed for the whole journey.

46 Clock time in Barbados is 4 hours behind GMT (Greenwich Mean Time). What time is it
a in London when it is 1 p.m. in Barbados
b in Barbados when it is 10 a.m. in London?

47 The time in Tokyo is GMT +9 and the time in Trinidad is GMT −4. Find
a the time in Tokyo when it is 8.30 a.m. in Trinidad
b the time in Trinidad when it is 9.15 p.m. in Tokyo.

48 The time in Jamaica is 11 hours behind the time in Mumbai. Christine wants to ring her cousin in Mumbai so that the local time in Mumbai is between 8 a.m. and 9 a.m. Between what times should she ring from Jamaica?

49 The time in Ottawa is GMT −5 and the time in San Francisco is GMT −8.
a What is the time in San Francisco when it is noon in Ottawa?
b What is the time in Ottawa when it is 5.45 p.m. in San Francisco?

50 The distance from Mumbai to London is approximately 4500 miles. An aeroplane leaves Mumbai at 5 a.m. local time and flies to London at an average speed of 500 mph. At what time (GMT) will it arrive in London if Mumbai time is $5\frac{1}{2}$ hours ahead of GMT?

51 An aeroplane leaves Rome at 8 a.m. to fly to Sydney, which is 10 500 miles away. If Rome time is 1 hour ahead of GMT and Sydney time is 10 hours ahead of GMT, find the local time of arrival in Sydney if the aeroplane travels at an average speed of 500 mph.

52 A motorist completed a journey of 20 km in 1 hour 20 minutes. He did x km at 10 km/h and the remainder of the distance at 30 km/h.
 a Form an expression in x for the time, in hours, he travelled at
  i 10 km/h ii 30 km/h.
 b i Form an equation in x.
  ii Hence find the time he spent travelling at 30 km/h.

53 A plane left Airport A at 0300 hr local time and arrived at Airport B at 2020 hr local time. The time at Airport B is 10 hours ahead of the time at Airport A and the distance between the two airports is 4800 km. Find the average speed of the plane. Give your answer correct to the nearest whole number.

Imperial units

Imperial units are an old system but are still used in some countries, mainly in the USA and to a lesser extent in the United Kingdom.

Units of length

The imperial units of length in common use are the **mile**, the **yard** (yd), the **foot** (ft) and the **inch** (in) where

$$1\text{ mile} = 1760\text{ yd}$$
$$1\text{ yd} = 3\text{ ft} = 36\text{ in}$$
$$1\text{ ft} = 12\text{ in}$$

The fathom is another imperial unit of length that is used mainly for maritime measures of depth.

Lengths smaller than 1 in are usually given as fractions of an inch, such as $\frac{1}{8}$ in.
The symbol ″ is often used as an abbreviation for 'inch'.

Units of mass

The imperial units of mass in common use are the **ton**, the **pound** (lb) and the **ounce** (oz) where

$$1\text{ ton} = 2240\text{ lb}$$
$$1\text{ stone} = 14\text{ lb}$$
$$1\text{ lb} = 16\text{ oz}$$

Another unit of mass is the hundredweight (cwt) but this is defined differently in the USA and the UK. In the USA, 1 cwt = 100 lb and in the UK, 1 cwt = 112 lb

Conversions between metric and imperial units

The conversions between metric and imperial units given below are approximate and are good enough for most purposes.

Units of length:

$$10\text{ cm} \approx 4\text{ inches}$$

So 1 in ≈ 2.5 cm or more accurately 1 in ≈ 2.54 cm

$$1\text{ m} \approx 39\text{ inches}$$

As 1 yd = 36 in, 1 m is very roughly equivalent to 1 yd

$$8\text{ km} \approx 5\text{ miles}$$

So 1 km ≈ 0.625 mile, i.e. 1 km is a bit longer than $\frac{1}{2}$ mile

Units of mass:

$1\,\text{oz} \approx 30\,\text{g}$

$1\,\text{kg} \approx 2.2\,\text{lb}$

$1\,\text{tonne} \approx 1\,\text{ton}$

This is accurate to 1 d.p. as 1 tonne = 1.016 tons to 3 d.p.

Tables and some calculators give more accurate conversions.

EXERCISE 4b

1 Which imperial units would it be most sensible to use to measure:
   a your height
   b the distance between Miami and Georgetown, Barbados
   c the thickness of a piece of window glass
   d the distance between street lamps?

2 What imperial units would you expect to use to measure the mass of:
   a a bus
   b a spoonful of sugar
   c a bag of mangoes?

3 Convert the given quantity to the units given in brackets
   a 2 ft 11 in (in)   b 9 yd 1 ft (ft)   c 2 ft 4 in (in)   d 36 in (ft)
   e 13 ft (yd and ft)   f 9 ft (yd)   g 29 in (ft and in)
   h 75 in (ft and in)   i 120 in (ft and in)

4 Convert the given quantity to the units given in brackets
   a 2 lb 6 oz (oz)   b 24 oz (lb and oz)
   c 1 lb 12 oz (oz)   d 4 lb 3 oz (oz)
   e 18 oz (lb and oz)   f 2 stone 3 lb (lb)
   g 36 oz (lb and oz)   h 4480 lb (tons)
   i 80 oz (lb)

5 How many yards are there in
   a $1\frac{1}{2}$ miles   b $\frac{1}{4}$ mile   c $\frac{1}{8}$ mile ?

6 I passed a sign on a road that said 'Road works – 1 mile ahead'. The next sign I passed said 'Roadworks – 800 yards ahead.' How far apart are the two signs?

7 Give the approximate equivalent of the first unit in the unit in brackets.
   a 3 kg (lb)   b 2 m (ft)   c 4 lb (kg)   d 9 ft (m)
   e 1.5 kg (lb)   f 5 m (ft)   g 3.5 kg (lb)   h 500 g (lb)
   i 100 g (oz)   j 12 kg (lb)   k 50 miles (km)
   l 100 miles (km)   m 30 lb (kg)   n 4 oz (g)
   o 300 km (miles)   p 240 km (miles)

8 In one catalogue a table cloth is described as measuring 4 ft by 8 ft. In another catalogue a different table cloth is described as measuring 1 m by 2 m. Which one is bigger?

9 The distance between London and Dover is about 70 miles. The distance between Calais and Paris is about 270 kilometres. Which is the greater distance?

10 A recipe requires 250 grams of flour. Roughly, how many ounces is this?

11 An instruction in an old knitting pattern says knit 6 inches. Mary has a tape measure marked only in centimetres. How many centimetres should she knit?

12 The instructions for repotting a plant say that it should go into a 10 cm pot. The flower pots that Tom has in his shed are marked 3 in, 4 in and 5 in. Which one should he use?

13 Ed knows his height is 4 ft 5 in. He needs to fill in a passport application form and has to give his height in metres. What should he enter for his height?

14 The doctor tells Mr Brown that he needs to lose 10 kilograms in weight. Mr Brown's scales at home show his weight now as 15 stone 61b. What will his scales show when he has lost the required weight?

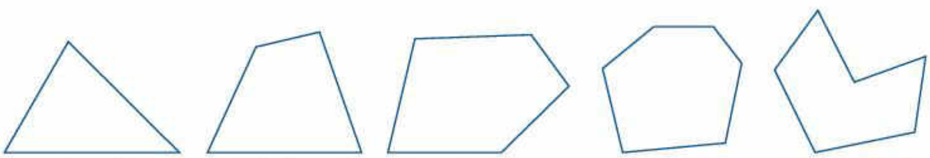
15 Arrange these weights in order of size with the lightest first.
2 oz, 50 g, 0.04 kg, $\frac{1}{4}$ lb

16 Arrange these lengths in order of size with the longest first.
25 cm, 8 inches, 25 mm, 1 inch

Polygons

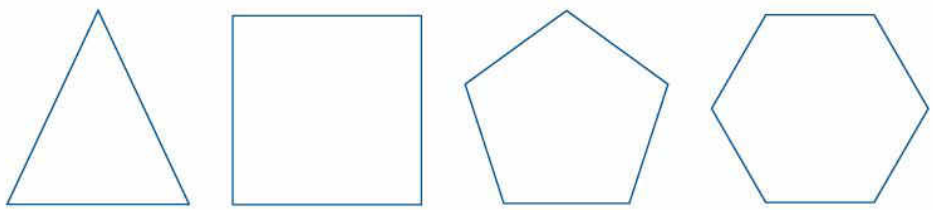
A **polygon** is a plane figure bounded by straight **line segments**. Some polygons have names:

'Plane' means flat.



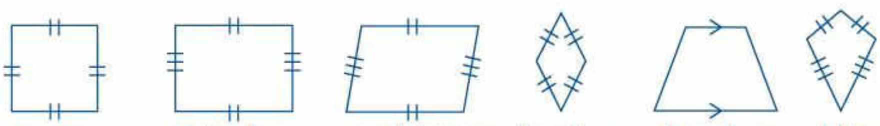
triangle **quadrilateral** **pentagon** **hexagon** **convex polygon**

A regular polygon has all of its sides equal, all of its interior angles equal and all of its exterior angles equal. These polygons are regular:



equilateral triangle **square** **regular pentagon** **regular hexagon**

There are several special **quadrilaterals**:



square: all four sides are equal, opposite sides are parallel, all angles are right angles

rectangle: opposite pairs of sides are equal and parallel, all angles are right angles

parallelogram: opposite pairs of sides are equal and parallel

rhombus: all four sides are equal, opposite pairs of sides are parallel

trapezium: one pair of opposite sides is parallel

kite: two pairs of adjacent sides are equal

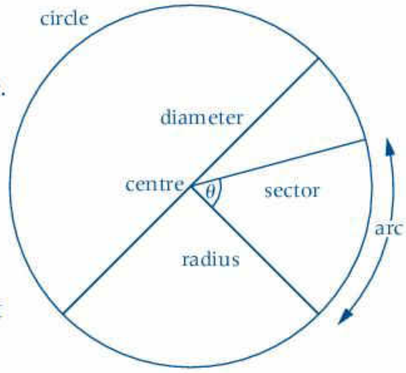
Circles

A **circle** is a closed curve that encloses a region of a plane so that every point on the curve is the same distance from one point, called the centre of the circle. A line segment from the centre to a point on the circle is called a **radius**. A line joining two points on the circle and passing through the centre is called a **diameter**.

Part of the circle is an **arc**.

The region enclosed by an arc and two radii is called a **sector**.

Perimeter



The **perimeter** is the length of the line enclosing a region of a surface.

The perimeter of a polygon is the sum of the lengths of its sides.

The perimeter of a circle is called the **circumference**, c. If the length of the radius is r, then

$$C = 2\pi r$$

The length of an arc depends on the angle at the centre of the sector it bounds. If that angle is θ, then

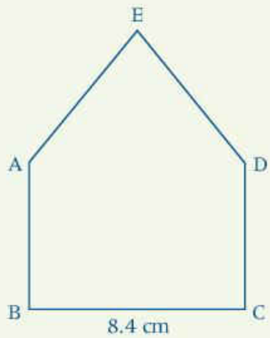
$$\frac{\text{length of arc}}{\text{circumference}} = \frac{\theta}{360} \quad \text{so} \quad \text{length of arc} = \frac{2\pi r\theta}{360} = \frac{\pi r\theta}{180}$$

EXERCISE 4c

Example:

In the diagram, which is not drawn to scale, AB, BC and CD are three sides of a square and AE and DE are two sides of an **equilateral** triangle.
Find the perimeter of the shape.

The perimeter = 5 × 8.4 cm
= 42 cm

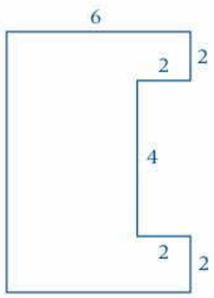


The sides of a square are all equal. The third side of the triangle is AD. As the triangle is equilateral, the two sides AE and DE are also 8.4 cm.

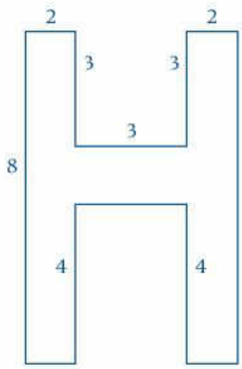
Find the perimeter of each shape in questions 1 to 4.

All measurements are in centimetres.

1



2



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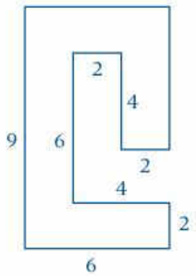
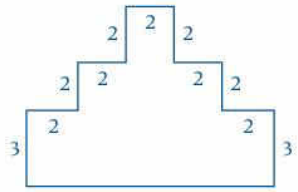
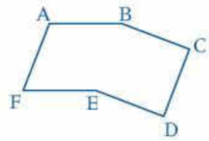


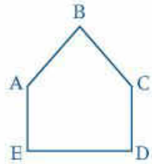
4



5 AB, AF and FE are three sides of a rhombus. BC, CD and DE are three sides of a square of side 4.5 cm. Calculate the perimeter.

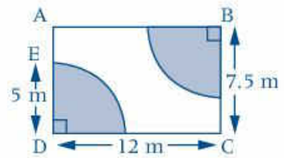


6 AB and BC are two sides of an equilateral triangle. AC and ED are opposite sides of a rectangle. ED = 9 cm and AE = 4 cm. Find the perimeter of this shape.



Example:

This diagram, not drawn to scale, shows a rectangular garden with two paved areas that are sectors of a circle of radius 5 m. The rectangle is 12 m long and 7.5 m wide. Find the perimeter of the unpaved section of the garden.



The length of one arc $= \frac{1}{4} \times 2\pi r = \frac{1}{4} \times 2\pi \times 5 = 7.853...$ m

The length of a straight short edge $= (7.5 - 5)$ m $= 2.5$ m

The length of a straight long edge $(12 - 5)$ m $= 7$m

Perimeter $= 2 \times 7.853... + 2 \times 2.5 + 2 \times 7$ m
$= 34.7$ m to 1 d.p.

Each sector is one quarter of the region inside a circle (they are called quadrants). Each arc is $\frac{1}{4}$ of the circumference of a circle of radius 5 m.

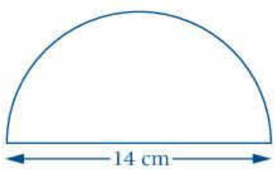
The perimeter is the length of two arcs, the two shorter sides and the two longer sides.

Take $\pi = \frac{22}{7}$,

7 Find the circumference of a circle whose radius is

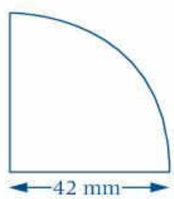
a 20 cm **b** 35 mm.

8 a



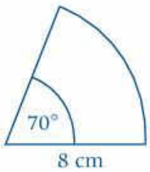
Find **i** the length of the arc
ii the perimeter of the shape.

b



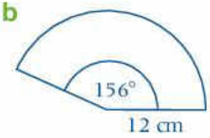
Find **i** the length of the arc
ii the perimeter of the shape.

9 **a**



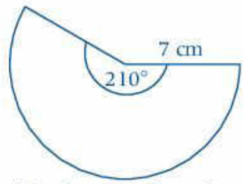
Find **i** the arc length
ii the perimeter.

b



Find **i** the arc length
ii the perimeter.

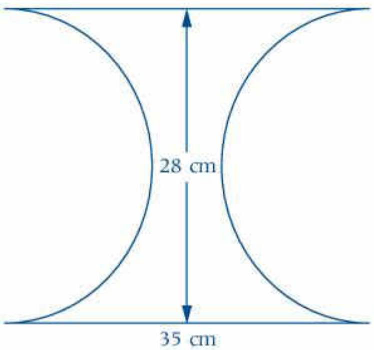
c



Find **i** the arc length
ii the perimeter.

In questions **10** to **13**, find the perimeter of each shape.

10



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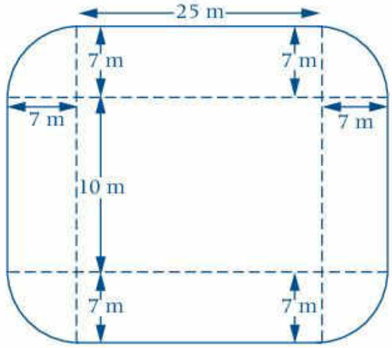

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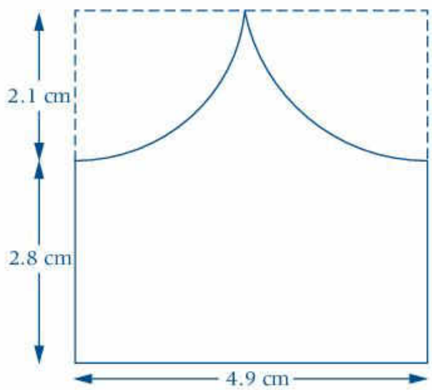
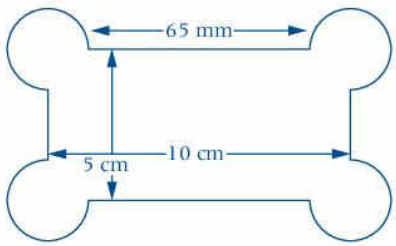


13



Each arc is $\frac{3}{4}$ of a circle

14 A cylinder lawn mower is 35 cm wide and has a diameter of 21 cm. Calculate

a the distance moved forward by the mower for each revolution of the cylinder

b the number of complete revolutions required to mow a straight run of 99 m.

Assume that the cylinder revolves without slipping.

15 A circular tablecloth is to be made for a circular table of radius 60 cm. If the overhang is to be 12 cm everywhere, find the circumference of the tablecloth.
Give your answer correct to the nearest whole number.

16 Calculate, in degrees, the angle subtended at the centre of a circle of radius 8.1 cm by an arc of length 20.7 cm.

> This means the angle between the radii of the sector bounded by the arc. Substitute the values you know into arc length $= \frac{\pi r\theta}{180}$ and solve the resulting equation for θ.

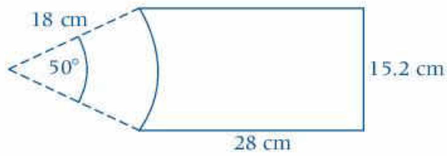
17 An arc of length 20 cm subtends an angle of 45° at the centre of a circle.
Find, in terms of π, the radius of the circle.

> This means do not substitute a numerical value for π but leave π in your answer.

18 The perimeter of a sector of a circle is 50 cm.
The radius of the circle is 15 cm.
Find the angle between the radii.

19 A piece of wire is in the form of a square of side 12 cm. The wire is opened out and bent into a circle. Find the radius of the circle.

20 The shape in the diagram is bounded by three sides of a rectangle and an arc of a circle. The radius of the circle is 18 cm and the arc subtends an angle of 30° at the centre of the circle. Calculate the perimeter of the shape.



Units of area

Area is a measure of a region of a surface. It is measured in standard sized squares.

The units in common use are the hectare (ha), the square kilometre (km^2), the square metre (m^2), the square centimetre (cm^2) and the square millimetre (mm^2), where

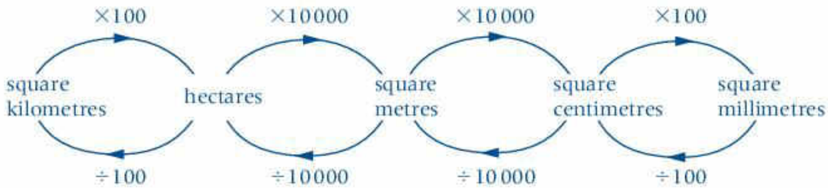
$$1 \text{ km}^2 = 100 \text{ hectares} = 10^6 \text{ m}^2$$

$$1 \text{ ha} = 10000 \text{ m}^2$$

$$1 \text{ m}^2 = 10000 \text{ cm}^2$$

$$1 \text{ cm}^2 = 100 \text{ mm}^2$$

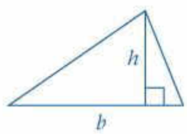
Imperial units of area in common use are square miles (sq miles), **acres**, square yards (sq yd) and square inches (sq in).
1 ha ≈ 2.47 acres



Areas of polygons and circles

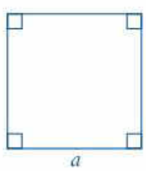
The area enclosed by a **triangle** is given by half the base times **perpendicular** height:

$$A = \frac{1}{2}bh$$



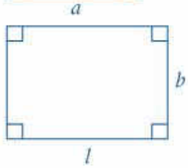
The area enclosed by a **square** is given by the square of the length of one side:

$$A = a^2$$



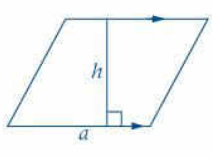
The area enclosed by a **rectangle** is given by the product of the length and the breadth:

$$A = lb$$

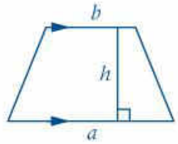


The area enclosed by a **parallelogram** is given by the length of one side times the perpendicular distance between that side and the side opposite it:

$$A = ah$$

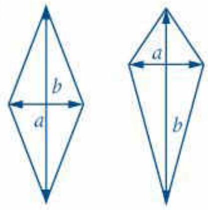


The area enclosed by a **trapezium** is given by half the sum of the lengths of the parallel sides times the perpendicular distance between them:



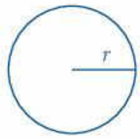
$$A = \frac{1}{2}(a + b)h$$

The area enclosed by a **rhombus** (and a kite) is given by half the product of the length of the diagonals:



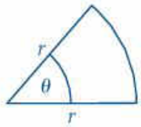
$$A = \frac{1}{2}ab$$

The area enclosed by a **circle** is given by:



$$A = \pi r^2$$

The area of a **sector** of a circle is the same fraction of the area of the whole circle as θ is of 360:

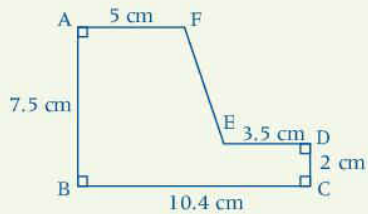


$$\frac{A}{\pi r^2} = \frac{\theta}{360} \quad \text{so} \quad A = \frac{\theta}{360} \times \pi r^2$$

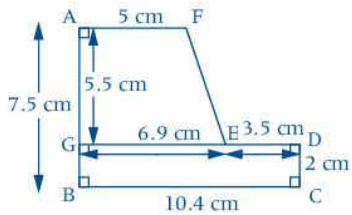
EXERCISE 4d

Example:

In this diagram, which is not drawn to scale, AB = 7.5 cm, BC = 10.4 cm, CD = 2 cm, DE = 3.5 cm and AF = 5 cm.
Find the area enclosed by ABCDEF.



We can divide this shape into a rectangle and a trapezium by drawing EG parallel to BC.



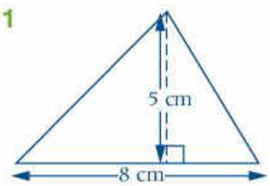
Area enclosed by BCDG $= 2 \times 10.4$ cm²
$= 20.8$ cm²

Area enclosed by AGEF $= \frac{1}{2}(5 + 6.9) \times 5.5$ cm²
$= 32.725$ cm²

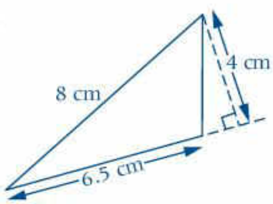
Area enclosed by ABCDEF $= 20.8 + 32.725$ cm²
$= 53.5$ cm² correct to 1 d.p.

Find the area of these triangles.

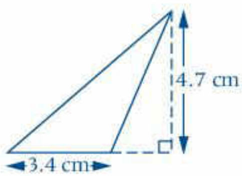
1



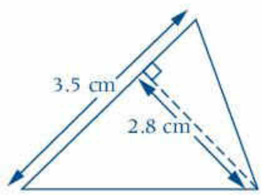
2



3

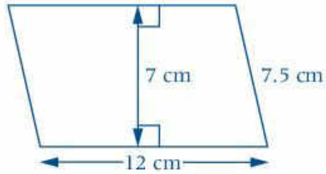


4



Find the area of these parallelograms.

5



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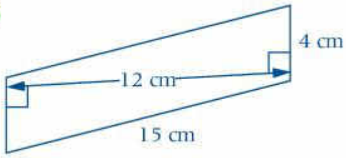

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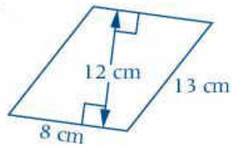

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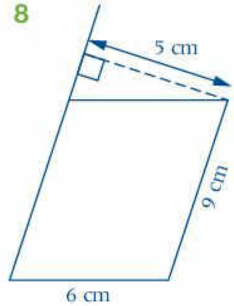
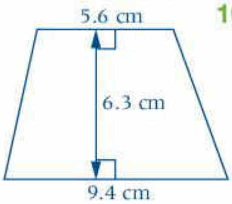
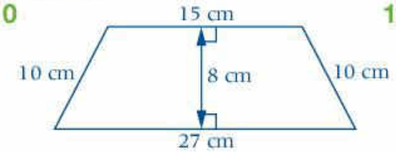


Find the area of these trapeziums.

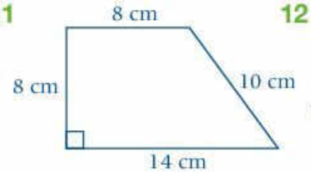
9



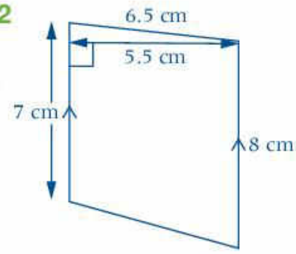
10



11

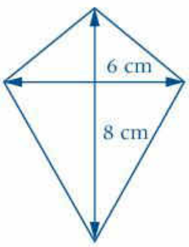


12

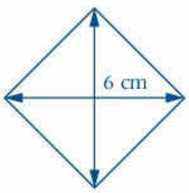


Find the area of each shape.

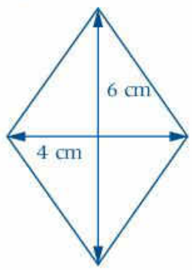
13 This is a kite.



14 This is a square.

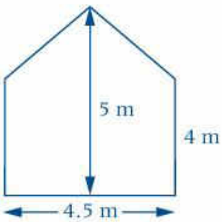


15 This is a rhombus.

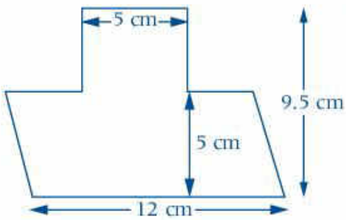


Each of these shapes can be divided into two equal triangles.

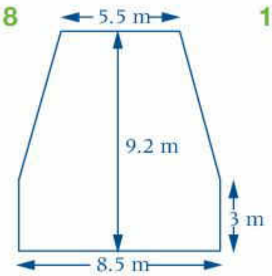
16



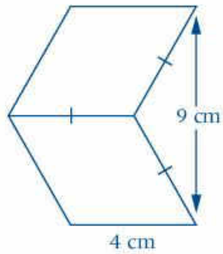
17



18

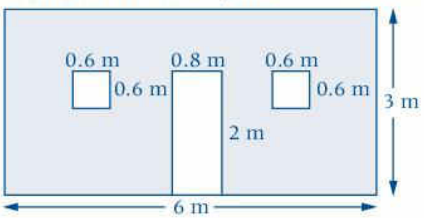


19

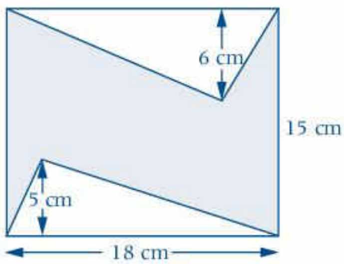


In questions 20 and 21, find the shaded area.

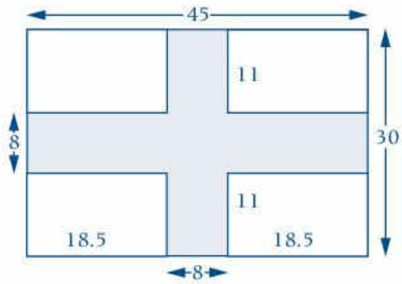
20



21



22 The diagram shows a flag. All dimensions are in centimetres.
Find **a** the perimeter of the cross, which is shaded.
b the unshaded area
c the shaded area.



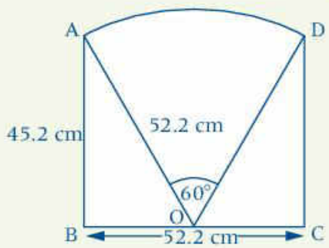
23 A rectangular lawn measuring 12 m long by 10.5 m wide is to be bordered on two adjacent sides by a path 0.75 m wide.
Find the area of this path.

Draw a diagram.

24 The page of a dictionary measures 24 cm by 17 cm with the text set out in two columns separated by a 5 mm margin. At the edges of the page are borders 1.5 cm wide except at the bottom of the page where it is only 1 cm wide.
Find **a** the area of text on one page
b the area on one page not used for text
c the total area of text if the dictionary has 1640 similar pages.

Example:

The diagram, which is not drawn to scale, shows a mirror frame. AD is an arc of a circle whose centre O is on BC. AB = 45.2 cm, BC = 52.2 cm. AO = 52.2 cm and angle AOD = 60°.
Find the area enclosed by the frame.



Area enclosed by ABO $= \frac{1}{2} \times 45.2 \times 26.1\ \text{cm}^2$
$= 589.86\ \text{cm}^2$

Area of the sector AOD $= \frac{1}{6} \times \pi \times 52.2^2$
$= 1426.72...\ \text{cm}^2$

This shape consists of two triangles that are identical and a sector of a circle. 60° is $\frac{1}{6}$ of 360° so the area of the sector is $\frac{1}{6}$ the area of the circle.

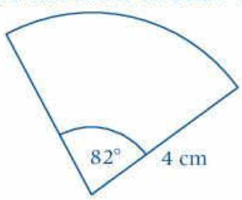
Area enclosed by the frame $= 2 \times 589.86 + 1426.72...\ \text{cm}^2$
$= 2610\ \text{cm}^2$ to 3 s.f.

For the remaining questions, use $\pi = 3.14$ and give answers correct to 3 significant figures. Assume that all curved lines are circles or arcs of circles.

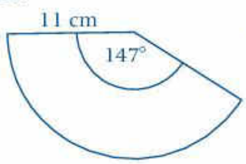
25 Find the area enclosed by a circle
a of radius 12 cm **b** of radius 5.4 cm **c** of diameter 13 cm.

26 Find the area of each sector.

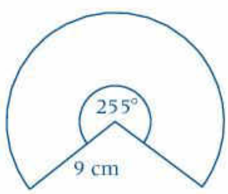
a



b



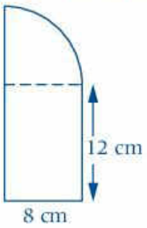
c



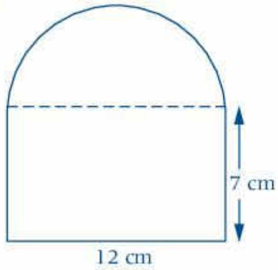
27 Find **a** the radius of a circle that encloses an area of 100 cm^2
b the diameter of a circle that encloses an area of 5.5 m^2.

In questions **28** to **30**, find the area of each shape.

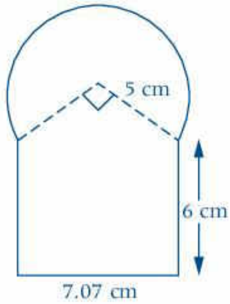
28



29

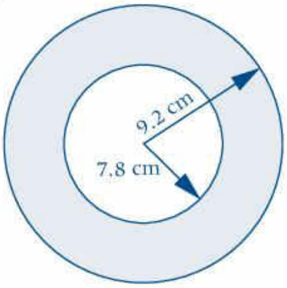


30

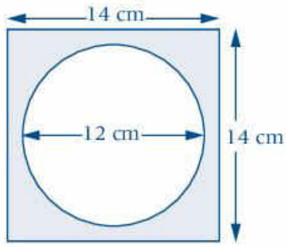


In questions **31** to **35**, find the area of the shaded part.

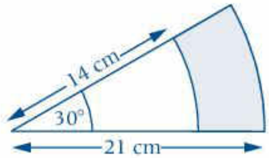
31



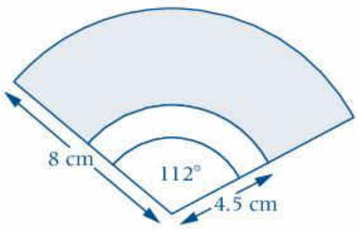
32



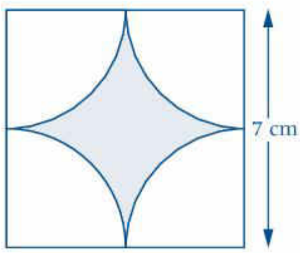
33



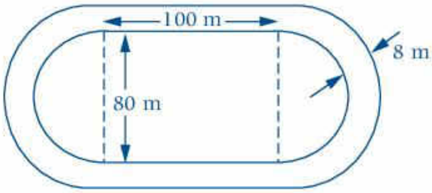
34



35

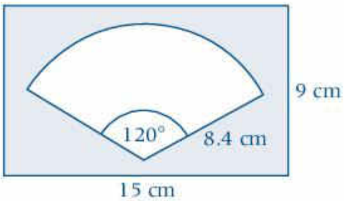


36 The running track in a stadium is 8 m wide and surrounds a rectangle measuring 100 m by 80 m, with semicircular ends.



Find
a the distance around the inside edge of the track
b the distance covered by a runner who runs on the outside edge of the track
c the total area covered by the track and the area within it
d the total area of the track.

37 A machine part, which is a sector of a circle radius 8.4 cm with an angle of 120° between the radii, is pressed out from a rectangular sheet of metal measuring 15 cm by 9 cm.
Calculate **a** the perimeter of the part
b the area of the part
c the area of sheet metal that remains.

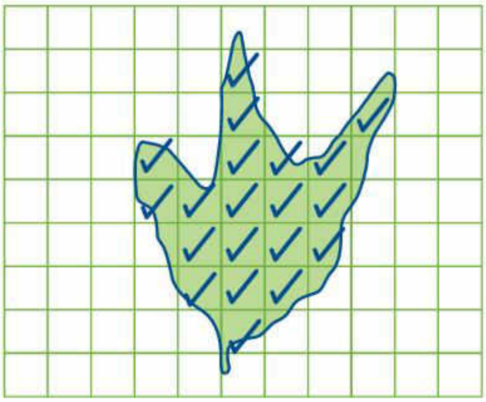


Areas of irregularly shaped plane figures

When a plane figure has an irregular shape we can use a square grid on a flat surface and count the squares inside the surface to estimate its area.

Sometimes the squares do not fit exactly on the area we are finding. When this is so we count a square if at least half of it is within the area we are finding, but exclude it if more than half of it is outside.

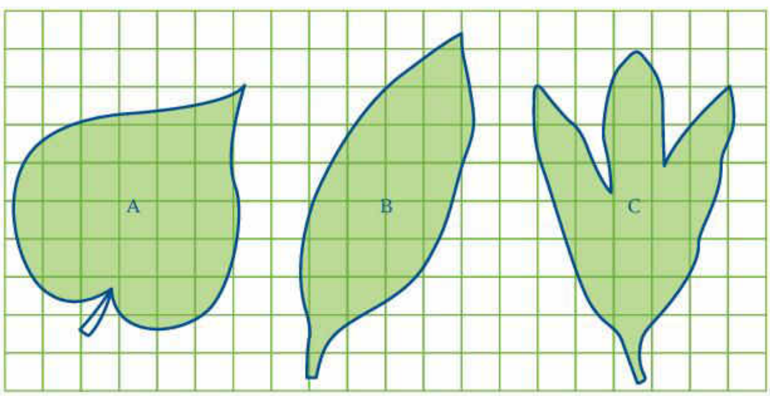
The diagram shows the outline of a leaf drawn on 1 cm squared paper reduced in size to fit this page.



We have ticked 20 squares, so the area of this leaf is about $20\,cm^2$.

EXERCISE 4e

This set of diagrams shows the outlines of three leaves that were drawn on 1 cm squared paper. The diagram has been reduced in size to fit this page.



1 By counting squares find the approximate area of
- **a** the leaf outline marked A.
- **b** the leaf outline marked B.
- **c** the leaf outline marked C.

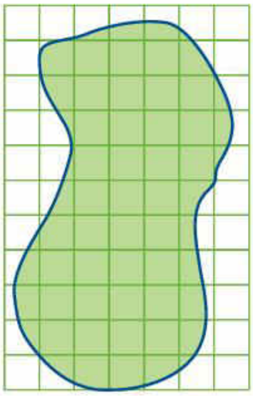
2 Which leaf has **a** the largest area **b** the smallest area?

3 Why are your answers to question **1** only approximate?

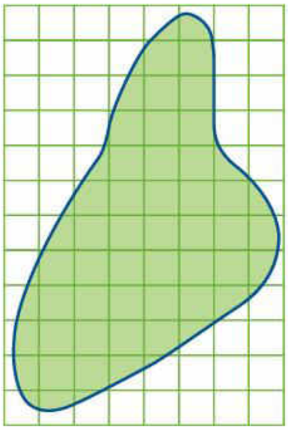
4 For each part of question **1**, decide if your answer is larger on smaller than the true area.

In the following questions, the figures were drawn on 1 cm squares, then the diagrams were reduced in size; so each square represents 1 cm^2. Find the area of the figures by counting squares.

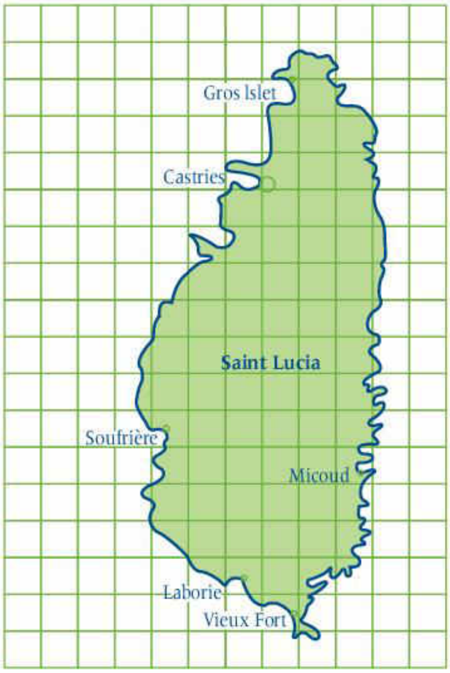
5



6



7 This diagram is a map of St Lucia.



a Find, roughly, the area of St Lucia as a number of grid squares.

b Each grid square represents an area of 9.48 square kilometres. Use this to find an estimate of the area of St Lucia in square kilometres.

Volume

Volume measures the amount of space occupied by a solid.
Volume is measured in standard sized cubes.
These are the cubic metre (m^3), the cubic centimetre (cm^3) and the cubic millimetre (mm^3).

Capacity

Capacity is another measure of volume. It is the amount that can be contained in a given space and is usually used for measuring volumes of liquid and gas. The units of capacity are the litre and the millilitre (ml), where

$$1 \text{ litre} = 1000 \text{ ml}$$
$$1 \text{ litre} = 1000 \text{ cm}^3$$

1 litre is the volume of 1 kg of pure water at 4°C and a pressure of 760 mm of mercury. 1 litre is not exactly 1000 cm^3 (this is derived from the definition of a metre) but is near enough for practical purposes.

The imperial measures of capacity are the fluid ounce, the **pint** and the **gallon** where

$$1 \text{ gallon} = 8 \text{ pints}$$
$$1 \text{ pint} = 20 \text{ fluid ounces.}$$

The gallon is defined differently in the USA where 1 gallon ≈ 3.8 litres. Assume that gallons in this book are UK gallons unless told otherwise.

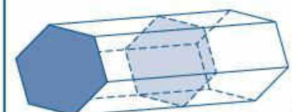
For an approximate conversion to metric units, use

$$1 \text{ gallon} \approx 4.5 \text{ litres}$$

Prisms

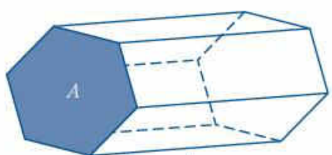
A right **prism** has two faces that are identical polygons and the other faces are rectangles. When a cut is made parallel to the ends of a right prism, we get the same shape as an end. This shape is called the **cross-section** and it is constant, i.e. the same throughout the prism.

This is a right prism.



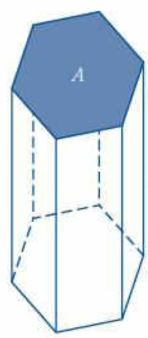
The volume of a prism = area of the cross-section × length.
So the volume of this prism is given by

$$V = Al$$



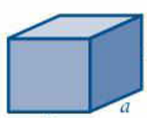
If the **prism** is standing on the cross-section, the length is the height, so

$$V = Ah$$



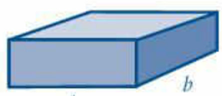
A **cube** is a right prism whose ends are squares and whose length is the same as its width. All the edges are the same length so the area of the cross-section is a^2.
The volume of a cube is given by

$$V = a^3$$



A **cuboid** is a right prism whose ends are rectangles or squares.
The volume of a cuboid is given by

$$V = lbh$$

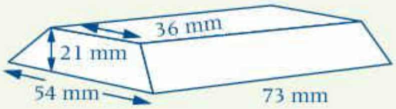


The **surface area** of a prism is the sum of the areas of all the faces.

EXERCISE 4f

Example:

This prism, not drawn to scale, represents a metal bar. The cross-section is a trapezium.



a Find the volume of the bar.

b The bar is melted down and recast into a cube. Find the length of one edge of the cube.

a Area of cross-section $= \frac{1}{2}(54 + 36) \times 21 \text{ mm}^2$
$= 945 \text{ mm}^2$

Volume of prism $= 945 \times 73 \text{ mm}^3$
$= 68\,985 \text{ mm}^3$
$= 69\,000 \text{ mm}^3$ correct to 2 s.f.

We first find the area of the cross-section.

The dimensions are given to 2 s.f., so we give the answer to 2 s.f.

b Volume of the cube $= 68\,985 \text{ mm}^3$

Edge of the cube $= \sqrt[3]{68\,985}$ mm $= 41.01\ldots$ mm
$= 41.0$ mm correct to 3 s.f.

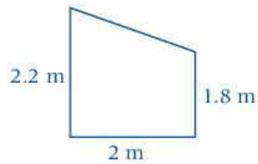
The volume of the cube = volume of the prism. So $a^3 = 68\,985 \text{ mm}^3$ (Use the uncorrected value for the volume so that errors are not compounded.)

Give answers that are not exact correct to 3 significant figures.

1 Find the volume of a prism
  a of length 32 mm and cross-sectional area 2.74 cm^2
  b of length 2.2 m and cross-sectional area 39.2 cm^2.

2 The cross-section of a trough is a trapezium with the parallel sides of lengths 80 cm and 50 cm.
If the trough is 2 m long and 25 cm deep, find its volume.

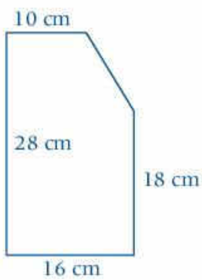
3 A lean-to glasshouse is 2 m wide and 3 m long. The heights of the back and front walls are 2.2 m and 1.8 m respectively.
Find its volume.



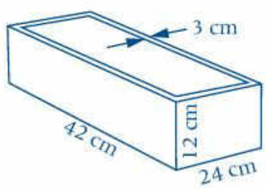
4 A classroom measuring 8 m by 6 m is to be used for a class of 32 pupils.
If 5 m^3 of airspace is allowed for each pupil, how high should the ceiling be?

5 How many litres of water can be stored in a rectangular tank measuring 1.2 m by 80 cm by 50 cm?

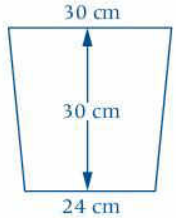
6 The diagram shows the cross-section of a kerbstone which is 1 m long.
Calculate its volume in cubic metres.



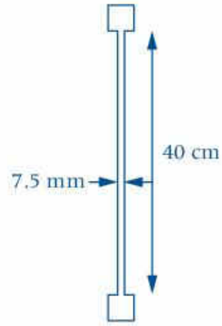
7 An open mould for making concrete blocks is shown in the diagram.
The external dimensions of the mould, which is everywhere 3 cm thick, are 42 cm by 24 cm by 12 cm.
Calculate **a** the volume of a concrete block
  b the volume of material used in making the mould.



8 The diagram shows the uniform cross-section of a horse trough which is a trapezium, the parallel sides of which have lengths 30 cm and 24 cm. If the trough is 30 cm deep and 2.5 m long, find the volume of water it will hold when full, giving your answer in cubic metres.



9 The diagram shows the cross-section through a garden fence panel. The centre section is 40 cm high and 7.5 mm thick. At the top and bottom is a square of side 2.5 cm. Calculate the area of cross-section and hence find the volume of wood needed to make a fence 2 m long.



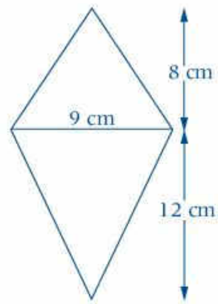
10 The diagram shows the cross-section of a uniform metal bar which is 3.5 m long.

Find

a its cross-sectional area in square centimetres

b its volume in cubic centimetres

c its mass, correct to the nearest kilogram, if 1 cm^3 of the metal has a mass of 8.2 g.



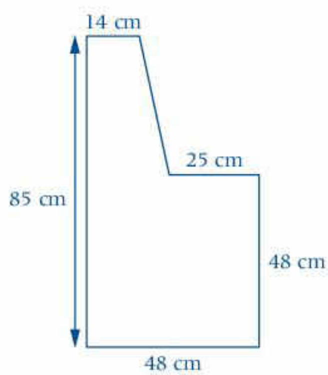
11 A closed rectangular box measures 12 cm by 8 cm by 6 cm and is made from wood 5 mm thick.

a Find the volume of wood used, giving your answer in cubic centimetres.

Another rectangular box which is twice as long, twice as wide and twice as high is to be made from wood of the same thickness.

b Calculate the volume, in cubic centimetres, of wood required for this second box.

12 The diagram shows the cross-section of a concrete end for a park seat. If the structures are 6 cm thick, calculate the volume of concrete used in the manufacture of one seat.



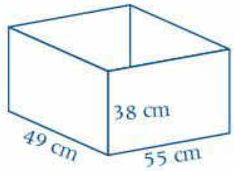
The seat has two ends!

Example:

This cuboid, not drawn to scale, represents an open tank.
The internal dimensions are 47 cm by 55 cm by 38 cm.

a Find the surface area of the inside of the tank.

b Find the capacity of the tank.



a The surface area of the inside
$= 47 \times 55 \text{ cm}^2 + 2(38 \times 55) \text{ cm}^2 + 2(38 \times 47) \text{ cm}^2$
$= 10\,337 \text{ cm}^2 = 10\,000 \text{ cm}^2$ correct to 3 s.f.

The internal faces are the rectangular base, two rectangular faces measuring 38 cm by 55 cm and two rectangular faces measuring 38 cm by 47 cm.

b The internal volume of the tank $= 47 \times 55 \times 38 \text{ cm}^3$
$= 98\,230 \text{ cm}^3 = 98.23...$ litres
$= 98.2$ litres correct to 3 s.f.

1 litre = 1000 cm^3

13 If the total surface area of a cube is 150 cm^2, find its volume.

14 A prism with a rectangular cross-section measuring 4 cm by 5 cm is 2.5 m long. Find

a the area of cross-section

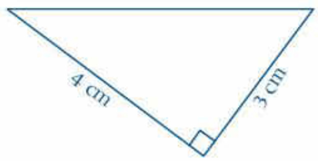
b the total surface area of the prism

c its volume.

15 The diagram shows the cross-section of a water channel which is 10 m long and open at the top.

a How many litres of water will it hold when full?

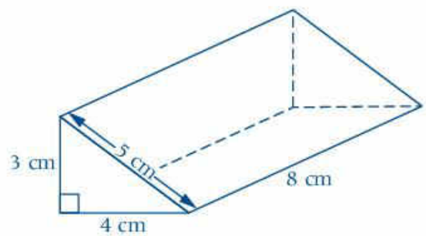
b Find, in square metres, the area of the channel in contact with water.



16 The cross-section of a wooden wedge is a triangle with sides 3 cm, 4 cm and 5 cm. The wedge is 8 cm wide. Find

a its volume

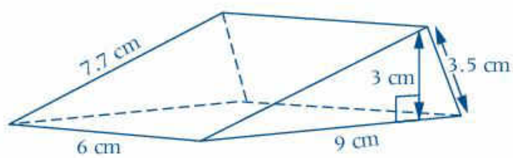
b its surface area.



17 The diagram shows a triangular prism. Use the given measurements to find

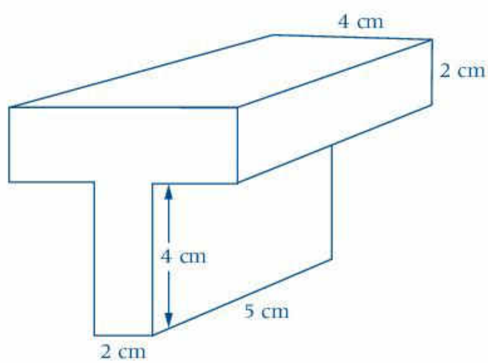
a its surface area

b its volume.

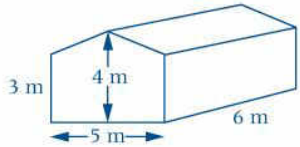


18 **a** Use the measurements given on the diagram to find the volume of wood used to make this capital letter T.

b The solid is to be painted. One tin of spray paint covers 1 m^2. How many tins are needed to give enough paint for 3 coats?



19 The cross-section of a building is a rectangle surmounted by a triangle. It is 5 m wide, 3 m high at the eaves and it is 4 m from the floor to the ridge.
   a If the building is 6 m long calculate its capacity in cubic metres.
   b Find the surface area of the building, excluding the roof and the floor.



Sources of error

A length can never be measured exactly. The accuracy of such a measurement depends on the accuracy of the measuring instrument. A school ruler can usually measure reliably to the nearest millimetre.

If the length of a nail is measured as 25 mm then we can assume this is correct to the nearest millimetre.

This means that if l mm is the actual length of the nail, then $24.5 \leqslant l < 25.5$.

The largest value of l that is less than 25.5 is 25.49999... which is 25.5 when corrected to any number of decimal places. For practical purposes we therefore assume that the greatest value of l is 25.5. So the length given as 25 mm has an error of up to plus or minus 0.5 mm.

When we calculate with corrected numbers, the errors can increase. For example, when we add 25 mm, to 5 mm, the smallest possible value is 24.5 mm + 4.5 mm = 29 mm and the largest possible value is 25.5 mm + 5.5 mm = 31 mm therefore 25 mm + 5 mm = 30 mm has an error of up to plus or minus 1 mm.

It follows that the greater the number of corrected numbers involved in a calculation, the greater the possible error in the final result.

ABCD MIXED EXERCISE 4

Several answers are given for these questions.
Write down the letter that corresponds to the correct answer.

1 The number of centimetres in 0.084 km is
   A 840   B 8400   C 84 000   D 84 0000

2 The number of millimetres in 0.76 m is
   A 7.6   B 76   C 760   D 7600

3 A book with 100 pages is 1.2 cm thick. The thickness of 1 leaf is
   A 0.024 mm   B 0.12 mm   C 0.24 mm   D 2.4 mm

A leaf has 1 page on each side.

4 The number of grams in 0.04 kg is
   A 0.4   B 4   C 40   D 400

5 A quarter to seven in the evening in 24-hour clock time is
   A 0645 hr   B 1645 hr   C 1815 hr   D 1845 hr

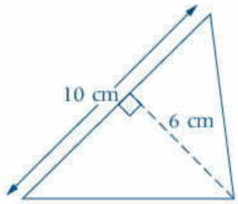
6 When 75 °C is converted into degrees Fahrenheit the temperature is
   A $9\frac{2}{3}$ °F   B $41\frac{2}{3}$ °F   C 135 °F   D 167 °F

7 The time in Barbados is 4 hours behind the time in London. When it is 3.30 p.m. in London the time in Barbados is
   A 11.30 a.m.   B 7.30 p.m.   C 6.30 p.m.   D 12.30 p.m.

8 One US dollar converts to 63 Jamaican dollars. The value of 500 Jamaican dollars in US dollars is
   A 7.94   B 79.4   C 315   D 31 500

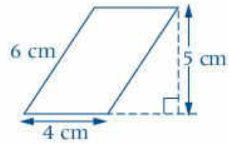
9 The area of this triangle is

A 24 cm^2 **B** 30 cm^2 **C** 48 cm^2 **D** 60 cm^2



10 The area of this parallelogram is

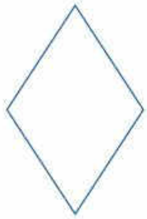
A 10 cm^2 **B** 12 cm^2 **C** 20 cm^2 **D** 24 cm^2



11 Which of the following statements about this rhombus are true?

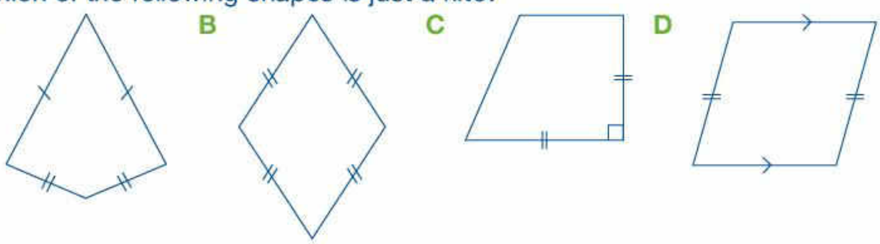
1 All the angles are equal.
2 The diagonals are equal.
3 The diagonals bisect each other at right angles.

A 1 only **B** 2 only **C** 3 only **D** 1 and 3



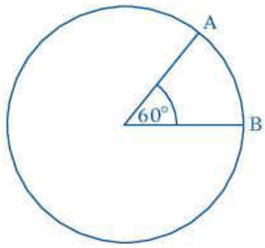
12 Which of the following shapes is just a kite?

A **B** **C** **D**



13 The circumference of this circle is 90 cm.
The length of the arc AB is

A 15 cm **B** 30 cm **C** 60 cm **D** 180 cm



14 A cube has the same volume as a cuboid measuring 2 cm by 4 cm by 8 cm. The length of an edge of the cube is

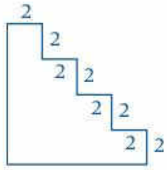
A 2 cm **B** 4 cm **C** 8 cm **D** 14 cm

15 Which of the following best describes a quadrilateral whose angles are all 90°?

A a trapezium **B** a parallelogram **C** a rectangle **D** a rhombus

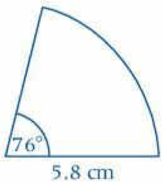
16 The area of this shape, where all the measurements are in millimetres, is

A 36 mm^2 **B** 40 mm^2
C 44 mm^2 **D** 48 mm^2

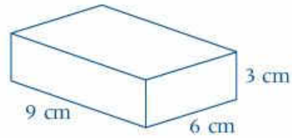


17 The area of this sector of a circle is

A 7.1 cm^2 **B** 17.6 cm^2
C 21.3 cm^2 **D** 22.3 cm^2

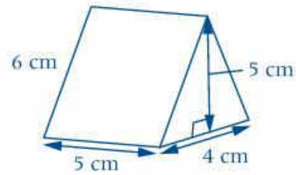


18 The total surface area of this rectangular block is



A 168 cm^2 **B** 189 cm^2
C 198 cm^2 **D** 222 cm^2

19 The volume of this prism is



A 50 cm^3 **B** 60 cm^3 **C** 100 cm^3 **D** 150 cm^3

20 The best equivalent distance in kilometres to 100 miles is
A 50 km **B** 73 km **C** 160 km **D** 200 km

21 An instruction in an old knitting pattern says knit 12 inches. To the nearest centimetres this is
A 6 cm **B** 24 cm **C** 30 cm **D** 60 cm

22 The nearest equivalent mass in grams to 4 oz is
A 50 g **B** 100 g **C** 150 g **D** 500 g

23 The nearest equivalent mass in pounds to 10 kg is
A 2.2 lb **B** 5 lb **C** 10 lb **D** 22 lb

24 An area of 1.5 hectares is equivalent to
A 2.5 acres **B** 3.7 acres **C** 4 acres **D** 5 acres

25 An old water tank holds 50 gallons. Its capacity in litres (to the nearest litre) is
A 45 litres **B** 227 litres **C** 450 litres **D** 500 litres

26 Ten squares each of side 30 cm to the nearest centimetre are laid end to end in a straight line. The greatest possible length of the line is
A 30.5 cm **B** 295 cm **C** 300 cm **D** 305 cm

27 A rectangle measures 2 cm by 1 cm, each length being correct to the nearest centimetre. The least possible value for the area of the rectangle is
A 0.5 cm^2 **B** 0.75 cm^2 **C** 1 cm^2 **D** 1.5 cm^2

PUZZLE

Which of the following circular objects, held at arm's length, will roughly cover the full moon?

A pea, a table-tennis ball, a cricket ball, a baseball.

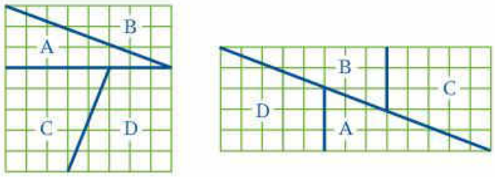
MATHS IS OUT THERE

The word metre was derived from the Greek word *metron*, meaning 'a measure'. The original definition of 1 metre was the length equal to one ten-millionth of the distance from the North Pole to the equator along a meridian.

Time for a library search: find the current definition of 1 metre.

INVESTIGATION

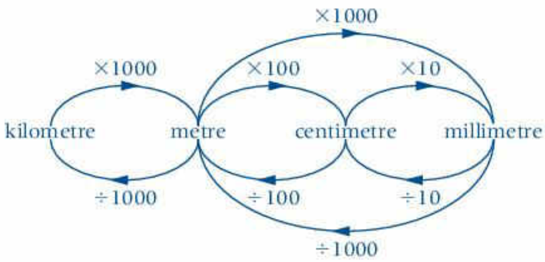
The diagram shows an area of 64 squares cut up and rearranged to make a 5 by 13 rectangle with an area of 65 squares.



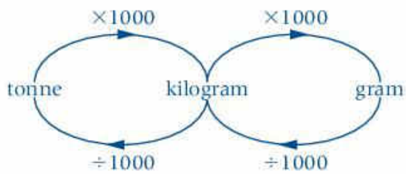
Investigate this and describe why it is an illusion.

IN THIS CHAPTER YOU HAVE SEEN THAT...

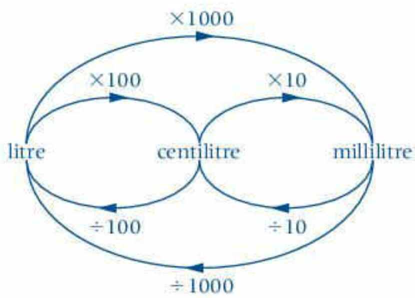
- the relationship between the main units of length are as given in the diagram:



- the relationship between the main units of mass are as given in the diagram:



- the relationship between the main units of capacity are as given in the diagram:

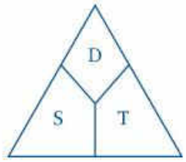


- 1 litre = 1000 cm^3
- temperature is measured in degrees Celsius or in degrees Fahrenheit.
  You can convert between them using $F = \frac{9 \times C}{5} + 32$ or $C = \frac{5}{9}(F - 32)$

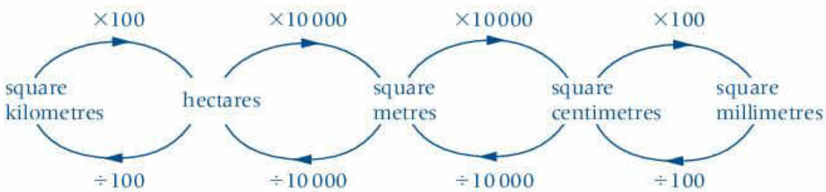
- speed is measured in kilometres per hour (km/h) or metres per second (m/s) where

  (average speed) = (total distance) ÷ time

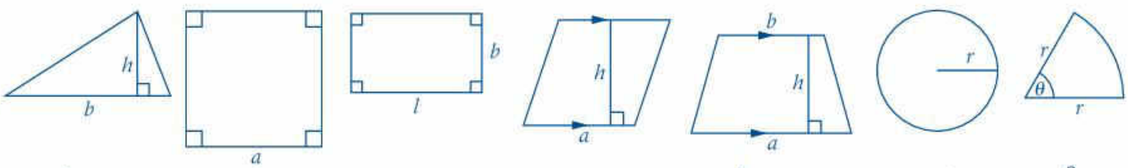
  This diagram gives relationships between distance, time and speed.
  Cover up the one you want to find



- the exchange rate gives the value of one unit of a currency as an amount in another currency, e.g. US$1 = JM$67.8
- a polygon is a plane figure bounded by straight line segments
- the length of an arc of a circle = $\frac{\pi r\theta}{180}$ where θ is the angle subtended by the arc at the centre of the circle
- the perimeter is the length of a line enclosing a region of a surface
- the relationships between the main units of area are as given in the diagram:

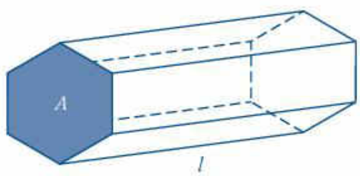


- formulas for working out areas are as given in the diagram:



- $A = \frac{1}{2}bh$ $A = a^2$ $A = lb$ $A = ah$ $A = \frac{1}{2}(a + b)h$ $A = \pi r^2$ $A = \frac{\theta}{360} \times \pi r^2$

- a prism has a uniform polygon as its cross-section.
  All the other faces are rectangles.
  Volume of a prism = area of cross-section × length
  (or area of cross-section × height if the prism is upright)



5 CONSUMER ARITHMETIC

AT THE END OF THIS CHAPTER YOU SHOULD BE ABLE TO...

1. Understand percentage increase and decrease.
2. Solve problems involving profit and loss.
3. Deal with problems about shopping including invoices, sales taxes, hire purchase and mortgages.
4. Understand how salaries and wages are calculated and how income tax and rates are calculated.
5. Understand utility bills.
6. Deal with problems on investments and insurance including simple and compound interest.

MATHS IS OUT THERE

Usury originally meant just the interest charged on a loan but it is now used to mean charging exorbitant or illegal rates of interest on loans.

BEFORE YOU START

you need to know:

✓ how to work with decimals and fractions
✓ how to work with percentages.

KEY WORDS

compound interest, cost price, endowment policy, insurance, interest, investment, loss, multiplying factor, percentage profit, percentage rate, principal, profit, returns, selling price, simple interest

Percentage increase and decrease

A percentage increase or decrease is always calculated as a percentage of the quantity before the change takes place.

If a bus fare of \$30 is to increase by 5%,

the increase is 5% of \$30 $= \frac{5}{100} \times \$30$.

The increased fare = 100% of \$30 + 5% of \$30

$= 105\%$ of \$30 $= \frac{105}{100} \times \$30$

In general, if a quantity N is increased by $a\%$,

the increase is $a\%$ of $N = \frac{a}{100} \times N$,

the increased quantity is $(100\% + a\%)$ of $N = \frac{100 + a}{100 \times N}$.

The fraction $\frac{100 + a}{100}$ is called the **multiplying factor**. It increases a quantity by $a\%$.

If the population of an island now is X people and is expected to decrease by 8% in the next ten years, the population in ten years' time = $(100\% - 8\%)$ of X

$= 92\%$ of $X = \frac{92}{100} \times X$.

In general, if a quantity P is decreased by $a\%$,

the decrease is $a\%$ of $P = \frac{a}{100} \times P$,

the decreased quantity is $(100\% - a\%)$ of $P = \frac{100 - a}{100} \times P$.

The fraction $\frac{100 - a}{100}$ is called the multiplying factor. It decreases a quantity by $a\%$.

Compound percentage change

Suppose a house is bought for \$100 000 and it increases in value (appreciates) at the rate of 10% per year.

After 1 year the value of the house is 110% of \$100 000 = \$110 000.

After another year the value will increase by 10% of its value at the start of the second year so the value of the house is then 110% of \$110 000 = \$121 000.

The actual increase in value gets bigger each year because we are finding 10% of an increasing amount.

After a third year the value will be 110% of \$121 000 = \$133 100.

This is an example of compound percentage change which is when a percentage increase or decrease happens more than once and the changes are compounded.

When a compound increase happens over more than a few intervals of time, the calculation can be shortened by using the formula given below for calculating the final quantity.

$$A = P\left(1 + \frac{r}{100}\right)^t$$

where A is the final amount, P is the initial amount, r is the interest rate per interval of time and t is the total number of time intervals. When the change is a decrease in value, the formula is $A = P\left(1 - \frac{r}{100}\right)^t$.

For example if a loan of \$500 is taken out and interest is charged at 1.5% per month, then (assuming no repayments are made) the formula shows that after 12 months

$A = 500\left(1 + \frac{1.5}{100}\right)^{12} = 500 \times 1.015^{12} = 597.81$

so \$597.81 is owed.

Reverse percentage problems

When we know the value of a quantity after a given percentage change, we can find the quantity before the change.

If a quantity Y includes an increase of $a\%$, then

$Y = \frac{100 + a}{100} \times$ (the quantity before the increase).

So (the quantity before the increase) $= Y \div \left(\frac{100 + a}{100}\right)$

For example, if a book costing \$400 is 3% more than it cost a year ago, then
\$400 $= \frac{103}{100} \times$ the cost a year ago.
So the cost a year ago = \$400 $\div \frac{103}{100}$

If a quantity X includes a decrease of $a\%$, then

$X = \frac{100 - a}{100} \times$ (the quantity before the decrease).

So (the quantity before the decrease) $= X \div \frac{100 - a}{100}$

If the 350 people now living in a village is 15% less than the number ten years ago, then
$350 = \frac{85}{100} \times$ (number ten years ago).
So (number ten years ago) $= 350 \div \frac{85}{100}$

Profit and loss

Many businesses are involved with buying goods in and then selling them.

- The price paid for goods bought in is called the **cost price**.
- The price goods are sold for is called the **selling price**.
- The **profit** (or **loss**) = selling price – cost price.
- The **percentage profit** (or loss) is the profit (or loss) as a percentage of the cost price.
- Percentage profit $= \frac{\text{profit}}{\text{cost price}} \times 100\%$

  and percentage loss $= \frac{\text{loss}}{\text{cost price}} \times 100\%$

Some accountants use a different way of calculating profit but this is the definition we use.

Sales tax

Sales tax is levied in many territories on goods sold and services provided. It is a percentage of the price to be paid and is added on to that price. Sales tax is sometimes called value added tax (VAT) and sometimes general consumption tax (GCT).

EXERCISE 5a

Example:

Mr Sampson bought a car for $400 000. The value had depreciated by 8% after 1 year. Calculate the value of the car after 1 year.

The value after 1 year = (100% − 8%) of the amount paid

$= \frac{92}{100} \times \$400\,000$

$= \$368\,000$

We want the decreased value of the car.

1 Bob earns $2500 a week. He gets a pay rise of 3%. Find his new weekly wage.

2 A car bought for $31 500 loses 76% of its value over 5 years. What is its value when it is five years old?

3 CD prices at TT$50 each are discounted by 25% in a sale. How much will I pay for six such CDs?

4 Christine's weight increased by 6% on holiday. She weighed 55 kg before she went. How many kilograms had she put on when she returned?

5 A retailer gives a discount of 5% if more than 5 jars of coffee are bought at the same time. If the marked price of one jar is $5.40, how much should I pay **a** for 3 jars **b** for 6 jars?

6 The number of students in a school has increased by 12% over the last year. Last year there were 350. How many more students are there this year?

Read the questions carefully to decide if the changed quantity or the change itself is required.

Give money answers that are not exact to the nearest cent.
Give other answers correct to 3 significant figures.

7 The volume of water increases by 4% when frozen. How many cubic centimetres of ice does $425\,\text{cm}^3$ of water make?

8 A lawn measures 8 m by 15 m. Ashley puts a flower bed in the middle of the lawn thereby reducing the area of grass by 30%. Find the area of the flower bed.

9 In a sale, the price of a TV set marked \$1800 is reduced by 15%. Find the reduction.

10 A mobile phone company predicts that its sales next year will rise by 42%. This year it sold 7850 phones. How many does it expect to sell next year?

Example:

Joe borrowed \$5000 from the bank. He paid it back in 12 monthly instalments of \$440. Calculate the percentage profit made by the bank.

Joe paid back $12 \times \$440 = \$5280$.

The bank made a profit of $\$5280 - \$5000 = \$280$

The percentage profit $= \frac{280}{5000} \times 100\% = 5.6\%$

The percentage profit is profit as a percentage of the original loan.

11 A retailer buys a camera for \$320 and sells it for \$368. Find his percentage profit.

12 The cost price of a bicycle is \$700 and the profit is 20% of the cost price. What is the selling price?

13 A shopkeeper buys a set of china for \$540 and adds 55% to give him a profit. VAT at 15% is then added to this total to give the selling price. Find the selling price.

14 An auctioneer arranges the transfer of a piece of furniture from Mr Martin's house to the saleroom at a cost of \$159. At auction the piece is sold for \$4530. If the auctioneer's commission is 12% of the selling price, how much could Mr Martin expect to receive from the sale after paying the commission and the cost of the transfer?

15 An article is bought for \$180. Find its selling price if it is sold
a at a gain of 10%
b at a loss of 10%.

16 In a sale, five articles are sold for the price of four. What percentage reduction is this?

17 Steven Arnold orders a new car believing that its price will be \$98 000. When the car arrives its price has increased by 9%, but the retailer agrees to allow him a discount of 10%.
How much more (or less) must he pay for the car?

18 A retailer bought 45 articles for \$50 580. How much should each be sold for to make a profit of 35%?

19 A company sells its sewing machines to customers at a price which gives the company a profit of 40%.
The company buys one brand of sewing machine for \$564.
What price will the customer pay for this machine?

20 A garden centre buys seats of two different types, paying the manufacturer \$160 for one type of seat and \$240 for the other. The garden centre sells the cheaper seat at a profit of 40% and the dearer seat at a profit of $33\frac{1}{3}$%.

a Calculate the total profit after selling 25 of the cheaper seats and 10 of the dearer seats.

b When the manufacturer increases all his prices by $12\frac{1}{2}$%, the garden centre raises the selling price of the cheaper seat to \$270 and the dearer seat to \$351. Find the percentage profit on each type of seat after the price rise.

Example:

A car, bought for \$120 000, depreciates each year by 20% of its value at the start of the year. Calculate the value of the car after 3 years.

Value after 1 year $= 80\%$ of \$120 000 $= \$\frac{80}{100} \times 120\,000 = \$96\,000$

Value after 2 years $= 80\%$ of \$96 000 $= \$\frac{80}{100} \times 96\,000 = \$76\,800$

Value after 3 years $= 80\%$ of \$76 800 $= \$\frac{80}{100} \times 76\,800 = \$61\,440$

Alternatively, using the formula, $A = 120\,000\left(1 - \frac{20}{100}\right)^3 = 61\,440$

Therefore the value after 3 years is \$61 440.

In this question, the loss gets less each year because we are finding a reduction in a reducing amount.

The value decreases each year so the interest rate is negative.

21 Eli bought a gold coin for \$250. What was its value 2 years later if it appreciated by $7\frac{1}{2}$% a year?

22 The BigSquash Company aim to increase production of their top-selling drink by 20% a year over the coming years. This year they produced 46000 litres. What production level are they aiming for the year after next?

23 Last February the total number of workers employed at a factory was 1200. By this February it had increased by 15%. Assuming that it continues to increase at the same rate what is the projected size of the workforce next February?

24 Andy bought a motorcycle for \$10 000. It depreciated by 20% during the first year and 15% of its value at the end of the first year during the second year. What was its value when it was 2 years old?

25 Mrs Eaton bought a house for \$450 000. It appreciated by 8% during the first year she owned it and 6% during the second year. How much more was its value 2 years after Mrs Eaton bought it?

26 The population of a village, which was 350 at the beginning of 2006, decreased by 10% during the year and by 40% in 2007. How many fewer people were there in the village at the end of 2007 than at the beginning of 2006?

27 Production figures in a factory increased by 15% and 12% respectively over successive years. The initial figures were 7000 units a year. What were the final figures?

28 Last year my second-quarter electricity bill was 8% more than it had been in the first quarter, and the third-quarter bill was 8% lower than the second quarter. The first-quarter bill showed that I had paid \$280. What did I pay for electricity in the third quarter?

29 The population of Beetown is 300 000. Over the next 5 years it is predicted that it will increase by 5% a year.
Estimate the population in 5 years' time.

30 Maria paid $12 000 for an oil painting as an investment and found that its value grew by 4% a year. What was its value 6 years later?

31 Viv bought a house for $80 000. Because of its unfortunate position its value depreciated at 5% per annum. What was its value 20 years later?

Example:

A. N. Electricals sells cell phones for $800. This price includes a sales tax of 15%. Calculate the price before the sales tax is added.

$\$800 = 115\%$ of the price excluding sales tax

Price $= \$800 \div \frac{115}{100} = \$800 \times \frac{100}{115} = \$695.65$ to the nearest cent

32 Mrs Malone sold a digital camera for $2950 thereby making a profit of 18%. What did she pay for it?

33 Mr Patton's salary was $6825 after a rise of 5%. What was it before the rise?

34 A printer costs $5000 including a sales tax of 16%. What is the price of the printer before the sales tax is added?

35 Tony Brown sold a damaged chair for $1199 thereby suffering a loss of 12%. What did he pay for it?

36 Barrie buys an article and sells it for $68.20 thereby making a profit of 24%. What did he pay for it?

37 Lena sells a book to Olly for $52.50 which is at a discount of 30% of what she had paid for it. What did the book cost Lena?

38 Peter pays $358.80 for a set of headlamps. The price includes sales tax at 15%. How much did he pay in sales tax? Express this as a percentage of the price Peter paid.

39 When the price of a cut-glass salad bowl is reduced by 30% it sells at $105. Calculate its original price.

40 If a shopkeeper sells a camera at $312, he makes a loss of 35%. What would it have to be sold for to make a profit of 35%?

41 A poultry farmer wants to rear 945 laying hens. Experience has shown that he can expect 70% of the eggs he incubates to hatch, and 10% of the chicks which hatch to die.
Assuming that there are equal numbers of hens and cockerels at each stage, how many eggs should he place in the incubator?

42 The cash price of some kitchen units, including VAT (sales tax) at 15%, is $4887.50. How much would they cost if the rate of VAT rose to 18%.

43 The cash price of a set of garden tools is $255.30, including GCT (sales tax) at 16.5%.
a Calculate the price before the GCT was added.
b What would be the selling price if the GCT rate was reduced to 12%?

44 A shopkeeper buys an article and marks the selling price to give a profit of 30% on the cost price. When the cost price of the article rises by 10%, she continues to sell at the same price.
What percentage profit on the new cost price is she then making?

45 When $8\frac{1}{2}\%$ of the pupils in a school are absent, 1098 pupils are present. How many pupils are there in the school?

46 Rena sat three tests, at three-monthly intervals. She scored 20% more in the second than in the first, and 20% more in the third than in the second. If she scored 72 in the third test, how many did she score in the first?

47 A store increases the price of a table by 10% before a sale then reduces it in the sale by 10%. How does the sale price compare to the price before the increase?

Invoices and shopping bills

A shopping bill gives details of the prices of the different items purchased and any sales tax.

This is a typical supermarket bill.

A B Stores
15 Arthur Street

1 kg sugar	\$0.96
Tastee Cheese	\$16.80
Cream Crackers	\$1.15
White Rice	\$12.50
Margarine	\$1.62
Colgate toothpaste	\$1.34
Balance due	\$34.37

Note that a bill shows the amount that has to be paid whereas a receipt shows that the amount has been paid.

An invoice is similar to a shopping bill. It is provided by a supplier of goods and services detailing the individual prices of items.

This is an invoice from a garage.

B C Motors
Stays ST
Kingston
Jamaica
GCT Reg #3457762

Invoice No. 50069332
Date 09/12/07

Description	Quantity	Unit price (\$)	Charge (\$)
Carry out service			1089.00
Plugs	4	172.00	688.00
Engine oil	3	250.00	750.00
Air filter	1	680.00	680.00
		Subtotal	3207.00
		Tax	529.16
		Total due	3736.16

EXERCISE 5b

Complete the following bills and write down the amounts represented by the letters.

1. 5 oranges at 60 c each A
   3 kg bananas at $1.42 per kg B
   2 litres milk at $1.35 a litre C
   $1\frac{1}{2}$ kg butter at $6.40 per kg D
   Total E

2. 3 cans orange juice at $1.70 a can A
   3 bottles of bitters at $5.75 B
   1 jar marmalade at $4.74 C
   2 jars cherry jam at $5.20 a jar D
   Total E

3. 3 brushes at $6.35 each A
   $2\frac{1}{2}$ litres of paint at $12.85 a litre B
   7 rolls of wallpaper at $2.55 a roll C
   5 sheets of sandpaper at 76 c per sheet D
   Subtotal E
   VAT at 15% F
   Total G

4. 4 electric light bulbs at $14.60 each A
   6 batteries at $1.34 each B
   1 saucepan $26.50 C
   8 blocks of soap at $2.84 a block D
   Subtotal E
   Sales tax at 12% F
   Total G

5. Copy and complete the table. Hence write down the amounts represented by the letters.

Article	**Quantity**	**Unit price ($)**	**Total cost ($)**
Skirts	3	64.99	A
Blouses	4	B	199.80
DVDs	C	25.50	127.50
Posters	9	6.40	57.60
		Subtotal	D
		VAT at 15%	86.98
		Grand total	E

6 Copy and complete this bill which shows Kenny's bill when he went to the ROADUSER shop. Hence write down the value represented by each letter.

Article	Quantity	Unit price ($)	Total ($)
Car mat	3	A	22.95
Car wash	2 litres	3.44/litre	B
Sponge	1	12.59	12.59
Oil	C litres	6.34/litre	22.19
		Subtotal	D
		VAT at 15%	E
		Grand total	F

7 Copy and complete this invoice which is for a car service at a garage. Hence write down the value represented by each letter.

Haywoods Garage Ltd
Wood Street
Kingston
Jamaica
GCT Reg #66350021

Invoice No. HE25004
Date 09/12/07

Description	Quantity	Unit price ($)	Charge ($)
Carry out service			1729.00
Plugs	4	185.00	A
Engine oil	3	B	435.00
Air filter	1	680.00	680.00
		Subtotal	C
		GCT @ 12.5%	D
		Total due	E

8 Copy and complete this invoice which shows the amount Ann Kenny has to pay for internet services. Hence write down the value represented by each letter.

Invoice 107702 **Date** 7/6/2007 **Invoice name** Sharon Green

Item	Description	Price	Qty
SVC#87	6/17/2007 to 7/17/2007 LightSpeed ADSL	$2,388.70	1
DTL#1312	Email s.green@kasnet.com	–	–
Tax#1	GCT ($2,388.70 × 16.5%)	A	1

Subtotal	Total taxes	Total	Paid
$2,388.70	B	C	$85

Please pay this amount: D

Example:

Mrs Taylor bought 6 bananas at $1.22 each, 5 apples at $2.36 each and some sweet potatoes at 76 c each. Her bill was $26.72.
Calculate the number of sweet potatoes that Mrs Taylor bought.

The cost of the 6 bananas $= 6 \times \$1.22 = \$7.32$

The cost of the 5 apples $= 5 \times \$2.36 = \$11.80$

The cost of the sweet potatoes $= \$26.72 - \$(7.32 + 11.80) = \$7.60$

The number of sweet potatoes bought $= 7.60 - 0.76 = 10$

The total cost of potatoes = total bill minus the cost of the bananas and apples. Divide this by the cost of each potato to give the number bought. Make sure the units are the same.

9 Mr Amanuah bought tickets to take his wife and three children to a concert. Tickets for adults cost $55.50 and tickets for children $35.50. He paid with five $50 notes. How much change did he get?

10 Mr Sospeter bought a set of four tyres for his car. Tyres cost $276 each to which was added 15% sales tax. How much did the transaction cost him?

11 Shamsa bought the following articles at the local shop:
3 magazines costing $7.50, $10.75 and $12.25 respectively, writing paper $7.45, envelopes $6.65, 3 pencils at $1.45 each and 2 pens at $3.55 each. She offered to pay with a $50 bill. By how much was this too much or too little?

12 Simon went to a cricket match. It cost him $12.55 to get there and back, $45 for admission and he spent $35.58 at the ground on refreshments.
   a How much did the day cost him?
   b If he started the day with $124.50, how much did he have when he got home?

13 Easter bought a new computer system. He paid $655 for the tower and keyboard, $165 for the printer, $175 for the scanner and $744.75 for extra software. By paying cash he was given a discount of 10%. How much did the new system cost him?

Utility bills and property tax

Utilities are commodities used in the home such as gas, electricity, water and fixed line telephones.

The charges for utilities are usually made up of a fixed charge for a period of time together with a charge for each unit used. The reading from a meter gives the number of units used.

Property tax is a tax levied on land and houses. This tax is usually based on the size and location of the property.

The unit of electricity is the kilowatt-hour (kWh). The unit of water is the cubic metre.
The unit for telephone charges is price per minute but this price varies depending on the type of phone call, e.g. local, cell phone, premium rate number.

EXERCISE 5c

Example:

The charges for electricity are given in this table:

Fixed charge	\$15 per month
Electricity charge	\$2.55 per kWh

The meter readings for the kWh used during July, August and September are given in this table:

Previous reading 30 June	Present reading 30 September
76 276	79 510

A tax of 16% of the total charges is added to the bill.

Calculate **a** the number of kWh used
b the total amount to be paid.

a The number of kWh used = 79 510 − 76 276 = 3234

b Fixed charge = 3 × \$15 = \$45

Electricity charge = 3234 × \$2.55 = \$8246.70

Total charges = \$45 + \$8246.70 = \$8291.70

Total amount to be paid = $\frac{116}{100}$ × \$8291.70 = \$9618.37 to 2 d.p.

1kW = 1000 W
A 25 W bulb uses 1 kWh in (1000 ÷ 25) h = 40 h, i.e. a 25 W bulb uses 1 unit every 40 hours.

1 **a** How many hours will each of the following run on 1 unit of electricity?
i a 100 W lamp **ii** a 12 W lamp
iii a 2 kW fire **iv** a 2.5 kW kettle

b How many units of electricity are used if
i a 150 W lamp burns for 10 h
ii a 2 kW fire is used for 5 h
iii a 3 kW kettle boils for 4 minutes
iv a 750 W iron is used for 4 hours?

c Calculate the number of kWh
i a 6 kW fire uses in 12 hours
ii a 100 W bulb uses in 90 hours.

2 Find the quarterly electricity bill for each household.

a

Name	Fixed charge	Number of units used	Cost per unit
Mr Evans	\$37.80	1774	\$0.11

b

Name	Fixed charge	Number of units used	Cost per unit
Mrs Kanhai	\$42.90	2071	\$0.09

3 The Smith household estimate that the numbers of units of electricity used for various appliances during a week are

Lighting 8 units

Electric stove 32 units

Refrigerator and freezer 20 units

Television 5 units

Heating 48 units.

If the fixed charge *per quarter* is $47.80 and a unit of electricity costs $0.12, calculate the cost of electricity for a week.

4 Mr Short's electricity bill for the quarter amounts to $269.93. The fixed charge is $47.50 and the price of electricity is 16.8 c a unit. How many units has he used?

5 Miss Deakin's electricity bill for a quarter amounts to $193.46. She has used 946 units of electricity costing 16.45 c per unit. Calculate the quarterly fixed charge.

6 Peter Glenn's electricity bills for a year total $674.76. The bills showed that he used 1084, 947, 713 and 828 units during the first, second, third and fourth quarters, respectively, of the year. If the fixed quarterly charge was $37.24, calculate the cost of electricity per unit, giving your answer in cents correct to three significant figures.

7 The charges for electricity on an island consist of a fixed fuel charge of 40 c per kWh and an energy charge calculated under three different schemes.

Scheme A Homes	13 c per kWh
Scheme B Schools	20 c per kWh
Scheme C Businesses	26 c per kWh

For a business the readings at the beginning and end of a quarter were 17 893 and 23 446 respectively. Calculate

a the number of kWh used

b the fixed fuel charge in dollars

c the energy charge

d the total amount, in dollars, the business had to pay for the electricity they used.

8 The charges for electricity on an island consist of a fixed fuel charge of 42 c per kWh and an energy charge calculated under three different schemes.

Scheme A Homes	18 c per kWh
Scheme B Schools	24 c per kWh
Scheme C Businesses	28 c per kWh

For a school the readings at the beginning and end of a quarter were 23 921 and 32 745 respectively. Calculate

a the number of kWh used

b the fixed fuel charge in dollars

c the energy charge

d the total amount, in dollars, the school had to pay for the electricity they used.

9 Mr Quentin used 105 cubic metres of water last quarter.
The water rates for domestic users were

$1.36 per cubic metre for the first 20 m^3
$1.15 per cubic metre for the next 50 m^3
$0.95 per cubic metre for amounts in excess of 70 m^3.

A discount of 6% was given on bills paid within one week of billing. Calculate the amount Mr Quentin paid for the quarter year, assuming that the bill was paid within 5 days of receiving it.

10 Last year Mrs Esther used 125 cubic metres of water.
Water rates for domestic users were

$2.00 per cubic metre for the first 50 m^3
$1.75 per cubic metre for the next 50 m^3
$1.50 per cubic metre for amounts in excess of 100 m^3.

A 5% discount is given on bills paid within two weeks of billing. Determine the amount Mrs Esther paid for the year, assuming that the bill was paid within a week.

11 A government department used 5342 cubic metres of water during a year. Water rates for government departments for that year were

$2.20 per cubic metre for the first 600 m^3
$2 per cubic metre for the next 600 m^3
$1.80 per cubic metre for amounts in excess of 1200 m^3.

Calculate the cost of water that year for the department.

12 A telephone bill gives the following details:

Monthly rental $52
Number of metered units used 5632
Number of operator-controlled units used 756
Government tax 18%
Cost per metered unit 21 c
Cost per operator-controlled unit 25 c

Calculate **a** the cost of the metered units used
b the cost of the operator-controlled units used
c the amount of government tax
d the total payable to clear the bill.

13 The table shows details of the long distance calls Mr Wong made last quarter.

Calls to	Duration in minutes	Fixed charge for 3 minutes	Charge for each additional minute
London	8	$24.00	$12.44
Quebec	21	$34.50	$8.50
Hong Kong	16	$52.25	$14.85

Calculate the cost of these calls.

14 Last quarter Miss Quezzo made the following calls:
75 calls under 3 minutes, 56 calls of 4 minutes and 23 calls of 8 minutes.

These calls were charged at the following rates:

calls less than 3 minutes $6.73
calls more than 3 minutes $6.73 plus $2.45 a minute

Calculate Miss Quezzo's bill if it includes a government tax of 15%.

15 The rateable value of a house is $8500. How much is due in rates when the rate is
a 34 c in the $ **b** 56 c in the $ **c** $0.44 in the $?

16 The total rateable value of the properties in a city is $260 000 000.
a How much income would there be from a rate of
i 5% **ii** $0.43 **iii** $0.65?
b What rate is needed to collect
i $117 000 000 **ii** $218 400 000?

Hire purchase and mortgages

Hire purchase (HP) is a way of spreading the cost of expensive items. It consists of an initial payment and then a fixed number of equal monthly payments.

Hire purchase is a relatively short-term loan repayable over a period ranging from a few months to about 5 years.

Buying on hire purchase is always more expensive than paying the full price to start with and the longer the repayment period, the more expensive it becomes. The goods bought on hire purchase may be repossessed if the repayments are not kept up.

A mortgage is a long-term loan of money raised on the value of a property and is paid back in monthly instalments over several years. The property on which the mortgage is arranged will be repossessed by the lender if repayments are not kept up.

> The main difference between a mortgage and hire purchase is that hire purchase repayments are fixed at the outset whereas mortgage repayments will vary as interest rates vary.

EXERCISE 5d

Example:

A plasma TV set costs $7500. A discount of 5% is made for payment by cash. Mrs Andrews buys the TV set on hire purchase. She has to pay a deposit of $500 and 12 monthly instalments of $610.

Calculate
a the total amount Mrs Andrews has to pay under the hire purchase agreement
b the difference between the total cost of buying the set on hire purchase and the cost of buying the set with cash.

a Total amount $= \$500 + (12 \times \$610) = \$7820$

b Cash price $= 95\%$ of $\$7500 = \$7125$

Difference between the hire purchase cost and the cash price
$= \$7820 - \$7125 = \$695$

1 A motorcycle is priced at $75000. If bought on hire purchase, the terms are: $\frac{1}{3}$ deposit + 36 monthly payments of $1800.50. Find the HP price.

2 Find the hire-purchase price of an article for which a deposit of $120 is needed plus 12 monthly payments of $26.

3 The cash price of a lawnmower is $420. HP terms require 25% deposit plus 24 monthly repayments of $16.80. Calculate the amount saved by paying cash.

4 A video camera is advertised at $1100. If bought on HP, the terms are 25% deposit + 12 monthly repayments of $77.50.
How much is saved by paying cash?

5 A grand piano is advertised at $8160. If bought on HP, the terms are 20% deposit plus 18 monthly payments of $426.
How much is saved by paying cash.

6 The cash price of a cut-glass water set is $256.50. The hire-purchase terms are $\frac{1}{5}$ deposit plus 52 weekly payments of $4.26.
How much is saved by paying cash?

7 A man's suit can be bought for $378 cash or for a deposit of $126 plus 12 monthly instalments of $22.68.
  a How much more does the suit cost if bought on the instalment plan compared with the cash price?
  b Express the additional cost as a percentage of the cash price.

8 A motorcycle is offered for sale at $1680. If bought on hire purchase, a deposit of $\frac{1}{4}$ is required together with 24 monthly payments of $64.47. Calculate the difference between the cash price and the hire purchase price.

9 The marked price of an electric cooker is $990. If bought for cash, a discount of $2\frac{1}{2}$% is given, but if bought on hire purchase, the terms are $\frac{1}{3}$ deposit plus 24 monthly payments of $33.60.
How much more does the cooker cost if bought on hire purchase?

10 A freezer may be bought for $1035 or for 24 monthly instalments of $41.28 following a deposit of $261. Find the total hire-purchase price. How much is saved by paying cash?

Example:

A hire-purchase agreement requires a down payment of 20% of the marked price. Interest of 10% is added to the balance which is then divided into 24 equal monthly instalments.

Mr Sharma buys a refrigerator priced at $1400.

Calculate a the exact monthly repayments under the hire-purchase agreement
b the percentage profit made by the hire-purchase company.

a Down payment = 20% of $1400 = $280
Balance = $1400 − $280 = $1120
Balance plus interest = 110% of $1120 = $1232
Each monthly instalment = $1232 ÷ 24 = $$51\frac{1}{3}$

$$51\frac{1}{3}$ = $51.333.... This would probably be rounded up to $51.34 giving the lender slightly more profit.

b The total hire purchase price = $280 + $1232 = $1512
The profit = $1512 − $1400 = $112
The percentage profit $= \frac{112}{1400} \times 100\% = 8\%$

11 The marked price of a three-piece suite is $7740. A 5% discount is offered for a cash sale, but if bought on HP, the deposit is $\frac{1}{3}$, followed by 18 monthly payments of $343.98. Find the cash difference in the two ways of paying for the suite, and express this difference as a percentage of the *cash* price, giving your answer correct to three significant figures.

12 A bus company is offered a second-hand coach for $26 760. Since it cannot afford to pay cash it has two options:

Option 1: 6 half-yearly payments of $4858

Option 2: A deposit of $\frac{1}{3}$ plus 12 three-monthly payments of $1558

Which option is the cheaper, and by how much?

13 Retiling a house will cost $22 800. If paid for on hire purchase, a deposit of $\frac{1}{5}$ is required together with 60 monthly payments of $425.60. Find the additional cost when bought on HP and express this as a percentage of the cash price.

14 A motorist decides to buy a new car, the list price of which is $29 760. If he sells his old car privately for $18 400 and then pays cash for the new car, he is given a discount of 12.5%. However, if he offers his car in part-exchange, it is valued at $16 000 and in addition he must make 36 monthly payments of $304.
How much will he save if he sells his car privately and pays cash?

15 A carpet, which is suitable for use in a lounge area measuring 5 m by 4 m, is offered for sale at $64.20 per square metre. Hire-purchase terms are as follows: 33.5% deposit, the balance to be increased by 12% and divided by 12 to give the monthly repayments for 1 year.
Find **a** the monthly repayments
b the increased cost if bought on hire purchase
c the increased cost expressed as a percentage of the cash price.

16 The cash price of a colour television set is $2695. On the instalment plan the deposit of 20% is followed by monthly payments of $195.79 for 1 year. For the second and subsequent years the set may be insured against failure for $224 p.a.

If the same set had been rented, the rental fee would have been $66.29 per calendar month for the first year and $62.72 for every additional month. Compare the hire-purchase costs with the rental costs over a 6-year period. Which is the cheaper and by how much?

17 A department store advertises a dining set at $10 500. Three different arrangements are offered if it is bought on credit:

i a deposit of $\frac{1}{3}$ plus 12 monthly payments of $638.75

ii a deposit of $\frac{1}{4}$ plus 12 monthly payments of $720.60

iii a deposit of $\frac{1}{5}$ plus 12 monthly payments of $758.65

Which of these arrangements is
a the cheapest **b** the most expensive?

Example:

Mr and Mrs Nichols take out a mortgage for $1 500 000. There is a fee of $500 for arranging the mortgage and the monthly repayments are $7500 for 25 years.

Calculate **a** the total cost of the mortgage
b the percentage profit made by the mortgage provider.

a Total cost $= \$500 + (25 \times 12 \times \$7500) = \$2\,250\,500$

b Profit $= \$2\,250\,500 - \$1\,500\,000 = \$750\,500$

Percentage profit $= \frac{750\,500}{1\,500\,000} \times 100\% = 50.03\%$ to 2 d.p.

18 A man borrows $280 000 from a building society in order to buy a house. The society charges $11.51 per calendar month per $1000 borrowed. What are the monthly repayments?

19 A country house is on sale for $150 000 cash. A bank offers an 85% mortgage. How much is needed as a deposit?

20 The price of a luxury apartment is $425 000. If a bank offers a 90% mortgage, calculate the deposit.

21 The sale price of a property is $225 000. What mortgage must be taken out if the deposit is
a 20% of the sale price **b** 15% of the sale price?

22 Mr and Mrs Raynor want to buy a house which is on sale for $250 000. Find the amount that must be borrowed if they have a deposit of
a 15% of the cash price **b** 17.5% of the cash price.

23 A married couple want to buy a house priced at $240 000. They are able to sell their present home for $90 000 and need to take out a mortgage for the difference.
If they take out a 20-year mortgage which costs $14.06 per $1000 per calendar month, calculate the total cost of the new house.
Express this cost as a percentage of the cost price.

24 A condominium is on sale for $325 000. Jake can buy the condominium by making a 15% deposit and taking a bank mortgage for the remainder.
Calculate **a** the deposit **b** the mortgage
c the total amount repaid to the bank, if monthly payments of $3403 are made over a 25-year period.

25 A country house can be bought for $150 000. It is possible to purchase the house by making a 12% deposit and taking a mortgage.
Work out **a** the deposit **b** the amount borrowed
c the total repaid to the bank, if monthly payments of $2216 are made for 15 years.

26 An apartment is priced at $195 000. An 85% mortgage can be obtained over a 20-year period.
Find **a** the deposit that must be found
b the mortgage needed
c the total amount of money to be repaid to the bank if each monthly payment is $2347
d the total amount that will be paid for the apartment.

27 A couple obtain a 90% mortgage on a house costing $180000. If the mortgage repayments over 20 years are $12.77 per month per $1000 borrowed, calculate
   a the amount borrowed
   b the annual repayments
   c the total cost of the house.

28 Mary Hughes obtains a 95% mortgage on a house costing $190000 and agrees to pay an interest rate of 15%. This means that her monthly repayments per $1000 borrowed may be either $16.61 for 10 years or $13.32 for 20 years. How much more would she pay if she chose to take the 20-year mortgage as opposed to the 10-year mortgage?

Salaries, wages and commission

When we work for someone else, we expect to be paid for that work.

A salary is an annual amount paid for a job which is divided into twelve equal amounts and paid monthly.

A wage is a weekly payment calculated as a fixed sum for each hour worked. There is usually an agreed number of hours in the working week and any time worked over this number is usually paid at a higher hourly rate.

Some jobs, such as sales, also pay commission or bonuses. A commission is usually an agreed percentage of the value of orders taken. A bonus is usually payable on an amount of work completed over and above a fixed quantity. Employees are not entitled to a bonus, which is given at the discretion of the employer.

Income tax

Income tax is payable on earnings. There is usually a tax free amount and all income above this (called taxable income) is taxed at a percentage determined by the government. Usually taxable income is split into bands so that higher earnings are taxed at a higher rate.

This percentage is called the tax rate.

EXERCISE 5e

Example:

A worker is paid $560 for a 35-hour week. Overtime is paid at 150% of the hourly rate. Calculate the number of hours overtime worked in a week in which the worker was paid $752.

The standard hourly rate of pay $= \$560 \div 35 = \$16$

The overtime rate $= 150\% \times \$16 = \$24$

The pay for overtime $= \$752 - \$560 = \$192$

The number of hours of overtime $= \$192 \div \$24 = 8$

1 Barry Crooks works a basic week of $37\frac{1}{2}$ hours and is paid \$6.25 per hour. Calculate his gross weekly wage.

2 John Giles works a basic week of 40 hours and is paid \$296. Calculate John's hourly rate of pay.

3 Colin Vincent works a basic week of 35 hours. Overtime is paid at time-and-a-half. If the basic hourly rate is \$6.20, how much does he earn in a week when he puts in 41 hours?

> Time-and-a-half means $1\frac{1}{2}$ times the hourly rate.

4 Rodney Hill works for a builder who pays \$9.50 per hour for a basic week of 36 hours. Overtime is paid at double time. How much will Rodney earn in a week when he works for 52 hours?

5 Sally Prescott's work-card showed that she worked $8\frac{1}{2}$ hours each day, Monday to Friday. If the basic day was 7 hours and overtime was paid at time-and-a-half, calculate her weekly wage when the basic rate was \$5.70 per hour. How much would her pay increase if the basic hourly rate rose by 10 c per hour?

6 A doctor earns an annual salary of \$84 996. What is her gross monthly salary?

7 Mr Grant is paid a gross monthly salary of \$3680 while Mr Gedge receives an annual salary of \$46 950. Which one earns more and by how much each month?

8 Walter earns \$3450 a calendar month. Deductions amount to 35% of his salary. Find his net monthly income.

> Net income means the amount after deductions are made.

9 Sarah's monthly net pay is \$1929.60. If deductions amount to 40% of her gross income, find her gross annual salary.

> Gross income means the amount before any deductions are made.

10 When deductions amounting to 38% of gross salary are deducted from Mrs Deakin's gross monthly pay, her take-home pay is \$1679.17. Calculate her gross annual salary.

11 A woman earns money by making boxes at home. She is paid \$9.20 a box for the first 50 boxes she makes in a week and 20% more than this on any boxes she makes over this number. Calculate how much she is paid in a week in which she makes 75 boxes.

Example:

The income tax in a certain island is calculated as follows.
The first \$48 000 of income each year is tax free.
Any income over this is taxed at 25%.
Calculate **a** the tax payable on a yearly income of \$500 000
b the gross annual income of a person who pays \$70 000 tax in one year.

a Taxable income = \$500 000 − \$48 000 = \$ 452 000
Tax payable = 25% of \$452 000 = \$113 000.

b \$70 000 = 25% of taxable income

$$\text{Taxable income} = \$70\,000 \div \frac{25}{100} = \$70\,000 \times 4 = \$280\,000$$

$$\text{Gross income} = \$280\,000 + \$48\,000 = \$328\,000$$

> Gross income = tax free income + taxable income

Per annum (p.a.) means each year

12 Mr Eaton earns \$38 000 p.a. He gets a tax free allowance of \$2100 and pays tax on his taxable income at the following rates: 10% on the first \$8000 and 35% on the remainder.
How much tax does he pay each month?

Use the information below for questions **13** to **18**.

Tax free allowances are as follows:

Single person: \$2000
Married person: \$3500
Child under 16: \$450
Child 16 or over in full-time education: \$750
A spouse who also works receives a single person's allowance.

The tax rates are 5% on the first \$14 000, 15% on the next \$9000, 35% on the next \$25 000 and 40% on the remainder.

Mortgage interest: 0%
National insurance: 0%

13 Mr Edwards earns \$45 700 a year. He is married and his wife does not work.
Calculate
a his taxable income
b his annual tax bill
c his net annual income if other deductions amount to \$7460.

14 Miss Owen is a secretary earning \$2230 a calendar month. Calculate her annual tax bill.

15 A married man, with one child aged 18 in full-time education, earns \$37 500 p.a.
Calculate
a his total tax-free income
b his taxable income
c his annual tax bill
d his net annual income if other deductions amount to \$5980.

16 Shirley Rees is a widow and earns \$28 000 a year. She lives with her two children aged 5 and 7. Calculate her monthly tax bill.

17 Mr Beynon earns \$42 000 p.a. Mr Beynon has a wife who does not work, two children aged 12 and 15 and a third child aged 18 in full-time education. In addition he pays \$1200 a month mortgage interest and \$35 a month National Insurance.
Calculate
a his total tax-free income
b his taxable income
c his annual tax bill
d his net annual income.

18 Mr and Mrs Peacock both work. They do not have children. Mr Peacock earns \$36 000 a year and Mrs Peacock \$2300 a calendar month.
Calculate
a their combined gross annual income
b the amount of tax Mrs Peacock pays each month
c Mr Peacock's annual tax bill if he pays \$900 a month mortgage interest.

19 Jo Deans's gross annual salary is $32 600. She pays 5% of her salary into a pension scheme and 35% of her salary goes in tax and other deductions.
Calculate
a her monthly pension contribution
b her net monthly salary.

20 Mr Gifford has deductions from his monthly salary of $2340 as follows:

6% pension contribution
$85 medical insurance
$150 National insurance
income tax 20% of gross salary

Calculate his net monthly income.

Investments and insurance

When we borrow money from a company such as a bank, they are lending us their money and we have to pay a sum over and above the amount borrowed. This is called **interest**.

When we save money by putting money in a savings account, we are lending our money to the company providing the account. They pay us interest.

The sum of money borrowed or lent is called the **principal** and is denoted by P.

The interest is usually quoted as a percentage payable per annum. It is called the **percentage rate** and is denoted by R.

Per annum (p.a.) means each year. An interest rate of 5% p.a. means 5% of the principal is paid each year.

There are two forms of interest.

Simple interest is paid on the same amount each time. The formula for working out simple interest, I, for a period of time, T, is

$$I = \frac{PRT}{100}$$

When the interest is added on to the principal each year, the principal will increase by larger amounts each year. This kind of interest is called **compound interest** where the amount, A after time t is given by $A = P\left(1 + \frac{r}{100}\right)^t$.

Investments are where we give our money (called **capital**) to a company in the hope that the sum of money will increase. The increases are called **returns**.

The simplest investments are into savings accounts where there is no risk to our capital but the returns are fairly low.

There are many other forms of investment, such as property, stocks and shares, artwork. These investments carry a risk because the value of these items can go down as well as up.

Insurance is a payment (called a premium) we make to a company who will then pay us for a defined loss. Insurance is available for almost any risk but the higher the risk, the greater the premium.

Some insurance is mandatory such as car insurance if we drive.

An **endowment policy** is a combination of a savings account and insurance. It is taken out for a fixed term (usually several years) and usually involves monthly payments for an agreed repayment at the end of the term or on the death of the policy holder if that occurs before the end of the policy.

EXERCISE 5f

Example:

The interest on a deposit account changed from 5% p.a. to 4.5% p.a. Find the difference in the yearly interest on a deposit of $750 000.

At 5%, the interest = 5% of \$750 000 = \$37 500

At 4.5%, the interest = 4.5% of \$750000 = $\frac{4.5}{100} \times \$750000 = \$33\,750$

The difference = \$37 500 − \$33 750 = \$3750

An easy way to find 5% of a quantity is to find 10% (divide by 10) then halve the result.

1 Find the simple interest on $20 000 invested for 3 years at 6% p.a.

2 Find the simple interest on \$640 invested for 7 years at $7\frac{1}{4}$% p.a.

3 a Find the simple interest on \$70 000 invested for 5 years at $7\frac{1}{2}$% p.a.
  b If the interest rate had been 8%, how much more interest would have been paid?

4 Find the simple interest on \$800 invested for 10 years at $3\frac{1}{2}$% p.a.

5 The simple interest on \$1200 invested for 2 years is \$108. What annual rate of interest is paid?

6 \$3500 amounts to \$4462.50 when invested for a number of years at $5\frac{1}{2}$% simple interest per annum. For how many years was the money invested?

Example:

Mr Adams invests \$30 000 in a 5-year deposit account that earns $8\frac{1}{2}$% p.a. simple interest, paid yearly.
a Calculate the total amount in the account at the end of the 5 years.
b How long would it take for the amount in the account to reach \$64 000?

a Simple interest = $\frac{PRT}{100} = \$\ \frac{30\,000 \times 8.5 \times 5}{100} = \$12\,750$

Amount in the account after five years = \$30 000 + \$12 750
= \$42 750

b For the account to amount to \$64 000, the simple interest has to be \$34 000.

So $34000 = \frac{30\,000 \times 8.5 \times T}{100}$

$34\,000 \times 100 = 30\,000 \times 8.5 \times T$

$T = \frac{340}{3 \times 8.5} = 13.33\ldots$

As the interest is paid yearly it would take 14 years for the amount to reach \$64 000.

The amount in the account will then be a bit more than \$64 000.

7 Steve Barnard invested \$20 000 in a 4-year deposit account that earned $7\frac{1}{2}$% p.a. simple interest.
  a Calculate the total amount in the account at the end of four years.
  b How many complete years would it take to pass \$30 000?

8 Mr Kanhai's bank pays interest of 6% p.a. on money he has on deposit. How much is in his account if the interest for 9 months is $25.20?

9 Mrs Kerada borrowed $7800 from her bank at $8\frac{1}{2}$% p.a. simple interest for 5 years.
Evaluate
- a the total paid in interest to the bank
- b the total amount repaid
- c the monthly repayments if the total was paid in equal monthly instalments over 5 years.

10 John Peters borrowed $5475 for 6 months at 7.7% p.a. Find
- a the sum John has to pay the bank as simple interest
- b the total amount paid to the bank to clear the loan
- c the regular monthly repayment if the total due is repaid in equal monthly instalments over the period of the loan.

11 Calculate the percentage rate per annum if simple interest of $216 is paid when $1200 is invested for 4 years.

12 Bank A pays simple interest of 6.4% p.a. whereas Bank B pays 6.55% p.a. Mrs Wood has $150 000 to invest. How much more will she get by investing for 4 years in Bank B rather than Bank A?

13 Sarah has $80 000 invested at 5.5% p.a. simple interest. The bank opens a new account paying 5.75% but charges $150 to move the investment from one account to the other. Sarah needs the money in a year's time. Would you advise her to move it or leave it where it is? Justify your answer.

14 Mrs Thomas took a bank loan for 6 months at 14% simple interest. How much did she borrow if the interest on her loan was $126?

Example:

Mrs Bolter placed $75 000 in a deposit account paying 7% p.a. compound interest. Calculate the amount in the account at the end of 2 years.

The amount after 1 year $= 107\%$ of $\$75\,000 = \frac{107}{100} \times \$75\,000$
$= \$80\,250$

The amount after 2 years $= \frac{107}{100} \times \$80\,250 = \$85\,867.50$

Find the compound interest payable, giving your answer correct to the nearest cent, on

15 $5200 invested for 2 years at 7% p.a.

16 $9540 invested for 2 years at 12% p.a.

17 $1400 invested for 5 years at 6.5% p.a.

18 $2500 invested for 8 years at 7.45% p.a.

19 $7900 invested for 10 years at $8\frac{3}{4}$% p.a.

20 $3000 invested for 15 years at 10% p.a.

21 $270 000 invested for 20 years at 6% p.a.

22 A couple want to borrow \$100 000 for a year. The bank offers to lend the money at 16% compound interest payable yearly, while a credit card company offers to lend at 2% per month simple interest payable monthly. Compare the total due to the bank with that due to the finance company by the end of the year.

23 A school borrows \$15 000 at 15% compound interest towards the cost of a minibus. They repay \$8000 after 1 year and the rest at the end of the second year. How much must they find at the end of the second year to clear the debt?

24 A motorist owns a car valued at \$10 000 and has \$80 000 cash in a building society. He debates whether or not to spend his cash on a new car costing \$90 000. If he leaves his money in the building society it will earn compound interest at 8%. On the other hand any car he owns depreciates by 20% of its value p.a. at the beginning of any year. Compare his financial position after 2 years if he buys the new car and sells the old car with his position if he does not.

25 A car bought for \$1 200 000 depreciates by 20% a year. What is its value when it is 3 years old?

26 A new caravan costs \$85 000 and depreciates by 10% a year. Calculate its value after 3 years.

27 Mrs Khan invests \$10 000 in an account that pays 5% compound interest whereas her husband invests \$10 000 in an account that pays 6.5% simple interest.
Calculate who receives more interest after five years and by how much.

28 Which is the better investment: an investment for 5 years at 5.5% compound interest or an investment for 5 years at 6.2% simple interest? Give a reason for your answer.

Example:

Mrs Dixon wants to insure the contents of her house, valued at \$300 000, against loss or damage.

Insurance company A quotes a premium of \$15 p.a. per \$1000 worth of property.

a Calculate the premium payable to this company.

Insurance company B quotes a premium of \$12 p.a. per \$1000 worth of property with an excess of \$1000.

An excess of \$1000 means the insurer reimburses any loss over and above \$1000.

Mrs Dixon has a fire in her house that destroys property valued at \$2000.

b Which insurance company offers better value for money under these circumstances?

a Premium payable to company A = \$15 × 300 = \$4500

b Premium payable to company B = \$12 × 300 = \$3600
To recover the cost of the damaged property, Mrs Dixon has to pay \$1000, making a payment of \$4600 in total.
Company A is better value when a claim is needed.

29 Find the annual premium on the contents of a house valued at \$250 000, if the rate is \$0.30 per annum per \$100 insured.

30 Find the cost of insuring a camera valued at \$350 under the 'all risks' section of a policy if the rate is 90c per \$100.

The table gives the premiums payable per person for holiday insurance. Use this table for questions **31** to **34**.

Number of days	Adult	Child under 16
Up to 10	\$100	\$60
11 to 17	\$120	\$70
18 to 31	\$150	\$80

31 Calculate the premium due to cover a family of three (father, mother and child aged 10 years) who go on a 12-day holiday.

32 Calculate the premium due to cover a family of six, including four children under 10 years, on a two-week holiday.

33 Mr and Mrs Keate and their children Joanne (aged 8), Jonathan (aged 11) and Martin (aged 17) propose taking a 3-week touring holiday.
Calculate the cost of holiday insurance for the family.

34 Peter and Betty Franka are taking their 18-year-old daughter to the United States for a 2-week holiday. Work out the total holiday insurance.

35 Calculate the annual insurance premium on a house valued at \$555 000 if the rate is \$1 per \$1000. What would the premium be in 2 years' time if the policy is index linked and the inflation rate is 10% per annum?

36 Mr and Mrs Caton live in a house valued at \$450 000. They value the contents of their home at \$180 000, but wish to cover \$30 000 of this for 'all risks'. The rates are

\$1 per \$1000 for the property
30c per \$100 for the contents
90c per \$100 for the 'all risks' section.

a Calculate the total premium due
i per annum **ii** per week.

b Supposing that the policy is index linked, what would be the annual premium in 2 years' time assuming an inflation rate of 10% per annum?

37 A company that sells motor insurance offers the following No Claims discounts:

1st renewal 30%
2nd renewal 40%
3rd renewal 50%
4th and subsequent renewals 60%

Fred Knight is quoted an annual premium of $700 when he takes out insurance on his car.
Calculate his premium on **a** the first renewal
b the fourth renewal.

For questions **38** to **40** use the No Claims discounts given in question **37**.

38 Robin Box is quoted an initial premium of $960 for third party cover for his motorcycle. After 2 years of accident-free motorcycling he makes an insurance claim during the third year. What will his renewal premium be for the fourth year?

39 A fully comprehensive insurance policy for Trevor Jones' car costs him $1350 for the first year.

a Assuming that there are no claims, how much will the premium be for the sixth year?

b If he decides to pay the premium in four equal quarterly instalments, 10% is added to the premium. Calculate his quarterly premium.

40 The annual rates quoted by an insurance company for insuring Mr Jerome's car were as follows:

Third party, fire and theft $900 Fully comprehensive $1550

How much more would he be paying on the sixth renewal if he had chosen to take out a fully comprehensive policy? Assume that no claims were made during the period.

ABCD MIXED EXERCISE 5

Several answers are given for these questions.

Write down the letter that corresponds to the correct answer.

1 If an article is bought for $66 and sold at a loss of 10%, its selling price is
A $52.80 **B** $63.50 **C** $60 **D** $59.40

2 In a sale, an article marked $550 was sold for $440. The reduction was
A 10% **B** 15% **C** 20% **D** 25%

3 Henry bought a car for $7750 and sold it for $9000. His percentage profit was
A 10% **B** 16.13% **C** 15% **D** 13.89%

4 The time that a 60 W electric light bulb will run on 1 kilowatt-hour of electricity is
A less than 16 h **B** 16 h
C 16 h 30 min **D** more than 16 h 30 min

5 In 8 hours the number of kilowatt-hours a 3 kW electric fire will use is
A 24 **B** 12 **C** $2\frac{2}{3}$ **D** $\frac{3}{8}$

6 One unit of electricity costs 16.77 c. The cost of running a 350 W television for 40 hours is
A 2.35 c **B** 23.5 c **C** $2.35 **D** $23.50

7 A painting bought for $3500 appreciated by 35% over two years. Its value then was
A $1225 **B** $4550 **C** $4725 **D** $4900

8 At 7.2% simple interest, $8500, in 2 years, will have a value of
A $9112 **B** $9724 **C** $1268.06 **D** $9768.06

9 The simple interest on $5500 invested for 4 years was $1210. The percentage rate to give this interest was
A 4% **B** 5% **C** 6% **D** $5\frac{1}{2}$%

10 A wholesaler sells 50 printers to a shopkeeper for $14 400. The shopkeeper sells them at a profit of $12\frac{1}{2}$%. The price he charges for each printer is
A $32.40 **B** $324 **C** $360 **D** $3240

11 The price of a computer including VAT at 15% is $920. The price before VAT is added is
A $782 **B** $800 **C** $905 **D** $1058

12 The marked price of a car was $210 000. A woman bought the car by paying a deposit of $10 000 and twelve monthly payments of $20 000. The amount she would have saved by paying cash for car was
A $10 000 **B** $20 000 **C** $30 000 **D** $40 000

13 The interest rate on an investment changed from $7\frac{1}{2}$% to 5% p.a. The difference in the interest earned on an investment of $50 000 in one year was
A $625 **B** $1250 **C** $2500 **D** $5000

14 The compound interest on £5000 invested for 2 years at 5% p.a. is
A $250 **B** $500 **C** $512.50 **D** $1025

15 A man bought a washing machine priced at $900 on a hire-purchase plan. The plan required a deposit of 20% then the remainder, increased by 15%, to be paid in 12 equal monthly instalments. Each instalment was
A $69 **B** $85 **C** $90 **D** $100

16 The electricity company charges $50 a month flat fee plus 50 c a unit for the first 200 units used then 25 c a unit for any extra units used. The bill for 300 units used in one month is
A $100 **B** $150 **C** $175 **D** $200

17 Mr Khan bought a desk for $832 which included sales tax of 4%. The price without the sales tax was
A $789 **B** $800 **C** $840 **D** $864

18 The simple percentage rate at which £1500 is invested to amount to $1725 in three years is
A 1 **B** $2\frac{1}{2}$ **C** 5 **D** $7\frac{1}{4}$

19 Mr Arnold is paid $10 an hour for a 35-hour week and $15 an hour for overtime. In a week in which he was paid $410, the number of hours overtime he worked was
A 3 **B** 4 **C** 6 **D** 10

20 A carpet priced at $25 000 may be bought under a hire-purchase plan by paying $500 immediately then paying $700 a month. How many months will payments need to be made to clear the debt?
A 12 **B** 24 **C** 35 **D** 50

INVESTIGATION

Imagine you have $100 000 to invest.

Investigate different ways in which you could invest the money and the advantages and disadvantages of each. For example, you could look at the different investments provided by a building society, options for investing in the stock market, or investing in a pension scheme.

MATHS IS OUT THERE

Find out about convertible investments – money, not cars!

IN THIS CHAPTER YOU HAVE SEEN THAT...

- a percentage increase or decrease is calculated on the quantity before any change is made
- the percentage profit (or loss) is the profit (or loss) expressed as a percentage of the cost price
- sales tax is added to the selling price of an article at a rate fixed by the government
- to increase a quantity by a% multiply it by $\frac{100 + a}{100}$.
  $\frac{100 + a}{100}$ is called the multiplying factor
- to decrease a quantity by a% multiply it by $\frac{100 - a}{100}$.
  $\frac{100 - a}{100}$ is called the multiplying factor
- to find a quantity before a percentage change, divide by the multiplying factor that changes it
- compound interest occurs when the simple interest r% for each period is added to the principal, P, to form a new principal for the next period and the amount A after time t is given by $A = P\left(1 + \frac{r}{100}\right)^t$
- utility bills are usually made up of a fixed charge for a period plus a charge for each unit used
- a kWh is 1 kilowatt used for 1 hour
- a salary is an annual amount paid for a job which is divided into twelve equal amounts and paid monthly
- a wage is a weekly payment calculated as a fixed sum for each hour worked for usually an agreed number of hours.

Multiple choice questions

Several possible answers are given.
Write down the letter corresponding to the correct answer.

1 If x is an odd number which of the following must be odd?
A $x - 1$ **B** $x + 1$ **C** $2x$ **D** $x + 2$

2 If $232_n = 67_{10}$ then n is
A 5 **B** 6 **C** 7 **D** 8

3 $9 + (2 + 7) = (9 + 2) + 7$ is an example of
A the distributive law **B** the associative law
C the commutative law **D** an identity

4 If $a = 5 \times 10^5$ and $b = 3 \times 10^4$ then $a + b =$
A 5.3×10^7 **B** 8×10^7 **C** 5.3×10^5 **D** 1.5×10^{10}

5 $\frac{1}{2} + \frac{1}{3} =$
A $\frac{1}{6}$ **B** $\frac{2}{5}$ **C** $\frac{5}{6}$ **D** $\frac{3}{2}$

6 The number 34.06791 written correct to 2 decimal places is
A 34.07 **B** 34.068 **C** 34.06 **D** 35

7 15% of \$240 is
A \$15 **B** \$18 **C** \$24 **D** \$36

8 $0.82 \times 5.99 = 4.9118$ so $8.2 \times 59.9 =$
A 0.49118 **B** 49.118 **C** 491.18 **D** 4911.8

9 If $5x - 2 \leqslant 2x + 5$ then
A $x \leqslant -\frac{7}{3}$ **B** $x < \frac{7}{3}$ **C** $x \leqslant \frac{7}{3}$ **D** $x < -\frac{7}{3}$

10 If $4 - 3x > x + 2$ then
A $x < -\frac{1}{2}$ **B** $x < \frac{1}{2}$ **C** $x < 3$ **D** $x < -3$

11 A and B are two sets such that $n(A) = 5$, $n(B) = 2$ and $(A \cap B) = \varnothing$.
$n(A \cup B) =$
A 7 **B** 5 **C** 3 **D** 0

12 A is the set $\{x: x < 4, x \in \mathbb{N}\}$. $A =$
A $\{2, 3, 4\}$ **B** $\{1, 2, 3\}$
C $\{0, 1, 2, 3\}$ **D** $\{0, 1, 2, 3, 4\}$

13 Which of the following sets is a proper subset of $\{1, 2, 3, 4\}$?
A $\{1, 2\}$ **B** $\{0, 1, 2\}$ **C** $\{1, 2, 3, 4\}$ **D** $\{1, 3, 5\}$

14 $\dfrac{(5 + 3)^2}{5^2 - 3^2}$ simplifies to
A 64 **B** 16 **C** 8 **D** 4

15 $\dfrac{(7 + 2)^3}{7^2 - 2^2}$ simplifies to
A $\frac{1}{5}$ **B** $\frac{49}{5}$ **C** $\frac{81}{5}$ **D** $\frac{81}{45}$

16 The simple interest due on a loan of \$2600 for 2 years at $4\frac{1}{2}\%$ is
A \$117 **B** \$234 **C** \$468 **D** \$936

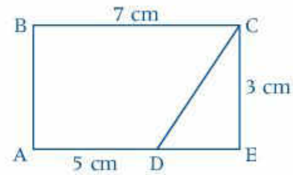
17 Charlie bought a toy car for \$30. He sold it, making a gain of 30% on the cost.
How many dollars did he gain?
A \$9 **B** \$24 **C** \$30 **D** \$60

18 The interest rate on Benny's deposit account is to be reduced from 2.5% this year to 2.2% next year.
How much less interest will Benny receive next year compared with this year on a deposit of \$20 000?
A \$6 **B** \$60 **C** \$600 **D** \$900

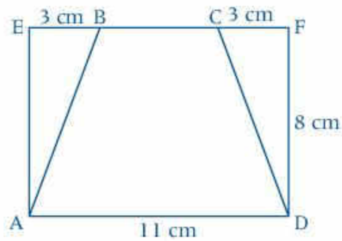
19 How many litres of water would a container whose volume is 48 cm^3 hold?
A 0.048 **B** 0.48 **C** 4.8 **D** 4800

20 The area of a square of side 0.4 m in square centimetres is
A 0.16 **B** 1.6 **C** 160 **D** 1600

21 ABCD is a trapezium which is not drawn to scale.
CDE is a triangle. Angles A, B and E are right angles.
The area of the trapezium is
A 18 cm^2 **B** 21 cm^2 **C** 27 cm^2 **D** 36 cm^2



22 ABCD is a trapezium which is not drawn to scale.
Angles A, E, F and D are right angles.
The area of the trapezium is
A 44 cm^2 **B** 64 cm^2 **C** 76 cm^2 **D** 88 cm^2



23 A circular hole of diameter 8 cm is cut from a circular metal plate with a diameter of 10 cm. The area of metal remaining, in cm^2, is
A 4π **B** 8π **C** 9π **D** 36π

24 A circular hole of diameter 4 cm is cut from a circular card with a radius of 6 cm.
The area of card remaining, in cm^2, is
A 16π **B** 20π **C** 24π **D** 32π

25 $\frac{4x-3}{6} - \frac{x+1}{3} =$
A $\frac{2x-5}{6}$ **B** $\frac{2x+5}{6}$ **C** $\frac{6x+1}{6}$ **D** $\frac{2x-1}{6}$

26 Correct to 3 significant figures, the value of $\frac{0.2 \times 1.3}{0.3}$ is
A 0.086 **B** 0.866 **C** 0.867 **D** 5.00

27 Given $P = \{3, 6, 7, 10\}$ and $Q = \{3, 4, 6, 9\}$, $P \cap Q =$
A $\{3, 6\}$ **B** $\{3, 4, 6, 7, 9, 10\}$ **C** $\{4, 7, 9, 10\}$ **D** $\{3, 6, 9\}$

28 Given $3x - 2 < 5x + 4$ then
A $x < -3$ **B** $x > -3$ **C** $x < \frac{1}{4}$ **D** $x > \frac{1}{4}$

29 One pound sterling (£) converts to 160 Jamaican dollars (J\$).
J\$ 10 000 =
A £625 **B** £62.5 **C** £6.25 **D** £0.016

30 An article bought for \$200 was sold for \$350. The percentage profit was
A 42.9% **B** 50% **C** 65% **D** 75%

General proficiency questions

1 An alloy consists of copper and zinc in the ratio 5 : 7. What weight of copper will be required to add to 931 kg of zinc? How much alloy will this give?

2 Find, in its simplest form, the ratio of 5.5 m : 250 cm : 1.5 m.

3 Factorise $12x^2y - 16xy^2$

4 Given that $x = 3$, $y = -5$, $z = 6$ find the value of
a $x + y + z$ **b** $xy + 3xz - 4yz$

5 Solve the equations
a $\frac{x + 6}{9} - \frac{x - 4}{4} = 1$ **b** $\frac{5(3x - 4)}{2} - \frac{2(3 - 2x)}{5} = 22$

6 When I add 50 to a number x the result is three times the original number. Find the number.

7 When I add 3 to two-thirds of a number the answer is the same as when I subtract 2 from seven-eighths of it. Find the number.

8 George walks the x km to town at 6 km/h but on the return trip reduces his speed to 5 km/h. In total he takes $2\frac{3}{4}$ hours. How far is his home from town?

9 A motorist travels frequently between two towns. When his usual average speed is x km/h he takes 4 hours but the last time he did the journey his average speed was reduced by 10 km/h and he took $\frac{4}{5}$ hour longer.
Find
a his usual average speed **b** the distance between the towns.

10 Tanya cycles to see her grandparents at 15 km/h and returns at 12 km/h, taking a total of $3\frac{3}{4}$ hours for the journey both ways. How far does she ride altogether?

11 Calculate
a $\dfrac{1\frac{2}{3} - \frac{5}{6}}{\frac{9}{10}}$ **b** $\dfrac{\frac{3}{4} \times 1\frac{2}{3}}{\frac{1}{2} + \frac{2}{3}}$

12 Solve the simultaneous equations
a $3p + 2q = 2$
$3p + 4q = 3$
b $7x - 3y = 1$
$5x + 3y = 5$

13 Viv is paid \$12 an hour for his normal work and \$18 an hour for overtime. One week he got paid \$708 for 52 hours work. How was his time divided between the normal rate and the overtime rate?

14 The sum of two numbers is 25. Three times the larger number is 3 bigger than five times the smaller number. Find the numbers.

15 Two numbers x and y, where x is smaller than y, differ by 5. Three times the larger number is 1 more than five times the smaller number. Find the numbers.

16 Solve the inequalities
a $11 \geqslant 2x + 7 \geqslant x + 2$ **b** $5 - 3x < 2x - 4 < 2$

17 Simplify
a $7x - 2y - 3x + 5y$ **b** $5(2a - 3b) - 3(a - 6b)$
c $2x - 4xy + 8x^2y$

18 Simplify

a $\frac{4a^2}{3} \times \frac{9a^3}{8}$ **b** $\frac{8x^2y}{3} \div \frac{4xy^2}{9}$ **c** $\frac{3p}{2q} - \frac{5p}{8q}$ **d** $\frac{3a - b}{5} - \frac{2a + b}{6}$

19 If $a = 3$, $b = -2$ and $c = 5$, find the value of

a $3a - 4b + c$ **b** $\frac{5a + 2b + c}{3a + 3b + c}$ **c** $\sqrt{a^2 - b^2 + 4c}$

20 Solve the equations

a $14 - 3x = 8$ **b** $3(5x - 2) - 2(3x + 1) = 10$

c $\frac{5 - x}{3} = 7$ **d** $\frac{5x + 7}{11} - \frac{3x - 1}{8} = 1$

21 The perimeter of a rectangle is 35 cm. Its length is 3.5 cm more than its width. Find its length.

22 Solve the simultaneous equations

a $5x + 2y = 4$, $x - y = 5$ **b** $4x + 4y = 5$, $x - 2y + 1 = 0$

23 3 pens and 5 pencils cost \$17 while 5 pens and 3 pencils cost \$15. If each pen costs \$$x$ and each pencil \$$y$ form two equations in x and y. Solve these equations to find

a the cost of a pen **b** the cost of a pencil.

24 Solve the inequalities and illustrate the solution on a number line.

a $6 - 5x \leqslant 2$ **b** $2x - 1 \leqslant x + 2 < 3x + 7$

25 **a** Convert into metres: **i** 60.24 km **ii** 49.7 cm.

b Convert into centimetres: **i** 3.6 m **ii** 7.31 mm.

c Convert into millimetres: **i** 13.5 cm **ii** 9.4 m.

d Convert into kilometres: **i** 830 m **ii** 9230000 mm.

26 Find, giving each answer in metres,

a 4 m − 264 cm + 930 mm **b** 7.2 m − 437 cm + 2600 mm

c 826 cm − 2.45 m + 200 mm **d** 2640 mm + 0.92 m − 356 cm.

27 How many concrete blocks, each 22 cm high, are required to reach a height of 1.76 m?

28 A woman returns from the supermarket with 6 bags of sugar, each of mass 2.2 kg; 12 cans of beans, each of mass 425 g; 4 packets of cereal, each of mass 440 g; and 8 jars of jam, each of mass 454 g. Calculate the total mass of her purchases.

29 Forty-eight tins of pineapple chunks together with their packaging have a mass of 22 kg. If the mass of the packaging is 928 g, calculate the mass of five tins of pineapple chunks.

30 Alma has \$10 000 to invest. Bank A offers 6.5% and Bank B 6.8%. How much more would Alma receive from Bank B than from Bank A if she invested her money for 1 year?

31 Copy and complete this bill.

$2\frac{1}{2}$ kg carrots at □ per kg	\$2.50
5 kg potatoes at □ per kg	\$1.20
□ packets of peas at 80 c per packet	\$6.40
9 oranges at 75 c each	\$□
Subtotal	\$□
VAT at 10%	\$□
To pay	\$□

32 A school ordered the following new textbooks:

75 English books at \$25 each
125 maths books at \$27.50 each
90 history books at \$35 each
430 exercise books at \$3.75 each

Because of the size of the order there was a discount of 10% but there was a charge of \$160 for delivery. How much did the order cost the school?

33 Work out the quarterly electricity bill for Mr Garner using the information given in the table.

Name	Fixed charge	Number of units used	Cost per unit
Mr Garner	45.30	1348	\$0.17

34 A food mixer may be bought by paying a deposit of \$87.10 together with 26 equal payments of \$13.02. If this is \$25.62 more than the cash price, find the cash price.

35 The cash price of an outfit is \$705.60. Alternatively, it may be paid for with a cash deposit of \$176.40 followed by 23 monthly payments of \$26.72. How much cheaper is it to pay cash?

36 A telephone operator works a basic week of 37 hours at \$18.75 an hour. What is her weekly wage?

37 When \$16 500 is invested for a certain number of years at 12% p.a. simple interest the interest paid is \$11 880. For how many years was the investment?

38 A man saved \$7500 in a year. The next year his savings were 10% greater than the first year, and the following year his savings were 5% greater than the second year. How much did he save over the three years?

39 Mr Carter has a taxable income of \$48 600. He has to pay tax on the first \$15 000 at 6% and 30% on the remainder. How much income tax is due for the year?

40 List the members of these sets.

a $A = \{\text{multiples of 12 less than 60}\}$
b $B = \{\text{factors of 50}\}$
c $C = \{x: -2 < x < 4, x \in \mathbb{Z}\}$
d $D = \{(x, y): y = 1 - x, -2 < x < 4, x \in \mathbb{Z}\}$

41 $P = \{a^2: 4 < a < 7, a \in \mathbb{N}\}$

a Find $n(P)$.
b List the proper subsets of P.
c $U = \{a: 20 < a < 40, a \in \mathbb{N}\}$. Find $n(P')$.

42 $U = \{u, v, w, x, y, z\}$, $A = \{v, w, y\}$, $B = \{u, v, w\}$.

a Draw a Venn diagram showing these sets and their elements.
b Write down the value of $n(A \cup B)'$.

43 M and N are two sets and $M \subset N$. $n(M) = 8$, $n(N) = 12$.
Find the value of

a $n\{M \cup N)$ **b** $n(M \cap N)$

44 A, B and C are three sets such that $A \cap B = \varnothing$, $A \cap C \neq \varnothing$ and $B \cap C \neq \varnothing$. Draw a Venn diagram to illustrate these sets.

6 RELATIONS, FUNCTIONS AND GRAPHS

AT THE END OF THIS CHAPTER YOU SHOULD BE ABLE TO...

1 Recognise a relation.
2 Describe a relation as a set of ordered pairs.
3 Use arrow diagrams to show relations.
4 Identify the domain and range in a relation.
5 Define a function as a one-to-one or many-to-one relation.
6 Use functional notation.
7 Represent linear functions graphically.

MATHS IS OUT THERE

Did you know that Archimedes' death contains just as much mathematical dedication as his life? Before he died in 212 BC, he requested that his tomb show a picture of a cylinder circumscribing a sphere within it.

BEFORE YOU START

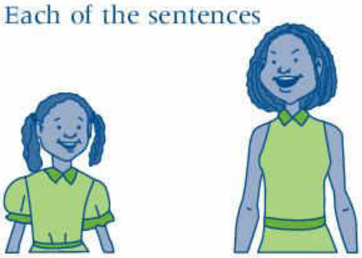
you need to know:
✓ how to use set notation
✓ how to plot points in a Cartesian plane.

KEY WORDS

binary relation, co-domain, conversion graph, domain, function, intercept, linear, mapping, ordered pairs, range, relation

Relations

Each of the sentences



Jane is the sister of Mary

Five is less than seven

Each of the sentence

Jane is the sister of Mary

and

Five is less than seven

involves a relation.

Each associates *two* objects:

(i) Jane and Mary, and **(ii)** 5and 7.

We therefore have a **binary relation**.

Consider the equation $y = 2x$, where x and y are integers.

This equation defines a relation, A, where

$$A = \{(1, 2)\ (2, 4)\ (3, 6)\ \ldots\}.$$

The values $x = 1$ and $y = 2$ satisfy the equation $y = 2x$.

We say that $(1, 2)$ is a member of A.

The pairs $(2, 4)$, $(3, 6)$, $(-4, -8)$ are also members of A.

However, $(1, 1)$, $(3, 7)$, $(-2, 9)$ are *not* members of A, because the value of y is not equal to twice the value of x.

A set of **ordered pairs** is a **relation**. This also called a mapping.

The set of all first components of the ordered pair in a relation is called the **domain** of the relation.
In the relation $\{(1, 3), (2, 4), (3, 5), (-2, 0)\}$, the domain is $\{1, 2, 3, -2\}$.

The set of all second components in the relation is called the **range**.
In the above relation, the range is $\{3, 4, 5, 0\}$.

The **co-domain** is a set that contains the range. The co-domain can be thought of as a universal set for the range, i.e. the range is a subset of the co-domain. In the above relation, the co-domain could be $\mathbb{R}$, where $\mathbb{R}$ is the set of all real numbers.

The co-domain must also contain only elements that are possible, even if the domain is changed. For example, for $\{(x, y): x = 1, 2, 3 \text{ and } y = x^2\}$ the range is $\{1, 4, 9\}$ but whatever values of x are chosen for $x \in \mathbb{R}$, values of $y\ (= x^2)$ are all greater than zero, so the co-domain cannot contain negative numbers.

EXERCISE 6a

Example:

Maximum temperature readings in Caribbean cities in August are given in the following table:

Kingston (Kgn)	32 °C
Bridgetown (B'town)	30 °C
Castries	31 °C
Montego Bay (Mobay)	30 °C

List the ordered pairs in the relation defined by the sentence 'city x has a temperature lower than city y.'
What are the domain and range of the relation?

If the relation involves the set, C, then C = {(Mobay, Castries),

(Mobay, Kgn), (B'town, Castries), (B'town, Kgn), (Castries, Kgn)}

Domain = {Mobay, B'town, Castries}

Range = {Castries, Kgn}

1 State the domain and range of the relations described by the following sets:
 a {(4, 3), (4, 2), (4, 1)}
 b {(5, 6), (6, 7), (7, 8), (8, 9)}
 c $\{(x, y): x, y \in \mathbb{N} \text{ and } x + y > 5\}$
 d $\{(x, y): x, y \in \mathbb{Z} \text{ and } xy = 6\}$

This means the set of ordered pairs (x, y) such that x and y are natural numbers and $x + y$ is less than 5.

$\mathbb{Z}$ = {integers}.

2 State which of the ordered pairs listed belong to the relation indicated:
 a $\{(x, y): x, y \in \mathbb{R} \text{ and } x^2 > y^2\}$; $(1, \frac{1}{3})$, $(1, -4)$, $(0, 1)$
 b $\{(x, y): x, y \in \mathbb{R} \text{ and } y > x + 3\}$; (0, 4), (1, 5), (3, 7)
 c $\{(x, y): x, y \in \mathbb{Q} \text{ and } y = \frac{x+2}{x+1}\}$; (0, 2), (2, 3), $\left(1, \frac{3}{2}\right)$

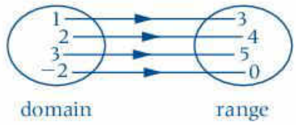
3 Given that the domain for each relation is {−4, −2, 0, 2, 4}, list the ordered pairs that belong to each relation and give a co-domain:
 a $\{(x, y): y = x^2\}$ **b** $\{(x, y): y = 2x + 1\}$ **c** $\{(x, y): y = x\}$

Diagrams representing relations

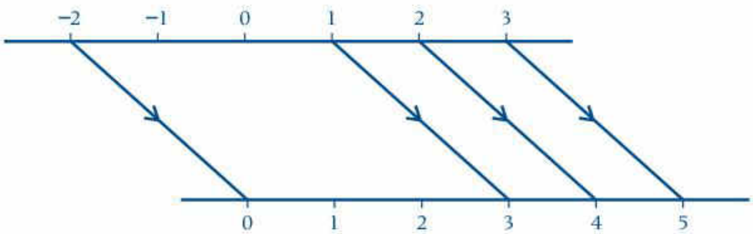
There are three main ways of representing a relation.

The relation {(1, 3), (2, 4), (3, 5), (−2, 0)} can be illustrated in one of three ways.

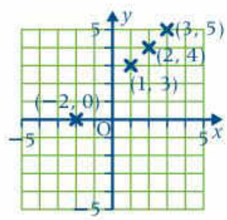
1. A 'bubble diagram' where the members of the domain are placed in one oval and the members of the range are placed in a second oval.



2. A pair of number lines where the members of the domain are placed on one number line and the members of the range are placed on another parallel number line.



3. Using the number pairs as coordinates and representing them as points in the *xy*-plane.



Types of relation

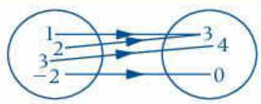
A one-to-one relation is such that

- the members of the domain are all different
- the members of the range are all different
- each member of the domain is paired with just one member of the range.

The relation shown above is one-to-one and the arrow diagrams show these properties clearly.

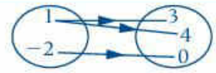
A many-to-one relation is such that

- the members of the domain are all different but the members of the range are not
- more than one member of the domain is paired with one member of the range.



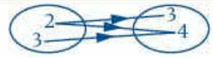
This diagram shows the many-to-one relation {(1, 3), (2, 3), (3, 4), (−2, 0)}.

A one-to-many relation is where one member of the domain is paired with more than one member of the range.



This diagram shows the one-to-many relation {(1, 3), (1, 4), (−2, 0)}.

A many-to-many relation is where one member of the domain is paired with more than one member of the range and more than one member of the domain is paired with one member of the range.



This diagram shows the many-to-many relation {(2, 3), (2, 4), (3, 4)}.

Functions

A relation that is either a one-to-one or many-to-one is called a **function**.

The relation {(0, 0), (1, 4), (2, 8), (3, 12)} is a function.

We can write this relation as $\{(x, y): y = 4x \text{ for } x \in \{0, 1, 2, 3\}\}$.

Because it is a function, we can describe it as $f: x \rightarrow 4x, x \in \{0, 1, 2, 3\}$ and we read it as 'the function such that x maps to $4x$ for $x = 0, 1, 2, 3$'.

We can describe it more briefly as $f(x) = 4x, x \in \{0, 1, 2, 3\}$ which we read as 'f of x equals $4x$ for $x = 0, 1, 2, 3$'.

A relation is also called a **mapping**. All functions are **mappings** but not all mappings are functions.

EXERCISE 6b

Example:

A relation is given by $\{(x, y): y^2 - x = 5, x \in \{-4, -1, 4\}\}$.

a List the relation as ordered pairs.

b State, with a reason, whether this relation is a function.

a When $x = -4$, $y^2 + 4 = 5$ so $y^2 = 1$, giving $y = \pm 1$
When $x = -1$, $y^2 + 1 = 5$ so $y^2 = 4$, giving $y = \pm 2$
When $x = 4$, $y^2 - 4 = 5$ so $y^2 = 9$, giving $y = \pm 3$
The relation is {(−4, −1), (−4, 1), (−1, −2), (−1, 2), (4, −3), (4, 3)}.

b The relation is not a function because the members of the domain are not all different.

Each value of x is paired with two values of y.

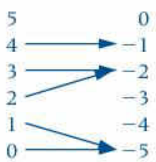
1 A and B are two relations where

$$A = \{(x, y): 5x + 3y = 15, x \in \{0, 3, 4, 5\}\}$$
$$B = \{(x, y): x^2 + y^2 = 25, x \in \{0, 3, 4, 5\}\}.$$

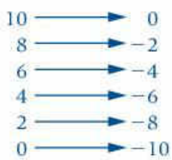
List the relations as ordered pairs and state whether they are functions.

2 List each of the following relations as ordered pairs and state which are functions.

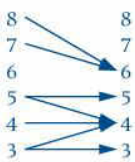
a



b



c



3 Given that $-3 > x > 3$, where $x \in \mathbb{Z}$, illustrate the following functions as mappings into the set of real numbers using diagrams like those in question 2.

a f: $x \to 3x$ **b** f: $x \to 5 - x$ **c** f: $x \to \frac{3}{x}, x \neq 0$ **d** f: $x^2 - 4$

Write down the range for each of the mappings above.

4 Draw an arrow diagram for the mapping defined below:

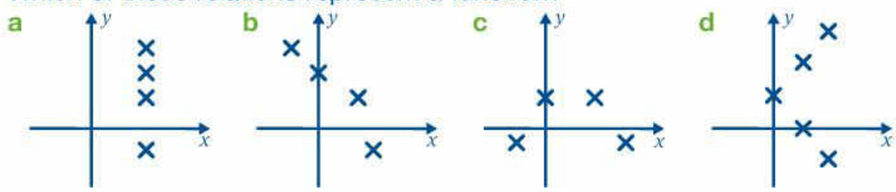
Mapping	Domain
$x \to 2x - 1$	$\{-2, -1, 0, 2, 3\}$

5 C and D are two relations where

$$C = \{(x, y): y^2 = 4x, x \in \{0, 1, 4, 9\}\}$$
$$D = \{(x, y): 3y = 2x + 1, x \in \{0, 1, 3, 5\}\}.$$

List the relations as ordered pairs and state whether or not they are functions.

6 Which of these relations represent a function?



Example:

For the function f: $x \to 3x$, $x \in \mathbb{R}$ (real numbers) find

a f(2) **b** f(0) **c** f(−4) **d** f(x − 1)

$f(x) = 3x$, then

a $f(2) = 3 \times 2 = 6$ **b** $f(0) = 3 \times 0 = 0$

c $f(-4) = 3 \times (-4) = -12$ **d** $f(x - 1) = 3 \times (x - 1) = 3x - 3$

f(2) means the value of f(x) when x is replaced by 2 and f(x − 1) means the value of f(x) when x is replaced by x − 1.

7 If $g(x) = 3x - 2$, $x \in \mathbb{R}$, state the value of each of the following.

a g(2) **b** g(0) **c** g(−2) **d** g(12)

e $g(x + 1)$ **f** $g(x^2)$ **g** $g(x + h)$

8 What is the image of −2 under each of the following mappings, where $x \in \mathbb{R}$?

a f: $x \to -x$ **b** f: $x \to x$ **c** f: $x \to 2x + 5$

d g: $x \to x^2 - 2$ **e** h: $x \to \frac{(3x - 2)}{4}$

9 What is the image of −3 under each of the following mappings, where $x \in \mathbb{R}$?

a g: $x \to -3x$ **b** g: $x \to \frac{1}{x}$ **c** g: $x \to \frac{x + 7}{4}$ **d** g: $x \to \frac{1 + x^2}{5}$

10 If g: $x \to \frac{3}{x + 1}$, $x \neq -1$, $x \in \mathbb{R}$, state the value of

a g(2) **b** g(−2) **c** $g\left(\frac{1}{2}\right)$

d g(4) **e** $g(x + 1)$ **f** $g(x^2)$

11 If h: $x \to 3x - \frac{6}{x}$, $x \neq 0$, $x \in \mathbb{R}$, state the value of

a h(2) **b** h(3) **c** h(−2) **d** h(12)

12 Given that $g: x \to \frac{4x-3}{1+2x}, x \neq -\frac{1}{2}, x \in \mathbb{R}$, find

a $g(1)$ **b** $g(-1)$ **c** $g\left(\frac{1}{2}\right)$ **d** $g(7)$

Why is $x \neq -\frac{1}{2}$?

13 $f(x) = \frac{2}{5+x}, x \neq -5$ where $x \in \mathbb{R}$.

a What is the image of -3 under this mapping?

b Find the value of x which maps $f(x)$ to **i** 2 **ii** $-\frac{1}{2}$

14 The functions g and h are defined by $g: x \to \frac{1}{x}, x \neq 0$ and $h: x \to \frac{3x-1}{x-3}, x \neq 3$

a Calculate

i $g\left(\frac{1}{2}\right)$ **ii** $h(5)$

b Calculate the value of x for which $h(x) = \frac{5}{3}$.

15 The functions f and g are defined by $f(x) = \frac{x}{2} + 1$ and $g(x) = 2x + 1$
Calculate

a $f(2)$ **b** $g(-3)$ **c** $f(5) \times g\left(\frac{1}{2}\right)$

16 The function f is defined by $f(x) = \frac{3(x-1)}{x}$

a Find **i** $f(2)$ **ii** $f(1)$ **iii** $f(-3)$

b Find the value of x for which f maps x to 2.

c Is there any value of x for which $f(x)$ is undefined?

What is the one number you cannot divide by?

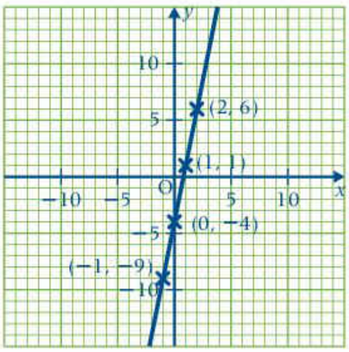
Linear functions

A **linear** function has the form $f(x) = ax + b$, where a and b are constants and $a \neq 0$.

If we let $y = f(x)$, then $y = ax + b$ and we can make a table of values to show some of the ordered pairs of the function which we can plot as points on the xy-plane to give a graphical representation of the function.

For example, for $y = 5x - 4$, $x \in \mathbb{R}$:

x	-1	0	1	2
y	-9	-4	1	6



These points all lie on a straight line.

Because x can have any real value, all the possible ordered pairs are represented by the points between and beyond the points we have plotted. This means that the line goes on for ever in both directions.

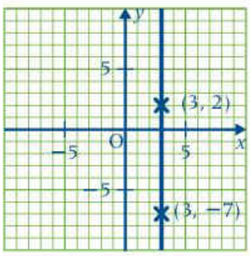
We say that $y = 5x - 4$ is the equation of a straight line.

All linear functions give straight lines when the ordered pairs are plotted on the xy-plane.

Straight line graphs

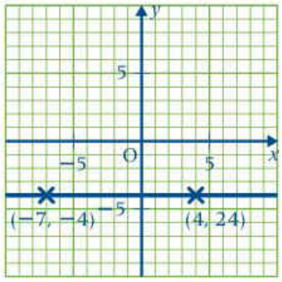
Straight lines can be anywhere on the xy-plane and at any angle to the x-axis.

Lines that are parallel to the y-axis have an equation of the form $x = k$ where k is a constant.



Every point on this line has an x-coordinate of 3. The equation of this line is $x = 3$.

Lines that are parallel to the x-axis have an equation of the form $y = c$ where c is a constant.



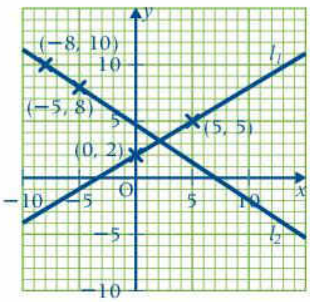
Every point on this line has a y-coordinate of -4. The equation of this line is $y = -4$

Sloping lines are at an angle to the x-axis.
We measure the slope by calculating the gradient.

If (x_1, y_1) and (x_2, y_2) are two points on the line,

$$\text{gradient} = \frac{y_1 - y_2}{x_1 - x_2} \quad \left(\text{or } \frac{y_2 - y_1}{x_2 - x_1}\right)$$

A positive gradient indicates a line that slopes up with respect to the positive direction of the x-axis, and a negative gradient indicates a line that slopes down.



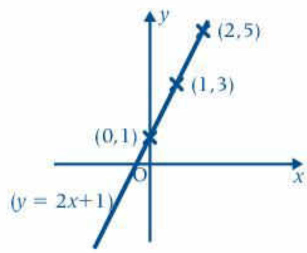
Gradient of l_1 = $\frac{5 - 2}{5 - 0} = \frac{3}{5} = 0.6$
This is positive and l_1 slopes up.
Gradient of l_2 = $\frac{10 - 8}{-8 - (-5)} = \frac{2}{-3} = -\frac{2}{3}$
This is negative and l_2 slopes down. Note that it is important to keep the coordinates in the correct order.

The standard equation of a straight line

The equation of any line can be written as

$$y = mx + c.$$

For example, $y = 2x + 1$ where $m = 2$ and $c = 1$



The points (1, 3) and (2, 5) are points on this line; these give the gradient as 2, i.e. the value of m gives the gradient.

When $x = 0$, $y = 1$; this is the value of y where the line crosses the y-axis. This is called the ***y*-intercept**, i.e. the value of c gives the y-intercept.

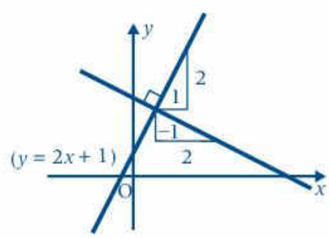
This means that we can read the gradient of a line and its intercept on the y-axis from the equation of the line, provided we write it in the form
$y = mx + c$.

To find the gradient and y-intercept of the line whose equation is $x + 2y = 1$, first rearrange it as $y = -\frac{1}{2}x + \frac{1}{2}$
Now we can see that the gradient is $-\frac{1}{2}$ and it crosses the y-axis where $y = \frac{1}{2}$

Parallel lines and perpendicular lines

Parallel lines have the same gradient.

In the diagram, the line $y = 2x + 1$ has a gradient of 2. We can see that a perpendicular to this line has a gradient of $-\frac{1}{2}$.



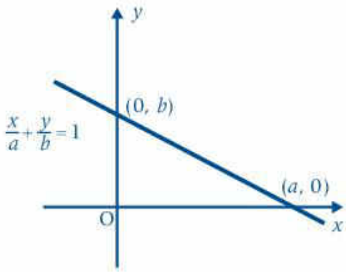
This is always true, so, if one line has a gradient m, then any line perpendicular to it has a gradient equal to $-\frac{1}{m}$.

This means that the product of the gradients of perpendicular lines is -1.

The intercept form of the equation of a straight line

When the equation of a line is written in the form $\frac{x}{a} + \frac{y}{b} = 1$,

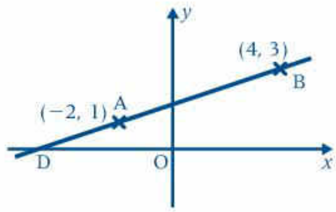
a is the intercept on the x-axis and b is the intercept on the y-axis because, when $y = 0$, $x = a$ and when $x = 0$, $y = b$.



EXERCISE 6c

Example:

In the diagram, which is not drawn to scale, AB is the straight line joining A and B.



a Calculate the gradient of AB.
b Find the equation of the line through A and B.
c Write down the x-coordinate of D, the point where the line crosses the x-axis.
d Find the equation of the line through A that is perpendicular to AB.
e Write down the equation of the line through (1, 1) that is parallel to AB.

a Gradient of AB $= \frac{3-1}{4-(-2)} = \frac{2}{6} = \frac{1}{3}$

b $y = \frac{1}{3}x + c$

When $x = 4, y = 3$ so $3 = \frac{1}{3}(4) + c$

$3 = \frac{4}{3} + c$ so $c = \frac{5}{3}$

The equation of the line is $y = \frac{1}{3}x + \frac{5}{3}$

We know the gradient so from $y = mx + c$, we have $y = \frac{1}{3}x + c$. We also know that (4, 3) is a point on the line, so $x = 4$ and $y = 3$ satisfies this equation. We can use this to find c.

c $y = 0$ where the line crosses the x-axis.

So $0 = \frac{1}{3}x + \frac{5}{3}$

$-\frac{5}{3} = \frac{1}{3}x$ so $x = -5$

The x-coordinate of D is -5

d The line perpendicular to AB has a gradient $\frac{-1}{\frac{1}{3}} = -3$

$y = -3x + c$ and when $x = -2, y = 1$ so $1 = -3(-2) + c$

$1 = 6 + c$ so $c = -5$

The equation of the line through A perpendicular to AB is

$y = -3x - 5$

e Any line parallel to AB has gradient $\frac{1}{3}$ and equation $y = \frac{1}{3}x + c$

When $x = 1$ and $y = 1$, $1 = \frac{1}{3} \times 1 + c$ so $c = \frac{2}{3}$

The equation of the line through (1, 1) parallel to AB is $y = \frac{1}{3}x + \frac{2}{3}$

1 Draw axes for x and y from -6 to $+6$ using the same scale on both axes.
Plot the points A(6, 3), B(5, 6), C(−3, 4) and D(−2, 1).
What is the name of the special quadrilateral ABCD?

2 Draw x and y axes for values of both x and y from -3 to 6.
Use 2 cm to represent 1 unit on both axes.
a Plot the points A(5, 5), B(−2, 5) and C(−2, −2).
b **i** Mark D so that ABCD is a square.
ii Write down the coordinates of D.
iii Write down the coordinates of the point of intersection of the diagonals.
c Find the gradient of **i** CA **ii** BD.

3 Draw x and y axes for values of both x and y from -8 to 10.
Use 1 cm to represent 1 unit on both axes.
a Plot the points A(−2, 9), B(6, 9) and C(6, −5).
b **i** Mark D so that ABCD is a rectangle.
ii Write down the coordinates of D.
iii Write down the coordinates of E, the point of intersection of the diagonals of the rectangle.
c Find the gradient of **i** DB **ii** AC.

4 For each of the following straight line equations, find the gradient and the coordinates of the point where it crosses the y-axis.
a $y = 2x + 6$ **b** $y = 3x - 7$ **c** $y = -5x + 2$
d $y = -\frac{1}{2}x + 3$ **e** $y = 4 - 3x$

5 Find the equation of the straight line
a with gradient 4, which crosses the y-axis at the point (0, 5)
b with gradient -3, which crosses the y-axis at the point (0, 2)
c with gradient $\frac{1}{2}$, which crosses the y-axis at the point (0, −4)
d with gradient $-\frac{3}{2}$, which crosses the y-axis at the point $\left(0, \frac{1}{2}\right)$.

6 Find the equation of the straight line
a with gradient 2, that passes through the point (2, 6)
b with gradient -3, that passes through the point (−4, 2)
c with gradient $-\frac{1}{2}$, that passes through the point (5, 3)
d with gradient $\frac{5}{3}$, that passes through the point (−6, −2).

7 Sketch the graphs of the straight lines whose equations are
a $x = 3$ **b** $y = 4$ **c** $x = -2$ **d** $y = 2x$
e $y = 2x + 3$ **f** $y = x - 4$ **g** $y = 4 - x$ **h** $y = -x - 4$

8 Sketch the graphs of the straight lines whose equations are
a $x + y = 1$ **b** $y - x = 3$ **c** $x + 2y = 4$ **d** $x - y = 2$
e $3x + y = 6$ **f** $x - 2y = 6$ **g** $x + y = 3$ **h** $y - 3x = 9$

9 Write down the gradient of a straight line that is perpendicular to the straight line with equation
a $y = 3x - 8$ **b** $2x - 3y = 7$ **c** $5x + 4y = 20$ **d** $2y + 7x = 8$

10 Write down the gradient of the straight line that is
a parallel to the line whose equation is $3y = 4x - 6$
b perpendicular to the straight line $2y = x + 5$
c perpendicular to the straight line $2x + 5y = 10$
d parallel to the line whose equation is $4x - 6y = 5$

11 Find the equation of the straight line that is

a perpendicular to the line $2x + 5y = 10$ and passes through the point $(2, 2)$

b parallel to the line $2y = 6x + 9$ and passes through the point $(-5, 4)$

c perpendicular to the line $2y - 5x = 20$ and passes through the point $(-3, 3)$

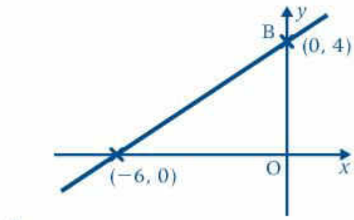
d parallel to the line $5x - 3y = 7$ and passes through the point $(2, 6)$.

12 **a** Find

i the gradient of AB

ii the equation of the line AB.

b Write down the equation of the line that passes through the origin and is perpendicular to AB.



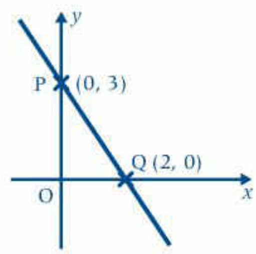
13 Find

a the gradient of PQ

b the equation of the line PQ

c the gradient of the line through O that is parallel to PQ

d the equation of this line.



14 For each equation find **i** the x-intercept **ii** the y-intercept.

a $y = 5x - 10$ **b** $3x + 4y = 12$ **c** $x + y = 6$ **d** $2x - 3y = 18$

15 Find, in the form $y = mx + c$, the equation of the straight line with x-intercept 3 and y-intercept -5

16 The coordinates of A and B are $(3, 7)$ and $(5, 3)$ respectively.
Find

a the gradient of AB

b the equation of AB

c the gradient of a line perpendicular to AB

d the equation of the line perpendicular to AB that passes through the point $(8, 10)$.

17 **a** Find the gradient of the straight line through the point $P(-1, -1)$ and $Q(2, 5)$.

b Find the equation of the line parallel to PQ that passes through the point $(-3, 5)$.

c Find the equation of the line perpendicular to PQ that passes through the point $(6, -2)$.

18 A straight line passes through the point $A(-1, -4)$ and $B(5, 1)$.
Find

a the equation of AB

b the equation of the straight line through A that is perpendicular to AB. Where does this line cross the axes?

19 The vertices of a quadrilateral ABCD are $A(-3, 2)$, $B(5, 6)$, $C(7, 2)$ and $D(-1, -2)$.

a Plot these points on a grid and find the equation of each side. What is the name of this special quadrilateral?

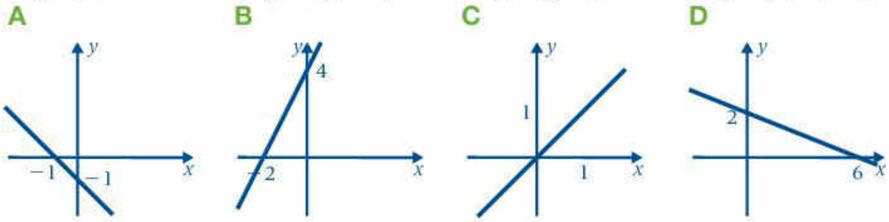
b Find the equation of the line through the point $(2, 2)$ and perpendicular to

i AB **ii** BC.

iii Write down the coordinates of the points where these lines cross the y-axis.

20 Pair up each equation with one of the graphs below.

a $y = x$ **b** $y = 2x + 4$ **c** $x + 3y = 6$ **d** $x + y + 1 = 0$



Example:

The cost of a visit by an engineer to repair a school photocopier is made up of a fixed call-out charge and a charge for the time taken to make the repairs.

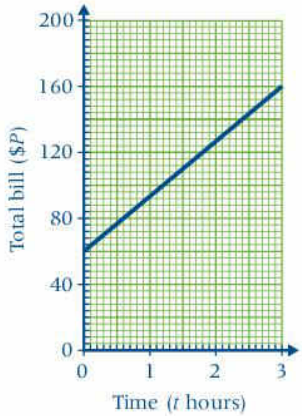
The graph shows the total bill ($*P*) for repairs lasting up to 3 hours.

a Write down the fixed call-out charge.

b How much is the charge for a repair that takes $1\frac{1}{2}$ hours?

c How long does a repair take that costs $80?

d Find the equation of the line in terms of *P* and *t*.



a $60

b $112

The fixed call-out charge is the amount charged when no time is spent on repairs.

Go up from $1\frac{1}{2}$ h to the line then across.

c 39 minutes

Each subdivision on the time axis represents $\frac{1}{10}$ h = 6 min.
Each subdivision on the vertical axis represents $4.

d Gradient $= \frac{160 - 60}{3} = 33\frac{1}{3}$. Intercept on *P*-axis = 60

The equation is $P = 33\frac{1}{3}t + 60$

The equation is in the form $p =$ (gradient) $\times t + c$.

21 The cost of hiring a car is made up of a fixed amount and a fee for each day it is hired.

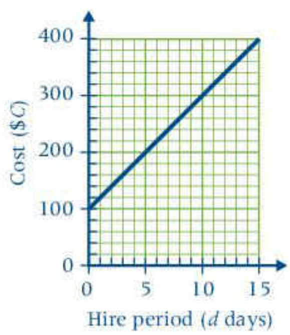
The graph shows the cost ($*C*) of hiring a car for up to 15 days.

a Write down the fixed amount.

b Calculate the fee for one day's hire.

c What is the bill for hiring the car for 7 days?

d Find the equation of the line.



22 **a** Use the values 0 °C = 32 °F and 20 °C = 68 °F to draw a graph to convert between degrees Fahrenheit (°F) and degrees Celsius (°C). Use a scale of 2 cm to represent 10° on both axes. Scale the Fahrenheit axis from 0 to 140 and the Celsius axis from −20 to 60.

b Use your graph to convert **i** 35 °F to °C **ii** 21 °C to °F.

c Find the equation of the line.

Graphs like this one, that can be used to convert a quantity measured in one unit to another unit are called **conversion graphs**.

23 **a** Use BD\$10 = JM\$33.97 to draw a graph to convert between B Barbadian dollars and J Jamaican dollars for sums of money up to BD\$100.

b Use your graph to convert

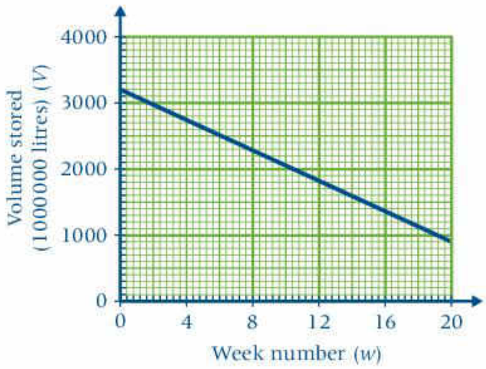
i BD\$55 to Jamaican dollars

ii JM\$750 to Barbadian dollars.

c Find the equation of the line.

> Use graph paper and choose the scales for the axes. Use scales where the subdivisions represent whole numbers such as 1 cm for \$10. Use BD\$0 = JM\$0 as the second point to draw the graph. Remember to label your axes.

24 This graph shows the amount of water in a reservoir over 20 weeks.



a Find the gradient of the line and explain what it represents.

b Find the equation of the line.

c The owners of the reservoir sent out a warning at the end of week 20 saying that the water would run out in another 2 months.
What assumptions have the owners made?

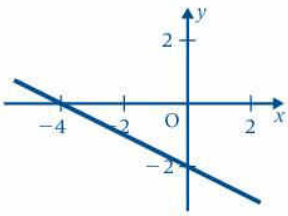
MIXED EXERCISE 6

Several answers are given for these questions.

Write down the letter that corresponds to the correct answer.

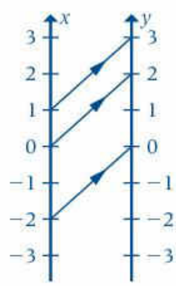
1 The relation shown in this graph is

A $y - 2x - 2 = 0$ **B** $2y + x + 2 = 0$
C $2y - x - 2 = 0$ **D** $2y + x + 4 = 0$



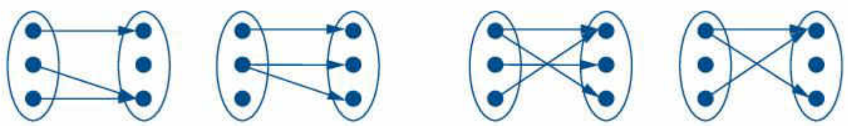
2 The relation shown in the diagram is

A $y = x + 2$ **B** $x = y + 2$ **C** $y = x - 2$ **D** $y = 2 - x$



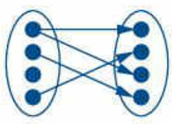
3 Which of these relations is a function?

A **B** **C** **D**



4 What type of relation is shown in the diagram?

A one-to-one **B** one-to-many
C many-to-one **D** many-to-many



5 If $f(x) = \frac{2x}{x+1}$ then f(1)=

A −1 **B** $\frac{1}{2}$ **C** 1 **D** 2

6 If $f(x) = 5 - x^2$ then $f(-2)$ =

A 1 **B** 3 **C** 9 **D** 49

7 The gradient of the straight line with the equation $7x - 14y = 4$ is

A −2 **B** $\frac{1}{2}$ **C** 2 **D** $-\frac{1}{2}$

8 The gradient of the straight line perpendicular to the line with the equation $6x - 2y = 5$ is

A $\frac{1}{3}$ **B** $-\frac{1}{3}$ **C** 3 **D** −3

9 The equation of the straight line passing through the point (2, −5) with gradient $\frac{1}{2}$ is

A $x - 2y = 12$ **B** $y = 2x - 9$
C $x + 2y = 12$ **D** $x - 2y + 8 = 0$

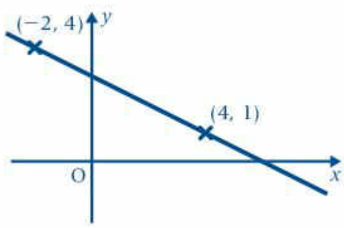
10 The equation of the straight line passing through the point (0, 5) that is parallel to the line $x + 4y = 8$ has the equation

A $x - 4y = 20$ **B** $4x + y = 20$
C $4x - y = 20$ **D** $x + 4y = 20$

11 The equation of the straight line passing through the point (3, 2) that is perpendicular to the line $2x - y = 6$ has the equation

A $x - 2y = 7$ **B** $x + 2y = 7$ **C** $x - 2y = 1$ **D** $x + 2y = 1$

For questions **12** to **15** refer to this diagram.



12 The gradient of the line is

A −2 **B** $-\frac{1}{2}$ **C** $-\frac{1}{3}$ **D** $-\frac{1}{4}$

13 The intercept on the y-axis is

A 1 **B** 2 **C** 3 **D** 4

14 The equation of the line is

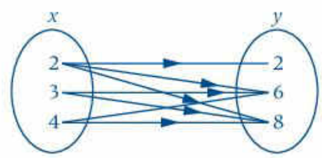
A $2x - y + 3 = 0$ **B** $x - 2y + 3 = 0$
C $x + 2y - 6 = 0$ **D** $x - 2y + 6 = 0$

15 The intercept on the x-axis is

A 2 **B** 4 **C** 5 **D** 6

16 The relation shown in the diagram is best described by

A x is less than y **B** x is less than or equal to y
C x is greater than y **D** x is greater than or equal to y



17 Which of the following sets lists the ordered pairs in the relation

$\{(x, y): y = x + \frac{1}{x}, x \in \{-1, \frac{1}{2}, 1\}\}$

A $\{(-1, -2), (\frac{1}{2}, 2\frac{1}{2}), (1, 2)\}$ **B** $\{(-2, -1), (3, \frac{1}{2}), (2, 1)\}$

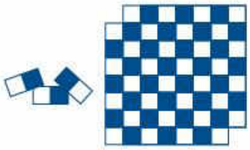
C $\{(-1, 0), (\frac{1}{2}, 1\frac{1}{2}), (1, 1)\}$ **D** $\{(-1, 1), (\frac{1}{2}, 3), (1, 1)\}$

INVESTIGATION

The squares have been removed from the opposite corners of this chess board.

Robin has 31 black and white dominoes, any one of which will cover two adjacent squares on this board.

Investigate whether or not he can cover all the squares on the board, and explain your answer.



MATHS IS OUT THERE

A straight line is probably the oldest geometric concept. It probably predates any concept of measurement or other geometric concepts such as angles.

The connection between a line drawn on a set of coordinate axes and an equation was established by Pierre de Fermat (1596–1650) and René Descartes (1601–1665).

IN THIS CHAPTER YOU HAVE SEEN THAT...

- a binary relation is a set of ordered pairs
- the domain is the set of the first components of the ordered pairs and the range is the set of the second components
- a relation may be one of four types: one-to-one, one-to-many, many-to-one or many-to-many
- a function is a relation that is either one-to-one or many-to-one
- a linear function, f, has the form $f: x \to ax + b$, where a and b are constants and $a \neq 0$
- the graphical representation of a linear function is a straight line
- the gradient of a straight line is $\frac{y_2 - y_1}{x_2 - x_1}$ where (x_1, y_1) and (x_2, y_2) are two points on the line
- the standard equation of a straight line is $y = mx + c$ where m is the gradient and c is the intercept on the y-axis
- parallel lines have the same gradient
- if one line has a gradient m, then any line perpendicular to it has a gradient equal to $-\frac{1}{m}$
- the intercept form of the equation of a straight line is $\frac{x}{a} + \frac{y}{b} = 1$, where a is the intercept on the x-axis and b is the intercept on the y-axis.

7 GEOMETRY AND TRIGONOMETRY

AT THE END OF THIS CHAPTER YOU SHOULD BE ABLE TO...

1. Identify plane figures possessing bilateral, rotational and translational symmetry.
2. Identify and describe the transformations of reflection, rotation, translation and enlargement.
3. Use the properties of parallel lines, angles and polygons to solve geometric problems.
4. Use the properties of congruent triangles and similar triangles.
5. Use Pythagoras' theorem and the sine, cosine and tangent of acute angles to solve problems in right-angled triangles.
6. Solve problems involving bearings, angles of elevation and depression.

MATHS IS OUT THERE

Euclid's book *The Elements* still has an influence on mathematics, 2000 years after it was written. This must make Euclid the greatest maths teacher of all time!

BEFORE YOU START

you need to know:
- ✓ how to work with decimals and fractions
- ✓ how to plot points on a set of x and y axes
- ✓ how to measure angles.

KEY WORDS

alternate angles, angle of depression, angle of elevation, angle of rotation, bearing, bilateral symmetry, centre of enlargement, centre of rotation, congruent, converse, corresponding angles, cosine, displacement, enlargement, equilateral triangle, glide reflection, hexagon, hypotenuse, interior angle, isosceles triangle, kite, mirror line, parallelogram, pentagon, polygon, quadrilateral, rectangle, reflection, rhombus, right-angled triangle, rotation, rotational symmetry, scale factor, sine, square, tangent of an angle, transformation, translation, translational symmetry, trapezium, triangle, vector, vertically opposite angles

Symmetry

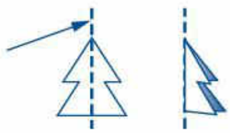
A plane shape has **bilateral symmetry** when it can be folded along a straight line so that one half fits exactly over the other half.

The straight line is called an axis of symmetry.

A plane shape can have more than one axis of symmetry.

This shape has bilateral symmetry.

This line is an axis of symmetry.



This square has 4 axes of symmetry.

A plane shape has **rotational symmetry** if it can be turned about a point to a new position and still look the same.

The number of times it can be turned in one revolution and still look the same is called the order of rotational symmetry.

This shape has rotational symmetry of order 2.

This shape has rotational symmetry of order 4.

A shape has **translational symmetry** when it can be divided by parallel straight lines into identical shapes.

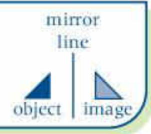
This shape has translational symmetry.

Transformations

A **transformation** changes the position and/or the size of a shape. The original shape is called the object and the transformation is called the image.

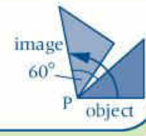
A **reflection** transforms the object by reflecting it in a line. The line is called the **mirror line** and the object and the image have bilateral symmetry about the mirror line.

The light triangle is the reflection of the dark triangle in the mirror line



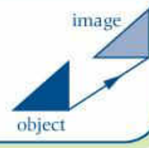
A **rotation** transforms the object by rotating it about a point. The point is called the **centre of rotation** and the amount of rotation is called the **angle of rotation**. A positive angle is an anticlockwise rotation and a negative angle is a clockwise rotation.

The light triangle is a rotation of the dark triangle by 60° anticlockwise about the point P.



A **translation** transforms an object by moving it without rotation or reflection. Every point on the object moves the same distance and in the same direction.

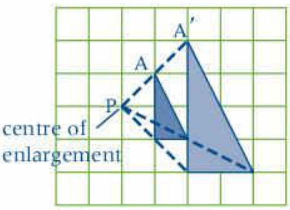
The light triangle is a translation of the dark triangle by the vector shown.



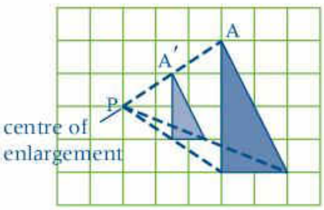
An **enlargement** transforms an object by making it larger or smaller by a given **scale factor**. This is done by drawing guide lines from a point P, called the **centre of enlargement**, to the vertices of the object. The guide lines are then extended to give the vertices of the image.

The centre of enlargement can be anywhere. It can be inside, on an edge, or outside the object.

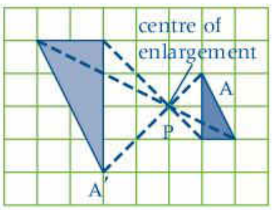
If A is a vertex on the object and A′ is the corresponding vertex on the image, then $PA' = (\text{scale factor}) \times PA$.



Scale factor = 2



Scale factor = $\frac{1}{2}$

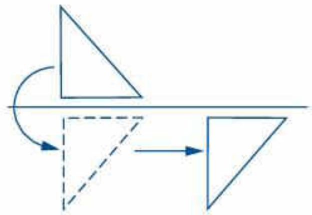


Scale factor = −2

Any length on the image = (scale factor) × the corresponding length on the object.
The area of the image = $(\text{scale factor})^2$ × the area of the object.

An area is the product of two lengths. If the area of the object is *ab*, the area of the image is $sa \times sb$, i.e. $s^2 \times ab$ where *s* is the scale factor.

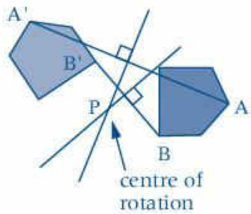
A **glide reflection** is a reflection in a mirror line followed by a translation in the direction of the mirror line.



Describing a transformation

We must give the mirror line to describe a reflection.
The mirror line is the perpendicular bisector of the line joining a point on the object to the corresponding point on the image.

We must give the centre and the angle of rotation to describe a rotation. We can find the centre of rotation by joining two pairs of corresponding vertices and finding where their perpendicular bisectors meet. The angle of rotation is the angle between lines drawn from P to a pair of corresponding vertices, angle BPB′.

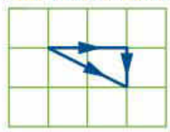


We must give the distance and direction of the shift to describe a translation.

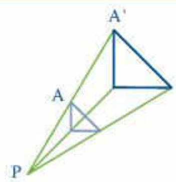
We can give the distance and direction with a column **vector**, such as $\binom{a}{b}$.

The top number, a, gives the **displacement** parallel to the x-axis and the bottom number, b, gives the displacement parallel to the y-axis.

The vector $\binom{2}{-1}$ represents a shift of 2 units along to the right and 1 unit down.



We must give the centre of enlargement and the scale factor to describe an enlargement. We can find the centre of enlargement, P, by drawing lines joining corresponding points on the object and image and finding where they meet. If A and A′ are corresponding points on the object and image, the scale factor is the value of $\frac{PA'}{PA}$.



To describe a glide reflection, we need to give the mirror line and the translation.

EXERCISE 7a

Example:

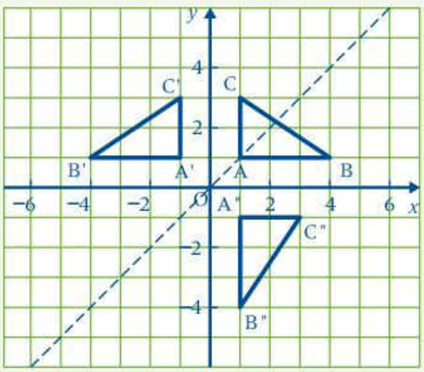
Use squared paper to draw a set of x and y axes and label them using a scale of 1 cm for 1 unit on both axes. Draw triangle ABC with A(1, 1), B(4, 1) and C(1, 3).

a Draw the reflection of triangle ABC in the y-axis and label it A′B′C′.

b Draw the reflection of triangle A′B′C′ in the line $y = x$ and label it A″B″C″.

c Describe the single transformation that maps triangle ABC to triangle A″B″C″.

a



b

c *A rotation about the origin by 90° clockwise.*

The reflection of C in the y-axis is C′ where CC′ is perpendicular to, and bisected by, the y-axis.

The reflection of A′ in the line $y = x$ is A″ where A′A″ is perpendicular to, and bisected by, the line $y = x$.

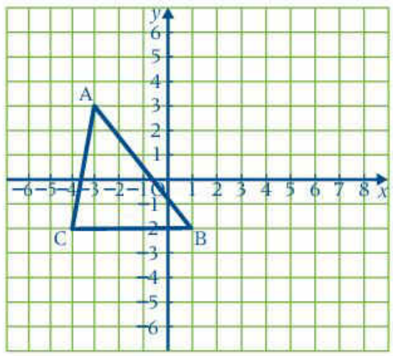
If you cannot 'see' the centre of rotation, draw the perpendicular bisectors of the lines joining AA″ and BB″ and find where they intersect. You do not need to repeat with a third pair of corresponding points as all the perpendicular bisectors are concurrent.

1 **a** The triangle ABC is translated using the vector $\begin{pmatrix}6\\3\end{pmatrix}$

Show this on a copy of this diagram.

Mark the transformed triangle A′B′C′.

b Draw the reflection of triangle A′B′C′ in the x-axis. Mark it A″B″C″ and write down the coordinates of its vertices.



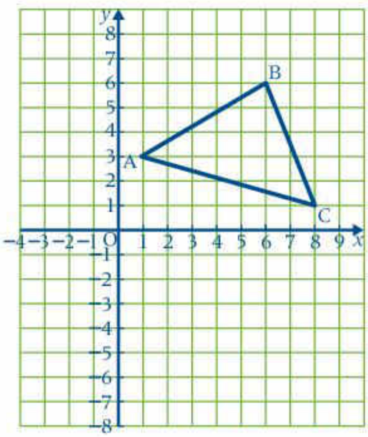
To find the image of A, translate it 6 units to the right and by 3 units up. Repeat with B and C.

2 **a** Draw the reflection of triangle ABC in the line $y = -1$.

Label it A′B′C′.

b Draw the image of triangle ABC under a rotation about O of 90° clockwise. Label the image A″B″C″.

c Draw the image of triangle ABC under an enlargement, scale factor 2 and centre of enlargement (6, 8). Label the image A‴B‴C‴.



Join O to A and rotate OA 90° clockwise about O to give the image of A. Repeat with B and C.

Join (6, 8) to A then extend the line so it is twice as long to give A‴. Repeat with B and C.

3 Triangle ABC is mapped to triangle A′B′C′ by a rotation of 90° anticlockwise about O.

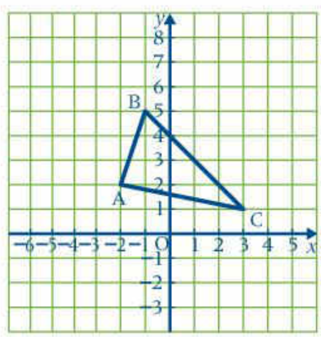
a Show this on a copy of the diagram.

Write down the coordinates of A′, B′ and C′.

b Draw the reflection of A′B′C′ in the y-axis.

Label it A″B″C″ and write down the coordinates of its vertices.

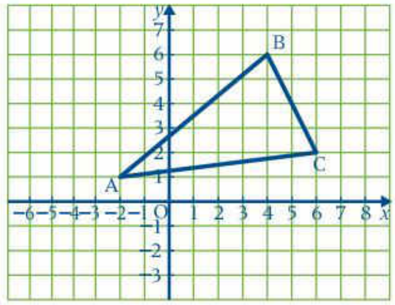
c Draw the image of triangle ABC under an enlargement by a scale factor of $1\frac{1}{2}$ from centre (3, 0). Label the image A‴B‴C‴.



4 **a** On a copy of the diagram draw the reflection of triangle ABC in the line $y = x$. Label it A′B′C′.

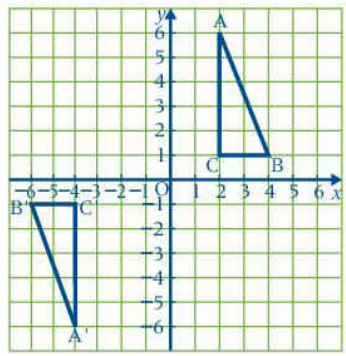
b Triangle A′B′C′ is translated using the vector $\begin{pmatrix} -6 \\ 0 \end{pmatrix}$ to give triangle A″B″C″.
Write down the coordinates of A″, B″ and C″.

c Draw the image of triangle ABC under an enlargement with scale factor $-\frac{1}{2}$ and centre P(2, 5). Label the image DEF.

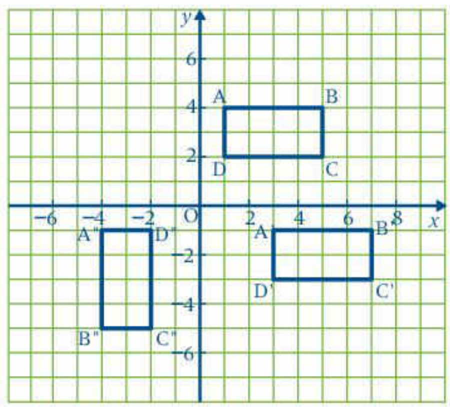


The scale factor is negative so you need to extend the guide lines backwards.

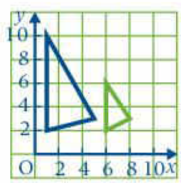
5 Triangle ABC is mapped by a rotation to triangle A′B′C′.
Give **a** the coordinates of the centre of rotation
b the angle of rotation.
State, with a reason, whether it is necessary to give the direction of rotation.



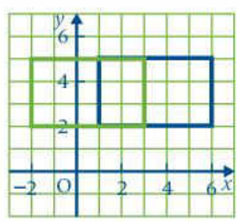
6 Describe completely the transformation that maps
a ABCD onto A′B′C′D′.
b ABCD onto A″B″C″D″.



7 The green triangle is an enlargement of the blue triangle.
Find
a the scale factor
b the coordinates of the centre of enlargement.



8 The blue rectangle can be mapped onto the green rectangle by a translation or a rotation or a reflection.
a If it is mapped by a translation, write down the vector.
b If it is mapped by a rotation, give the centre and angle of rotation.
c If it is mapped by a reflection give the equation of the mirror line.

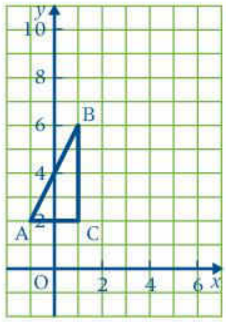


9 A(−1, −2), B(−3, −4) and C(−5, −1) are the vertices of a triangle ABC. The triangle is reflected in the line which passes through the points G(−3, 3) and H(3, −3) to give triangle A′B′C′.
Write down the coordinates of A′, B′ and C′.

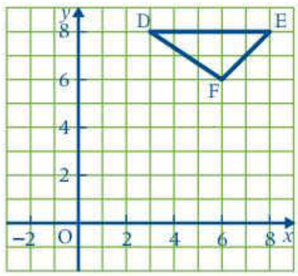
10 A(2, 1), B(3, 5) and C(6, 5) are the vertices of triangle ABC. The triangle is rotated about O so that the image of A is A′(1, −2) and the image of B is B′(5, −3).
Write down **a** the angle of rotation **b** the coordinates of C′.

11 Quadrilateral ABCD is mapped onto quadrilateral A′B′C′D′ with vertices A′(−2, −3), B′(1, −5), C′(5, −4) and D′(2, −2) under a reflection in the line $y = -2$.
Write down the coordinates of the vertices of ABCD.

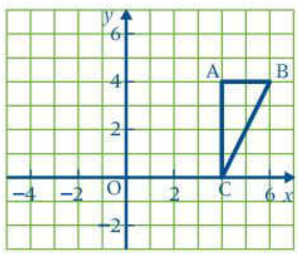
12 **a** Reflect triangle ABC in the line $x = 2$.
b Translate the resulting triangle using the vector $\begin{pmatrix} 0 \\ 3 \end{pmatrix}$.
Mark this triangle A′B′C′ and write down the coordinates of its vertices.



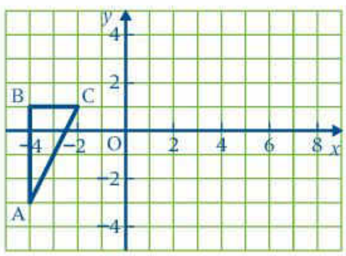
13 **a** Reflect triangle DEF in the line $y = 5$.
b Translate the resulting triangle using the vector $\begin{pmatrix} -5 \\ 0 \end{pmatrix}$.
Mark this triangle D′E′F′ and write down the coordinates of its vertices.



14 **a** Reflect triangle ABC in the line $y = x$.
b Translate the resulting triangle using the vector $\begin{pmatrix} -4 \\ -4 \end{pmatrix}$.
Mark this triangle A′B′C′ and write down the coordinates of its vertices.

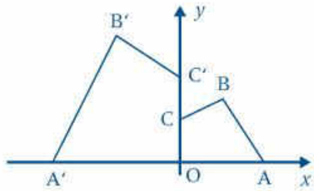


15 **a** Reflect triangle ABC in the line $y = -x$.
b Translate the resulting triangle using the vector $\begin{pmatrix} 5 \\ -5 \end{pmatrix}$.
Mark this triangle A′B′C′ and write down the coordinates of its vertices.



Example:

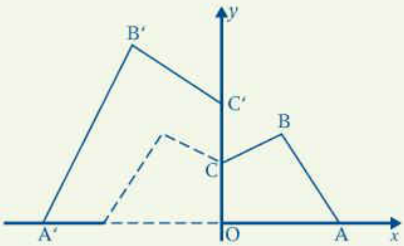
In the diagram, $OA' = 2OA$ and quadrilateral OABC is transformed to quadrilateral OA′B′C′ by a transformation M followed by a transformation N.



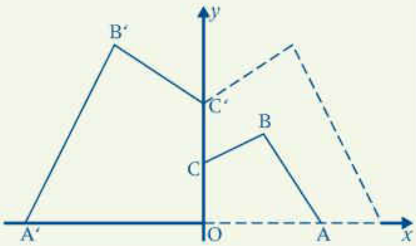
a Describe **i** M **ii** N.

b The area of OABC is $10\,cm^2$. Find the area of OA′B′C′.

We can get to OA′B′C′ by reflecting OABC in the y-axis then enlarging the image by a scale factor 2, centre O.



Alternatively we can enlarge OABC by a scale factor 2, centre O and then reflect the image in the y-axis.



a **i** *M is a reflection in the y-axis.*

ii *N is an enlargement, centre O, scale factor 2.*

OR

i *M is an enlargement, centre O, scale factor 2.*

ii *N is a reflection in the y-axis.*

b *Area OA′B′C′* $= (2^2) \times 10\,cm^2$
$= 40\,cm^2$

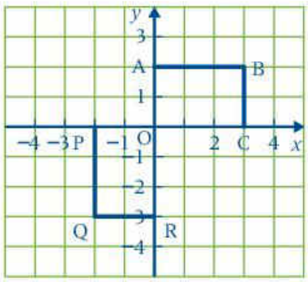
The scale factor is 2, so the area of the image is 2^2 times the area of the object.

16 In the diagram OABC can be mapped onto OPQR by a transformation M followed by another transformation N.

a Describe fully **i** the transformation M

ii the transformation N.

b Describe fully the single transformation that will map OABC onto OPQR.



17 **a** Describe the transformation that maps

i triangle P onto triangle Q

ii triangle Q onto triangle R.

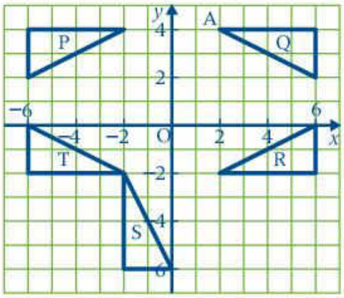
b Describe the single transformation that maps P onto R.

c Describe the transformation that maps

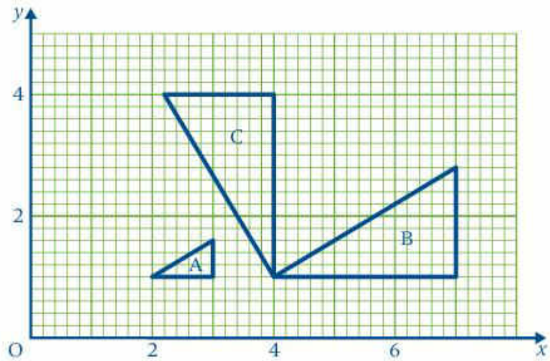
i triangle R onto triangle S

ii triangle S onto triangle T.

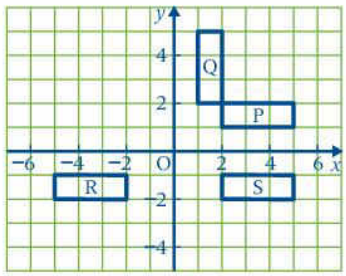
d Describe the single transformation that maps R onto T.



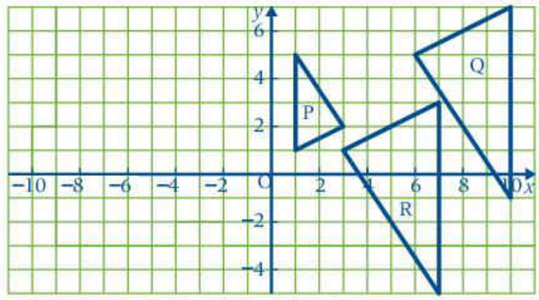
18 **a** Describe the transformation that maps
   i triangle A onto triangle B
   ii triangle B onto triangle C.
b Is there a single transformation that maps A onto C? Justify your answer.



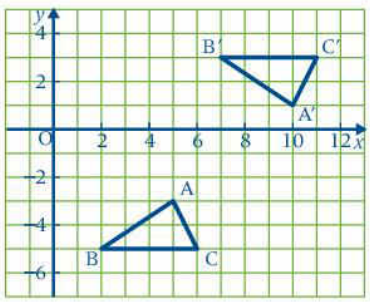
19 **a** Describe the transformation that maps
   i P onto Q
   ii Q onto R
   iii R onto S.
b Describe the single transformation that maps P onto S.



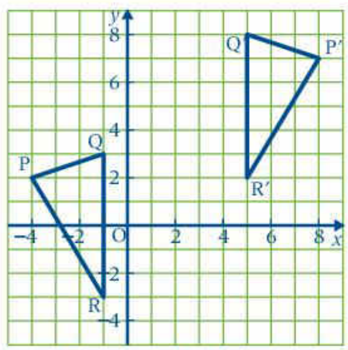
20 **a** Describe the transformation that maps
   i triangle P onto triangle Q
   ii triangle Q onto triangle R.
b Draw the image of triangle P under a rotation of 180° about O. Label the image S.
   i Write down the coordinates of the vertices of S.
   ii Describe the transformation that maps S onto R.



21 Describe the transformation that maps triangle ABC onto triangle A′B′C′.



22 Describe the transformation that maps triangle PQR onto triangle P′Q′R′.

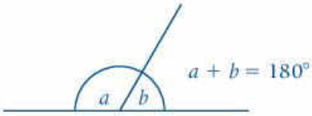


Parallel lines

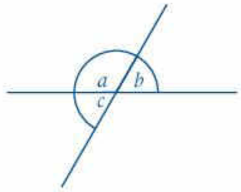
There are two fundamental axioms from which most other geometric facts can be deduced.

An axiom is a statement that is self-evidently true, such $1 + 1 = 2$.

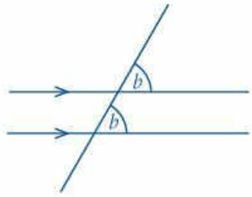
The first is that a pair of angles on a straight line add up to 180°.



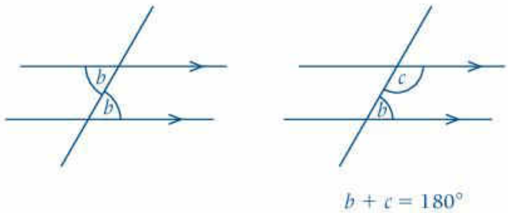
From this it follows that **vertically opposite angles** are equal,
because, in the diagram, $a + b = 180°$
and $a + c = 180°$
so $b = c$



The second is that two lines are parallel when a line that crosses them (called a transversal) makes equal angles with the two lines. These angles are called **corresponding angles**.



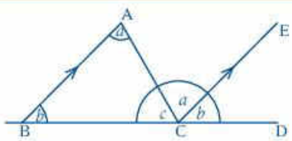
From this it follows that **alternate angles** are equal and the **interior angles** are supplementary (add up to 180°). (We can prove these results using the facts above. Try writing out the proofs.)



Triangles

A **triangle** is a region of a plane enclosed by three straight lines.

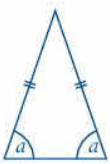
> The sum of the interior angles of a triangle is 180°.
>
> An exterior angle is equal to the sum of the two interior opposite angles.



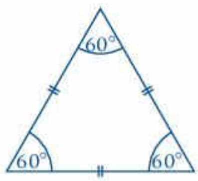
CE is parallel to BA,
so $\angle ACE = a$ (alternate angles)
and $\angle ECD = b$ (corresponding angles)
$c + \angle ACE + \angle ECD = 180°$ ($\angle$s on a st. line)
so $a + b + c = 180°$ and $\angle ACD = a + b$.

There are three special triangles:
A **right-angled triangle** has one angle equal to 90°.

An **isosceles triangle** has two sides equal and the angles opposite those sides are equal.

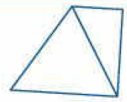


An **equilateral triangle** has all three sides equal and all three angles are 60°.



Quadrilaterals

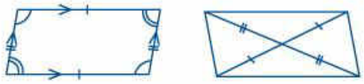
A **quadrilateral** is a region of a plane bounded by four straight lines. The sum of the interior angles of a quadrilateral is 360°.



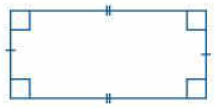
A quadrilateral can be divided into two triangles. The sum of the interior angles of each triangle is 180°, so the sum of the interior angles of the quadrilateral is 360°.

There are six special quadrilaterals:

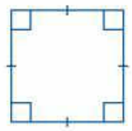
A **parallelogram** has opposite angles equal, opposite sides equal and parallel, and the diagonals bisect each other.



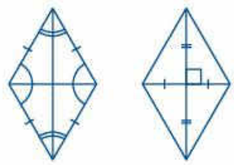
A **rectangle** has opposite sides equal and all four angles are right angles.



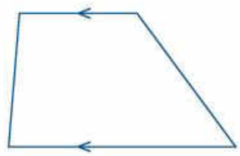
A **square** has all four sides equal and all four angles are right angles.



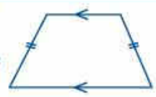
A **rhombus** has all four sides equal, opposite angles equal, opposite sides parallel, and the diagonals bisect each other at right angles.



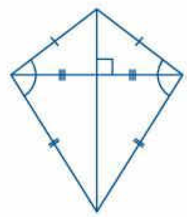
A **trapezium** has just one pair of opposite sides parallel.



A trapezium with the other pair of sides equal is called an isosceles trapezium.



A **kite** has one pair of opposite angles equal, two pairs of adjacent sides equal, the diagonals cut at right angles, and just one diagonal bisects the other. The diagonal that is bisected is called the base diagonal.



A kite is two isosceles triangles with the same base length joined at their bases.

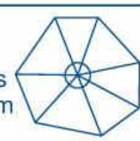
Polygons

A **polygon** is a region of a plane bounded by straight lines.
A polygon with three sides is called a triangle, a polygon with four sides is called a quadrilateral, a polygon with five sides is called a **pentagon** and a polygon with six sides is called a **hexagon**.

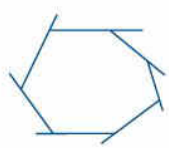
The sum of the interior angles of a polygon with n sides is $(n-2) \times 180°$.

The sum of the exterior angles of a polygon is 360°.

When each vertex of an n-sided polygon is joined to a point inside the polygon, n triangles are formed. The sum of the angles in these triangles is $(n \times 2)$ rt-angles. The sum of the angles round the point is 4 rt-angles. So the sum of the interior angles of the polygon is $(2n - 4)$ rt-angles.



The sum of each pair of interior and exterior angles is 180° or 2 rt-angles so the sum of all the interior and exterior angles is $2n$ rt-angles.
The sum of the exterior angles
$= 2n$ rt-angles $- (2n - 4)$ rt-angles
$= 4$ rt-angles $= 360°$.

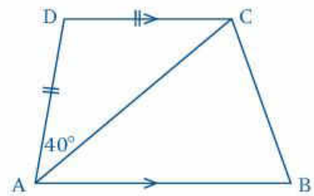


A regular polygon has all sides the same length and all interior angles equal and all exterior angles equal.

EXERCISE 7b

Example:

In the diagram, AD = DC, AB and DC are parallel and ∠DAC = 40°.
Calculate **a** ∠ADC **b** ∠BAC
Give reasons for each statement you make.

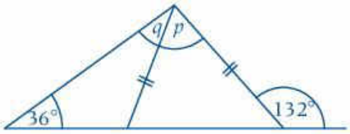


a ∠DCA = 40° (equal base angles of an isosceles triangle)
∠ADC + ∠DCA + 40° = 180° (angle sum of a triangle)
∴ ∠ADC + 40° + 40° = 180° so ∠ADC = 100°

b ∠BAC = ∠DCA (alternate angles)
∴ ∠BAC = 40°

In questions **1** to **14**, find the angles marked with letters.
Give reasons for each statement you make.

1



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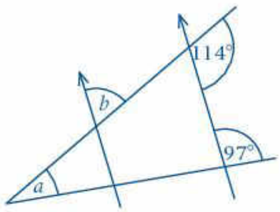

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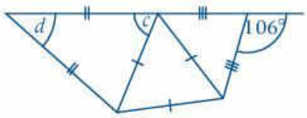

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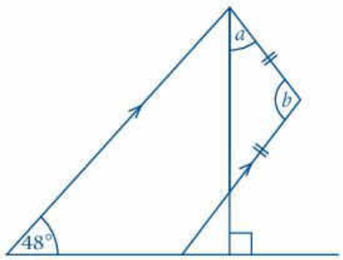

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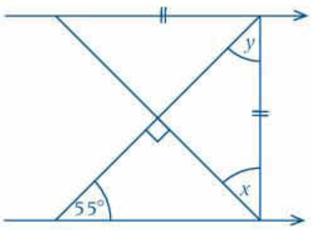

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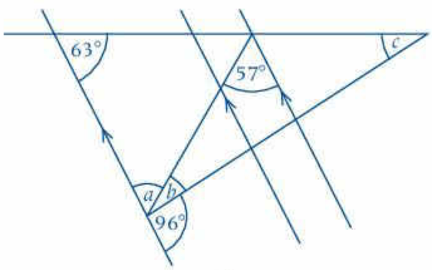

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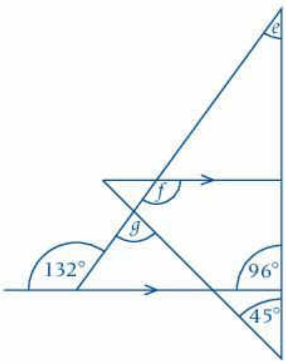

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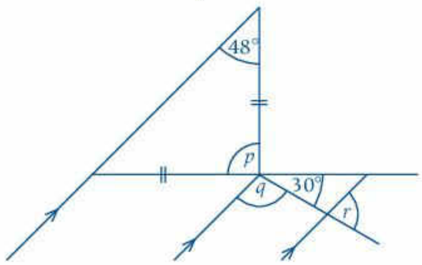
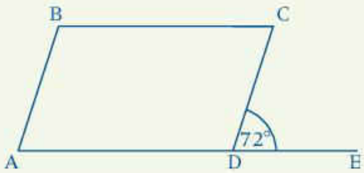


Example:

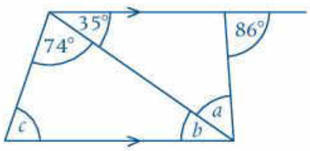
ABCD is a parallelogram. ∠CDE = 72°.
Find **a** ∠DAB **b** ∠ABC
Give reasons for your statements.



a ∠DAB = 72° (AB is parallel to DC and corresponding angles are equal)

b ∠ADC = 180° − 72° = 108°
(angles on a straight line are supplementary)
∴ ∠ABC = 108°
(opposite angles of a parallelogram are equal)

9



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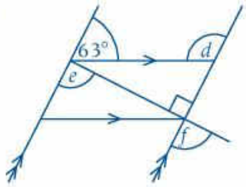

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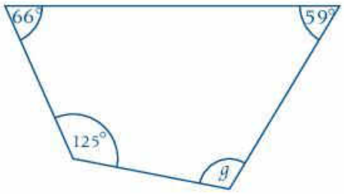

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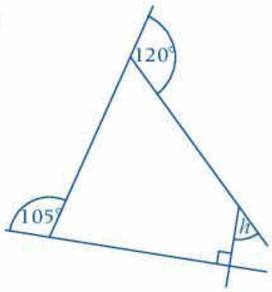

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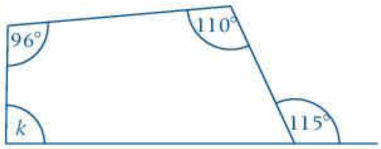

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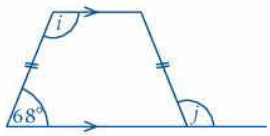


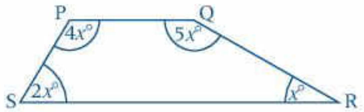
15 **a** If one angle of a parallelogram is 56°, find an adjacent angle.
b In a quadrilateral two opposite angles are each 115°.
i What is the sum of the other two opposite angles?
ii The diagonals intersect at right angles. What is the name of this quadrilateral?
c In a quadrilateral every exterior angle is 90° and two adjacent sides are of different lengths. What is the name of this quadrilateral?

16 ABCD is a square and BEC an equilateral triangle on the outside of the square. Calculate the size of **a** ∠DCE **b** ∠DEC.

17 ABCD is a parallelogram. ∠ABC = 76° and ∠ABD = 44°. DB is produced to E such that BE = AB and CB produced meets AE in F. Find the angles in **a** triangle ABE **b** triangle ADE.

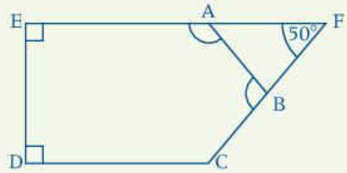
DB produced means DB extended from B.

18 **a** Find the value of x.
b What is the name of this quadrilateral?



Example:

In the diagram, $\angle AED = \angle EDC = 90°$, $\angle AFB = 50°$ and $\angle EAB = \angle ABC$.
Calculate **a** $\angle DCB$ **b** $\angle EAB$
Give reasons for your statements.



a $90° + 90° + 50° + \angle DCB = 360°$
(EFCD is a quadrilateral and the angle sum is 360°)
$\therefore\ 230° + \angle DCB = 360°$ so $\angle DCB = 130°$

b $130° + 90° + 90° + \angle EAB + \angle ABC = 540°$
(ABCDE is a pentagon so angle sum = $(2n - 4)$ rt-angles with $n = 5$: $(2 \times 5 - 4) \times 90° = 6 \times 90° = 540°$)
But $\angle EAB = \angle ABC$
so $130° + 90° + 90° + 2\angle EAB = 540°$
$310° + 2\angle EAB = 540°$
$2\angle EAB = 230°$
$\angle EAB = 115°$

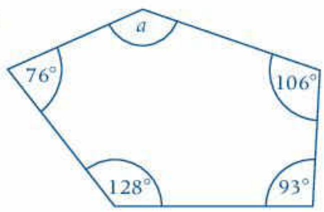
19 A regular polygon has 8 sides.
Find **a** the exterior angle **b** the interior angle.

20 The interior angle of a regular polygon is 144°.
a Find an exterior angle.
b How many sides does the polygon have?

21 ABCDEF is a regular hexagon.
Work out the angles in **a** triangle ABC **b** triangle ADE.

In questions **22** to **25**, find the angles denoted by the letters.
Give reasons for each statement you make.

22



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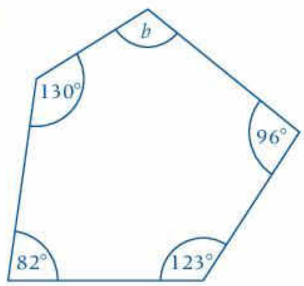

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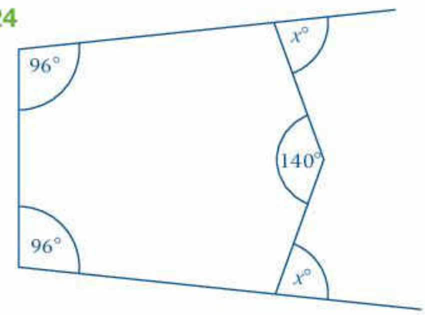

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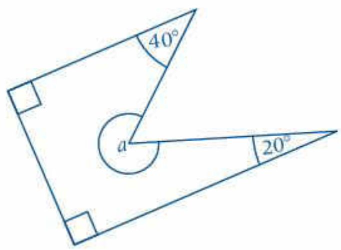


26 ABCDE is a regular pentagon with centre O.
Find the angles in
a triangle OAB **b** triangle ABC.

27 A polygon has *n* sides. Three of its interior angles are 90° and each of the remaining angles is 162°. Find *n*.

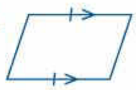
28 In a regular polygon, each interior angle is 90° more than each exterior angle.
Calculate the number of sides in the polygon.

29 Calculate the exterior angle of a regular polygon with 20 sides.

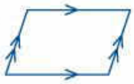
Necessary and sufficient conditions to prove that a quadrilateral is a parallelogram

To prove that a quadrilateral is a parallelogram, you need to show

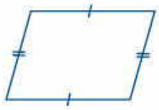
either that one pair of opposite sides are equal and parallel



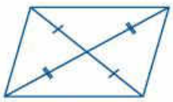
or that both pairs of opposite sides are parallel



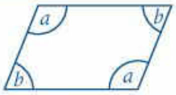
or that both pairs of opposite sides are equal



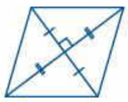
or that the diagonals bisect each other

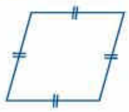


or that both pairs of opposite angles are equal.



To prove that a quadrilateral is a rhombus you need to show
either that the diagonals bisect at right angles
(if only one diagonal bisects the other, it is a kite)
or that all four sides are equal.

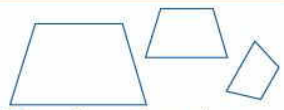




Similar shapes

Similar shapes are the same shape but not usually the same size. This means that one shape is an enlargement of the other, although it may be turned round or over compared with the first.

To prove that two shapes are similar, we need to show that they are equiangular *and* that their corresponding sides are in the same ratio.



These three trapeziums are similar.

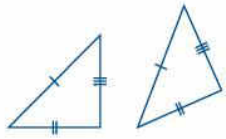
Similar triangles are a special case because the sizes of the three angles fixes the shape.
So to prove that a pair of triangles are similar we need to show that either
- the three angles of one triangle are equal to the three angles of the other, or
- that the corresponding sides are in the same ratio.

Congruent triangles

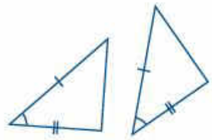
Congruent figures are the same shape *and* size.

To prove that two triangles are congruent, we need to show that one of these conditions is satisfied.

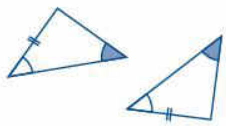
- The lengths of the three sides of one triangle are equal to the lengths of the three sides of the other triangle. (SSS)



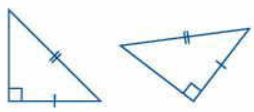
- The lengths of two sides of one triangle are equal to the lengths of two sides of the other triangle and the angles between these two sides are equal. (SAS)



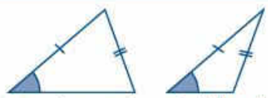
- Two angles of one triangle are equal to two angles of the other triangle and one pair of corresponding sides are the same length. (AAS) or (ASA)



- The two triangles are right-angled and the lengths of the sides opposite the right angles and another pair of sides are equal. (RHS)(The side opposite the right angle is called the **hypotenuse**.)



Note that RHS is the only case where we can use two pairs of equal sides and a pair of corresponding equal angles that are not between the equal sides to show congruency. If the angles are not 90°, the triangles may not be congruent.

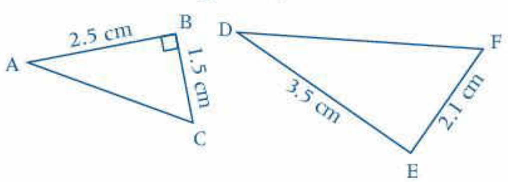


These triangles have two pairs of sides the same length and a pair of corresponding angles equal but they are not congruent.

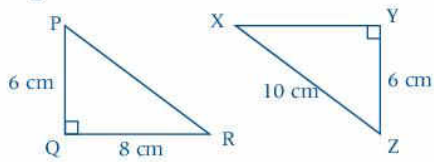
EXERCISE 7c

1 For each pair of triangles, determine whether they are congruent, or similar but not congruent, or neither similar nor congruent.

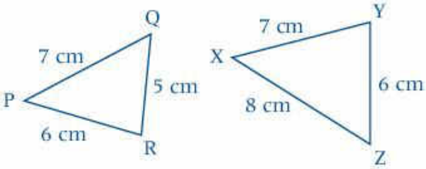
a



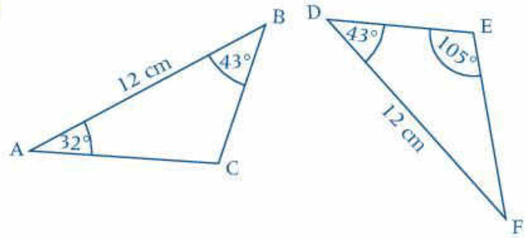
b



c



d



Example:

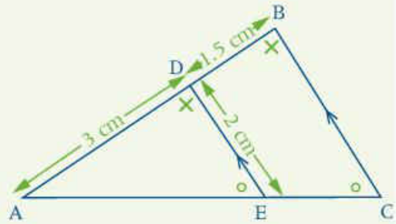
In the diagram, BC and DE are parallel,

AD = 3 cm, DB = 1.5 cm and DE = 2 cm.

a Find the length of BC.

b The area of triangle ADE is 2.5 cm^2.

Find the area of triangle ABC.



Start by marking the diagram with the lengths you know and any facts that you can see.

a $\angle ADE = \angle ABC$ (corresponding angles)

$\angle AED = \angle ACB$ (corresponding angles)

$\angle BAC$ is the same for both triangles

$\therefore \triangle ADE$ is similar to $\triangle ABC$

$\therefore \frac{AB}{AD} = \frac{BC}{DE}$ so $\frac{4.5}{3}$ 1.5 $= \frac{BC}{2}$

$\therefore BC = 1.5 \times 2 = 3$ cm

To find BC we need to show that $\triangle$s ABC and ADE are similar, then we can use the fact that their sides are in the same ratio.

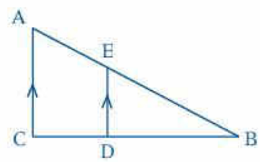
b $\frac{AB}{AD} = \frac{1.5}{1} = \frac{3}{2}$

Area $\triangle ABC = \left(\frac{3}{2}\right)^2 \times 2.5\ cm^2$

$= \frac{9 \times 2.5}{4\ cm^2} = 5.625\ cm^2$

The ratio AB/AD is the scale factor that enlarges $\triangle$ADE to $\triangle$ABC.

Use this diagram for questions **2** and **3**.



2 In the diagram, AB = 13.2 cm, BE = 5.5 cm, BD = 4.5 cm and DE = 3.5 cm.

a Show that $\triangle$s ABC and BDE are similar.

b Find **i** BC **ii** AC.

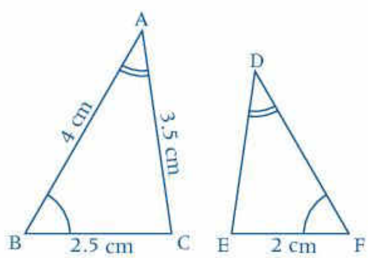
3 In the diagram, BC = 4.4 cm, DE = 3 cm, AC = 5.5 cm and BE = 3.6 cm.

Find **a** BD **b** DC **c** AB.

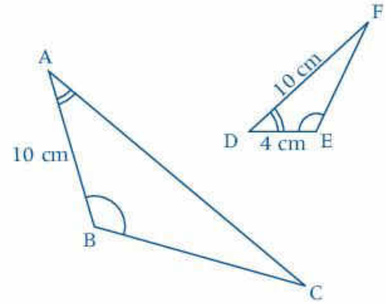
4 ABC and DEF are two similar triangles with $\angle ABC = \angle DFE$ and $\angle BAC = \angle EDF$. AB = 4 cm, AC = 3.5 cm, BC = 2.5 cm and EF = 2 cm.

a Copy and complete the statement $\frac{AB}{DF} = \text{—} = \text{—}$.

b Find DE and DF.

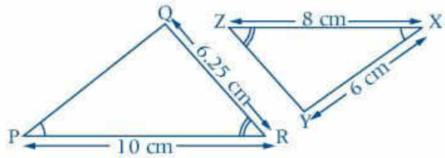


5 ABC and DEF are two similar triangles with $\angle A = \angle D$ and $\angle B = \angle E$. AB = 10 cm, DE = 4 cm and DF = 10 cm.

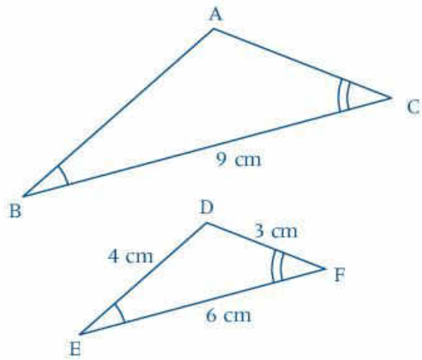


- **a** Copy and complete the statement $\frac{\ }{AB} = \frac{DF}{DE}$
- **b** Calculate AC.
- **c** Calculate the ratio $\frac{BC}{EF}$

6 In triangles PQR and XYZ, $\angle P = \angle X$, $\angle R = \angle Z$, PR = 10 cm, QR = 6.25 cm, XZ = 8 cm and XY = 6 cm.
Find **a** PQ **b** YZ



7 ABC and DEF are similar triangles with $\angle ABC = \angle DEF$ and $\angle ACB = \angle DFE$. BC = 9 cm, DE = 4 cm, DF = 3 cm and EF = 6 cm.

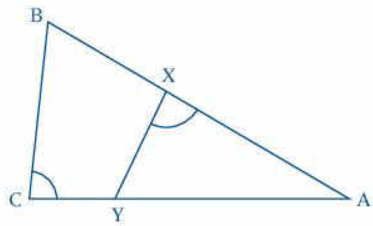


- **a** Copy and complete the statement $\frac{AB}{\ } = \frac{AC}{DF} = \frac{\ }{EF}$.
- **b** Calculate AB and AC.
- **c** Find the ratio of the area of triangle ABC to the area of triangle DEF.

8 ABC and DEF are two similar triangles with $\angle ABC = \angle DEF$ and $\angle BAC = \angle EDF$. AB = 4 cm, AC = 3.5 cm, BC = 2.5 cm and EF = 2 cm.

- **a** Copy and complete the statement $\frac{AB}{DE} = \text{—} = \text{—}$
- **b** Find DE and DF.

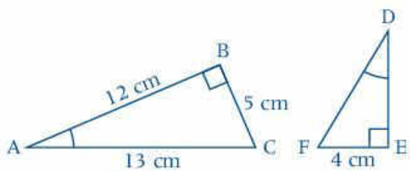
9 X and Y are points on the sides AB and AC respectively of a triangle ABC such that $\angle AXY = \angle ACB$.



Calculate

- **a** AX and AY if AB = 35 cm, BC = 25 cm, AC = 30 cm and XY = 15 cm
- **b** AB and AC if BC = 1.8 cm, AY = 1.2 cm, XY = 0.6 cm and AX = 0.9 cm
- **c** AB, AX and the ratio $\frac{\text{area}\triangle ABC}{\text{area}\triangle AXY}$ if BC = 5.6 cm, AC = 8.4 cm, AY = 8 cm and XY = 4 cm.

10 Triangles ABC and DEF are similar with $\angle B = \angle E = 90°$, $\angle A = \angle D$, AB = 12 cm, AC = 13 cm, BC = 5 cm and EF = 4 cm.



- **a** Copy and complete the statement $\frac{BC}{EF} = \frac{AC}{\ } = \frac{\ }{DE}$
- **b** Find **i** DE **ii** DF.
- **c** Find the ratio of the area of $\triangle ABC$ to area $\triangle DEF$.

11 ABC is a triangle in which AB = 10 cm and AC = 7.5 cm. X is a point on AB such that AX = 4 cm and Y is a point on AC such that XY is parallel to BC. Calculate

a the length of AY

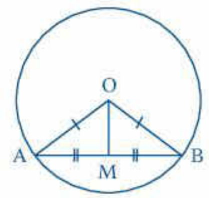
b the ratio of the areas $\frac{\triangle AXY}{\triangle ABC}$

c the ratio of the area of the triangle AXY to the area of trapezium XYCB

d the ratio of the areas $\frac{\triangle AXY}{\triangle BXY}$

Example:

In the diagram, O is the centre of the circle and M is the midpoint of the chord AB.
Prove that OM is perpendicular to AB.



In $\triangle AOM$ and $\triangle BOM$,

AM = MB (M is the midpoint of AB)

AO = BO (they are radii)

OM is the same for both,

$\therefore$ $\triangle AOM$ and $\triangle BOM$ are congruent (SSS).

$\therefore \angle AMO = \angle BMO$

But $\angle AMO + \angle BMO = 180°$ (angles on a straight line)

$\therefore \quad \angle AMO = \angle BMO = 90°$

$\therefore$ OM is perpendicular to AB

To prove that OM is $\perp$r to AB, we need to show that $\angle AMO = \angle BMO = 90°$.
This means we need to show that $\triangle$s AMO and BMO are congruent.

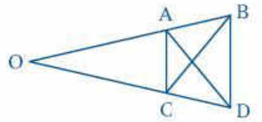
You can use this result, without proving it, to solve problems involving circles.

The **converse** is also true and can be used, i.e. the perpendicular bisector of a chord goes through the centre of the circle.

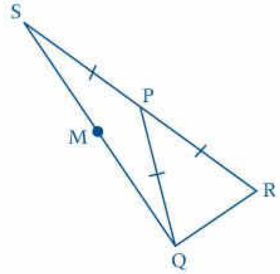
12 Prove that the midpoint of a side of a square is equidistant from the ends of the opposite side.

13 ABCD is an isosceles trapezium with AB parallel to DC and AD = BC.
Prove that triangles ABD and ABC are congruent.

14 OAB and OCD are two straight lines. OB = OD and AB = CD.
Prove that $\angle OBC = \angle ODA$.

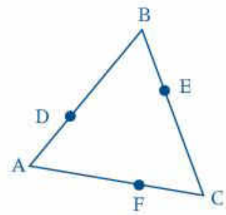


15 PQR is an isosceles triangle in which PQ = PR.
RP is produced to S so that PS = PR.
M is the midpoint of QS.
Prove that PM is perpendicular to QS.

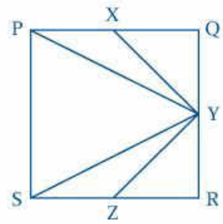


16 M is the midpoint of the side AC of triangle ABC. The line through M parallel to BC meets AB in E and the line through M parallel to AB meets BC in F.
Prove that ME = CF and MF = AE.

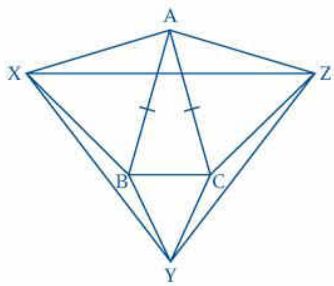
17 ABC is an equilateral triangle, and D, E, F are points in AB, BC, CA respectively such that AD = BE = CF.
Prove that triangle DEF is also equilateral.



18 PQRS is a square and X, Y, Z are the midpoints of the sides PQ, QR, RS respectively.
Prove that **a** triangles PQY and SRY are congruent
b triangles QXY and RZY are congruent
c triangles PXY and SZY are congruent.



19 ABC is an isosceles triangle in which AB = AC. Equilateral triangles XAB, YBC and ZAC are drawn on the sides of triangle ABC as shown in the diagram. Prove that triangles BXY and CYZ are congruent and hence show that triangle XYZ is isosceles.



20 In triangle ABC, D is a point on AB such that AD = 4 cm and DB = 6 cm, and E is a point on AC such that AE = 3 cm and EC = 4.5 cm. If F is a point on BC such that EF is parallel to BC prove that BDEF is a parallelogram.

21 ABCDEF is a regular hexagon. Prove that triangles ABC and DEF are congruent. Hence show that in quadrilateral ACDF the opposite sides are equal and the quadrilateral contains a right angle. What special name do you give to this quadrilateral?

22 ABCDEFGH is a regular octagon. Use congruency to prove that ACEG is a square.

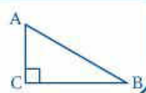
23 In triangle ABC, X is a point on AB and Y is a point on AC.
AX = 4.2 cm, BX = 5.6 cm, AY = 4.5 cm and AC = 10.5 cm.
Prove that BXYC is a trapezium.

Pythagoras' theorem

Pythagoras' theorem states that in a right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the two shorter sides.

The converse of Pythagoras' theorem states that if the square of the longest side of a triangle is equal to the sum of the squares of the other two sides, the angle opposite the longest side is a right angle.

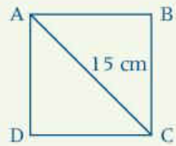
In $\triangle ABC$, $AB^2 = AC^2 + BC^2$



EXERCISE 7d

Example:

In the diagram, ABCD is a square and AC = 15 cm.
Find the length of a side of the square.



$AB^2 + BC^2 = 15^2$ (Pythagoras' theorem)
$AB = BC$ so $2AB^2 = 225$
$\therefore AB^2 = 112.5$
so $AB = \sqrt{112.5} = 10.60\ldots$
The length of a side of the square = 10.6 cm to 3 s.f.

> ABCD is a square so $\triangle ABC$ has a right angle at B and we can use Pythagoras' theorem. Remember that the hypotenuse is opposite the right angle.

1 The length of one side of a rectangle is 20 cm and each of its diagonals measures 52 cm. Calculate the perimeter of the rectangle.

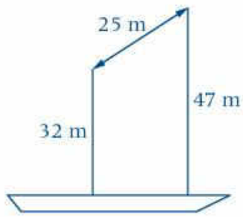
2 The diagonals of a rhombus are 9 cm and 16.8 cm. Calculate the perimeter of the rhombus.

3 The diagonal of a square measures 26.9 cm.
Find **a** the length of a side **b** its area.

4 The side of a rhombus is 12 cm, and one diagonal is 20 cm. Find the length of the other diagonal.

5 M is the midpoint of the side BC in an equilateral triangle ABC. BC = 16 cm. Calculate AM.

6 The two vertical masts of a sailing ship have heights 47 m and 32 m. If the distance between their highest points is 25 m, find the distance between the masts on the deck of the ship.



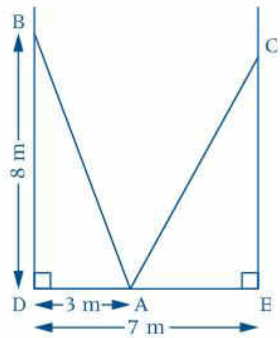
7 Find the length of the longest thin stick that can be placed in a rectangular box measuring 12 cm by 8 cm by 7 cm.

8 Determine whether or not each triangle contains a right angle. If it does, state which angle is the right angle.

> Use the converse of Pythagoras' theorem

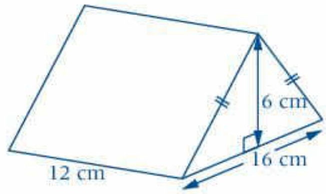
a $\triangle ABC$ in which AB = 7.2 cm, AC = 5.4 cm and BC = 9 cm
b $\triangle PQR$ in which PQ = 6.6 cm, QR = 4.5 cm and PR = 4.5 cm
c $\triangle XYZ$ in which XY = 16.9 cm, XZ = 15.6 cm and YZ = 6.5 cm

9 The diagram shows a ladder AB resting on horizontal ground DAE in a narrow street 7 m wide.
The foot of the ladder A is 3 m from the base of a vertical wall DB and touches the wall at a point 8 m above the ground.
a How long is the ladder?
b The ladder is now rotated about A so that it rests against a vertical building on the opposite side of the street at a point C.
How high is C above the ground?



Example:

This triangular prism is 12 cm long.
The cross-section is an isosceles triangle whose base is 16 cm long and whose height is 6 cm.
Calculate the total surface area of the prism.



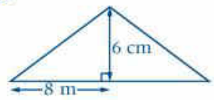
Area of a triangular face $= \frac{1}{2} \times 16 \times 6\,\text{cm}^2$
$= 48\,\text{cm}^2$

The faces of the prism are the two triangular ends, the rectangular base and the two rectangular sloping sides.

Area of base $= 12 \times 16\,\text{cm}^2 = 192\,\text{cm}^2$

Length of side of sloping face $= \sqrt{8^2 + 6^2}\,\text{cm}$
$= \sqrt{100}\,\text{cm} = 10\,\text{cm}$

The cross-section is divided into two congruent rt-angled triangles by the height. We can use Pythagoras' theorem in one of these to find the length of the sloping edge.

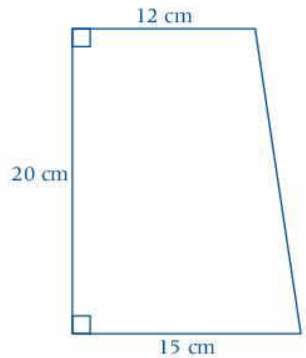


Area of sloping face $= 12 \times 10\,\text{cm}^2 = 120\,\text{cm}^2$

Total surface area $= (2 \times 48 + 192 + 2 \times 120)\,\text{cm}^2$
$= 528\,\text{cm}^2$

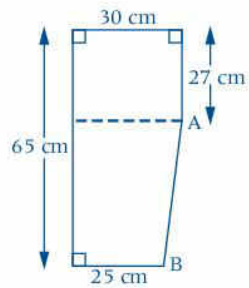
10 From a point A, a boy walks 350 metres due north to a point B. At B he turns east and walks 420 metres to C where he turns and walks 760 metres in a southerly direction to a fourth point D.
How far is D from the starting point?

11 The diagram shows the dimensions of a wooden book-end which is to be made by a class of twenty pupils.

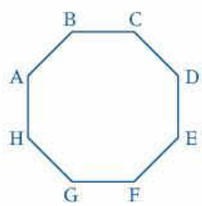


Find **a** its perimeter
b its volume if the wood is uniformly 1.5 cm thick
c the total volume of wood used by the class if each pupil makes two book-ends.

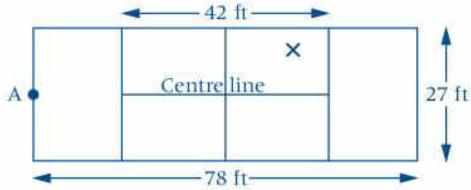
12 The diagram shows the section through a kitchen wall cupboard. AB represents the sliding glass door.
Find AB correct to the nearest millimetre.



13 ABCDEFGH is a regular octagon.
AB = 5 cm.
Find the area of the octagon.



14 The diagram shows a tennis court measuring 78 feet by 27 feet. A tennis player standing at A attempts to serve into the rectangle marked X.



a If he strikes the ball at a height of 9 feet above the ground, find the longest straight path which the ball may travel if it is to strike the ground within X.

b If it travels at 180 feet per second how long will it take from the time it leaves the racquet until it strikes the ground?

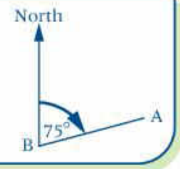
Bearings

A **bearing** is a direction. The bearing of a point A from a point B is the direction we need to go from B to get to A.

The three-figure bearing of A from B is the clockwise angle between the north direction at B and the line segment AB.

If the angle is less than 100°, a preceding zero is added to give three figures.

The bearing of A from B is 075°.

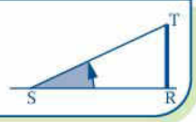


Angles of elevation and depression

An **angle of elevation** is the angle we need to look up to see an object above us.

The angle of elevation of a point T from a point S is the angle between the horizontal at S and the line segment ST.

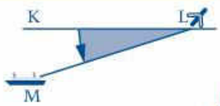
The angle of elevation of T from S is ∠RST.



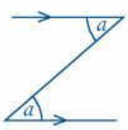
An **angle of depression** is the angle we need to look down to see an object below us.

The angle of depression of a point M from a point L is the angle between the horizontal through L and the line segment LM.

The angle of depression of the ship, M, from the aircraft L is ∠KLM.



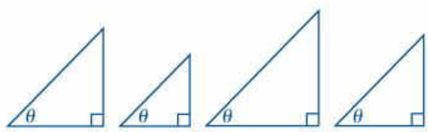
Note that the angle of elevation of a point A from a point B is equal to the angle of depression of B from A.



Trigonometry

All right-angled triangles containing another equal angle are similar.

These triangles are all similar:



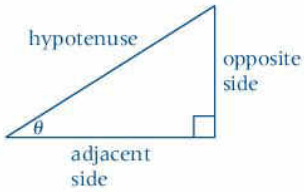
This means that the ratio of the side opposite θ to the hypotenuse is the same in every right-angled triangle containing an angle θ. This ratio is called the **sine** of angle θ and it is written as $\sin\theta$.

θ is a Greek letter pronounced 'theta'. It is often used to denote an angle.

We can also take the ratio of the side adjacent to θ and the hypotenuse; this is called the **cosine** of angle θ and is written as $\cos\theta$.

We can also take the ratio of the side opposite θ and the side adjacent to θ which is called the **tangent** of angle θ and is written as $\tan\theta$.

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}}$$
$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}}$$
$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}}$$



We can use these ratios in right-angled triangles

- to find an angle when we know two sides
- to find a side when we know one side and an angle.

EXERCISE 7e

Example:

A ship, S, is 2.5 km from another ship, P, on a bearing of 115°. Calculate the distance that S is east of P.

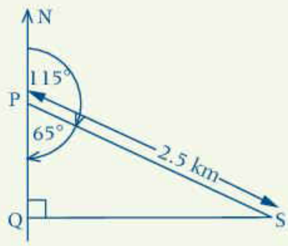
$\angle QPS = 180° - 115° = 65°$

In $\triangle PQS$, $\sin 65° = \dfrac{SQ}{2.5}$

$\therefore SQ = 2.5 \times \sin 65°$

$= 2.265\ldots$

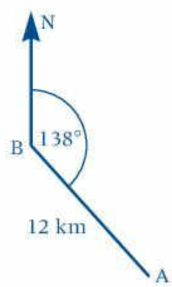
S is 2.27 km east of P (to 3 s.f.).



Draw a diagram showing all the information given. QS is the line giving the distance that S is east of P. $\triangle PQS$ has a right angle at Q. We know the hypotenuse and we want the side opposite $\angle P$, so we use sine.

In this exercise give lengths correct to 3 significant figures and angles correct to the nearest degree.

1 A town A is 12 km from a village B on a bearing of 138°.
How far is A **a** south of B **b** east of B?



2 The angle of depression of a boat which is directly out to sea as observed from the top of a vertical cliff 50 m high is 32°. Find the distance of the boat from the foot of the cliff.

3 The angle of elevation of the top of a building from a point 200 m from the foot of the building is 35°. How high is the building?

4 A man walks 70 metres away from the base of a vertical building and finds that the elevation of its top is 30°.

a How high is the building?

b He continues his walk away from the building in the same straight line until he reaches another point from which the elevation of the top is 25°. How far has he walked from the foot of the building?

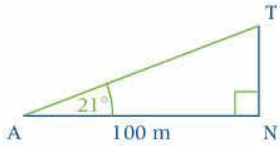
5 A village P is 20 km from a village Q on a bearing of 165°.
Calculate **a** how far Q is west of P **b** how far Q is north of P.

Example:

The diagram shows a vertical tower, TN, standing on horizontal ground.
A is a point on the ground 100 m from the foot of the tower.
The angle of elevation of the top, T, of the tower from A is 21°.

a Calculate the height of the tower.

b An instrument placed 2.5 m above A is used to measure the angle of elevation of T.
Calculate the reading on the instrument.



a $\tan 21° = \frac{TN}{100}$

$\therefore TN = 100 \times \tan 21° = 38.38...$

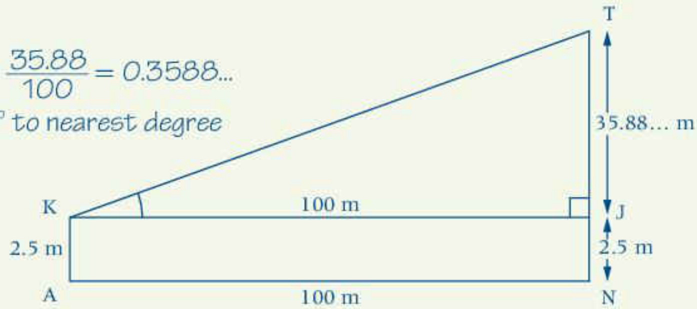
The height of the tower is 38.4 m to 3 s.f.

∠ANT = 90° as TN is vertical and AN is horizontal. Mark all the facts on the diagram.
∠A is known. TN is opposite ∠A and AN is adjacent to ∠A, so the ratio to use is tan.

b In △TJK,

$\tan \angle K = \frac{35.88}{100} = 0.3588...$

∠K = 20° to nearest degree



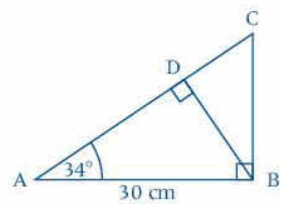
Draw a new diagram and label it with the new information. △TKJ has a rt-angle at J and TJ = TN − JN
(Use the uncorrected value of TN to calculate TJ.)

6 A girl observes that the angle of depression of the base of a tree outside her window is 23° and the angle of elevation of its top is 52°. If her eye is 5 metres above ground level, find the height of the tree.

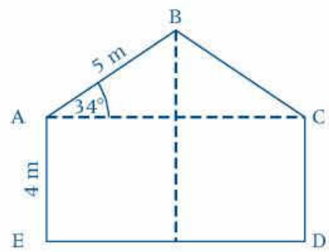
7 From the top of a building 25 metres high the angles of depression of the near side and far side of a road which runs parallel to the building, are 63° and 53° respectively. How wide is the road?

8 The angles of depression of the bow and stern of a ship from the bridge, which is 20 metres above the level of the deck, are 22° and 14° respectively.
How long is the deck?

9 In the diagram AB = 30 cm, ∠ABC = ∠ADB = 90° and ∠BAC = 34°.
Calculate **a** BD **b** AD **c** ∠ACB **d** DC.

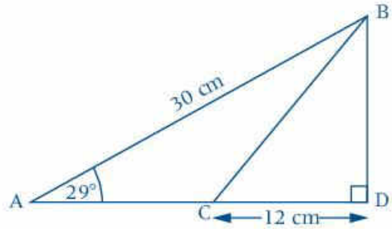


10 The diagram shows the cross-section through a shed.
AB = BC = 5 metres, AE = CD = 4 metres and ∠BAC = 34°.
Calculate
a AC **b** the height of the ridge B above the ground.



11 ABC is a triangle and D is the foot of the perpendicular from B to AC produced. If ∠BAC = 29°, AB = 30 cm and CD = 12 cm, calculate

a BD **b** AC **c** ∠BCD.



12 The diagram shows the relative positions of three points A, B and C.

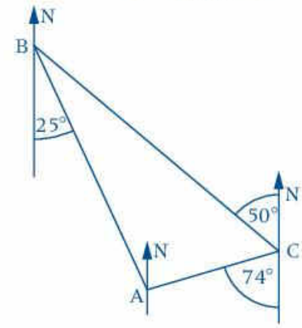
a Find the bearing of

i C from B **ii** A from B **iii** B from A **iv** B from C.

b Find the three angles of triangle ABC.

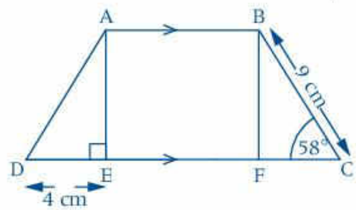
c If BC = 4 km, how far is

i B west of C **ii** C south of B?



13 ABCD is a trapezium with AB parallel to DC.
E and F are the feet of the respective perpendiculars from A and B to DC.
∠BCD = 58°, BC = 9 cm and DE = 4 cm.
Calculate

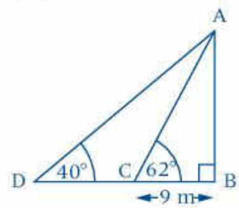
a BF **b** FC **c** ∠ADE **d** AD.



14 AB represents a flagpole. B, C and D are points on level ground such that ∠ACB = 62°, ∠ADB = 40° and CB = 9 metres.
Calculate

a the height of the flagpole AB

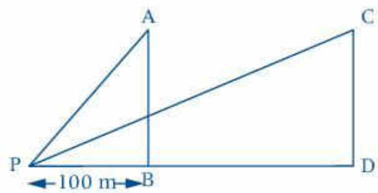
b the distance of D from the foot of the flagpole.



15 The diagram shows the two similar supporting towers of a bridge. When the tops of the towers A and C are viewed from a point P, situated 100 m from the base of A such that P, B and D are in a horizontal straight line, their angles of elevation are 39° and 8° respectively.
Calculate

a the height of a tower

b the horizontal distance between them.



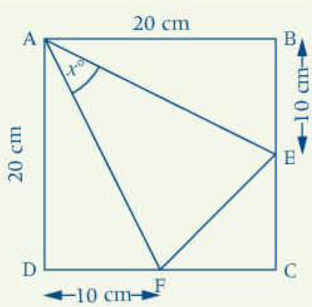
16 From a look-out point A, 200 m above the sea, a coastguard sights two boats, B and C directly out to sea. The angle of depression of B is 23° and the angle of depression of C is 13°.
Find the distance between the boats.

Example:

In the diagram, ABCD is a square with sides 20 cm long.
E and F are the midpoints of BC and DC respectively.
Calculate

a the length of AE **b** the value of x.

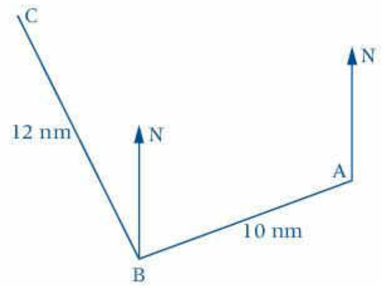
a $AE^2 = 20^2 + 10^2$ (Pythagoras' thm)
$AE = \sqrt{500} = 22.36...$
AE is 22.4 cm long to 3 s.f.



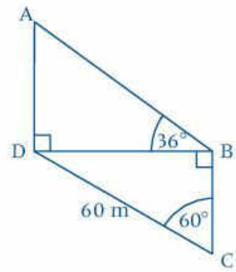
Add other facts to the diagram that you know: this helps us identify right-angled triangles, congruent triangles and similar triangles.

b In $\triangle ABE$, $\tan \angle BAE = \frac{10}{20} = 0.5$
$\angle BAE = 26.565...^\circ$
$\angle FAD = \angle BAE$ ($\triangle$s ABE, ADF are congruent (SAS))
$\therefore x + 2 \times 26.565 = 90$
$x = 90 - 53.13... = 36.86... = 36.9^\circ$ to 1 d.p.

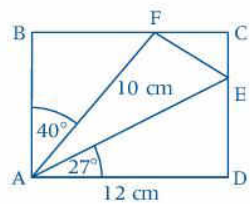
17 A ship sails 10 nautical miles from a point A on a bearing of 250° to a point B. At B it changes course and sails a further 12 nautical miles on a bearing of 330°.
Find
a the distance of B south of **i** A **ii** C.
b the distance of B **i** west of A **ii** east of C.
c Hence find the distance of A **i** east of C **ii** south of C.
d Find the distance and bearing of C from A.



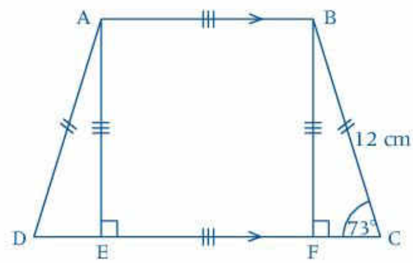
18 In the diagram, ABCD represents a field. The diagonal BD is perpendicular to AD and BC, $\angle BCD = 60°$, $\angle ABD = 36°$ and CD = 60 metres. Find
a BD
b AB
d the area of the field in square metres.



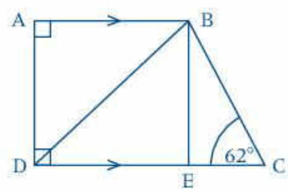
19 ABCD is a rectangle with AD = 12 cm. E is a point on CD and F a point on BC such that $\angle EAD = 27°$, $\angle BAF = 40°$ and AF = 10 cm.
Find
a BF **b** AB **c** ED **d** CE **e** FC **f** $\angle CFE$.



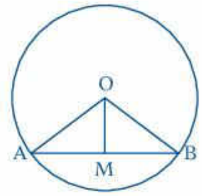
20 ABCD is an isosceles trapezium with AB parallel to DC.
ABFE is a square, $\angle BCD = 73°$ and BC = 12 cm.
Find the length of **a** AE **b** DC.



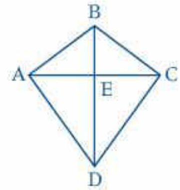
21 ABCD is a trapezium in which BC = 12 cm, DC = 20 cm and $\angle BCD = 62°$. Calculate
a the length of **i** BE **ii** EC **iii** DE
b the size of angle ABD.



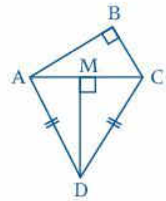
22 AB is a chord in a circle, centre O. $\angle AOB = 116°$,
M is the midpoint of AB and OM = 7 cm.
Find
a the radius of the circle
b the length of AB.



23 The diagonals of a kite ABCD intersect at E.
AC = 18 cm, AD = 15 cm and ∠ABC = 88°.
Find
a BC
b BD
c ∠EDC.



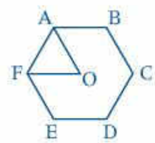
24 In the quadrilateral ABCD, AB = 6.4 cm, BC = 4.8 cm, ∠ADC = 66° and AD = CD. M is the foot of the perpendicular from D to AC.
Find
a the length of AC
b the lengths of the sides of triangle AMD
c the area of the quadrilateral.



25 ABCD is a rhombus whose diagonals intersect at E. BD = 14 cm and ∠ABC = 54°.
Find
a the length of the diagonal AC
b the length of a side of the rhombus
c the area of the rhombus.

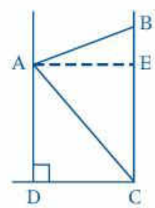
26 P, Q and R are, respectively, points on the sides AB, BC and AD of a square ABCD. AP = 15 cm, PB = 8 cm, PQ = 10 cm and PR = 24 cm.
Find
a the length of **i** BQ **ii** AR **iii** QR
b the area of **i** triangle PQR **ii** trapezium RQCD.

27 O is the centre of a regular hexagon with each side measuring 8 cm. Find the radius of the smallest circle in which this hexagon can be drawn.

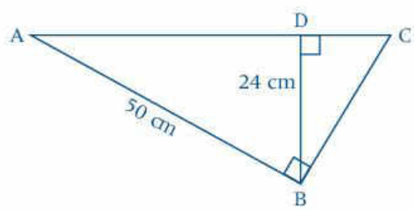


28 A, B and C are three towns. The bearing of B from A is 062° and the bearing of C from A is 325°.
AB = 16 kilometres and AC = 22 kilometres
a Find the distance of B **i** north of A **ii** east of A.
b Find the distance of C **i** north of A **ii** west of A.
c Find the distance from B to C.
d Calculate the bearing of C from B.

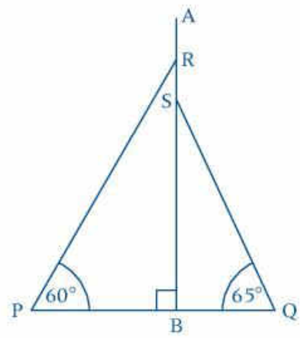
29 A girl looks out from her bedroom window at a building on the other side of the street. She observes that the angle of elevation of the top of the building opposite is 43° and the angle of depression of its bottom is 58°.
The street is level and 30 metres wide.
Find
a the height of the girl above street level
b the height of the building opposite.



30 ABC represents the cross-section through a water-trough which is 2 metres long. ∠ABC = 90°, AB = 50 cm and the depth of the trough at its deepest point is 24 cm.
Calculate
a **i** AD **ii** angle BAC
b the width of the trough AC
c the length of the other sloping side BC
d the volume of water in the trough when full. Give your answer in cubic metres.



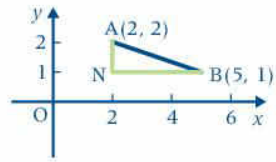
31 A flagpole AB, standing on level ground, is supported by two wires PR and QS which make angles of 60° and 65° respectively with the ground. R is attached to the flagpole 1 metre from the top of the pole and S is 1 metre below R. PB = 8 m and PQ = 14 m.
Find
a the height of the flagpole
b the total length of wire required for the two supports if an extra 1.5 m is needed to tie them off.



Coordinate geometry

The length of a line segment

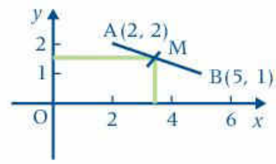
Pythagoras' theorem can be used to find the length of a line segment when we know the coordinates of its end points.



In the diagram,
AN = 1 (this is the difference in the y-coordinates of A and B)
NB = 3 (this is the difference in the x-coordinates of A and B)
Therefore $AB^2 = 1^2 + 3^2$ (Pythagoras' theorem)
$= 10$
Therefore $AB = \sqrt{10}$

For any two points $A(x_1, y_1)$ and $B(x_2, y_2)$, $AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

The midpoint of a line segment



In the diagram, M is the midpoint of the line segment from A(2, 2) to B(5, 1)

The x-coordinate of M is halfway between the x-coordinate of A and the x-coordinate of B. This is the average of 2 and 5, i.e. $x = \frac{1}{2}(2 + 5) = 3.5$

Similarly, the y-coordinate of M is $\frac{1}{2}(2 + 1) = 1.5$

The coordinates of the midpoint of the line between any two points $A(x_1, y_1)$ and $B(x_2, y_2)$ are given by $\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$

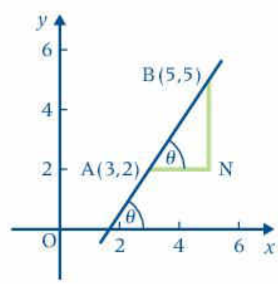
The angle between a line and the x-axis

We can use the tangent of an angle to find the size of the angle that a line makes with the x-axis.

In the diagram, A and B are two points on a line.

If θ is the angle the line makes with the positive direction of the x-axis, then $\tan\theta = \frac{BN}{AN} = \frac{3}{2} = 1.5$
so $\theta = 56°$ to the nearest degree.

Notice that the gradient of this line is $\frac{BN}{AN}$.



Therefore the tangent of the angle that a line makes with the positive direction of the x-axis is equal to the gradient of the line.

EXERCISE 7f

1 Write down the coordinates of the midpoint of the line segment joining the points
   a (0, 0), (8, 10)   b (−6, 1), (2, 5)
   c (0, 5), (−6, −3)   d (−1, −5), (7, −8).

2 Find the distance between the points
   a (−2, 3), (3, 1)   b (−5, 4), (3, 2)
   c (3, −7), (7, 3)   d (−1, −5), (−12, −6).

3 a Plot the points A(7, 5), B(5, 7), C(−1, −1).
   b Prove that ABC is an isosceles triangle.
   c Find the length of its base.
   d Find the angle between
      i CA and the x-axis   ii CB and the x-axis.

The angle between a line and the x-axis is, by convention, the angle with the positive direction of the x-axis.

4 Show that the points (2, 0), (6, 2), (4, 6) and (0, 4) are the vertices of a square.
   Find the coordinates of the point of intersection of the diagonals.

5 Prove that the points (4, 2), (−3, 5), (−5, 0) and (2, −3) are the vertices of a parallelogram.
   Find the coordinates of the point where its diagonals intersect.

6 Given the points P(−2, 4), Q(4, 2), R(6, −4) and S(0, −2), show that PQRS is a rhombus and write down the coordinates of the point where the diagonals intersect.

7 a Show that the points A(7, 5), B(6, −2), C(−1, −1) and D(0, 6) lie on a circle centre (3, 2).
   b Find the radius of the circle.
   c Find the angle that AB makes with the x-axis.

8 a Plot the points A(−3, 5), B(6, 5), C(6, −2) and D(−3, −2).
   b Find the length of i AB ii BC iii CD iv DA.
   c What is the name of this special quadrilateral?
   d Write down the coordinates of the midpoint of i AC ii BD.

9 A, B, C, D are respectively the points (−2, 3), (4, 5), (7, 1), (−2, −2).
   a Find the length of i AD ii BC.
   b Show that AB and DC are parallel.
   c Find the coordinates of the midpoint of i AB ii DC.
   d What special name do you give to this quadrilateral?

10 A, B, C are the points (−3, 0), (7, 4), (4, −3).
   a Find the length of i AB ii BC iii AC.
   b What are the coordinates of M, the midpoint of AB?
   c Work out the gradient of i AB ii CB.
   d Find the length of i AM ii BM iii CM.
      Hence show that M is equidistant from A, B and C.

11 A, B, C, D are the points (0, 2), (2, 5), (5, 3), (4, −5).
   a Find the length of i AB ii BC iii CD iv AD.
   b Find the coordinates of
      i M the midpoint of AC ii N the midpoint of BD.
   c What name can you give to the quadrilateral ABCD?
   d What can you say about the lengths of the sides of quadrilateral ABCN?

ABCD MIXED EXERCISE 7

Several answers are given for these questions.

Write down the letter that corresponds to the correct answer.

1 Which of the four transformations

i reflection **ii** translation **iii** rotation **iv** glide reflection

are congruency transformations?

A **i**, **ii** and **iv** **B** **ii**, **iii** and **iv** **C** **i**, **ii** and **iii** **D** all of them

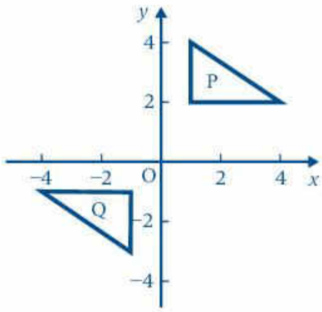
2 The transformation that maps triangle P onto triangle Q is

A a reflection in the line $y = x$

B a rotation of 180° about the origin

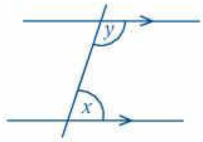
C a translation described by the vector $\begin{pmatrix} -3 \\ 2 \end{pmatrix}$

D none of these



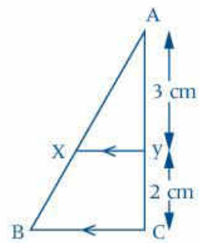
3 In this figure, the fact relating x and y is

A $x = y$ **B** $y = 90° - x$ **C** $x + y = 180°$ **D** $x = y - 90°$



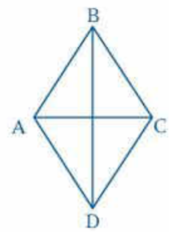
4 In the diagram, the area of triangle AXY is $3\,\text{cm}^2$.
The area of triangle ABC is

A $5\,\text{cm}^2$ **B** $8\frac{1}{3}\,\text{cm}^2$ **C** $12\,\text{cm}^2$ **D** $25\,\text{cm}^2$



5 In the rhombus, AC = 6 cm and BD = 8 cm.
The length of AB is

A 5 cm **B** 6 cm **C** 8 cm **D** 10 cm

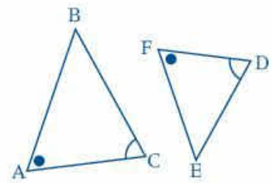


6 The length of the line segment joining A(2, 1) and B(4, 6) is

A $\sqrt{7}$ **B** 5 **C** $\sqrt{29}$ **D** 7

7 In these two triangles ∠A = ∠F and ∠C = ∠D
It follows that

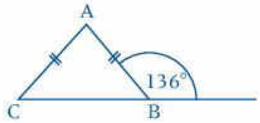
A $\frac{AB}{FE} = \frac{ED}{BC}$ **B** $\frac{AB}{FE} = \frac{AC}{FD}$ **C** $\frac{AB}{DE} = \frac{BC}{EF}$ **D** $\frac{AB}{DE} = \frac{AC}{DF}$



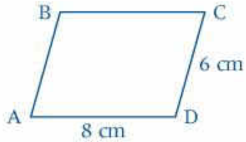
8 A square has a perimeter of 36 cm. Its area is

A $18\,\text{cm}^2$ **B** $36\,\text{cm}^2$ **C** $42\,\text{cm}^2$ **D** $81\,\text{cm}^2$

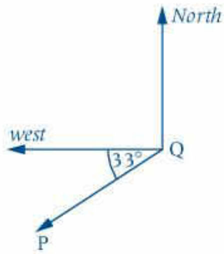
9 In this triangle, the size of angle BAC is
A 44° **B** 88° **C** 92° **D** 136°



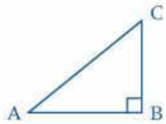
10 ABCD is a parallelogram. Its area, in square centimetres, is
A 24 sin ∠A **B** 24 cos ∠A **C** 48 sin ∠A **D** 48 cos ∠A



11 The bearing of P from Q is
A 303° **B** 237° **C** 213° **D** 123°



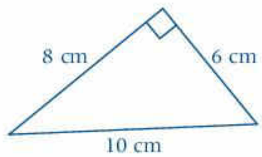
12 In triangle ABC, cos ∠C is
A $\frac{AB}{BC}$ **B** $\frac{AB}{AC}$ **C** $\frac{BC}{AB}$ **D** $\frac{BC}{AC}$



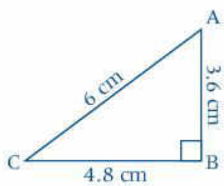
13 A ship was travelling on a bearing of 270°.
Another way of describing its direction is
A north **B** south **C** east **D** west

14 The best description of a quadrilateral with all its sides equal is
A a rectangle **B** a parallelogram **C** a kite **D** a rhombus

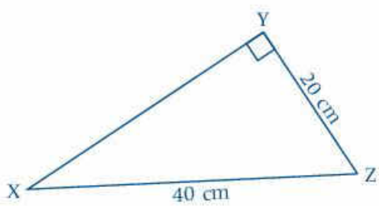
15 The area of this triangle is
A 60 cm^2 **B** 30 cm^2 **C** 24 cm^2 **D** 12 cm^2



16 In this triangle, sin ∠A simplifies to
A 0.6 **B** $\frac{4}{3}$ **C** 0.75 **D** $\frac{4}{5}$

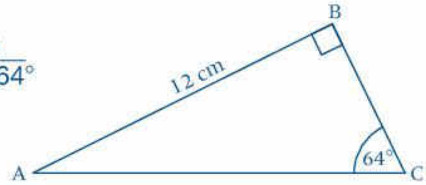


17 In this triangle, angle X is
A 20° **B** 30° **C** 45° **D** 60°



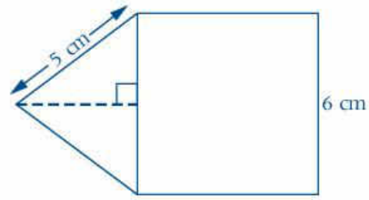
18 In this triangle, the length of BC is

A 12 tan 64° **B** $\frac{12}{\tan 64°}$ **C** $\frac{12}{\cos 64°}$ **D** $\frac{12}{\sin 64°}$



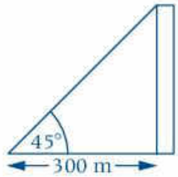
19 The figure shows an isosceles triangle drawn on one side of a square. The area of the figure is

A 30 cm² **B** 48 cm² **C** 51 cm² **D** 66 cm²



20 A man standing 300 m from the base of a tower measures the angle of elevation of the top of the tower as 45°. The height of the tower is

A 100 m **B** 150 m **C** 300 m **D** 600 m

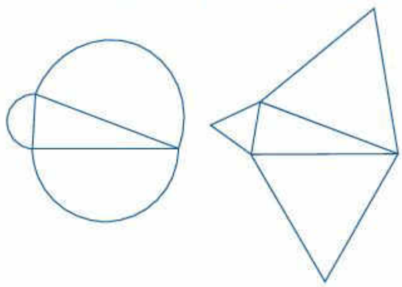


INVESTIGATION

Pythagoras' theorem relates areas of the squares drawn on the sides of a right-angled triangle.

Investigate whether a similar relationship exists between the areas of other shapes drawn on the sides of a right-angled triangle.

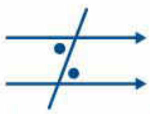
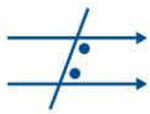
For example, semicircles or equilateral triangles:

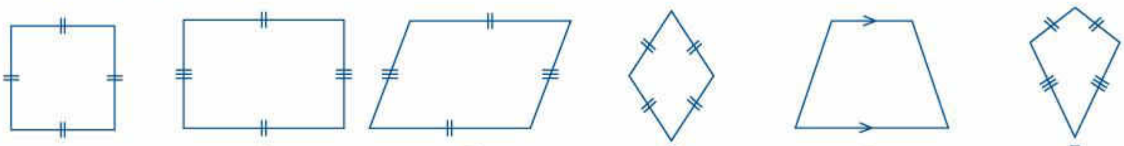


MATHS IS OUT THERE

Trigonometric ratios were first tabulated by Hipparchus of Bithynia (190–120 BC) to help with astronomy calculations.

IN THIS CHAPTER YOU HAVE SEEN THAT...

- a plane shape has bilateral symmetry when it can be folded along a straight line so that one half fits exactly over the other half
- a plane shape has rotational symmetry when it can be turned about a point to a new position and still look the same
- a shape has translational symmetry when it can be divided by parallel straight lines into identical shapes
- a reflection transforms an object by reflecting it in a line
- a rotation transforms an object by rotating it about a point
- a translation transforms an object by moving it without rotation or reflection
- an enlargement transforms an object by making it larger or smaller by a given scale factor. When the scale factor is greater than 1, the image is larger; when the scale factor is less than 1, the image is smaller; and when the scale factor is negative the image is also rotated by 180°
- vertically opposite angles are equal 
- corresponding angles are equal 
- alternate angles are equal 
- interior angles are supplementary (add up to 180°) 
- the sum of the interior angles of a triangle is 180° and an exterior angle is equal to the sum of the two interior opposite angles
- there are six special quadrilaterals:



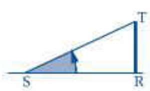
A	B	C	D	E	F
square: all four sides are equal, opposite sides are parallel, all angles are right angles	rectangle: opposite pairs of sides are equal and parallel, all angles are right angles	parallelogram: opposite pairs of sides are equal and parallel	rhombus: all four sides are equal, opposite sides are parallel	trapezium: one pair of opposite sides is parallel	kite: two pairs of adjacent sides are equal

- the necessary and sufficient condition for a quadrilateral to be a parallelogram is one of the following:
  - one pair of opposite sides are equal and parallel
  - both pairs of opposite sides are parallel
  - both pairs of opposite sides are equal
  - the diagonals bisect each other
  - both pairs of opposite angles are equal

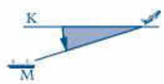
- a polygon is a region of a plane bounded by straight lines
- the sum of the interior angles of a polygon with n sides is $(n - 2) \times 180°$ and the sum of the exterior angles of any polygon is $360°$
- to show that two triangles are similar (i.e. the same shape) you need to prove either
  - the triangles are equiangular

  or

  - the corresponding sides are in the same ratio
- to show that two triangles are congruent (i.e. the same shape and size) you need to show that one of these conditions is satisfied:
  - the lengths of the three sides of one triangle are equal to the lengths of the three sides of the other triangle (SSS)
  - the lengths of two sides of one triangle are equal to the lengths of two sides of the other triangle and the angles between these two sides are equal (SAS)
  - two angles of one triangle are equal to two angles of the other triangle and one pair of corresponding sides are the same length (AAS) or (ASA)
  - the two triangles are right-angled and the lengths of the sides opposite the right angles and another pair of sides are equal. (RHS)
- Pythagoras' theorem states that in a right-angled triangle, the square of the hypotenuse is equal to the sum of the squares of the two shorter sides. The converse of Pythagoras' theorem states that if the square of the longest side of a triangle is equal to the sum of the squares of the other two sides, the angle opposite the longest side is a right angle
- the three-figure bearing of A from B is the clockwise angle between the north direction at B and the line segment AB
- an angle of elevation is the angle we need to look up to see an object above us

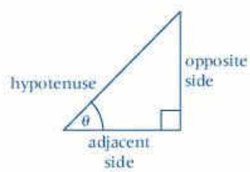


- an angle of depression is the angle we need to look down to see an object below us



- in a right-angled triangle

  $\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}}$, $\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}}$, $\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}}$



- the length of the line between the points $A(x_1, y_1)$ and $B(x_2, y_2)$ is given by $AB = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$
- the coordinates of the midpoint of the line between two points $A(x_1, y_1)$ and $B(x_2, y_2)$ are $\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}$

  the tangent of the angle a line makes with the positive direction of the x-axis is equal to the gradient of the line.

8 STATISTICS

AT THE END OF THIS CHAPTER YOU SHOULD BE ABLE TO...

1 Construct an ungrouped frequency table from a given set of data.
2 Draw and use pie charts, bar charts and line graphs.
3 Calculate the mean, mode and median for a set of data.
4 Understand the difference between theoretical and experimental probability.
5 Understand the meaning of mutually exclusive events and independent events.
6 Use contingency tables

MATHS IS OUT THERE

This is the message on the voice mail of a statistician 'Supposing that the universe doesn't end in the next 20 seconds, the odds of which I'm still trying to calculate, you can leave your name, phone number and message, and I'll probably get back to you.'

BEFORE YOU START YOU NEED TO KNOW:

✓ how to work with fractions, decimals and percentages
✓ the relationship between the size of a sector of a circle and the angle it contains at the centre of the circle.

KEY WORDS

bar chart, categorical data, central tendency, continuous, data, discrete, expectation, experiment, experimental probability, fair, frequency, mean, median, mode, numerical data, outcome, pie chart, probability, random, raw data, sample space, sector, theoretical probability, unbiased

Categorical and numerical data

Categorical data is information about different categories, such as preferred colours or the first names of students.

Numerical data can be counted or measured. For example, the numbers of students in each class or the weights of the students in a class.

Numerical data may be **discrete** or **continuous**.

Discrete data has distinct and exact values, whereas continuous data can have any value within a range.

The numbers of eggs laid by the hens in a yard are discrete; a hen can only lay 0, 1, 2, ... eggs. Discrete data might not have whole number values; shoe sizes are discrete but they come in half sizes as well as whole sizes.

The masses of the eggs are continuous; a mass of 20 grams, to the nearest gram, is in the range 19.5 g up to 20.5 g.

Measures of central tendency

This list of test marks (out of 10) is **raw data**:

2, 4, 2, 6, 3, 8, 3, 3, 5, 6

We can organise it by writing the marks in order:

2, 2, 3, 3, 3, 4, 5, 6, 6, 8

We can choose to represent these marks by an average mark.

There are three different types of average: the mean, the mode and the median. They are collectively called measures of **central tendency**.

The **mean** of a set of values is calculated from $\frac{\text{sum of the values}}{\text{number of values}}$.

The mean mark of the set given above is

$$\frac{2 + 2 + 3 + 3 + 3 + 4 + 5 + 6 + 6 + 8}{10} = \frac{42}{10} = 4.2$$

The mean is often called the average. The correct term is the arithmetic mean (there are other means such as the geometric mean).

The **mode** is the value that occurs most often.

The modal mark of the set given above is 3.

There might be no mode or there might be more than one mode

The **median** is the middle value when the values have been arranged in ascending or descending order of size. For n values, the middle value is the $\frac{n+1}{2}$ th value.

When there is an even number of values, the median is the average of the two middle values.

The middle values of the marks above are 3 and 4, so the median mark is 3.5

The mean, mode and median each have advantages and disadvantages.

The mean has the advantage that it includes all the values, it is also useful for comparing related data.

We can use the mean to compare several sets of test marks, even when the numbers of marks are different for each set.

The main disadvantage of the mean is that it can be affected by a few very large or very small values (called outliers).

The mean of the marks 0, 6, 8, 8, 9 is 6.2 which is not really representative of the marks because the one mark of 0 pulls down the value of the mean.

The advantage of the median is that it is not affected by outliers. The disadvantage is that it does not take account of all the values, so may not be a good representative of the data.

The median of the marks 0, 6, 8, 8, 9 is 8 which is more representative than the mean.
However, the median of the marks 0, 1, 2, 7, 8, is 2 which is not a good representative.

The mode is useful in circumstances where we need to know the most frequent value. However, it is not always a good representative of the data.

A shopkeeper who sells different brands of trainers is interested in the modal brand and the modal size.
He can then order more of that brand, and order more of the modal size.

EXERCISE 8a

Example:

These are the scores awarded to the contestants in a debating competition.

5.7, 9.8, 6.4, 5.9, 8.2, 6.9, 5.9

a Find **i** the mean **ii** the mode **iii** the median.

b The marks of two more contestants are added to the list.
These marks are 5.0 and 9.5
Explain what effect adding these two marks has on
i the mean **ii** the mode **iii** the median.

a i The mean score $= \frac{5.7 + 9.8 + 6.4 + 5.9 + 8.2 + 6.9 + 5.9}{7}$

$= \frac{48.8}{7} = 6.97$ to 3 s.f.

ii 5.7, 5.9, 5.9, 6.4, 6.9, 8.2, 9.8

The median is 6.4

iii The mode is 5.9

Arrange the marks in order of size.

b i The mean mark will increase because the mean of the added marks is 7.25 which is greater than the mean of the original set of marks.

ii The mode will not change because 5.9 is still the only mark that occurs more than once.

iii The median will not change because one mark is higher and one lower than the original median, so 6.4 is still the middle mark.

In this exercise give any answers that are not exact correct to 3 significant figures.

1 Find the mean, median and mode of each of the following sets of numbers.

a 5, 8, 4, 7, 11, 6, 4

b 4.9, 2.6, 3.4, 2.7, 2.6, 1.2, 2.3, 3.2

c 14, 13, 14, 15, 12, 15, 14, 13

d 0.6, 0.7, 0.4, 0.7, 0.8, 0.7, 0.6, 0.4

2 The heights (to the nearest cm) of some boys are

158, 164, 161, 162, 169, 167, 170, 166, 158, 164

Find **a** the median height **b** the modal height **c** the mean height.

3 The heights (to the nearest centimetre) of some girls are

145, 152, 150, 149, 158, 162, 155

a Find **i** the median height **ii** the mean height.

b The heights of the tallest girl and the shortest girl are removed from the list. Explain whether this changes

i the median **ii** the mean.

4 Mr Ebdon watches his fellow workmates arriving by car one morning. He counts the number of occupants in each car and obtains the following list:

1, 2, 2, 2, 1, 1, 4, 3, 2, 1, 2, 1, 1, 5, 2, 1, 2, 3, 1, 1

a How many cars are there in the survey?

b How many workmates does he count altogether?

c What is the mean number of occupants per car?

d What is the modal number of occupants per car?

e What is the median number of occupants per car?

5 Some of the patients who attended a doctor's surgery one morning had their blood pressure taken. The diastolic blood pressures recorded were 82, 88, 69, 76, 84, 90, 75, 62, 80, 84, 93, 79 and 88

a How many patients had their blood pressure taken?

b Find the mean diastolic blood pressure for the group.

c What was the modal diastolic blood pressure?

d What percentage of the group had a diastolic blood pressure greater than 80?

e Find the median diastolic blood pressure.

6 A small business employs 10 people. The basic weekly wages of the three senior employees are \$1500, \$1260 and \$1200, and the basic wage for the remainder is \$768
  a How many employees receive a basic weekly wage of \$768?
  b Find the total cost of the basic weekly wages.
  c What is the mean basic weekly wage?
  d What is the modal basic weekly wage?
  e State, with a reason, which is the fairest average to use if you were telling a friend about the level of wages paid by the business.

7 In successive rounds a golfer took the following numbers of strokes: 73, 92, 71, 72, 72, 75, 82
  a State, with an explanation, which of the three averages, mean, median or mode, would be best to use to describe his 'average' score.
  b In his next round, what is the most he should score to reduce his mean score? State, with a reason, whether this score would affect his modal score.

8 a Make a list of 6 numbers, 5 of which are smaller than the mean.
  b Make a list of 5 numbers, 4 of which are greater than the mean.

9 During last year the number of days taken off sick by the staff in a computer centre were 4, 23, 0, 1, 0, 10, 0, 3, 18, 35
  a i How many staff are employed at the computer centre?
    ii Find the total number of days lost.
  b Find i the mean ii the mode iii the median.
  c The owner of the centre had 3 days off sick last year. This number was not included in the list above. If it is included, explain how it affects i the mean ii the median iii the mode.

10 The ages of a family of seven children are 17.5, 15.11, 13.8, 14.7, 12.1 and the twins are 10.2
  a Find i the mean ii the mode iii the median
  b Next year the eldest child is leaving home. Explain what effect this will have on
    i the mean ii the mode iii the median.

An age of 17.5 means 17 years and 5 months.

11 The scores, out of 20, awarded by eight judges in an Art competition were 16, 12, 9, 16, 18, 13, 8, 14
  a Work out i the mean ii the mode iii the median.
  b Which two scores can be removed without changing the value of the median?
  c If the score of 12 is removed, will the value of the mean go up or down? Justify your answer.

Example:

The average height of 20 students is 1.4 m. The average height of another 10 students is 1.1 m. Calculate the average height of the 30 students.

The total height of the 20 students is $20 \times 1.4\,\text{m} = 28\,\text{m}$

The total height of the 10 students is $10 \times 1.1\,\text{m} = 11\,\text{m}$

The total height of the 30 students is $28\,\text{m} + 11\,\text{m} = 39\,\text{m}$

The mean height of the 30 students is $\frac{39}{30}\,\text{m} = 1.3\,\text{m}$

We cannot average the two means because they are for different numbers of heights.

To find the mean of 30 heights, we need the sum of the 30 heights.

As (mean) $= \frac{\text{sum of values}}{\text{number of values}}$, we have (sum of values) = (mean) × (number of values), which we can use to find the sums of the heights of each group.

12 In a game of darts, three throws make one turn. After 12 turns Rohan has a mean score of 24. How many does he need to score on his next turn to raise his mean score to 26?

13 The mean weight of the 13 people in a slimming class is 84 kg. If the mean weight of the 6 heaviest people is 91 kg find
- a the total weight of the 13 people
- b the total weight of the 6 heaviest
- c the total weight of the other 7
- d the mean weight of these 7

14 Over a four-week period the mean number of rejects per day from an automatic lathe is 12. (Assume that the lathe runs for 6 days each week.) This is thought to be unsatisfactory, so the lathe is serviced ahead of schedule. As a result, during the next seven weeks, the mean number of daily rejects is reduced to 2.5
- a Find the total number of rejects during
  - i the first four-week period
  - ii the next seven-week period
  - iii the full period of the study.
- b What is the mean number of daily rejects for the period of the study?
- c Estimate the number of potential rejects that have been 'saved' as a result of the service.

15 The mean daily takings in a grocery shop from Monday to Friday were \$6960, while the mean daily takings from Monday to Saturday were \$8166
- a How much was taken over the five days Monday to Friday?
- b How much was taken over the six days Monday to Saturday?
- c How much was taken on Saturday?

16 Three groups of students are given the same test. The mean mark for the first group of 22 is 58, the mean mark for the second group of 23 is 51 and the mean mark for the third group of 25 is 78
What is the average test score for all the students taking the test?

17 To be allowed to take part in the cricket-ball throwing competition in a school sports event, competitors must throw an average of at least 55 metres over three throws. In his first two throws, Ashley throws 58 m and 49 m. How far must he throw at his third attempt to make sure of qualifying?

18 Last season Tom Smith's batting average in three-day games was 21.5 in 16 completed innings and in four-day games it was 42.8 in 5 completed innings.
- a What was his average for the season?
- b In one game he was given out when on 56 to close his side's innings.
  How would your answer to part **a** change if his batting partner had been out to close the innings while Tom was still on 56?
  Explain whether it matters whether this innings was in a three-day match or a four-day match.

Frequency tables

This list of marks is raw data:
1, 4, 2, 5, 3, 4, 3, 5, 4, 4, 5, 3, 5, 2, 4, 2, 3, 4

We can make more sense of it if we make a frequency table.

For an ungrouped frequency table, we list each different value against the number of times it occurs – this is called its **frequency**.

This frequency table shows the marks in the list.

Mark	1	2	3	4	5
Frequency	1	3	4	6	4

It tells us that there is one 1, three 2s, and so on, in the list.

The sum of the frequencies gives the total number of marks.

We can also use a frequency table for categorical data.

This table shows the numbers of students using different modes of transport to get to school.

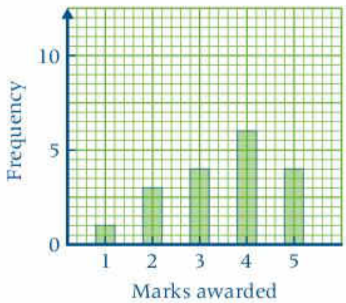
Walk	Bicycle	Bus
20	15	8

Bar charts

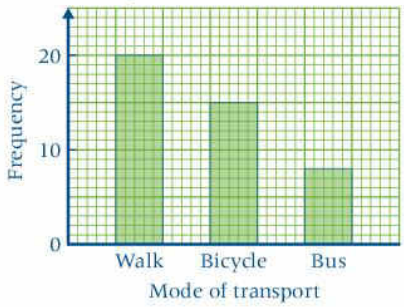
We can illustrate an ungrouped frequency table showing discrete data with a **bar chart**. A bar is drawn to represent each category or value. The height of the bar represents the frequency.

Notice that the bars are all the same width and they are the same distance apart.

This bar chart shows frequencies of the different marks awarded in a test



This bar chart shows frequencies of the different modes of transport of students travelling to school



EXERCISE **8b**

Example:

This frequency table shows the marks, out of 5, scored by a sample of students in a test.

Mark	0	1	2	3	4	5
Frequency	1	2	2	8	15	10

a Calculate the mean mark.

b Find the median and the mode.

a

Mark, x	0	1	2	3	4	5	
Frequency, f	1	2	2	8	15	10	Total 38
fx	0	2	4	24	60	50	Total 140

Mean mark $= \frac{140}{38} = 3.7$ to 1 d.p.

We need to find the sum of all the marks. We can do this by multiplying each mark, x, by its frequency, f, and then adding these values. Adding another row to the table helps keep track of the calculations. The number of marks is the sum of the frequencies.

b There are 38 marks, so the middle mark is the $(39 \div 2)$th mark.

This is the average of the 19th and 20th marks.

These are both 4, so the median mark is 4

The modal mark is 4

Adding the frequencies shows that the 1st mark is 0, the 2nd and 3rd marks are 1, the 4th and 5th marks are 2, the 6th to 13th marks are 3 and the 14th to 28th marks are 4.

1 The table shows the number of workers off sick during the last week of June.

Number of days absent	0	1	2	3	4	5
Frequency	50	6	3	2	1	2

Find **a** the mode **b** the median **c** the mean.

2 The table shows the scores of people in a quiz.

Score	0	1	2	3	4	5	6	7	8	9	10
Frequency	2	2	1	5	4	12	16	9	9	6	4

a How many people took part in the quiz?
b Find the mean, median and modal score.

3 The number of letters in the words in the first few paragraphs of a copy of Jane Austen's *Pride and Prejudice* are given in the table.

Number of letters	1	2	3	4	5	6	7	8	9	10	11	12	13
Frequency	7	24	16	17	10	3	2	5	1	2	3	1	1

a How many words were in these paragraphs?
b Find the mean, mode and median number of letters.
c Try doing the same thing for two paragraphs from your local newspaper. Are there any differences between the lengths of the words used in the two pieces of prose?

4 In a store, sales are placed in one of four categories: electrical (E), floor coverings (C), fabrics (F) and bedding (B). The list given below records the sales on one day:

F C B C E B E E F C E F E C B E
C B C E F F E C B C E E F C B F
B E C E F B F F E E E C E F B B

a Show this data in a frequency table.
b Draw a bar chart to illustrate the data.
c How many more electrical sales than bedding sales were there?
d How many more sales were there for fabrics than for floor coverings?
e How many sales were there altogether?

5 In a factory making radios a sample of 50 radios is selected at random each day to test for defects. The number with defects found on 40 consecutive days is listed below.

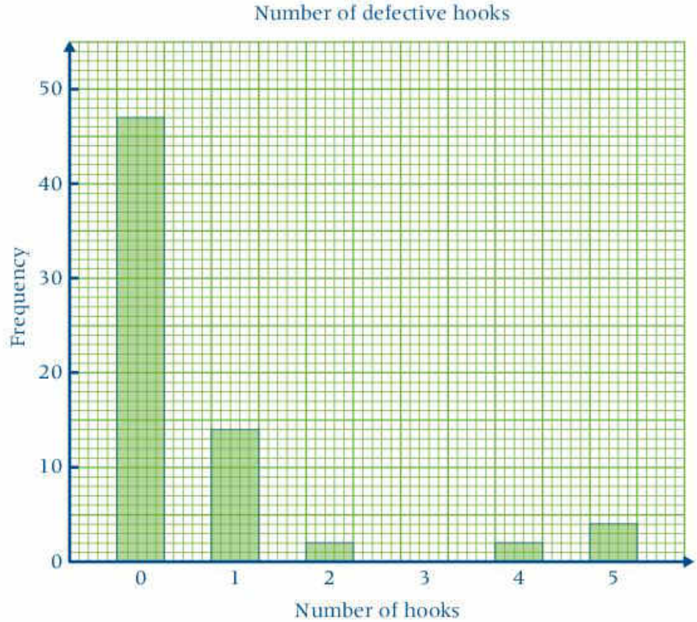
2 0 8 4 7 3 6 4 7 3 8 3 7 2 0 7 3 1 4 3
3 8 7 3 6 4 5 6 4 6 3 7 8 5 6 1 7 0 5 0

a On how many days were there more than 6 radios with defects?
b Draw a bar chart to illustrate this data.
c Find the mean, mode and median for this data.

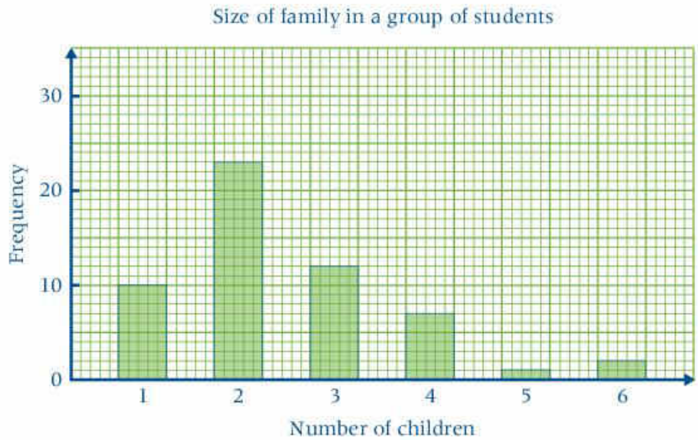
'Selected at **random**' means that all of the radios made have an equal chance of being selected.

Make a frequency table before trying to draw the bar chart.

6 A machine makes brass picture hooks and automatically sorts them into packs of five. A small number of packs are selected at random and the number of defective hooks in each pack noted. The results for one day's sampling are shown in the bar chart.
a How many packs contained
  i no defective hooks
  ii 2 defective hooks?
b How many packs were checked?
c How many defective hooks were there altogether?
d Find the mean, the mode and the median for this data.



7 This bar chart shows the number of children in the families of a group of students.
a How many students were asked about the number of children in their family?
b Find the mean, median and modal number of children in the families of the group.



8 A group of students were asked to name their favourite fruit. Their replies were recorded in this table.
a How many students were asked to name their favourite fruit?
b What fraction of the group chose
  i a banana ii none of the named fruits?
c Draw a bar chart to illustrate this information.

Fruit	Orange	Pineapple	Peach	Banana	Apple	Other
Frequency	13	12	23	15	7	30

9 The frequency table shows the number of goals scored by the teams in a league one weekend.

Number of goals	0	1	2	3	4	5	6
Frequency	8	11	6	7	2	3	1

a Draw a bar chart to illustrate this data.
b i How many teams are there in the league?
  ii How many goals were scored altogether?
c Find the mean, median and modal number of goals scored.

10 The frequency table shows the scores in the last round of a golf tournament.

Score	66	67	68	69	70	71	72	73	74	75
Frequency	3	4	7	9	6	8	10	7	4	2

a How many competitors were left in the tournament at the end?
b How many had a round under 70?
c Draw a bar chart to illustrate this data.
d Find the mean, median and modal score.

Line graphs

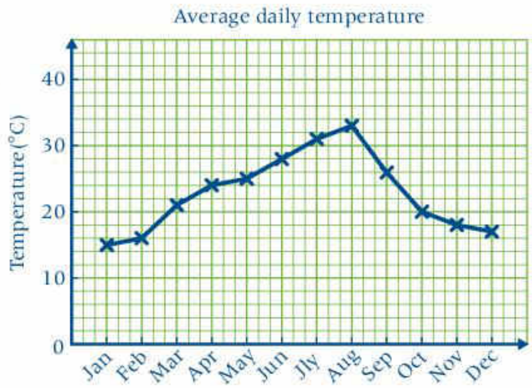
A line graph is used to illustrate how the value of a quantity changes over time.

Temperature, share prices, rainfall, profits, are examples of quantities whose values vary over time.

This table shows the average daily temperature at noon, in degrees Celsius, for each month of a year on an island.

Jan	Feb	Mar	Apr	May	Jun	Jul	Aug	Sep	Oct	Nov	Dec
15	17	21	24	25	28	31	33	26	20	18	17

We can draw a line graph by plotting the temperatures against the months and joining the points with straight lines.



The lines are there to show how the quantity is changing. Values in between points have no meaning.

Pie charts

Pie charts are used to illustrate how the total amount is shared between different categories.

A pie chart is a circle divided into **sectors**.
The area of each sector represents the fraction that the amount in a category is of the whole.

This table shows how the total amount spent by an island government was shared between various categories.

Education	Health	Tourism	Defence	Police
$150 000 000	$110 000 000	$20 000 000	$70 000 000	$10 000 000

To draw a pie chart, we need to find the size of each sector.

The total amount spent = \$360 000 000.

The fraction spent on education is $\frac{150\,000\,000}{360\,000\,000} = \frac{5}{12}$

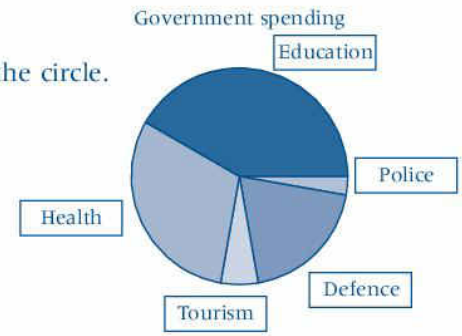
So the area of the sector representing education is $\frac{5}{12}$ the area of the circle.

We know that $\frac{\text{area of sector}}{\text{area of circle}} = \frac{\text{angle at the centre}}{360°}$

so $\frac{5}{12} = \frac{\text{angle at the centre}}{360°}$

∴ the angle at the centre of the sector showing education is $\frac{5}{12} \times 360° = 150°$

The angles for the other sectors can be calculated in the same way and the angles can be drawn to make the pie chart.



EXERCISE 8c

Example:

The line graph shows the value of the Jamaican Stock Exchange market index over one month.

a Write down an estimate for the value of the market index on 26 March.

b Explain why it is not possible to give a value for 6 April.

The scale on the horizontal axis is a 6-day interval.

a *\$90 500*

b *There is no point plotted for 6 April.*

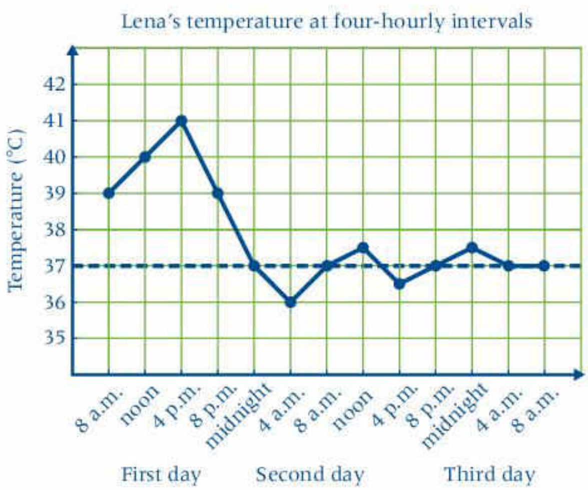
This means that the market was closed on that day so no trading took place.

1 Lena was admitted to hospital as an emergency. Her temperature was taken at 4-hourly intervals and a record kept on a chart which is shown here.

a What was Lena's
 i lowest temperature
 ii highest temperature?

b Was the highest temperature shown on the chart necessarily her highest temperature?

c By the third day Lena was feeling much better. What do you think the dashed line represents?

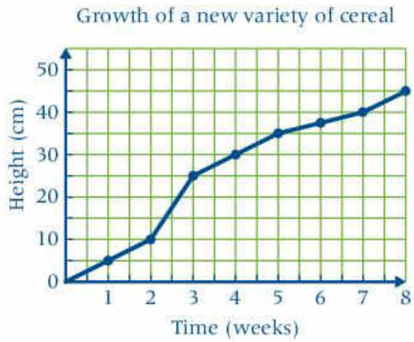


2 Joyce measured the height of a plant at the end of each week for 8 weeks. Her values are shown on the graph.

a How high was the plant after
- **i** 2 weeks
- **ii** 3 weeks?

b How much did the plant grow
- **i** in the 3rd week
- **ii** from the end of the 2nd week to the end of the 7th week?

c During which week did the plant grow
i most **ii** least?



3 This line graph shows the quarterly sales figures for a manufacturing company.

a What were the 1st quarter sales in
i 2003 **ii** 2007?

b What were the 3rd quarter sales in
i 2004 **ii** 2006?

c Find the difference between the 2nd quarter sales and the 4th quarter sales in
i 2003 **ii** 2007

d In which year was there the greatest difference between the poorest quarter and the best quarter?

e Describe the sales pattern in these figures.

f Suggest a product that could give a sales pattern like this.

4 This table shows the annual rainfall on an island from 1998 to 2006.

Year	1998	1999	2000	2001	2002	2003	2004	2005
Rainfall, mm	550	490	580	610	580	540	620	590

a Plot these values on a graph.

b Join the points with straight lines.

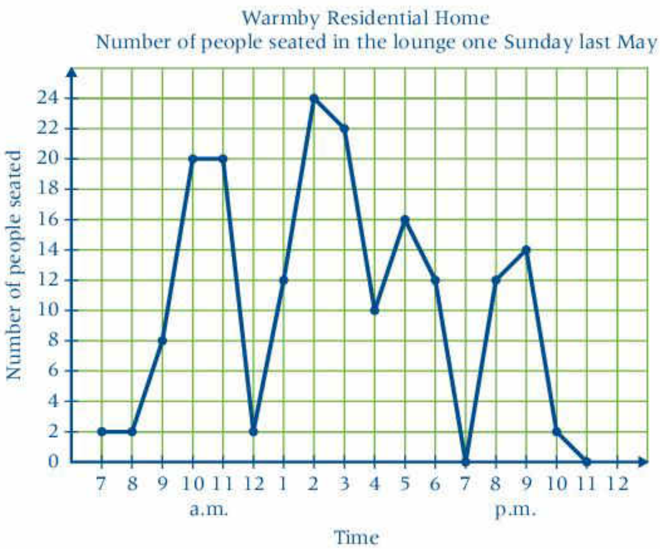
c Is it true that the annual rainfall on the island is tending to increase? Give a reason for your answer.

5 The number of people sitting in the lounge of a residential home was counted at hourly intervals one Sunday and the results recorded on this graph.

a How many people were sitting in the lounge at **i** 10 a.m. **ii** 11 p.m.?

b Jason said that the graph shows there were 10 people sitting in the lounge at 11.30 a.m.
Explain whether Jason is correct.

c Suggest an explanation for the pattern.



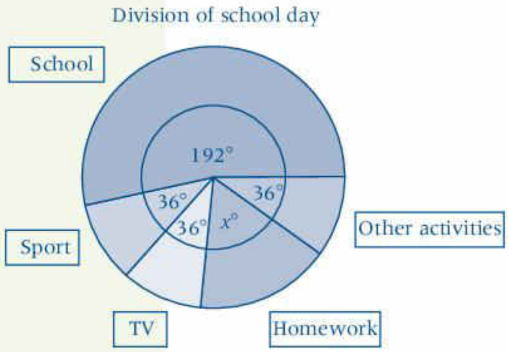
Example:

A 12-year-old boy is awake for a number of hours a day. The pie chart shows how his time is divided between different activities on a school day.

a Calculate the value of x.

b Calculate the fraction of his time awake spent on sport.

c He spends $2\frac{1}{2}$ hours on homework. Calculate the number of hours the boy is awake.

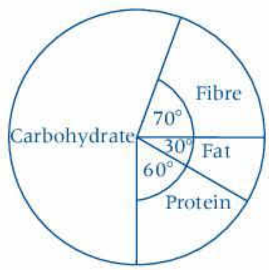


a $192 + 3 \times 36 + x = 360$
so $x = 360 - 192 - 108 = 60$

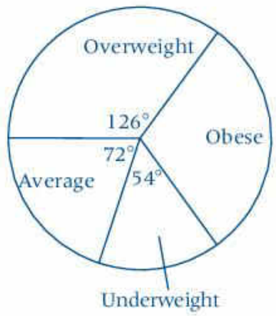
b The sector representing sport
$= \frac{36}{360} = \frac{1}{10}$ of the area of the circle.
The boy spends $\frac{1}{10}$ of his time awake on sport.

c He spends $\frac{60}{360} = \frac{1}{6}$ of his time on homework.
So $\frac{1}{6}$ of his time awake $= 2\frac{1}{2}$ hours.
$\therefore$ his time awake $= 6 \times 2\frac{1}{2}$ hours $= 15$ hours.

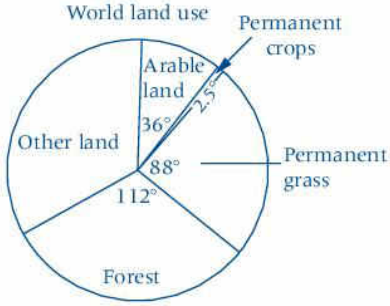
6 This chart shows the proportion, by weight, of various nutrients in a packet of oat cereal.
a What fraction of the nutrients is
i fat ii fibre iii carbohydrate?
b How many grams of protein are there in a serving of
i 100g ii 36g?
c How many grams of carbohydrate are there in a serving of
i 100g ii 36g?



7 180 adults were weighed and placed into one of four categories: underweight, average, overweight, obese (i.e. grossly overweight). The pie chart shows the proportion of adults falling into each category.
a What fraction of the group were
i overweight ii obese iii above average?
b How many of these adults were
i underweight
ii either above average weight or below average weight?
c What percentage of the group were
i obese ii not above average weight?



8 The pie chart shows how all the land in the world was being used in the early 2000s.
If the total area of the land mass in the world is 200 million square kilometres, find how many million square kilometres was
a forest
b being used as arable land.



9 Compuco's total income of $50 million is divided between costs, taxes and profits as shown in the pie chart.

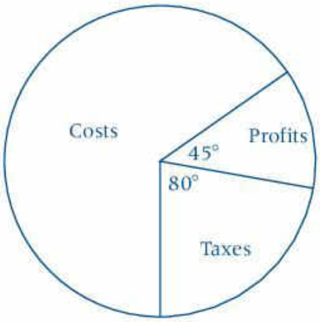
a What fraction of the income is allocated to **i** taxes **ii** costs?

b What is the ratio of the profit to the costs?

c How much do Compuco pay in taxes?

d If taxes are reduced by one third and the costs remain unchanged, what is the ratio of profits to taxes?

e Explain why a bar chart would not be a good way of illustrating this data.



Example:

The table gives the percentages of a 24-hour day that a TV station broadcasts different categories of programmes.

News and current affairs	5%
Documentaries	15%
Children's programmes	20%
Chat and game shows	30%
Movies	30%

a Calculate the number of hours spent broadcasting movies.

b Draw a pie chart to show this information.

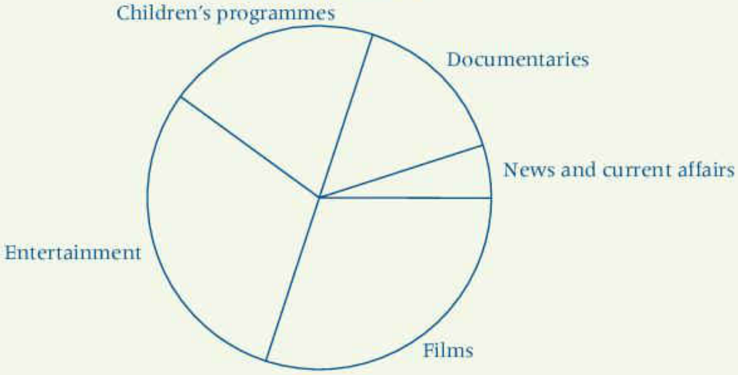
c Explain why a bar chart is not an appropriate way of illustrating this information.

a 30% of $24\text{ h} = \frac{30}{100} \times 24\text{ h} = 7.2\text{ h}$

b

News and current affairs	5% of $360° = 18°$
Documentaries	15% of $360° = 3 \times 18° = 54°$
Children's programmes	20% of $360° = 2 \times 36° = 72°$
Chat and game shows	30% of $360° = 3 \times 36° = 108°$
Movies	30% of $360° = 108°$

First calculate the angles of the sectors.
We are given the percentages that each category is of the 24 h, so we need these percentages of 360°.
$10\% = \frac{1}{10}$



Start by drawing a circle and one radius. Make the radius at least 5 cm. Use a protractor to make an angle of 18° with this radius to give the first sector, then continue round the circle to give the other sectors. You can use the angle at the centre of the last sector as a check. Do not forget to add a title for the pie chart and a label for each sector.

c The total time is fixed and it is the proportions of that time that are important, not the actual times spent on each category.

10 The sales of petrol from five service stations are shown in the table. Construct a pie chart to illustrate this data.

Petrol station	Adcot	Burley	Crossway	Deighton	Eden
Sale (litres 000s)	85	30	120	130	35

11 Each $1 received from sales at Busicom Industries is divided as follows: raw materials 28 c, wages 33 c, plant and machinery 8 c, advertising 9 c, the remainder being profit.

a Construct a pie chart to illustrate this information.

b Explain why a bar chart would not be a suitable way of illustrating this data.

12 The populations, in millions, of India, China and the United States in 1995 were given as: India 844, China 1116 and the United States 253.

a Draw a pie chart to illustrate this data.

b During the decade to the year 2005 the population of India grew by 170 m, the population of China grew by 100m and the population of the United States grew by 20 m.
Draw a pie chart to illustrate the populations of the three countries in the year 2005.

c Would bar charts be a better way of illustrating the data? Explain your answer.

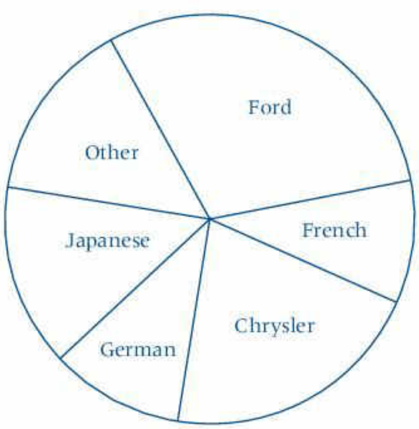
13 The table shows the known oil reserves.

Region	**Number of barrels (thousand million)**
America (North and South)	160
Middle East	660
Former Soviet Union	60
Africa	60
Asia and Australasia	50
Europe	10

a Construct a pie chart to show this data.

b Would a bar chart be a better way of illustrating the data? Explain your answer.

14 The cars that pass through Paramount Car Auctions are placed in one of six categories for record purposes. This pie chart shows the number of cars in each category for sales that took place in 2005.



a What fraction of the cars sold were

i Fords **ii** Japanese **iii** German?

b If 340 German cars went through these auction rooms, how many of the cars auctioned were

i from Chrysler **ii** Japanese?

c By 2007 the total number of cars passing through these auction rooms had increased to 4420 and the percentage in each category had changed to: French 15%, Ford 25%, Chrysler 20%, German 12% and Japanese 20%.

i What percentage were there in the 'Other' category?

ii Draw a pie chart to represent this data.

d Comparing 2007 with 2005, which category had
  i held its market share steady **ii** increased its market share?

e In which category had the number of cars sold increased but the market share decreased?

15 A student estimated how he had spent the previous 24 hours. The fraction of time he spent in various activities is shown in the table.

Activity	Sleeping	Eating	School	Home study	Sport	Other
Fraction of 24 hours	$\frac{1}{3}$	$\frac{1}{8}$	$\frac{1}{4}$	x	$\frac{1}{12}$	$\frac{1}{12}$

a Calculate the number of hours spent on Home study.

b Calculate the angles in a pie chart that would be used to represent the number of hours spent **i** sleeping **ii** on sport.

c Draw a pie chart to represent this data.

Probability

Words such as 'highly likely' and 'little chance' are used to describe the likelihood of events taking place.

If we toss a coin there are two possible results: a head or a tail.

The act of tossing the coin is called an **experiment**.

The possible results are called **outcomes**.

The set {head, tail} is the set of all the possible outcomes.

> The set of all possible outcomes in any experiment is called the **sample space**.

Probability is a measure of the likelihood of an outcome.

We use P(E) to denote the probability of an event E occurring.

$$P(E) = \frac{\text{number of outcomes which result in event E}}{\text{the number of equally likely outcomes}}$$

> Probability can be given as a fraction, a decimal or a percentage.

If an outcome is impossible, there are no ways in which it can happen so the probability of it happening is zero, i.e. $P(E) = 0$

If an outcome is certain, the number of ways it can happen equals the number of equally likely outcomes so the probability that it happens is 1, i.e. $P(E) = 1$

In general, $0 \leqslant P(E) \leqslant 1$

> If we roll an ordinary unbiased six-sided die, the probability of getting a 3 is $\frac{1}{6}$ because there is only one way of getting a 3 and there are 6 equally likely outcomes.
> The probability of getting a 9 is zero, and the probability of getting a score of one to six is 1.

So the probability of getting a head if we toss a fair coin is

$$\frac{\text{number of ways of getting a head (1)}}{\text{number of equally likely outcomes (2: a head or a tail)}} = \frac{1}{2} = 0.5$$

> If a coin, or a die, is described as **fair**, or **unbiased**, the outcomes are equally likely.

This is a **theoretical probability** because we do not actually toss the coin and we assume that it is unbiased.

If we actually toss a coin many times, we can use the outcomes of the experiment to find an experimental probability for getting a head with that coin.

This table shows the outcomes of such an experiment.

Total number of tosses	**Number of heads**
500	295

The fraction $\frac{\text{number of heads}}{\text{total number of tosses}} = \frac{295}{500} = 0.59$ is called the relative frequency of a head. We can use this value as an estimate for the probability of getting a head when this coin is tossed. This is called the **experimental probability**.

The theoretical probability of a head with an unbiased coin is 0.5. $0.59 > 0.5$ so it may be that a head is slightly more likely with this coin.

In general, for any experiment,

$$\text{relative frequency of a particular outcome} = \frac{\text{number of times that outcome occurs}}{\text{total number of outcomes}}$$

We can use relative frequency as the experimental probability that a particular outcome will occur. Remember that experimental probability is an estimate for the theoretical probability.

EXERCISE 8d

Example:

One tile is chosen at random from a bag containing 25 green and 35 blue tiles. What is the probability that the tile will be green?

There are 25 ways of choosing a green tile out of a total choice of (25 + 35) tiles.

Probability of a green tile $= \frac{25}{60} = \frac{5}{12}$

'At random' means that every possible choice is as likely as any other choice.

In this exercise, assume that all the possible outcomes are equally likely.

1 What is the probability of
  a rolling an ordinary 6-sided die and getting a 3
  b tossing a coin and getting a tail
  c the sun will set tonight
  d you will never be given any homework?

2 If one letter is chosen at random from the letters in the word BIKE what is the probability that the letter is E?

3 If one letter is chosen at random from the letters in the word FIGURE what is the probability that the letter is G?

4 A letter is chosen at random from the word CARIBBEAN. What is the probability that the letter is
  a A b C c a vowel d a consonant
  e after R in the alphabet
  f one of the first five letters in the alphabet?

5 A fair 6-sided die is rolled. What is the probability of getting
  a a prime number b an even number
  c a negative number d 7
  e a number greater than 2 f 6?

6 A card is chosen at random from an ordinary pack of 52 playing cards. What is the probability that the card chosen is
  a an Ace b a black card
  c a Jack, Queen or King
  d a red card greater than 2 but smaller than 8?

7 What is the probability that if a number is chosen at random from the numbers from 30 to 50 inclusive it is
- a a multiple of 7
- b a prime number
- c a multiple of 3 and 7?

8 The numbers from 1 to 100 are written on separate pieces of paper. One number is then drawn at random. What is the probability that the number is
- a a prime number
- b a square number
- c a multiple of 8
- d neither a square number nor a cube number?

9 A box contains an equal number of triangles, squares, rectangles, parallelograms and pentagons.
If one shape is taken out at random, find the probability that it
- a is a quadrilateral
- b is a triangle
- c has either 4 or 5 sides
- d is neither a quadrilateral nor a pentagon?

10 200 tickets, numbered from 1 to 200, were sold for a raffle. Mr Mahto bought 5 and Mrs Mahto bought 10. The winning ticket is drawn at random.
What is the probability that
- a it belongs to Mr Mahto
- b it belongs to either Mr or Mrs Mahto
- c the number on the ticket
  - i has two digits ii is greater than 100?

11 A box contains a set of snooker balls. They consist of 15 reds and one each of white, yellow, green, brown, blue, pink and black. If one ball is removed at random from the box, what is the probability that it is
a red b not red c blue d neither red nor black?

12 A letter is taken at random from the word PREPOSSESSIONS. What is the probability that it is
a S b O c M d a vowel e not a vowel?

13 In a vehicle park there are 100 vehicles of which 80 are cars, 15 lorries and the rest buses. If they are all equally likely to leave, what is the probability that the first vehicle to exit is
a a car b a lorry c not a lorry?

14 At the end of a day a market trader has a wad of bank notes. He counts them and finds he has two at \$50, twenty-eight at \$20, thirteen at \$10 and nine at \$5.
He takes one note at random from this wad towards paying his assistant.
- a Is the note more likely to be a \$20 note than any other?
- b What is the probability that the value of the note he gives is
  - i \$10 ii \$50 iii not \$20?

15 A class of business studies students is made up of 5 male students and 10 female students. Sue Powell, the lecturer, has the habit of going round the class at random asking different students questions.
What is the probability that the next student she asks is
a male **b** female?

16 On a day-trip, 38 senior citizens, 24 of whom are women, fly to a nearby island. The boarding cards for the aircraft are given out at random at the airport.
What is the chance that the ticket for the best seat goes to a man?

Example:

This frequency table shows the marks, out of 5, scored by students in a test.

Mark	0	1	2	3	4	5
Frequency	1	2	2	8	15	10

One of the students who took the test is chosen at random.
What is the probability that the student scored less than 3 in this test?

There are 5 students who scored less than 3.

There are marks for 38 students listed.

The probability that a student scored less than 3 $= \frac{5}{38}$

Less than 3 means 0, 1 or 2. One student scored 0, two scored 1 and two scored 2.

$\frac{\text{number of ways of choosing a student who scored less than 3}}{\text{number of ways of choosing any student}}$

17 If you take a card from an ordinary shuffled pack of 52 playing cards, and repeat this 60 times, how many times would you expect
a to get a spade
b not to get a heart
c to get a black card?

The probability of getting a head when a coin is tossed is $\frac{1}{2}$. If the coin is tossed 20 times, you would expect (but not necessarily get) a head on $\frac{1}{2}$ of these tosses, i.e.you wouldexpect $\frac{1}{2} \times 20$ heads.
In general, if the probability of an outcome is p on one occasion, the expected number of times the outcome will occur on n occasions is pn.This is called the **expectation**.

18 Over the years the annual cricket match between Jamestown and Charlestown has had fine weather four times out of five.
a **i** What is the probability of fine weather for next year's match?
ii Is your answer to **i** theoretical or experimental?
b How many matches can be expected to have bad weather over the next decade?

19 The Blood Transfusion Service held a session at the Grenford Park Industrial Estate. The table shows how the donors were divided into different blood groups.
a If a donor is chosen at random, what is the probability that the donor's blood group is
i A **ii** AB **iii** not O?
b How many people would be expected to be blood group B from a different sample of 248 people?

Blood group	Number of donors
O	56
A	44
B	15
AB	9

20 A hardware store sells fluorescent tubes. The present stock is given in the table.

Length (cm)	25	35	45	55	65	85
Number	5	12	10	4	3	2

A tube is chosen at random from this stock.
What is the probability that the length of the chosen tube is

a 45 cm **b** 35 cm
c at least 45 cm **d** at most 35 cm?

21 For which of the following events can you *calculate* the probability that the event happens? Which must be found by *experiment*?

a Scoring 3 when an ordinary 6-sided die is rolled.
b Dropping a thumb tack to land point up.
c Getting the higher score when you play darts with a friend.
d Picking up exactly 10 sweets when you select a handful of sweets in a supermarket.

Independent events

If we toss a coin twice, it is possible that the first toss lands heads up and the second toss also lands heads up. However the way the first toss lands has no influence on the way the second toss lands. Two events like this where both can happen but each has no effect on the other are called **independent** events.

When two coins are tossed (shown as green and black), there are four possible outcomes:

		Coin	
		H	T
Coin	H	HH	TH
	T	HT	TT

This shows that the probability that both coins land heads up is $\frac{1}{4}$.

The probability that the black coin lands heads up is $\frac{1}{2}$ and the probability that the green coin lands heads up is $\frac{1}{2}$. Now $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

Therefore P(black coin is heads AND green coin is heads)
= P(black coin is heads) × P(green coin is heads).

This is true for any two independent events A and B, i.e.

$$P(A \text{ and } B) = P(A) \times P(B)$$

Mutually exclusive events

When just one coin is tossed, it can land heads up or tails up but it cannot land both heads up and tails up.

Events like this are called **mutually exclusive** events.

When a die is rolled, it could score either a five or a six but not both.

There are six possible outcomes and a score of five or six are two possible outcomes.

Therefore P(5 or 6) $= \frac{2}{6} = \frac{1}{3}$ but P(5) $= \frac{1}{6}$ and P(6) $= \frac{1}{6}$. Now $\frac{1}{6} + \frac{1}{6} = \frac{1}{3}$

so P(die scores either a 5 or a 6) = P(die scores 5) + P(die scores 6)

This is true for any two mutually exclusive events A and B, i.e.

$$P(A \text{ or } B) = P(A) + P(B)$$

EXERCISE 8e

In questions **1** to **6** decide whether the events described are mutually exclusive, independent or neither.

1 Mona and Clive each buy a ticket for a raffle and one of them wins first prize.

2 Two coins are tossed.
- **a** The first coin lands heads up or tails up.
- **b** Both coins land heads up.

3 A coin is tossed and a die is rolled.
- **a** The coin lands heads up and an even number is scored on the die.
- **b** A three or a six is scored on the die.

4 A blue bag and a red bag each contain a large number of coins, some of which are counterfeit. One coin is selected at random from each bag.
- **a** The coin taken from the blue bag is counterfeit or not counterfeit.
- **b** Both coins are counterfeit.

5 Hartfield Airport has 100 scheduled flights due to depart on Saturday.
- **a** Two or three flights are cancelled.
- **b** One flight is cancelled because the plane is faulty and another flight is cancelled because of a hurricane at its destination.

6 A box contains six blue pens and three red pens. One pen is removed at random.
- **a** The pen is put back then a pen is removed again.
- **b** The pen is not put back and another pen is removed.

In questions **7** to **14** the events are mutually exclusive.

7 A card is selected at random from an ordinary pack of 52. What is the probability that the card is
a a red ace **b** a black king **c** a red ace or a black king?

8 Emily rolls an ordinary die once. What is the probability the number shown is
a 2 **b** 3 or 4 **c** 2, 3 or 4?

9 A card is drawn at random from the 12 court cards (jacks, queens and kings). What is the probability that the card is
- **a** a black jack
- **b** a red queen
- **c** either a black jack or a red queen?

10 Garon is looking for his house key. The probability that it is in his pocket is $\frac{5}{9}$, while the probability that it is in his car is $\frac{1}{13}$. What is the probability that
- **a** the key is either in his pocket or in his car
- **b** the key is somewhere else?

11 When Mrs George goes shopping the probability that she returns by bus is $\frac{3}{7}$, in a taxi $\frac{1}{7}$, on foot $\frac{5}{14}$. What is the probability that she returns
a by bus or taxi **b** by bus or on foot **c** by none of these ways?

12 Abigail has a bag containing discs of four different colours. One disc is removed at random. The table shows the probabilities of choosing three of the four colours.

Colour	red	white	blue	pink
Probability	$\frac{2}{7}$	$\frac{2}{9}$	$\frac{1}{4}$	

Abigail removes one disc at random. What is the probability that this disc is

a red or white
b white or blue
c red, white or blue
d pink?

13 Rajev has a pack of playing cards with some cards missing. There are 45 cards in the pack. He knows that all the clubs and hearts are in his pack. One card is drawn at random from the pack. What the probability that this card is not a club or a heart?

14 Leta rolls an ordinary die. What is the probability that the number on the die is

a an even number
b a prime number
c either even or prime?

Your answer to part **c** should not be the sum of the answers to parts **a** and **b**. Why not?

In questions **15** to **20** the events are independent.

15 Two dice are tossed. Find the probability of getting a double six.

16 Jack has two tubes of sweets. Each tube contains 10 red sweets and 30 sweets of other colours. Jack takes one sweet, chosen at random, from each tube. Find

a the probability that he takes a red sweet from a tube
b the probability that he takes a sweet that is not red from a tube
c the probability that both the sweets he takes are not red.

17 The probability that Jacinta will win the girls' 100 m race is $\frac{2}{5}$ and the probability that Liam will win the boys' 100 m race is $\frac{3}{5}$. What is the probability that

a both of them will win their events
b neither of them will win their event?

18 A mother has an equal chance of giving birth to a boy or a girl. Holly plans to have two children.

a What is the probability that the first is a girl?
b What is the probability that both are boys?
c What is the probability that neither is a boy?

19 The probability that Olivia will have to wait before she can cross Westgate Street is $\frac{1}{3}$ and the probability that she will be able to cross High Street without waiting is $\frac{1}{4}$.

What is the probability that

a she does not have to wait to cross Westgate Street
b she has to wait to cross High Street
c she can cross both streets without waiting?

20 A bag contains three red sweets and two green sweets. Alvita takes one sweet at random and eats it. She then takes another sweet, also at random.

Explain why, if both sweets removed are red, the events are not independent.

In questions **21** to **25** the events may be either independent or mutually exclusive.

21 A red die and a blue die are rolled. Find the probability of getting
- **a** a 5 or a 6 on the red die
- **b** a 1 or a 2 on the blue die
- **c** a 2 on both dice
- **d** an even number on both dice.

22 A card is drawn at random from an ordinary pack of 52 playing cards. What is the probability that the card is
a a two **b** a red ace **c** a two or a red ace?

23 When Kim goes to the cinema the probability that she returns on foot is $\frac{2}{3}$, by bus $\frac{1}{6}$ and in a friend's car $\frac{1}{6}$.

What is the probability that she returns
- **a** by bus or in a friend's car
- **b** on foot or by bus?

24 The probability that Wesley will complete the 5000 km race is 0.9 and the probability that Aaron will complete it is 0.6. What is the probability that both Wesley and Aaron will complete the 5000 km race?

25 A pack of cards is cut, reshuffled and cut again. What is the probability that
- **a** the first card cut is an ace or a king
- **b** the second card cut is an ace or a king
- **c** both cards cut are aces?

Contingency tables

Sometimes situations arise where people (or objects) are classified according to two or more sets of characteristics. Tables that show this information are called **contingency tables**.

For example, this contingency table shows one day's sales of tins of baked beans and jars of instant coffee from two supermarkets.

	Baked beans	Instant coffee	Total
Ascos	256	145	401
Blacks	494	127	621
Total	750	272	1022

This is the total number of items sold in both stores.

Tables like this can be used to find probabilities.

For example Rashid went into both supermarkets that day and bought just one tin of baked beans.

The probability that he bought it in Blacks

= (number of baked beans sold by Blacks) ÷ (number of baked beans sold by both stores)

$= \frac{494}{750} = 0.66$ correct to 2 d.p.

EXERCISE 8f

1 An investigation into the colour blindness in a mixed college gave the following information.

	Colour-blind	Not colour-blind	Total
Boys	25	725	750
Girls	20	400	420
Total	45	1125	1170

Use the information given in the table to answer the questions that follow.

What is the probability that

a a boy chosen at random is i colour-blind ii not colour-blind

b a scholar chosen at random is not colour-blind

c a girl chosen at random is colour-blind?

2 A survey of the sporting interests of the members of a sports' club provided the club officials with the preferred sport of its members.

	Cricket	Hockey	Total
Left-handed	30	15	45
Right-handed	180	140	230
Total	210	155	365

Use the information given in the table to answer the following questions:

What is the probability that

a a member chosen at random prefers i cricket ii hockey

b a member chosen at random in right-handed

c a cricketer is left-handed

d a member chosen at random is a left-handed hockey player?

3 The table gives the pass/fail details of the learners at a Driving School when they sat their driving test for the first time.

	Passed	Failed	Total
Male	*a*	15	40
Female	35	*b*	50
Total	60	30	*c*

a Copy the table and fill in the values for *a*, *b* and *c*.

b What is the probability that a male driver chosen at random failed first time?

c What is the probability that a female driver chosen at random passed first time?

d What is the probability that a learner driver chosen at random failed first time?

e One female and one male driver are selected at random from this group. What is the probability that they both passed their driving test?

4 A survey of 640 senior citizens asked each of them the question 'Do you wear a hearing aid?' Details from the survey are shown in the table.

	Wear a hearing aid	Do not wear a hearing aid	Total
Male	120	120	*a*
Female	*b*	*c*	*d*
Total	280	*e*	*f*

a Copy the table and fill in the value for each letter *a*–*f*.

b What is the probability that a female chosen at random wears a hearing aid?

c What is the probability that a male chosen at random does not wear a hearing aid?

d What is the probability that a senior citizen chosen at random wears a hearing aid?

5 This table shows the numbers of pupils in a school who either own or do not own a tablet.

	Owns a tablet	Does not own a tablet	Total
Girls	400	150	*a*
Boys	420	230	*b*
Total	820	380	*c*

a Write down the value for each of the letters *a*, *b* and *c*.

b What is the probability that a boy chosen at random owns a tablet?

c What is the probability that a girl chosen at random owns a tablet?

d What is the probability that a pupil chosen at random does not own a tablet?

e What is the probability that a pupil chosen at random owns a tablet?

6 Mrs Peacock and Mrs Essex both keep six chickens. Mrs Peacock keeps White Leghorns because she likes white eggs while Mrs Essex keeps Marans because she prefers brown eggs. The table shows the numbers of eggs their hens lay over the same period.

	Small	Medium	Large	Total
Mrs Peacock's hens	5	15	125	
Mrs Essex's hens	8	12	80	100
Total				

a Copy and complete the table.

b What is the probability that one of Mrs Peacock's eggs chosen at random was small?

c What is the probability that an egg chosen at random from all the eggs was small?

d What is the probability that an egg chosen at random from all the eggs was not a large egg?

e What is the probability that one of Mrs Essex's eggs chosen at random was a medium egg?

f Are you more likely to get a large egg from Mrs Peacock than from Mrs Essex?

g One of Mrs Essex's eggs is chosen at random. What is the probability that it is either small or medium in size?

h One of Mrs Peacock's eggs is chosen at random and one of Mrs Essex's eggs is chosen at random. What is the probability that they are both large eggs?

7 At Queenstown High School students choose one subject from the option group showing Spanish, French and German. The choices made by Third Year students are shown in the table.

Subject

	Spanish	French	German	Total
Boys	65	b	c	120
Girls	a	40	10	e
Total	115	80	d	220

a Determine the value of each letter in the table.
What is the probability that

b a boy chosen at random decides to study Spanish

c a girl chosen at random decides to study French

d a boy chosen at random does not choose French

e a girl chosen at random does not choose German

f a student chosen at random chooses French

g a student chosen at random does not choose Spanish?

8 A survey of living accommodation was conducted in two towns A and B.

Some of the results are given in the table.

	House	Bungalow	Condominium	Total
Town A		50		200
Town B	115	55		250
Total	205		140	

a Copy and complete the table.

b Of those who replied how many

i lived in a condominium

ii in Town B lived in a bungalow

iii in Town A did not live in a bungalow?

c What is the probability that if a person living in Town A is chosen at random they

i lived in house

ii did not live in a condominium?

d What is the probability that a person chosen at random lived in

i Town B

ii a house in Town B

iii a bungalow in Town A

iv a house or a bungalow in either town?

9 The members of Youth Clubs in Albany and Falmouth were asked which drink their parents preferred: tea, coffee or water? Some of the results are shown in the table.

Youth Club	Coffee	Tea	Water	Total
Albany		30		90
Falmouth			8	60
Total		42	18	

a Copy and complete the table.

b From both Youth Clubs, how many parents preferred
i tea **ii** coffee **iii** neither tea nor coffee?

c What is the probability that if a parent from Albany is chosen at random they
i prefer water **ii** prefer coffee?

d What is the probability that if a parent from Falmouth is chosen at random they prefer
i coffee **ii** tea?

e What is the probability that a parent from Albany and a parent from Falmouth, both chosen at random, prefer water?

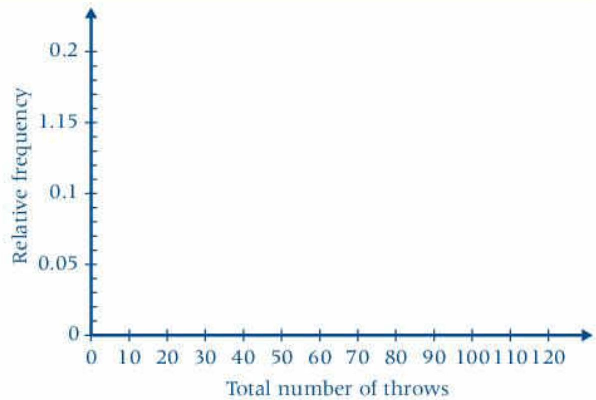
INVESTIGATION

1 a Throw an ordinary 6-sided die 10 times and count the number of times the score is 6. Work out the relative frequency of a 6, correct to 2 decimal places.

Repeat the experiment again and again until the total number of throws is 120. Record your results on a table like this:

Number of throws	10	20	30	40	50	60
Total number of 6s						
Relative frequency of a 6						
Number of throws	70	80	90	100	110	120
Total number of 6s						
Relative frequency of a 6						

Plot the results on a graph like this one:



As the number of throws gets larger, describe what happens to the relative frequency.

b What is the theoretical probability of getting a 6 when a die is thrown?
How does this value compare with the value you got in part **a**?

2 a Toss a coin 100 times and count the total number of heads after every 10 tosses. Record the results in a table like this one:

Number of tosses	10	20	30	40	50
Number of heads					
Relative frequency					
Number of tosses	60	70	80	90	100
Number of heads					
Relative frequency					

Plot the values you get for the relative frequency against the total number of tosses, on a graph like the one you drew in the previous question.

b What is the theoretical probability of getting a head when a coin is tossed?
Does your experimental value agree with this?
If it does not can you suggest a reason?

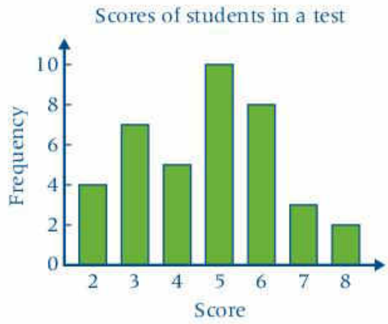
3 a If you choose a card at random from an ordinary pack of playing cards what is the probability of getting a heart?
b Shuffle an ordinary pack of 52 playing cards. Cut the pack, and note the suit of the card. Do this 40 times.
Find the relative frequency of a heart.
How does this value compare with the value you got in part **a**?
c Repeat part **b** two or three times again.

ABCD MIXED EXERCISE 8

Several answers are given for these questions.

Write down the letter that corresponds to the correct answer.

1 The mean weight of 12 students in a class is 47.3 kg. A student who weighs 48.4 kg leaves the class while a student who weighs 47 kg joins it. The mean weight of the students in the class is now
A 47 kg **B** 47.3 kg
C less than 47.3 kg **D** greater than 47.3 kg



This bar chart shows the scores of a group of students in a test. Use it for questions **2** to **4**.

2 The number of students who sat the test was
A 35 **B** 36 **C** 39 **D** 40

3 The modal score was
A 3 **B** 4 **C** 5 **D** 6

4 The median score was
A 3 **B** 4 **C** 5 **D** 6

5 In a bag containing blue and green marbles there are 14 blue marbles. If the probability of choosing a blue marble is $\frac{7}{19}$, the number of green marbles in the box is
A 24 **B** 14 **C** 12 **D** 7

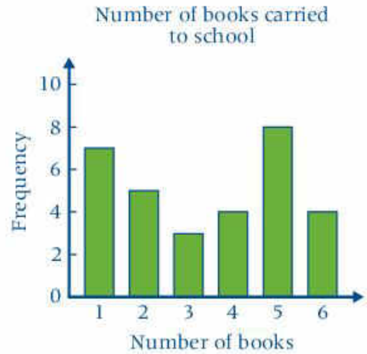
6 A letter is chosen at random from the word PARALLELOGRAM. The probability that it is L is
A $\frac{1}{13}$ **B** $\frac{2}{13}$ **C** $\frac{3}{13}$ **D** $\frac{4}{13}$

7 The pie chart shows the number of household pets owned by the families of a group of pupils. If 27 pupils belong to families with 3 pets, the total number of pupils in the group is
A 360 **B** 300 **C** 270 **D** 162

8 This frequency table shows the number of stamps on the letters received by an office one morning.

Number of stamps	1	2	3	4	5	6
Frequency	44	27	16	8	2	3

The mean number of stamps per letter was
A 1 **B** 2 **C** 2.04 **D** 2.06



This bar chart shows the number of books the pupils in a class carried to school one day. Use it to answer questions **9** and **10**.

9 Which of these statements is true?
A the mode is bigger than both the median and the mean
B the mean is bigger than the mode and the median
C the mode is smaller than the mean and the median
D the median is bigger than the mode and the mean

10 The total number of books brought to school by these pupils was
A 100 **B** 106 **C** 110 **D** 116

11 A card is chosen at random from a fair pack of 52 playing cards. The probability that it is a king or a queen is
A $\frac{1}{13}$ **B** $\frac{2}{13}$ **C** $\frac{4}{13}$ **D** $\frac{5}{26}$

12 The mean of 10 numbers is 43. One of the numbers is 52. The mean of the other numbers is
A 43 **B** 42 **C** 41 **D** 40

13 An ordinary six-sided die is tossed several times.
The scores are recorded in this table.

Score	1	2	3	4	5	6
Frequency	1	1	4	3	1	2

The relative frequency of 6 is

A $\frac{1}{6}$ **B** $\frac{2}{11}$ **C** $\frac{2}{9}$ **D** 2

14 A pie chart is drawn to represent the information in the table.

Distribution of population by age on an island (thousands)

Under 5	**5–10**	**11–20**	**21–65**	**Over 65**
26	20	30	210	74

The angle of the sector representing 11–20-year-olds is

A 12° **B** 30° **C** 120° **D** 360°

MATHS IS OUT THERE

A geologist, a doctor and a statistician are aiming darts at the bullseye on a dart board. The geologist throws first and hits the top of the board. The doctor throws next and hits the bottom of the board. The statistician says 'That's got it!'

IN THIS CHAPTER YOU HAVE SEEN THAT...

- discrete data has distinct and exact values
- continuous data can have any value within a range
- the mean of a set of values = $\frac{\text{sum of the values}}{\text{number of values}}$
- the mode is the value that occurs most often
- the median is the middle value, or the mean of the two middle values when the values have been arranged in order of size. For n values, the middle value is the $\frac{n+1}{2}$ th value
- an ungrouped frequency table lists each value against its frequency
- bar charts are used to illustrate ungrouped discrete data
- a line graph illustrates the changes in a quantity over time
- a pie chart is a circle divided into sectors. It is used to illustrate how a total is shared between different categories so that the area of each sector represents the fraction that the category is of the whole
- if a category is the fraction $\frac{a}{b}$ of the whole, the angle at the centre of the sector is $\frac{a}{b} \times 360°$
- probability is a measure of the likelihood of an outcome.
- theoretical probability that an event E occurs is given by
  $P(E) = \frac{\text{number of outcomes which result in E occurring}}{\text{the number of equally likely outcomes}}$

  This can be given as a fraction, a decimal or a percentage.
- experimental probability is the relative frequency of an outcome when an experiment is repeated several times where
  relative frequency = $\frac{\text{number of times that outcome occurs}}{\text{total number of outcomes}}$
- mutually exclusive events cannot both happen
- independent events have no influence on each other.

9 ALGEBRA 2

AT THE END OF THIS CHAPTER YOU SHOULD BE ABLE TO...

1. Work with operations that are defined.
2. Multiply out a pair of brackets.
3. Factorise quadratic expressions.
4. Solve quadratic equations by factorisation, completing the square and by using the formula.
5. Change the subject of a formula.

BEFORE YOU START you need to know:
- ✓ how to simplify algebraic expressions
- ✓ the laws of indices
- ✓ how to factorise an algebraic expression when the terms have a common factor.

KEY WORDS counter-example, difference between two squares, expand, operation, quadratic expression, roots, subject of a formula

MATHS IS OUT THERE

al'Khwarizmi who lived from about AD 780 to AD 850, was an Arab mathematician who worked in Baghdad. His most important book introduced what became known as **algebra**. The word derives from *al-jabr*, which was the most important word in the title of his book.

Binary operations

A **binary operation** is a rule for combining two elements to produce another element. The addition of real numbers is an example of an operation because it combines two numbers to give another number, for example
$2.1 + 3.6 = 5.7$
The union and intersection of sets are further examples of operations.

We can define other operations. For example, if a and b are real numbers, we can define an operation such as: combine a and b to give ab^2...

Any symbol can be used to denote an operation as long as it does not have a known meaning. We can write $a * b$ means ab^2.

The operations of addition, subtraction, multiplication and division of real numbers are familiar rules for combining numbers.

EXERCISE 9a

Example:

a and b are real numbers and $a * b = ab^2$.

a Find **i** $3 * 4$ **ii** $(3 * 4) * 2$

b Explain whether this operation is **i** commutative **ii** associative.

c Find the value of c given that $3 * c = 75$

a i $3 * 4 = 3 \times 4^2 = 3 \times 16 = 48$
ii $(3 * 4) * 2 = 48 * 2 = 48 \times 2^2 = 48 \times 4 = 192$

Substitute 3 for a and 4 for b in $a * b = ab^2$.

b i $3 * 4 = 48$ and $4 * 3 = 4 \times 3^2 = 36$
$3 * 4 \neq 4 * 3$ so this operation is not commutative.
ii $(3 * 4) * 2 = 192$
$3 * (4 * 2) = 3 * (4 \times 2^2) = 3 * 16 = 3 \times 16^2 \neq 192$
so this operation is not associative.

Alternatively, if the operation is commutative then $a * b = b * a$, i.e. $ab^2 = ba^2$ which is true only if $a = b$.
The easiest way to show that a statement is untrue is to use a numerical example that demonstrates it is untrue. This is called a **counter-example**.

c $3 * c = 3c^2$
$\therefore 3c^2 = 75$
$c^2 = 25$ so $c = \pm 5$

Substitute 3 for a and c for b in $a * b = ab^2$.
Remember that a positive number has two square roots, one positive and the other negative.

In this exercise the letters stand for real numbers.

1 If $a \oplus b$ means $2a - b$ find
a $2 \oplus 1$ **b** $2 \oplus 2$ **c** $\frac{1}{2} \oplus 3$

2 Given that $p \, \Delta \, q$ means $p^2 - q^2$ find
a $3 \, \Delta \, 2$ **b** $4 \, \Delta \, 3$ **c** $-3 \, \Delta \, 2$

3 Given that $x * y$ means $2x^2 + y^2$ find
a $-2 * 3$ **b** $\frac{1}{2} * \frac{3}{4}$ **c** $-2 * -4$

4 If $a \otimes b$ means $\frac{ab}{a + b}$ find
a $2 \otimes 4$ **b** $-3 \otimes 4$ **c** $\frac{1}{2} \otimes \frac{3}{4}$

5 If $p \bullet q$ means $2p + q$ find
a $2 \bullet 3$ **b** $(2 \bullet 3) \bullet 4$ **c** $2 \bullet (4 \bullet 3)$

6 Given that $a \, \omega \, b$ means a^2b^2 find
a $3 \, \omega \, 2$ **b** $(3 \, \omega \, 2) \, \omega \, 4$ **c** $3 \, \omega \, (2 \, \omega \, 4)$

7 Given that $x \wedge y$ means xy^2 find
a $3 \wedge 2$ **b** $(3 \wedge 2) \wedge -1$ **c** $3 \wedge (2 \wedge -1)$

8 If $5 \, \Omega \, 2$ means $\frac{x}{2y}$ find
a $5 \, \Omega \, 2$ **b** $(5 \, \Omega \, 2) \, \Omega \, -1$ **c** $5 \, \Omega \, (2 \, \Omega \, -1)$

9 If $p \blacktriangleright q$ means $3p + 2q$ find
a i $2 \blacktriangleright 4$ ii $(2 \blacktriangleright 4) \blacktriangleright 2$ **b** i $(2 \blacktriangleright 4) \blacktriangleright 3$ ii $2 \blacktriangleright (4 \blacktriangleright 3)$

10 If $a * b$ means $(a - b)^2$ find
a i $5 * 3$ ii $3 * 5$ iii $(5 * 3) * 2$
b Explain whether this operation is i commutative ii associative.

To show that an operation is not commutative or associative all you need is one numerical example that shows this. However to prove that it is, you need to show the general case, i.e. in terms of the letters given.

11 $a \sim b$ means 'the difference between the two numbers'.
a Find i $3 \sim 2$ ii $2 \sim 3$ iii $(2 \sim 3) \sim 4$
b Explain whether this operation is i commutative ii associative.

The difference between two numbers is always positive, e.g. the difference between 5 and 8 is 3

12 $p \leftrightarrow q$ means 'the larger of p and q'.
  a Find i $7 \leftrightarrow 9$ ii $9 \leftrightarrow 7$ iii $(7 \leftrightarrow 9) \leftrightarrow 11$
  b Explain whether this operation is i commutative ii associative.

13 $a \circ b$ means $(a - 2b)$
  a Find i $4 \circ 2$ ii $2 \circ 4$ iii $(4 \circ 2) \circ 3$ iv $4 \circ (2 \circ 3)$
  b Explain whether this operation is i commutative ii associative.

14 $c \veebar d$ means the sum of c and d
  a Find i $7 \veebar 4$ ii $4 \veebar 7$ iii $(4 \veebar 7) \veebar 3$
  b Explain whether this operation is i commutative ii associative.

15 $p \odot q$ means $\frac{2p}{q}$
  a Find i $5 \odot 4$ ii $4 \odot 5$ iii $(5 \odot 4) \odot 2$ iv $5 \odot (4 \odot 2)$
  b Explain whether this operation is i commutative ii associative.

16 $a \blacktriangle b$ means a^2b
  a Find $2 \blacktriangle 6$ b Find c if $c \blacktriangle 2 = 18$

17 $a \nabla b$ means $3a - 4b$
  a Find $5 \nabla 2$ b Find k if $k \nabla 3 = 6$

18 $p \lozenge q$ means 'the cube of the difference between p and q'.
  a Find $4 \lozenge 3$ b Find c if $c \lozenge 5 = 8$

19 $c \bigstar d$ means $\frac{1}{c} + \frac{1}{d}$
  a Find $2 \bigstar 3$ b Find e if $3 \bigstar e = \frac{7}{12}$

20 a and b are positive whole numbers. $a \sqcup b$ means $\frac{a}{b} - \frac{b}{a}$
  a Find $5 \sqcup 3$ b Find c when $c \sqcup 2 = 1.5$

21 p and q are whole numbers. $p \sqcap q$ means $p + q^2$
  a Find $4 \sqcap 6$ b Find x if $x \sqcap 4 = 20$ c Find y if $3 \sqcap y = 28$

22 $a \circledcirc b$ means the positive square root of the product of a and b.
  a Find $4 \circledcirc 9$ b Find c if $c \circledcirc 16 = 8$ c Find y if $9 \circledcirc y = 3$

23 Given $x * y = xy$ determine
  a $2 * (3 * 5)$ b whether the operation is associative.

24 If $x \wedge y = \frac{xy}{x^2 - y^2}$ show that $x \wedge y$ is not commutative.

25 Is the operation $x * y = x^2 + 4y^2 - 4xy$ commutative or non-commutative?

  Justify your answer.

The product of two brackets

The product $(a + b)(c + d)$ means that every term in the first bracket is multiplied by every term in the second bracket.

The process of finding this product is called multiplying out, or **expanding**, the brackets.

Expanding is best done in this order:

$(a + b)(c + d) = ac + ad + bc + bd$ (① $a \times c$, ② $a \times d$, ③ $b \times c$, ④ $b \times d$)

It can also be done using a table:

	c	d
a	ac	ad
b	bc	bd

$= ac + ad + bc + bd$

EXERCISE 9b

Example:

Expand $(2p + q)(p - 2r)$

$(2p + q)(p - 2r) = 2p^2 - 4pr + qp - 2qr$

Multiply $2p$ by each term in the second bracket then multiply q by each term in the second bracket.

Expand

1 $(x + y)(x + 3z)$ **2** $(x + 2y)(x + 4z)$ **3** $(x + 3y)(x + 5z)$

4 $(x + 4y)(x + 5z)$ **5** $(a + 5b)(a + 7c)$ **6** $(a + 4b)(a + c)$

7 $(a + 3b)(a + 6c)$ **8** $(a + 2b)(a + 9c)$ **9** $(x - 2y)(x - 3z)$

10 $(x - 4y)(x - 5z)$ **11** $(x - 3y)(x - z)$ **12** $(x - 7y)(x - 6z)$

13 $(x + 2y)(x - 7z)$ **14** $(a - 3b)(a + 5b)$ **15** $(a + 10b)(a - c)$

16 $(a - 5c)(a + 6b)$ **17** $(ax - c)(bx + 3d)$ **18** $(p + 2q)(r - 4s)$

Example:

Expand

a $(x + 3)(x - 5)$ **b** $(2x - 7)(3x - 1)$ **c** $(5 - x)(2 + x)$

a $(x + 3)(x - 5) = x^2 - 5x + 3x - 15$

$= x^2 - 2x - 15$

b $(2x - 7)(3x - 1) = 6x^2 - 2x - 21x + 7$

$= 6x^2 - 23x + 7$

c $(5 - x)(2 + x) = 10 + 5x - 2x - x^2$

$= 10 + 3x - x^2$

Simplify

Remember that the product of two negative numbers is positive.

With practice you should be able to miss out the intermediate step and just write down the answer.

Expand

19 $(x + 5)(x - 7)$ **20** $(a + 4)(a - 1)$ **21** $(x - 3)(x + 5)$

22 $(a - 2)(a + 7)$ **23** $(x - 2)(x + 3)$ **24** $(x - 4)(x - 5)$

25 $(x + 3)(x - 1)$ **26** $(x - 7)(x - 6)$ **27** $(x - 2)(x - 7)$

28 $(a + 3)(a - 5)$ **29** $(a - 10)(a - 1)$ **30** $(a + 5)(a - 6)$

31 $(x + 1)(x + 9)$ **32** $(x + 9)(x - 7)$ **33** $(x + 9)(x + 8)$

34 $(2x + 5)(x - 3)$ **35** $(x + 12)(3x - 2)$ **36** $(2p - 11)(p + 2)$

37 $(4n + 6)(5n - 7)$ **38** $(7a - 5)(2a + 1)$ **39** $(8x - 3)(x + 15)$

40 $(5q + 3)(3q - 10)$ **41** $(7c - 8)(c + 1)$ **42** $(5p + 7)(5p - 1)$

43 $(4a - 3)(3a + 10)$ **44** $(10b - 7)(2b - 9)$ **45** $(4q - 9)(5q + 2)$

46 $(5p - 3)(p + 12)$ **47** $(2p - 7)(7p - 8)$ **48** $(x + 6)(x - 11)$

49 $(5 - x)(7 - x)$ **50** $(3 + a)(5 - a)$ **51** $(10 - 5a)(1 - 2a)$

52 $(5 + 4a)(6 - a)$ **53** $(x - 1)(9 + 2x)$ **54** $(6 - x)(x - 7)$

55 $(9 - 2x)(8 + x)$ **56** $(3 + 5m)(7 - m)$ **57** $(5a - 8b)(a - 6b)$

Example:

Expand

a $(2x + 3)^2$ **b** $(2x - 3)^2$ **c** $(2x + 3)(2x - 3)$

a $(2x + 3)^2 = (2x + 3)(2x + 3)$
$= 4x^2 + 6x + 6x + 9 = 4x^2 + 12x + 9$

b $(2x - 3)^2 = (2x - 3)(2x - 3)$
$= 4x^2 - 6x - 6x + 9 = 4x^2 - 12x + 9$

c $(2x + 3)(2x - 3) = 4x^2 - 6x + 6x - 9$
$= 4x^2 - 9$

These three examples are special cases of the general results:
$(x + a)^2 = x^2 + 2xa + a^2$
$(x - a)^2 = x^2 - 2xa + a^2$
$(x + a)(x - a) = x^2 - a^2$
You can use these general cases to find $(2x + 3)^2$ by substituting $2x$ for x and 3 for a in $x^2 + 2xa + a^2$:
$(2x)^2 + (2x)(3) + 3^2 = 4x^2 + 6x + 9$

Expand

58 $(2x + 5)^2$

59 $(2x - 5)^2$

60 $(2x + 5)(2x - 5)$

61 $(5n + 6)^2$

62 $(7a - 5)^2$

63 $(8x - 3)(8x + 3)$

64 $(5q + 3)(5q - 3)$

65 $(7c - 8)^2$

66 $(3x - 7)^2$

67 $(3x + 7)^2$

68 $(3x + 7)(3x - 7)$

69 $(6n + 5)^2$

70 $(7p - 4)^2$

71 $(4x - 7)(4x + 7)$

72 $(5q + 4)(5q - 4)$

73 $(5q - 11)^2$

74 $(2x + 3)^2 - 12x$

75 $(2x - 1)^2 + 4x$

76 $(2x + 5)(2x - 5) + 25$

77 $(5n + 4)^2 + 30n$

78 $(6a - 5)(3a - 4) + 15a$

79 $(5x - 3)(8x + 3) - 24x$

80 $(2x - 3)(3x + 4) - 5(x + 3)$

81 $(7x - 8)(2x - 3) + 4(x + 2)$

82 $3(x - 2)^2$

83 $5(x - 2)(x + 2)$

84 $4 - 5(x - 2)^2$

85 $4(2x - 1)(x + 2)$

86 $3(x - 4)^2 - 2(x - 3)$

87 $7x + 5(x + 4)(x - 4)$

88 $5(x - 1)^2 + 3(x - 2)^2$

89 $3(2x + 1)(3x - 1) - (x - 1)^2$

Factorising by grouping

Expressions such as $pq + pr + sq + sr$ can be factorised by grouping the terms in pairs; the first two terms have a common factor p and the second two terms have a common factor s, so

$\overline{pq + pr} + \overline{sq + sr} = p(q + r) + s(q + r).$

Now we can see that $(q + r)$ is a common factor of the expression, so

$pq + pr + sq + sr = (q + r)(p + s).$

EXERCISE 9c

Example:

Factorise

a $2p - 2pq - q + q^2$ **b** $sc - sd + td - tc$

a $2p - 2pq - q + q^2 = 2p(1 - q) - q(1 - q)$
$= (1 - q)(2p - q)$

b $sc - sd + td - tc = s(c - d) + t(d - c)$
$= s(c - d) - t(c - d) = (c - d)(s - t)$

$-q \times -q = q^2$

$(d - c) = -(c - d)$

Factorise

1 $ab + 3a + 3b + 9$

2 $ab + b + ac + c$

3 $x^2 + xy + xz + yz$

4 $ab - ac + db - dc$

5 $ab - 5a + 2b - 10$

6 $pq - 3q + 4p - 12$

7 $ab - ac - db + dc$

8 $x^2 + 2xy - 2x - 4y$

9 $a - ab + b - b^2$

10 $2ab + 2a - b^2 - b$

11 $2s - 5t - 2st + 5t^2$

12 $6y^2 - 9y - 2xy + 3x$

13 $ab - ac - 3c + 3b$

14 $2p - pq + 3q - 6$

15 $3x - xy + 4y - 12$

16 $x^2 - xy + x - y$

17 $3p^2 + pq - 3p - q$

18 $ab - a - b + 1$

Factorising quadratic expressions

A **quadratic expression** has the form $ax^2 + bx + c$ where $a \neq 0$.

When $(x + 4)(x + 2)$ is expanded it gives the quadratic expression $x^2 + 6x + 8$. We can sometimes reverse this process by starting with a quadratic expression and expressing it as the product of two factors. We do this by observing the pattern when two brackets are expanded:

$$(mx + n)(px + q) = mpx^2 + (mq + np)x + nq$$

This is the product of the x-terms in each bracket.

This is the sum of the products of the x-term in each bracket and the number term in the other bracket.

This is the product of the number terms in each bracket.

- When n and q are both positive, the signs in the quadratic are all positive.

Thus to factorise $x^2 + 5x + 6$, we know that the signs in both brackets are + and we want two numbers whose product is 6 and whose sum is 5, so $x^2 + 5x + 6 = (x + 2)(x + 3)$.

- When n and q are both negative, the middle term in the quadratic is negative and the last term is positive.

Thus to factorise $x^2 - 5x + 6$, we know that the signs in both brackets are − and we want two numbers whose product is 6 and whose sum is 5, so $x^2 - 5x + 6 = (x - 2)(x - 3)$.

- When either n or q is negative and the other is positive, the last term in the quadratic is negative and the middle term may be either positive or negative.

Thus to factorise $x^2 - 5x - 6$, we know that the sign in one bracket is + and the sign in the other bracket is −. This time we want two numbers whose product is −6 and whose sum is −5, so $x^2 - 5x - 6 = (x - 6)(x + 1)$.

EXERCISE 9d

Example:

Factorise

a $x^2 + 13x - 30$ **b** $4 + x^2 - 5x$ **c** $6 + x - x^2$

a $x^2 + 13x - 30 = (x + 15)(x - 2)$

We need two numbers whose product is −30 and whose sum is 13: 15 and −2

b $4 + x^2 - 5x = x^2 - 5x + 4$
$= (x - 4)(x - 1)$

First rearrange to the form $ax^2 + bx + c$.

c $6 + x - x^2 = (3 - x)(2 + x)$

When the x^2 term is negative, treat it as the last term. We need two numbers whose product is 6 and whose difference is 1. Check that the signs are the right way round by expanding the brackets.

Factorise

1 $x^2 + 12x + 35$ **2** $x^2 - 10x + 24$ **3** $x^2 + 10x + 21$

4 $x^2 - 7x + 12$ **5** $x^2 + 12x + 32$ **6** $x^2 + 2x - 15$

7 $x^2 + 4x - 32$ **8** $x^2 - 12x + 35$ **9** $x^2 + 13x - 30$

10 $x^2 - 22x + 120$ **11** $x^2 - 7x + 10$ **12** $x^2 - 2x - 63$

13 $x^2 + 11x - 26$ **14** $x^2 - 2x - 35$ **15** $x^2 + 6x - 91$

16 $x^2 - 14x - 51$ **17** $x^2 - 13x + 36$ **18** $x^2 + 5x - 36$

19 $24 + x^2 - 10x$ **20** $9 + x^2 - 6x$ **21** $10 + x^2 - 7x$

22 $15 + x^2 + 8x$ **23** $63 + x^2 - 16x$ **24** $40 + x^2 + 13x$

25 $10 + x^2 - 11x$ **26** $28 + x^2 - 11x$ **27** $1 - x - 2x^2$

28 $14 - 5x - x^2$ **29** $6 - x - x^2$ **30** $35 + 2x - x^2$

31 $63 + 2x - x^2$ **32** $10 - 9x - x^2$ **33** $84 + 5x - x^2$

34 $72 - x - x^2$ **35** $30 + x - x^2$ **36** $70 + 3x - x^2$

Always check your answer by expanding the brackets. You can do this in your head if you are confident that you will not make mistakes.

Example:

Factorise

a $x^2 - 25$ **b** $4x^2 - 9y^2$

a $x^2 - 25 = (x - 5)(x + 5)$

b $4x^2 - 9y^2 = (2x - 3y)(2x + 3y)$

Both these examples have the form $a^2 - b^2$. This is called the **difference between two squares**. We know that the expansion of $(a - b)(a + b)$ gives $a^2 - b^2$ so we can use this to factorise the difference between two squares.

Factorise

37 $x^2 - 36$ **38** $x^2 - 49$ **39** $x^2 - 100$

40 $x^2 - 1$ **41** $x^2 - 121$ **42** $x^2 - 64$

43 $x^2 - 4$ **44** $a^2 - 1$ **45** $p^2 - 81$

46 $x^2 - 144$ **47** $4x^2 - 49$ **48** $9x^2 - 25$

49 $25x^2 - 36$ **50** $9x^2 - 100$ **51** $4x^2 - 25y^2$

52 $9x^2 - 16y^2$ **53** $4x^2 - 49y^2$ **54** $36p^2 - 49q^2$

55 $100a^2 - 81b^2$ **56** $121x^2 - 64y^2$ **57** $\frac{x^2}{4} - \frac{y^2}{9}$

58 $4x^2 - \frac{y^2}{4}$ **59** $\frac{a^2}{25} - \frac{b^2}{16}$ **60** $\frac{4x^2}{25} - 1$

Example:

Factorise $2x^2 - 8x + 8$

$$2x^2 - 8x + 8 = 2(x^2 - 4x + 4)$$
$$= 2(x - 2)(x - 2) = 2(x - 2)^2$$

Each term has a common factor of 2. Start by taking out this factor, then factorise the quadratic.

Factorise

61 $2x^2 - 12x + 18$ **62** $12x^2 + 10x - 12$ **63** $20x^2 - 80x + 75$

64 $3x^2 + 6x - 8$ **65** $3x - 2 + 9x^2$ **66** $16 + 30x^2 - 52x$

67 $5x^2 - 125$ **68** $4x^2 - 36$ **69** $30 - 4x - 2x^2$

70 $36 - 6x - 6x^2$ **71** $2x^2 + 126 - 16x$ **72** $5x^2 + 5x - 10$

When neither the coefficient of x^2 nor the number term is 1, we can find the factors of $ax^2 + bx + c$ by finding two numbers whose product is ac and whose sum is b.

Example:

Factorise

a $2x^2 + 7x + 5$ **b** $6x^2 - 17x + 12$ **c** $8 - 2x - 15x^2$

a $2x^2 + 7x + 5 = 2x^2 + 5x + 2x + 5$
$= x(2x + 5) + (2x + 5)$
$= (2x + 5)(x + 1)$

We want two numbers whose product is 10 and whose sum is 7: these are 5 and 2. We start by splitting the x term into $5x$ and $2x$.
Then we factorise the first two terms and the second two terms. This gives a common factor of $(2x + 5)$ which we can take out.

b $6x^2 - 17x + 12 = 6x^2 - 9x - 8x + 12$
$= 3x(2x - 3) - 4(2x - 3)$
$= (2x - 3)(3x - 4)$

The two numbers whose product is 72 and whose sum is -17 are -9 and -8. When factorising the second pair of terms we want the terms in the bracket to be the same as the terms in the first bracket, so take -4 out. Remember that to get $+12$. we multiply -4 by -3.

c $8 - 2x - 15x^2 = 8 + 10x - 12x - 15x^2$
$= 2(4 + 5x) - 3x(4 + 5x)$
$= (4 + 5x)(2 - 3x)$

We want two numbers whose product is -120 and whose sum is -2. These are 10 and -12

Factorise

73 $2x^2 + 7x - 15$ **74** $6x^2 - 13x - 5$ **75** $2x^2 - 9x + 4$

76 $2x^2 - 5x - 3$ **77** $3x^2 - 14x + 8$ **78** $4x^2 - 4x - 15$

79 $9x^2 - 9x - 10$ **80** $3x^2 - 28x + 49$ **81** $35x^2 - 11x - 6$

82 $4x^2 - x - 3$ **83** $6x^2 + 19x - 7$ **84** $12x^2 + 16x - 3$

85 $40x^2 - 17x - 12$ **86** $20x^2 + x - 12$ **87** $8x^2 + 30x - 27$

88 $4x^2 + 12x - 27$ **89** $16x^2 - 40x + 21$ **90** $81x^2 - 27x + 2$

91 $19x + 14x^2 - 3$ **92** $14x + 8x^2 - 15$ **93** $64x^2 + 7 + 64x$

Example:

Factorise

a $9 - (x + 1)^2$ **b** $2(x - 2)^2 - x + 2$

a $9 - (x + 1)^2 = (3 - (x + 1))(3 + (x + 1))$
$= (3 - x - 1)(3 + x + 1)$
$= (2 - x)(4 + x)$

b $2(x - 2)^2 - x + 2 = 2(x - 2)^2 - (x - 2)$
$= (x - 2)(2x - 4 - 1)$
$= (x - 2)(2x - 5)$

$9 - (x + 1)^2$ is the difference between two squares.

$-x + 2$ can be written as $-(x - 2)$.
Then $(x - 2)$ is a common factor.

Factorise

94 $4 - (x + 2)^2$ **95** $1 - (2x + 1)^2$ **96** $(2x - 1)^2 - 9$

97 $(x - 3)^2 - 16$ **98** $9(x + 1)^2 - 4$ **99** $25 - 9(2x - 1)^2$

100 $3(x + 2)^2 + x + 2$ **101** $(3x - 1)^2 + 6x - 2$ **102** $(5x + 2)^2 - 5x - 2$

103 $(x - 1)^2 - 2x + 2$ **104** $9(x - 3)^2 - 25$ **105** $4(x + 1)^2 - 9(x - 1)^2$

Example:

Show that $x^2 - 4x + 7 = (x - 2)^2 + 3$

RHS $= (x - 2)^2 + 3 = x^2 - 4x + 4 + 3$
$= x^2 - 4x + 7 =$ LHS
$\therefore x^2 - 4x + 7 = (x - 2)^2 + 3$

This is an example of an algebraic identity. The right-hand side is another way of writing the left-hand side.
To show that the two sides are identical, start with one side and rearrange it. In this case, the right-hand side is easier to work with.

Show that

106 $\frac{1}{x} - \frac{1}{y} = \frac{y - x}{xy}$ **107** $(x - 2)^2 - (x + 4)^2 = -12(x + 1)$

108 $\frac{a^2 - b^2}{a - b} = a + b$ **109** $x^2 + 6x + 4 = (x + 3)^2 - 5$

110 $\frac{1}{(x + 2)(x + 1)} = \frac{1}{(x + 1)} - \frac{1}{(x + 2)}$

Simplification of algebraic expressions

When the numerator and/or the denominator of an algebraic fraction is a quadratic expression, and it can be factorised, it may be possible to cancel a common factor.

EXERCISE 9e

Example:

Simlify

a $\dfrac{a+1}{a^2-1}$ b $\dfrac{2x^2-5x+2}{x-2}$

a $\dfrac{a+1}{a^2-1} = \dfrac{\cancel{(a+1)}^1}{(a-1)\cancel{(a+1)}^1}$

$= \dfrac{1}{a-1}$

b $\dfrac{2x^2-5x+2}{x-2} = \dfrac{(2x-1)\cancel{(x-2)}^1}{\cancel{(x-2)}^1}$

$= 2x-1$

Factorise the denominator and put brackets round the numerator. (This makes it easier to see common factors.)

When the denominator is 1, there is no need to write it as a fraction.

Simplify

1 $\dfrac{x+y}{x^2-y^2}$
2 $\dfrac{x-2}{x^2-x-6}$
3 $\dfrac{x-4}{x^2+2x-24}$
4 $\dfrac{x-4}{x^2+3x-28}$
5 $\dfrac{a+1}{a^2+3a+2}$
6 $\dfrac{2x+1}{2x^2+5x+2}$
7 $\dfrac{x-7}{3x^2-23x+14}$
8 $\dfrac{3x+9}{x^2+x-6}$
9 $\dfrac{2x^2-5x-3}{2x+1}$
10 $\dfrac{6x^2+13x-5}{2x+5}$
11 $\dfrac{49x^2+7x-2}{7x-1}$
12 $\dfrac{20x^2+23x-21}{4x+7}$
13 $\dfrac{6x^2+13x+5}{2x+1}$
14 $\dfrac{x^2-x-12}{x+3}$
15 $\dfrac{2x^2-13x-7}{x-7}$
16 $\dfrac{2x^2+7x-15}{x^2+x-20}$
17 $\dfrac{6x^2-19x+10}{3x^2+19x-14}$
18 $\dfrac{6x^2-13x-5}{8x^2-18x-5}$

Quadratic equations

A quadratic equation has the form $ax^2 + bx + c = 0$.

There are at most two values of x that satisfy a quadratic equation. These values are called the **roots** or solutions of the equation.

The process of solving a quadratic equation starts, if necessary, by first rearranging the equation so that it is in the form $ax^2 + bx + c = 0$.

There are three algebraic methods for solving a quadratic equation.

Quadratic equations can also be solved graphically. This method is covered in Chapter 14.

Solution by factorisation

When $ax^2 + bx + c$ factorises, the equation becomes the product of two factors that are equal to zero.

For example, $x^2 - 7x + 6 = 0$ becomes $(x-6)(x-1) = 0$.
The product of two numbers is zero if and only if one of the numbers is itself zero.
Hence $(x-6)(x-1) = 0$ only if $x - 6 = 0$ or if $x - 1 = 0$,
i.e. if $x = 6$ or if $x = 1$.

Not all quadratic expressions factorise.
$ax^2 + bx + c$ will factorise only if two rational numbers exist whose product is ac and whose sum is b. When no such numbers exist, the quadratic expression does not factorise.

EXERCISE 9f

Example:

Solve the equation $11x - 2x^2 = 12$

$11x - 2x^2 = 12$

$0 = 12 - 11x + 2x^2$

Start by rearranging the equation to the form $ax^2 + bx + c = 0$.

$2x^2 - 11x + 12 = 0$

$(2x - 3)(x - 4) = 0$

Factorise.

either $2x - 3 = 0 \Rightarrow x = \frac{3}{2}$

or $x - 4 = 0 \Rightarrow x = 4$

The symbol $\Rightarrow$ means 'gives' or 'implies'.

The solutions are $x = \frac{3}{2}$ and $x = 4$

Solve the equations

1 $x^2 + 4x - 45 = 0$ **2** $x^2 - 19x + 84 = 0$ **3** $2x^2 - 9x + 4 = 0$

4 $2x^2 - 5x - 3 = 0$ **5** $3x^2 - 14x + 8 = 0$ **6** $3x^2 + 7x + 2 = 0$

7 $6x^2 + x - 2 = 0$ **8** $6x^2 - 7x - 5 = 0$ **9** $10x^2 - 13x - 3 = 0$

10 $8x^2 + 14x - 15 = 0$ **11** $14x^2 + 19x - 3 = 0$ **12** $6x^2 - 23x + 15 = 0$

13 $9x^2 - 39x + 36 = 0$ **14** $15x - 2x^2 = 27$ **15** $49 + 3x^2 = 28x$

16 $7 + 11x - 6x^2 = 0$ **17** $4x^2 - 4x = 15$ **18** $9x + 10 = 9x^2$

19 $8 - 12x^2 = 10x$ **20** $18x^2 - 39x = 45$ **21** $6x^2 + 4 = 11x$

Solution by completing the square

Not all quadratic equations will factorise but this does not mean that such equations have no solutions.

We can solve the equation $(x - 1)^2 = 20$ by taking the square root of both sides, i.e. $x - 1 = \pm\sqrt{20} \Rightarrow x = 1 \pm\sqrt{20}$

When we cannot factorise, we can solve a quadratic equation by transforming it into a form like $(x - 1)^2 = 20$

Consider $x^2 - 6x - 3 = 0$

Start by rearranging so that the x^2 and x terms are on one side of the equals sign and the number is on the other side: $x^2 - 6x = 3$

Now add ($\frac{1}{2}$ the coefficient of x)^2 to both sides to make the left-hand side a perfect square:

$$x^2 - 6x + 9 = 12 \Rightarrow (x - 3)^2 = 12$$

This process is called 'completing the square' and it uses the fact that $(x + a)^2 = x^2 + 2ax + a^2$ i.e. provided that the coefficient of x is 1, the number term in the expansion is the square of half the coefficient of x.

We can now solve the equation by taking the square root of both sides:

$x - 3 = \pm\sqrt{12} \Rightarrow x = 3 - \sqrt{12}$ and $x = 3 + \sqrt{12}$

$x = 3 - 3.464...$ and $x = 3 + 3.464...$

$x = -0.46$ and 6.46 correct to 2 decimal places.

EXERCISE 9g

Solve these equations. Give answers correct to 3 significant figures.

1 $(x - 2)^2 = 8$ **2** $(x + 1)^2 = 5$ **3** $(2x - 3)^2 = 6$

4 $(x - 1)^2 - 3 = 0$ **5** $(6x - 5)^2 - 5 = 0$ **6** $(x + 4)^2 + 3 = 8$

Example:

a Write $2x^2 + 8x + 3$ in the form $a(x + b)^2 + c$.

b Hence solve the equation $2x^2 + 8x + 3 = 0$ giving answers correct to 2 decimal places.

a $2x^2 + 8x + 3 = a(x + b)^2 + c$

$= a(x^2 + 2bx + b^2) + c$

$= ax^2 + 2abx + ab^2 + c$

$\therefore \quad a = 2$

$2ab = 8$ so $b = 2$

and $ab^2 + c = 3$ so $2(2)^2 + c = 3$ so $c = -5$

$\therefore \quad 2x^2 + 8x + 3 = 2(x + 2)^2 - 5$

Since $a(x + b)^2 + c$ is another way of writing $2x^2 + 8x + 3$, the two expressions are identical. This means that they must have the same number of x^2s, the same number of xs and the same number term.

Expanding and simplifying $a(x + b)^2 + c$ means we can then compare these terms and hence find the values of a, b and c.

b $2x^2 + 8x + 3 = 0$

$2(x + 2)^2 - 5 = 0$

$2(x + 2)^2 = 5$

$(x + 2)^2 = 2.5$

$x + 2 = \pm\sqrt{2.5}$

$x = -2 + 1.581...$ and $x = -2 - 1.581...$

$x = -0.42$ and $x = -3.58$ correct to 2 d.p.

In questions **7** to **12**, write each expression in the form $a(x + b)^2 + c$.

7 $2x^2 + 12x + 5$ **8** $4x^2 + 16x + 13$ **9** $2x^2 - 12x + 23$

10 $5x^2 - 50x + 20$ **11** $7x^2 + 70x - 50$ **12** $4x^2 - 4x - 3$

In questions **13** to **18**, write each equation in the form $a(x + b)^2 + c = 0$.

Hence solve the equation.

13 $2x^2 - 12x + 9 = 0$ **14** $3x^2 - 18x + 7 = 0$ **15** $4x^2 + 16x - 5 = 0$

16 $7x^2 + 84x - 48 = 0$ **17** $16x^2 - 24x - 15 = 0$ **18** $5x^2 + 10x - 4 = 0$

Solve the following equations using the method of completing the square.

19 $x^2 + 4x = 9$ **20** $x^2 - 6x = 5$ **21** $2x^2 + 4x = 5$

Solution by the formula

We can use the method of completing the square to solve the general quadratic equation $ax^2 + bx + c = 0$

$$x^2 + \frac{b}{a}x + \frac{c}{a} = 0$$

Divide both sides by a then subtract $\frac{c}{a}$ from both sides.

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = -\frac{c}{a} + \left(\frac{b}{2a}\right)^2$$

Add ($\frac{1}{2}$ the coefficient of x)^2 to both sides to complete the square on the left-hand side.

$$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$$

$$\left(x + \frac{b}{2a}\right)^2 = \frac{-4ac + b^2}{4a^2}$$

Combine the fractions on the right-hand side.

$$x + \frac{b}{2a} = \pm\sqrt{\frac{-4ac + b^2}{4a^2}}$$

Take the square root of both sides.

$$x = -\frac{b}{2a} \pm \sqrt{\frac{-4ac + b^2}{4a^2}}$$

Subtract $\frac{b}{2a}$ from both sides.

$$= -\frac{b}{2a} \pm \sqrt{\frac{b^2 - 4ac}{2a}}$$

Simplify the right-hand side.

$$\therefore \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

We can now use this to solve any quadratic equation provided that $b^2 - 4ac \geq 0$.

If $b^2 - 4ac < 0$, $\sqrt{b^2 - 4ac}$ is the square root of a negative number which does not exist as a real number. When this occurs, the equation has no real roots.

EXERCISE 9h

Example:

Solve the equation $3x^2 - 2x - 4 = 0$

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(3)(-4)}}{2(3)} = \frac{2 \pm \sqrt{4 + 48}}{6}$$

$$= \frac{2 \pm \sqrt{52}}{6} = \frac{2 + 7.211...}{6} \text{ or } \frac{2 - 7.211...}{6}$$

$\therefore$ $x = 1.54$ or -0.87 correct to 2 d.p.

From the equation, $a = 3$, $b = -2$ and $c = -4$. Substitute these values into the formula. Placing the numbers in brackets helps to avoid mistakes.

Solve these equations. Give your answers correct to 2 decimal places.

1 $2x^2 + 5x - 4 = 0$

2 $2x^2 - 7x - 3 = 0$

3 $3x^2 + 5x + 1 = 0$

4 $7x^2 - 3x - 2 = 0$

5 $5x^2 + 8x - 2 = 0$

6 $4x^2 + x - 7 = 0$

7 $3x^2 - 2x - 4 = 0$

8 $10x^2 + 7x - 1 = 0$

9 $2x^2 + 11x + 8 = 0$

10 $6x^2 + 10x + 3 = 0$

11 $2x^2 + x = 9$

12 $4x^2 + 9x + 3 = 0$

Rearrange the equations in the form $ax^2 + bx + c = 0$ before substituting into the formula.

13 $3x^2 + 4 = 8x$ **14** $2x^2 - 7x + 2 = 0$ **15** $7x^2 + 4x = 2$

16 $3x^2 = 5x + 4$ **17** $5x^2 - 7 = 7x$ **18** $4x + 3 - 8x^2 = 0$

19 $6x^2 + 13x + 4 = 0$ **20** $3x = 7 - 9x^2$ **21** $9 - 2x - 3x^2 = 0$

22 Which of these quadratic equations have solutions and which do not? Give reasons for your answers.

a $2x^2 + 3x + 4 = 0$ **b** $2x^2 + 5x - 7 = 0$

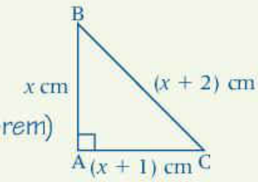
c $3x^2 + 2x + 5 = 0$ **d** $5x^2 - 5x + 2 = 0$

Example:

In triangle ABC, $\angle A = 90°$, AB $= x$ cm, AC $= (x + 1)$ cm and BC $= (x + 2)$ cm.

a Show that $x^2 - 2x - 3 = 0$

b Find the length of AB.



Start by drawing a diagram.

a $(x + 2)^2 = x^2 + (x + 1)^2$ (Pythagoras' theorem)

$x^2 + 4x + 4 = x^2 + x^2 + 2x + 1$

$0 = x^2 - 2x - 3$, i.e. $x^2 - 2x - 3 = 0$

b $x^2 - 2x - 3 = 0$

$(x - 3)(x + 1) = 0$

$x = 3$ or $x = -1$

The length of AB is 3 cm.

Check whether a quadratic equation will factorise; this is easier than using the formula.

When you solve an equation, always check whether the solutions give sensible answers to the problem. In this case, $x = -1$ is a solution of the equation, but it is not a solution of the problem because the length of a line cannot be negative.

Give answers that are not exact correct to 2 decimal places.

23 In a trapezium the parallel sides are of length x cm and $(x + 2)$ cm. If the distance between these parallel sides is $\frac{1}{2}x$ cm and the area of the trapezium is 15 cm², form an equation in x and solve it to find the dimensions of the figure.

24 The sum of the first n natural numbers is $\dfrac{n(n + 1)}{2}$. Find n if the sum of the first n numbers is 300.

25 The sum of the squares of three consecutive positive whole numbers is 302. Find them.

Let the smallest number be x.

26 One side of a rectangle is two metres longer than the other. If the length of a diagonal is 8 metres, find the sides.

Draw a diagram and allocate a letter for one of the unknown lengths.

27 One side of a rectangle is 5 m shorter than the other. If the length of a diagonal is 14 metres, find the sides.

28 Find the number such that ten times the number subtracted from five times its square gives 600.

29 The hypotenuse of a right-angled triangle is 15 cm. Find the lengths of the other two sides if their sum is 21 cm.

30 A square of side x cm is removed from a rectangular sheet of cardboard. The sheet is $2x$ cm long and is 3 cm longer than it is wide. Find the length of a side of the square if the cardboard that remains has an area of 24 cm².

31 The parallel sides of a trapezium are $(x - 3)$ cm and $(x + 5)$ cm long. The perpendicular distance between the parallel sides is x cm and the area of the trapezium is $30\,\text{cm}^2$.
Find
a the distance between the parallel sides
b the lengths of these sides.

32 The hypotenuse of a right-angled triangle is 19 cm, and the sum of the lengths of the other two sides is 24 cm. Find these sides.

33 A rectangular picture measuring 30 cm by 20 cm is surrounded by a rectangular frame whose area is $216\,\text{cm}^2$ and whose width is constant. Find the width of the frame.

34 Tickets are available for a concert at two prices, the dearer ticket being \$5 more than the cheaper ticket. Find the price of each ticket if a group can buy 2 more of the cheaper tickets than the dearer tickets for \$200. (The group cannot buy a mixture of cheap and dear tickets.)

35 The area of the page of a book is $216\,\text{cm}^2$. If the length is half as long again as the width, find the dimensions of the page.

36 The area of the page of a book is $300\,\text{cm}^2$. If the length is 5 cm more than the width, find the dimensions of the page.

37 A straight-sided plane figure with n sides has $\frac{n(n-3)}{2}$ diagonals.
If such a figure has 54 diagonals, how many sides does it have?

38 A building-site foreman has 20 metres of fencing with which to erect three sides of a rectangular pen, the fourth side being an existing wall. If the area of the pen is $40\,\text{m}^2$, find its dimensions.

39 A rectangular lawn measuring 10 metres by 8 metres is surrounded by a path of uniform width x metres.
If the area of the path is $60\,\text{m}^2$, find x.

40 The sum of a number and its reciprocal is 4. Find the two possible values for this number.

41 A stone thrown vertically into the air is h metres above the ground after t seconds, where $h = 30t - 5t^2$.
a When is the stone 40 metres above the ground?
Explain the two answers.
b Find the time taken for the stone to reach the ground.

42 A wooden block is x cm long, $\frac{1}{2}x$ cm wide and $\frac{1}{3}x$ cm high. If the total surface area of the block is $72\,\text{cm}^2$ find x.

43 A circular lawn of radius r metres is surrounded by a uniform path one metre wide. If the area of the path is one-tenth the area of the lawn, find the radius of the lawn.

44 A rectangular lawn measuring 5 m by 4 m is surrounded by a path of width x m. The area of the path is the same as the area of the lawn. Form an equation in x and solve it. How wide is the path?

45 In a rhombus, one diagonal is 1.5 cm longer than the other. If the area of the rhombus is $20\,\text{cm}^2$ find the lengths of the diagonals and hence find the length of a side.

46 The altitude of a triangle is 3 cm more than the base. Its area is $24\,\text{cm}^2$. Find the length of the base.

47 A circular lawn, of diameter 10 m, is surrounded by a path of uniform width. The area of the path is 50% more than the area of the lawn. How wide is the path?

48 If the average speed of a car is increased by 8 km/h, the time taken for a 240-km journey is reduced by one hour. Find the original average speed of the car.

Changing the subject of a formula

Consider the formula for the area of a triangle: $A = \frac{1}{2}bh$. A is the subject of this formula because A is given in terms of the other variables.

The **subject of a formula** is the variable on its own (i.e. the coefficient is 1 and the power is 1) on one side of the equals sign.

We can change the subject to another variable by thinking of the formula as an equation and 'solving' it for that other variable.

When we derived the formula for solving a quadratic equation, we did this by making x the subject of $ax^2 + bx + c = 0$.

These steps to change the subject of a formula will work in most cases:

- expand any brackets and eliminate any fractions
- if there is a square root, isolate it on one side of the equals sign and then square both sides
- collect the terms containing the new variable on one side of the equals sign and all the other terms on the other side
- factorise the terms containing the new variable if necessary and divide by the factor that does not involve the new subject
- if the new subject is squared, take the square root of both sides remembering that there are two square roots, one positive and one negative.

EXERCISE 9i

Example:

a Make a the subject of the formula $a + b = c - 2a(b - c)$.

b Make r the subject of the formula $H = \frac{1}{r} + \frac{1}{s}$

a

$$a + b = c - 2a(b - c)$$

Expand the bracket.

$$a + b = c - 2ab + 2ac$$

Collect all the terms containing a on one side of the equals sign.

$$a + 2ab - 2ac = c - b$$

Factorise the left-hand side.

$$a(1 + 2b - 2c) = c - b$$

Divide both sides by $1 + 2b - 2c$.

$$a = \frac{c - b}{1 + 2b - 2c}$$

b

$$H = \frac{1}{r} + \frac{1}{s}$$

Eliminate the fractions by multiplying every term by the LCM of the fractions; this is rs.

$$rsH = rs \times \frac{1}{r} + rs \times \frac{1}{s}$$

Cancel the fractions.

$$rsH = s + r$$

$$rsH - r = s$$

$$r(sH - 1) = s$$

$$r = \frac{s}{sH - 1}$$

Change the subject of each formula to the letter shown in brackets.

1	$v = u + at$	(t)	2	$N = a + bc$	(c)
3	$y = mx + c$	(m)	4	$A = \pi r(3r + 2h)$	(h)
5	$S(1 + \mu\mu') = W$	(μ')	6	$A = \pi r^2 + 2rh$	(h)
7	$a + 2b = c - 3a(b - c)$	(c)	8	$p - 3q = r + 3p(q - r)$	(q)
9	$P = \frac{A - B}{C}$	(A)	10	$Q = \frac{s - r}{R}$	(R)
11	$p + q = \frac{1}{r}$	(r)	12	$\frac{p + q}{3} = r$	(q)
13	$\frac{a + b}{b} = C$	(b)	14	$a + \frac{a}{m} = b$	(m)
15	$\frac{1}{p + q} = r$	(p)	16	$I = \frac{PRT}{100}$	(T)
17	$x = \frac{1}{y - z}$	(z)	18	$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{1}{r}$	(a)
19	$\frac{W}{P} = \frac{2a}{b - c}$	(c)	20	$x = \frac{h(2a + b)}{3(a + b)}$	(b)
21	$S = \frac{a}{1 - r}$	(r)	22	$V = \pi r^2 \left(h + \frac{r}{3}\right)$	(h)
23	$\frac{x + a}{b} = \frac{x + b}{a}$	(x)	24	$\frac{2x + y}{z} = \frac{x + 2z}{y}$	(x)
25	$\frac{T_1}{T_2} = \frac{2W + 3w}{2W + w}$	(w)	26	$\frac{a - b}{b + c} = \frac{b - a}{c + a}$	(c)

Example:

a Make r the subject of the formula $T = 1 - \sqrt{\frac{2r}{3g}}$.

b Make u the subject of the formula $v^2 = u^2 + 2as$.

a $T = 1 - \sqrt{\frac{2r}{3g}}$

$T - 1 = -\sqrt{\frac{2r}{3g}}$ — First isolate the square root term on one side of the equals sign.

$(T - 1)^2 = \frac{2r}{3g}$ — Place the left-hand side in brackets then square both sides.

$3g(T - 1)^2 = 2r$ — Multiply both sides by $3g$.

$\frac{3g(T - 1)^2}{2} = r$ — Divide both sides by 2.

b $v^2 = u^2 + 2as$

$v^2 - 2as = u^2$ — Subtract $2as$ from both sides to isolate u^2.

$u = \pm\sqrt{v^2 - 2as}$ — Take the square root of both sides.

Change the subject of each formula to the letter shown in brackets.

27	$V = \pi r^2 h$	(r)	28	$A = b + cn^2$	(n)
29	$A = \pi(R^2 - r^2)$	(R)	30	$v^2 = u^2 - 2as$	(u)
31	$f = \frac{\sqrt{3}}{m + 4M}mg$	(M)	32	$A = \pi r\sqrt{(h^2 - r^2)}$	(h)

33 $T = 2\pi\sqrt{\frac{l}{g}}$ (g)

34 $A = \sqrt{s(s-a)(s-b)(s-c)}$ (c)

35 $C = \sqrt{p-q}$ (p)

36 $L = \sqrt{a^2 - b^2}$ (b)

37 $n = \sqrt{\frac{p^2 + 4q^2}{3p^2 - q^2}}$ (p)

38 $F = P\sqrt{(1 + \mu^2)}$ (μ)

39 $T = \frac{\pi}{2}\sqrt{\frac{ma}{\lambda}}$ (l)

40 $T = 2\pi\sqrt{\frac{k^2 + h^2}{gh}}$ (g)

41 $a = \frac{b}{a + c^2}$ (c)

42 $a = b\sqrt{(x^2 - y^2)}$ (y)

43 $\sqrt{(x^2 + y^2)} = \sqrt{(y^2 + z^2)}$ (z)

44 $\frac{A}{B} = \sqrt{\frac{3a^2 - 4b^2}{2b^2 - 5a^2}}$ (a)

MIXED EXERCISE 9

Several answers are given for these questions.

Write down the letter that corresponds to the correct answer.

1 If a ✪ b means $5a - 3b$ the value of 5 ✪ 3 is

A 2 **B** 15 **C** 16 **D** 22

2 If $p \star q$ means $\frac{3p}{q^2}$ the value of $2 \star -1$ is

A -6 **B** -3 **C** 3 **D** 6

3 $(2x + 5)(3x - 1) =$

A $6x^2 + 17x - 5$ **B** $6x^2 - 13x + 5$
C $6x^2 - 17x + 5$ **D** $6x^2 + 13x - 5$

4 $(5x - 3)^2 =$

A $25x^2 - 15x + 9$ **B** $25x^2 - 30x + 9$
C $25x^2 + 15x - 9$ **D** $25x^2 - 30x - 9$

5 $15 + 2x - x^2 =$

A $(3 - x)(5 - x)$ **B** $(3 + x)(5 - x)$
C $(3 - x)(5 + x)$ **D** $(x - 3)(5 + x)$

6 $12 + 2x - 30x^2 =$

A $2(3 - 5x)(2 + 3x)$ **B** $2(3 - 5x)(2 - 3x)$
C $2(3 + 5x)(2 - 3x)$ **D** $(3 + 5x)(2 - 3x)$

7 $\frac{2x^2 - 3x - 14}{x + 2} =$

A $2(x + 7)$ **B** $2x + 7$
C $2x - 7$ **D** none of these

8 Given that $3x^2 + 8x - 3 = 0$, $x =$

A 3 and $\frac{1}{3}$ **B** -3 and $\frac{1}{3}$ **C** 3 and $-\frac{1}{3}$ **D** -3 and $-\frac{1}{3}$

9 Given that $6x^2 - 11x + 4 = 0$, $x =$

A $-\frac{1}{2}$ and $-\frac{4}{3}$ **B** $-\frac{1}{2}$ and $\frac{4}{3}$ **C** $\frac{1}{2}$ and $-\frac{4}{3}$ **D** $\frac{1}{2}$ and $\frac{4}{3}$

10 If $2x^2 + 4x + 5$ is written in the form $a(x + b)^2 + c$, the values of a, b and c are, respectively

A 2, 2, 1 **B** 2, −1, 3 **C** 2, 1, 3 **D** 2, 2, 3

11 If $3(x - 3)^2 + 7$ is written in the form $ax^2 + bx + c$, the values of a, b and c are, respectively

A 3, 9, 34 **B** 3, 18, 48 **C** 3, 18, 34 **D** 3, −18, 34

12 $9x^2 - 9x - 4 =$

A $(3x - 1)(3x + 4)$ **B** $(3x + 1)(3x - 4)$
C $(9x - 1)(x - 4)$ **D** $(9x + 1)(x - 4)$

13 Given that $(x + 2)^2 = 18$, $x =$

A $2 \pm \sqrt{18}$ **B** -2 ± 9
C $-2 \pm \sqrt{18}$ **D** none of these

14 Given that $3x^2 + 5x - 7 = 0$, $x =$

A $\frac{5 \pm \sqrt{109}}{6}$ **B** $\frac{-5 \pm 109}{6}$ **C** $\frac{5 \pm \sqrt{59}}{6}$ **D** $\frac{-5 \pm \sqrt{109}}{6}$

15 The sum of the two solutions of the quadratic equation $5x^2 - 4x + 3$ is

A $-\frac{4}{5}$ **B** $-\frac{4}{3}$ **C** $\frac{4}{3}$ **D** $\frac{4}{5}$

16 If $a - b = b - c + 2$ then $b =$

A $\frac{a + c}{2} - 2$ **B** $\frac{a + c}{2} + 2$ **C** $\frac{a + c - 2}{2}$ **D** $\frac{a + c + 2}{2}$

17 If $\frac{2}{a} + \frac{3}{b} = \frac{1}{c}$ then $a =$

A $\frac{3bc}{b - 2c}$ **B** $\frac{2bc}{b - 3c}$ **C** $\frac{2bc}{b + 3c}$ **D** $2c + \frac{2b}{3}$

18 If $D = \frac{\sqrt{2a}}{b}$ then $a =$

A $\frac{Db}{2}$ **B** $\frac{D^2b^2}{4}$ **C** $\frac{Db}{4}$ **D** $\frac{D^2b^2}{2}$

19 If $a^2 + b^2 - 4 = c^2$ then $a =$

A $4 \pm \sqrt{(c^2 - b^2)}$ **B** $2 - b + c$
C $\pm \sqrt{(4 - b^2 + c^2)}$ **D** $2 \pm \sqrt{(c^2 - b^2)}$

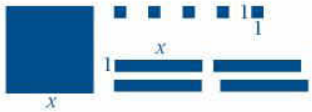
20 If $a = \sqrt{(b^2 - 3c^2)}$ then $c =$

A $\pm\frac{\sqrt{b^2 - a^2}}{3}$ **B** $\frac{b - a}{3}$ **C** $\pm\sqrt{\frac{b^2 - a^2}{3}}$ **D** $\pm\frac{\sqrt{a^2 - b}}{3}$

MATHS IS OUT THERE

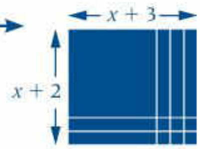
Squares and rectangles can be used to demonstrate factorisation of a quadratic expression of the form $x^2 + ax + b$.

Start with a square of side x units, some squares of side 1 unit and some rectangles of side x units by 1 unit:



Then $x^2 + 5x + 6$ can be represented by

These can be rearranged as showing that $x^2 + 5x + 6 = (x + 3)(x + 2)$.
Try showing the factorisation of $2x^2 + 5x + 2$ using shapes like these.



PUZZLE

A farmyard was full of sheep and hens. Altogether there were 50 heads and 140 legs. How many sheep were there?

IN THIS CHAPTER YOU HAVE SEEN THAT...

- a binary operation is a rule for combining two elements to produce another element
- a quadratic expression has the form $ax^2 + bx + c$ where a, b and c are constants (i.e. represent known numbers) and $a \neq 0$
- $ax^2 + bx + c$ will factorise if two integers exist whose product is ac and whose sum is b
- a quadratic equation $ax^2 + bx + c = 0$ can be solved by
  - factorisation if $ax^2 + bx + c$ factorises
  - by expressing $ax^2 + bx + c$ in the form $a(x + h)^2 + k$
  - by using the formula $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- the subject of a formula is the single letter given in terms of the other letters and numbers in the formula.

10 CIRCLE THEOREMS

AT THE END OF THIS CHAPTER YOU SHOULD BE ABLE TO...

1 Identify and name the different parts and shapes associated with a circle.

2 State the relationship between an angle subtended by an arc at the circumference of a circle and an angle subtended by the same arc at the centre of a circle.

3 Identify angles subtended by the same arc at the circumference of a circle and know the relationship between them.

4 State the size of an angle in a semicircle.

5 Identify a cyclic quadrilateral and state its properties.

6 State the properties of tangents to circles.

7 State and be able to use the alternate segment theorem.

MATHS IS OUT THERE

Squaring the circle, that is, using ruler and compasses to construct a square that has an area equal to the area of a given circle, has been a problem that mathematicians have tried to solve since time immemorial.
In 1880 the German mathematician Carl Lindemann (1852–1939) proved that no such construction is possible.

BEFORE YOU START

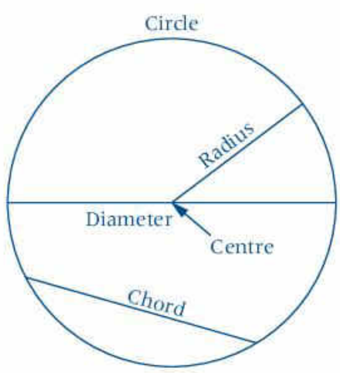
you need to know:
- ✓ the properties of angles formed by a pair of parallel lines and a transversal
- ✓ the angle properties of a triangle and a quadrilateral
- ✓ the properties of equilateral and isosceles triangles
- ✓ the properties of the special quadrilaterals
- ✓ how to show that triangles are congruent
- ✓ Pythagoras' theorem
- ✓ how to use trigonometry in right-angled triangles
- ✓ how to find the areas of triangles and sectors of a circle.

KEY WORDS

alternate segment, arc, chord, circle, circumference, cyclic quadrilateral, diameter, major arc, major segment, minor arc, minor segment, point of contact, radius, sector, segment, semicircle, tangent to a circle

Definitions

Every point on a **circle** is the same distance from its centre.
This distance is the **radius**.
The length of the circle is called the **circumference**.
The straight line segment joining two points on the circle is a **chord**.
A chord through the centre of a circle is a **diameter**.
From this it follows that the length of a diameter is twice the length of the radius. (The diameter is the largest possible chord.)

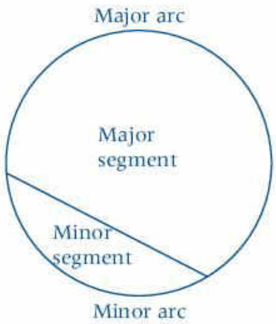


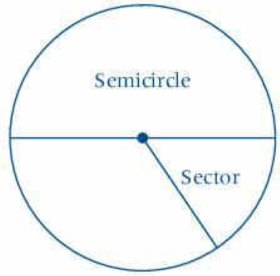
Any part of the circle is called an **arc**. If it is less than half the circle it is a **minor arc**; if it is more than half the circle it is a **major arc**.

A chord divides a circle into two **segments**. The smaller segment is called a **minor segment** and the larger is called a **major segment**.

A diameter divides a circle into two equal segments called **semicircles**.

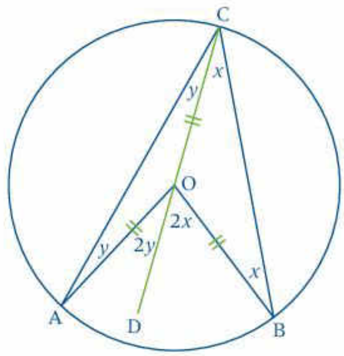
The region inside a circle bounded by an arc and two radii is called a **sector**.





Angles in circles

In this diagram, O is the centre of the circle, $\angle AOB$ is an angle at the centre subtended by (standing on) the arc AB and $\angle ACB$ is an angle at the circumference subtended by the same arc.



$OA = OB = OC$ (radii of the circle)

In ΔOAC, $\angle OCA = \angle OAC$ (the base angles of an isosceles triangle)

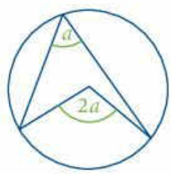
$\therefore \quad \angle AOD = 2\angle OCA$ (exterior angle of a triangle property)

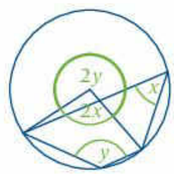
Similarly from ΔOBC, $\angle BOD = 2\angle OCB$

$\therefore \quad \angle AOB = 2\angle ACB$

This is the first theorem giving a relationship between angles in circles, it states that:

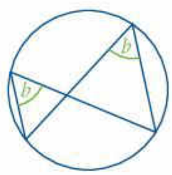
> the angle at the centre of a circle is twice the size of any angle at the circumference subtended by the same arc.

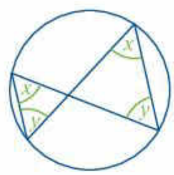




From this it follows that:

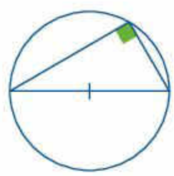
> any two angles subtended by the same arc at the circumference of a circle are equal in size,





It is also follows that:

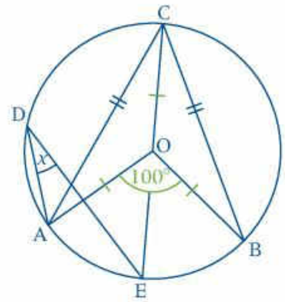
> the angle in a semicircle is a right angle.



EXERCISE 10a

Example:

In the diagram, O is the centre of the circle, AC = CB, COE is a straight line and ∠AOB = 100°. Find the value of *x* giving reasons for your statements.



∠ACB = 50° (angle at centre property)

Δs ACO and BCO are congruent (SSS)

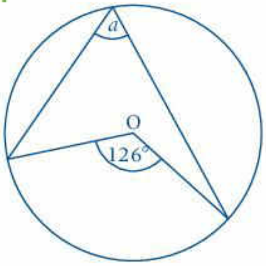
∴ ∠ACO = ∠BCO so ∠ACO = 25°

∴ x = 25° (angles subtended by the same arc)

In this exercise give reasons for every statement that you make.
O represents the centre of the circle.

In questions **1** to **5**, find the value of each angle marked with a letter.

1



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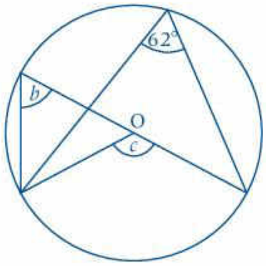

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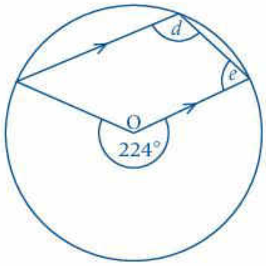

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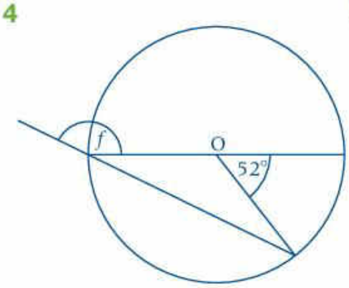
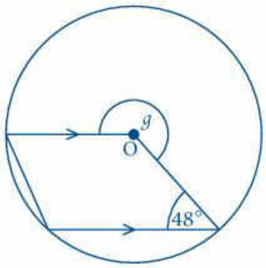
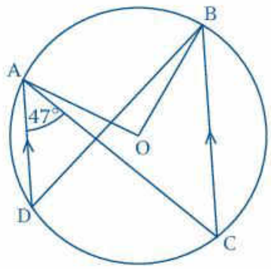


5

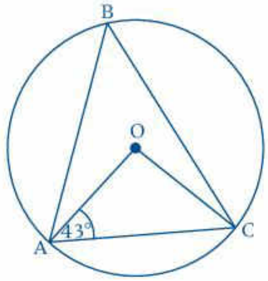


6 In the diagram AD is parallel to BC and ∠DAC = 47°.
Find ∠AOB.

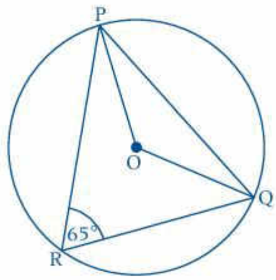


7 Triangle ABC is inscribed in a circle, centre O. ∠OAC = 43°.
Find ∠ABC.

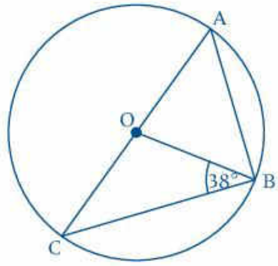
Inscribed means that the vertices are on the circle.



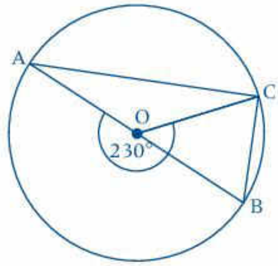
8 PQR is a triangle inscribed in a circle, centre O. ∠PRQ = 65°.
Find
a ∠POQ **b** ∠OPQ.



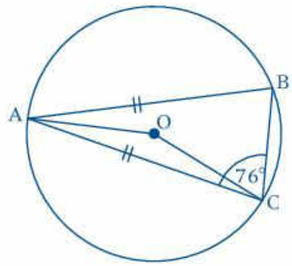
9 ABC is a triangle inscribed in a circle, centre O.
AC is a diameter and ∠OBC = 38°.
Find ∠OAB.



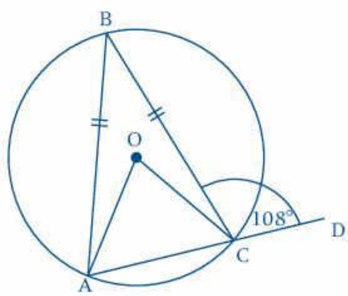
10 ABC is a triangle inscribed in a circle, centre O.
AB is a diameter and reflex angle AOC = 230°.
Find
a ∠ABC **b** ∠ACB.



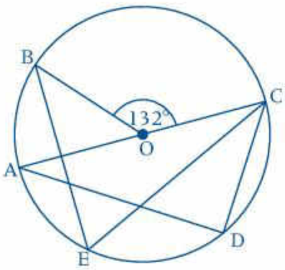
11 In triangle ABC, AB = AC and ∠ACB = 76°.
Find
a ∠AOC **b** ∠BAC.



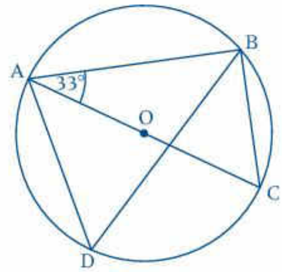
12 In triangle ABC, AB = BC. If AC is produced to D, ∠BCD = 108°.
Find
a ∠ABC **b** ∠AOC.



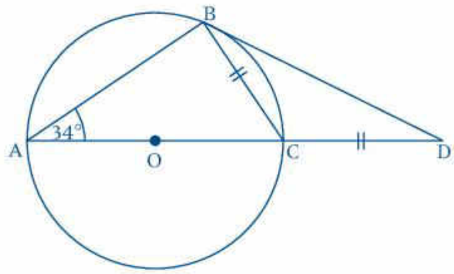
13 A, B, C, D and E are points on a circle, centre O.
AC is a diameter and ∠BOC = 132°.
Find
a ∠ADC **b** ∠BEC.



14 AC is a diameter and $\angle BAC = 33°$.
Find $\angle ADB$.



15 In the diagram, the diameter AC is produced to D and in triangle BCD, BC = CD. $\angle BAC = 34°$.
Find $\angle BDC$.



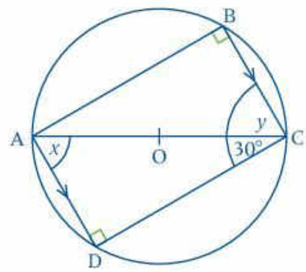
Example:

In the diagram, O is the centre of the circle,

AD is parallel to BC and $\angle ACD = 30°$.

a Calculate the values of x and y giving reasons for your statements.

b Prove that ABCD is a rectangle.



a $\angle ADC = 90°$ (angle in a semicircle)

$x = 60°$ (angle sum of a triangle is 180°)

$y = 60°$ (x and y are alternate angles)

b $\angle BAC = 180° - (90° + y) = 30°$
(angle sum of ΔABC is 180°)

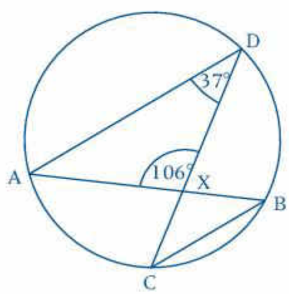
$\therefore$ $\angle BAC = \angle ACD$ and these are alternate angles with respect to AB and DC so AB is parallel to DC.
In quadrilateral ABCD,
AD is parallel to BC (given) and AB is parallel to DC
$\angle B = \angle D = 90°$ (angles in a semicircle)
so ABCD is a rectangle.

When a proof is asked for, we must give reasons for each step of the argument.

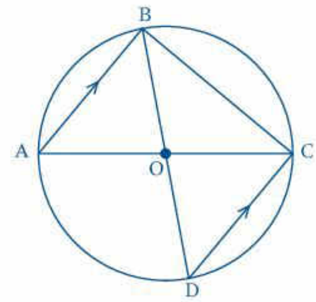
When you solve geometric problems, mark all facts that you know on the diagram. In particular mark all equal angles and the sizes of all angles you know. Mark the radii – this helps identify isosceles triangles.

16 Two chords, AB and DC, intersect at X.

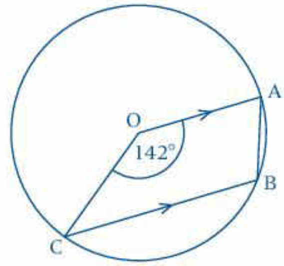
$\angle ADC = 37°$ and $\angle AXD = 106°$.
a Find **i** $\angle ABC$ **ii** $\angle DCB$
b What can you conclude about AD and CB?



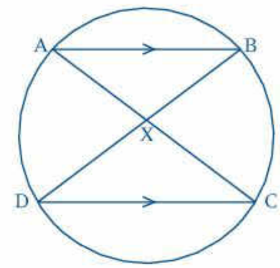
17 In the diagram, AC is a diameter and AB is parallel to DC.
Prove that BD passes through the centre of the circle.



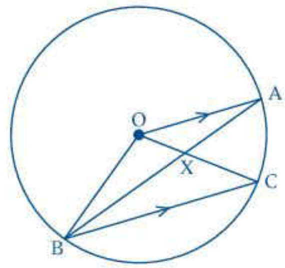
18 In the diagram, OA is parallel to CB and ∠AOC = 142°.
Find ∠OAB.



19 AB and DC are two parallel chords. AC and BD intersect at X.
Prove that
a AX = XB **b** ∠AXD = 2∠ACD.

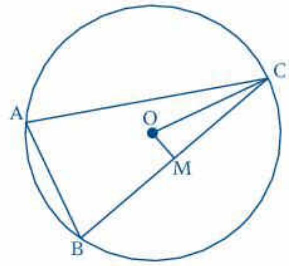


20 OA is a radius of a circle, centre O.
A chord BC is parallel to OA. ∠OCB = 38°.
a Find **i** ∠OAB **ii** ∠ABC.
b Prove that AB bisects ∠OBC.

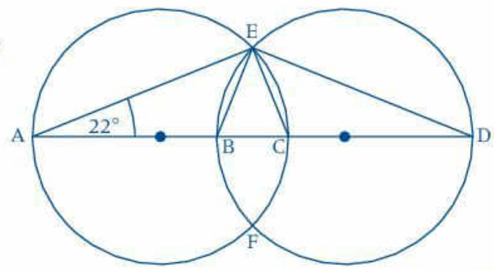


21 ABC is a triangle inscribed in a circle, centre O.
M is the midpoint of BC.
Prove that ∠BOM = ∠BAC.

Hint: Join OB.



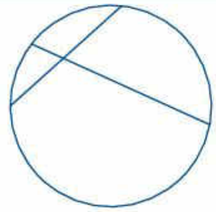
22 Two circles with the same radii intersect at E and F. ABCD is a straight line which passes through the centre of each circle. EB = EC and ∠EAD = 22°.
a Prove that triangles AEC and EBD are congruent.
Hence show that AE = ED.
b Find **i** ∠AEB **ii** ∠BEC **iii** ∠ECD.



INVESTIGATION

Draw a circle with two intersecting chords.

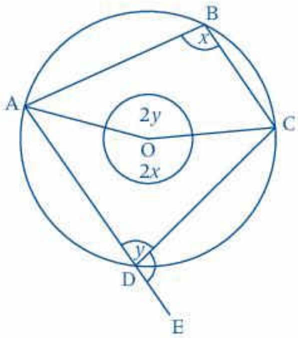
Investigate the ratio into which the point of intersection of the chords divides their lengths.



Cyclic quadrilaterals

A **cyclic quadrilateral** has all four vertices on a circle.

In the diagram, ABCD is a cyclic quadrilateral, O is the centre of the circle and ADE is a straight line.



The reflex angle AOC and x are both subtended by the major arc ADC.
$\therefore$ reflex angle AOC $= 2x$ (angle at the centre property)
Similarly obtuse angle AOC $= 2y$.
$2x + 2y = 360°$ (angles round a point)
$\therefore x + y = 180°$

This shows that:

> the opposite angles of a cyclic quadrilateral are supplementary. Conversely, when the opposite angles of a quadrilateral are supplementary, the quadrilateral is cyclic.

Also at D, $\angle EDC + y = 180°$ (angles on a straight line)

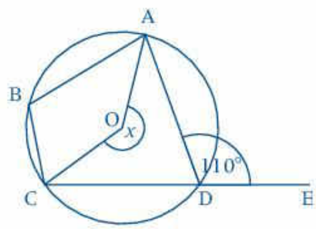
So it follows that $\angle EDC = x$, i.e.:

> an exterior angle of a cyclic quadrilateral is equal to the interior opposite angle of the quadrilateral.

EXERCISE **10b**

Example:

In the diagram, O is the centre of the circle, CDE is a straight line and $\angle ADE = 110°$. Find the value of x.



$\angle ABC = 110°$ (exterior angle property of a cyclic quadrilateral)

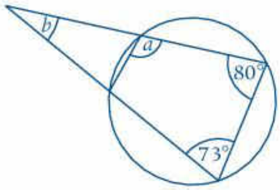
$x = 220°$ (angle at the centre $= 2 \times$ angle at the circumference subtended by the major arc CDA)

In this exercise give reasons for every statement that you make.

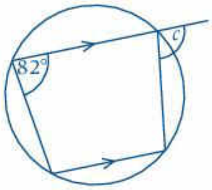
O represents the centre of the circle.

In questions **1** to **3**, find the value of each angle marked with a letter.

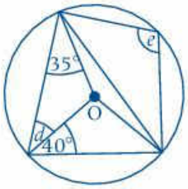
1



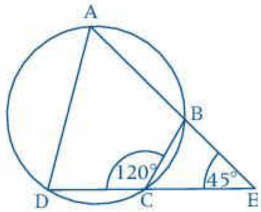
2



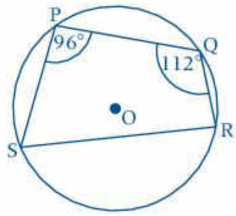
3



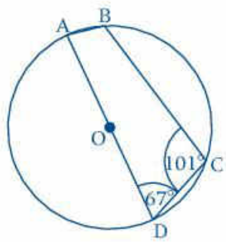
4 ABCD is a cyclic quadrilateral.
AB and DC produced meet at E.
∠BEC = 45° and ∠BCD = 120°.
Find
a ∠BAD **b** ∠ADC.



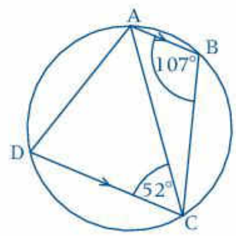
5 PQRS is a cyclic quadrilateral in which ∠SPQ = 96° and ∠PQR = 112°.
Find
a ∠PSR **b** ∠QRS.



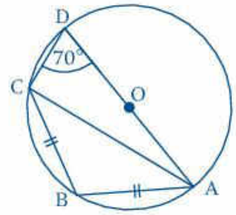
6 ABCD is a cyclic quadrilateral such that AD is a diameter. ∠BCD = 101° and ∠ADC = 67°.
Find
a ∠BAD **b** ∠ABC.



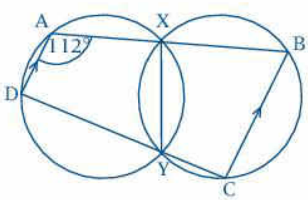
7 ABCD is a cyclic quadrilateral in which AB is parallel to DC.
∠ABC = 107° and ∠ACD = 52°.
Find
a ∠ACB **b** ∠DAC.



8 ABCD is a cyclic quadrilateral with ∠ADC = 70°, AB = BC, and AD a diameter.
Find
a ∠BAC **b** ∠DAC.

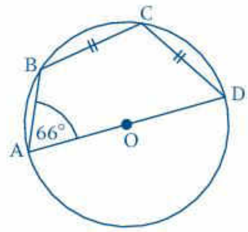


9 Two circles AXYD and XBCY intersect at the points X and Y. AD is parallel to BC, AXB is a straight line and ∠DAX = 112°.
a Find **i** ∠XYD **ii** ∠ABC.
b Prove that DYC is a straight line.



For part **a** do not assume that DYC is a straight line.

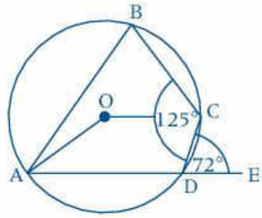
10 ABCD is a cyclic quadrilateral in which BC = CD, AD is a diameter and ∠DAB = 66°. Find ∠ADC.



Hint: Join BD.

11 ABCD is a cyclic quadrilateral in a circle, centre O. AD is produced to E, ∠CDE = 72° and ∠BCD = 125°. Find

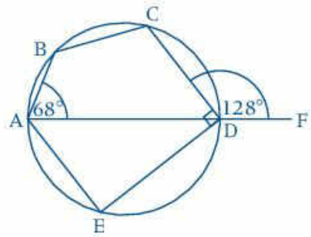
a ∠ABC **b** ∠AOC **c** BAD.



12 ABCDE is a pentagon inscribed in a circle. AD is a diameter which is produced to F, ∠CDE = 90°, ∠BAD = 68° and ∠CDF = 128°.

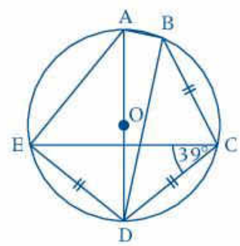
a Find **i** ∠ABC **ii** ∠ADE.

b Prove that CE is a diameter.



13 ABCDE is a pentagon inscribed in a circle, centre O. ∠ECD = 39°, BC = CD = DE, and AD is a diameter. Find

a ∠DBC **b** ∠BCE **c** ∠EAB **d** ∠ADB.

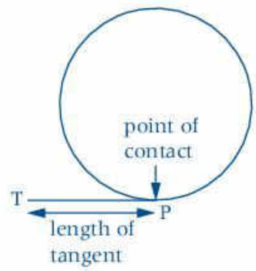


Tangents to circles

A **tangent to a circle** is a straight line that touches the circle but does not cross it.

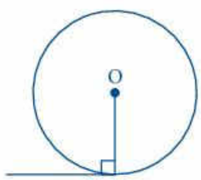
The point where the tangent touches the circle is called the **point of contact**.

The length of a tangent is the length of the line segment from a point on the tangent to the point of contact.

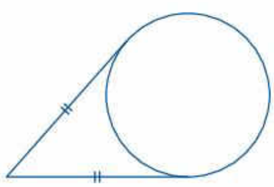


Tangents to circles have the following properties.

- A tangent is perpendicular to the radius of the circle drawn through the point of contact.



- The two tangents drawn from an external point to a circle are equal in length. (This is proved in the following Example.)



EXERCISE 10c

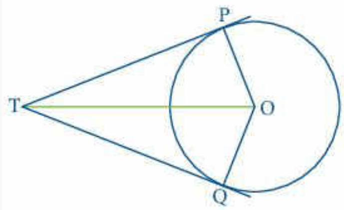
Example:

In the diagram, TP and TQ are tangents to the circle and O is the centre of the circle.

a Prove that TP = TQ.

b Show that TPOQ is a cyclic quadrilateral.

c The radius of the circle is 7 cm and the length of TP is 18 cm. Calculate the size of angle PTQ.



a In triangles TPO and TQO,

$\angle TPO = \angle TQO = 90°$ (angle between tangent and radius)
PO = QO (radii of the circle)
TO is common to both triangles.
Therefore triangles TPO and TQO are congruent (RHS).
Therefore TP = TQ.

b $\angle TPO = \angle TQO = 90°$ (angle between tangent and radius)
∴ P and Q lie on the circle drawn using TO as the diameter.
So TPOQ is a cyclic quadrilateral.

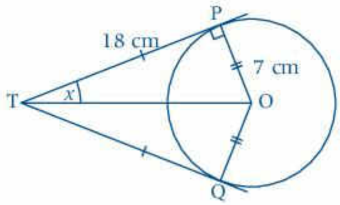
c Triangle TPO has a right angle at P.

$\therefore \tan x = \frac{7}{18} = 0.3888...$

$x = 21.250...°$

$\angle PTQ = 2 \times 21.250...°$

$= 43°$ to the nearest degree

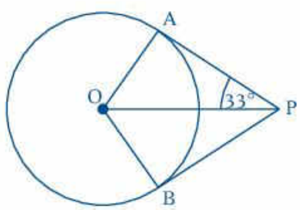


Mark all the facts you know on the diagram. Triangles TPO and TQO are congruent (PO = QO, radii; TP = TQ, tangents from a point and TO is common) so PTQ = 2*x*.

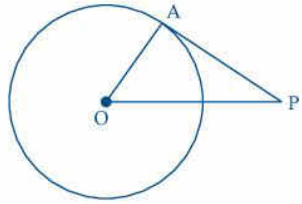
Give lengths correct to 3 significant figures and angles correct to the nearest degree.

You will need Pythagoras' theorem and trigonometry for these questions.

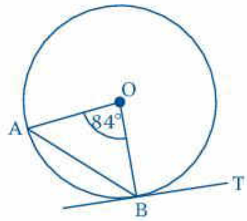
1 PA and PB are tangents to a circle, centre O.
$\angle APO = 33°$.
Find
a $\angle OPB$ **b** $\angle AOB$.



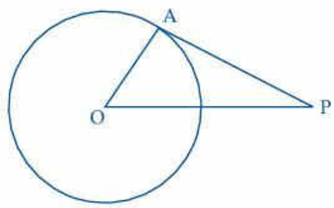
2 A tangent from P to the circle touches it at A. The radius of the circle is 3.2 cm and P is 5.8 cm from the centre of the circle. Calculate the length of the tangent.



3 AB is a chord in a circle, centre O. $\angle AOB = 84°$. Find the obtuse angle between the chord and the tangent.



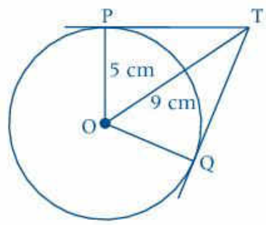
4 PA is a tangent from an external point P to a circle, centre O. PA = 15 cm and OP = 18 cm. Find the radius of the circle.



5 TP and TQ are tangents from T to a circle, centre O. OT = 9 cm and the radius of the circle is 5 cm. Find

a the length of a tangent

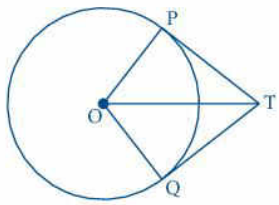
b $\angle PTQ$.



6 TP and TQ are two tangents from an external point T to a circle, centre O, radius 8 cm. OT = 13 cm. Find

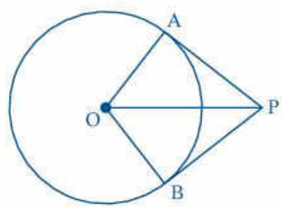
a the size of the angle between the tangents

b the length of a tangent.

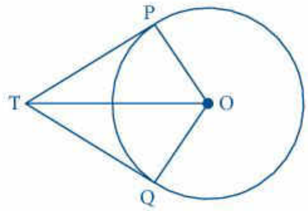


7 The angle between the tangents PA and PB is 68°. O is the centre of the circle and OA = 7.5 cm. Find the length of

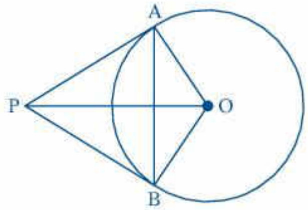
a PA **b** OP.



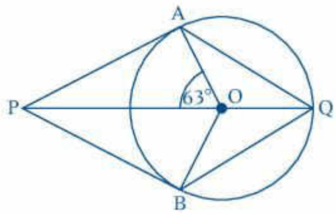
8 TP and TQ are two tangents from an external point T to a circle, centre O, radius 5.5 cm. ∠POQ = 112°.
Find the length of
a TQ
b OT.



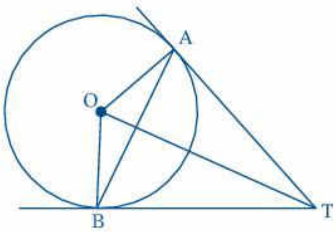
9 PA and PB are the tangents from an external point P to a circle, centre O.
OP = 9 cm and AP = 7.5 cm.
Find
a the radius of the circle
b i ∠AOP   ii ∠AOB
c the length of the chord AB.



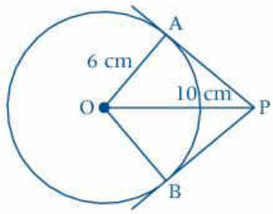
10 PA and PB are tangents to a circle, centre O.
PO produced meets the circle again at Q.
∠AOP = 63°.
a Find   i ∠AQB   ii ∠APB.
b Show that triangle OAQ and BOQ are congruent.
Hence prove that AQ = BQ.



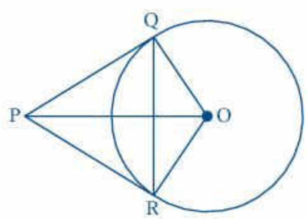
11 TA and TB are tangents to a circle, centre O, radius 8 cm.
AB = 10 cm.
Find
a ∠AOB
b the length of each tangent
c the area of quadrilateral OATB.



12 PA and PB are the tangents from P to a circle, centre O, radius 6 cm. OP = 10 cm.
Find
a the length of each tangent
b ∠APB
c the area of triangle AOP.

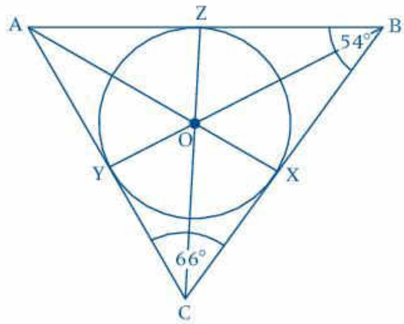


13 PQ and PR are tangents from P to a circle, centre O.
QR = 8 cm and ∠QPR = 60°.
Find
a ∠PQR
b the radius of the circle
c the area of   i triangle OPQ   ii quadrilateral PQOR.

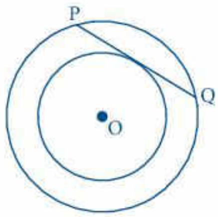


14 A circle, centre O, radius 6 cm, is inscribed in a triangle ABC. The sides of the triangle touch the circle at X, Y and Z. $\angle ABC = 54°$ and $\angle BCA = 66°$. Find the lengths of the sides of the triangle.

Find the lengths of the tangents from each vertex to the circle.



15 Two circles have the same centre. Their radii are 14 cm and 20 cm. A tangent to the inner circle intersects the outer circle at P and Q. Calculate the length of PQ.

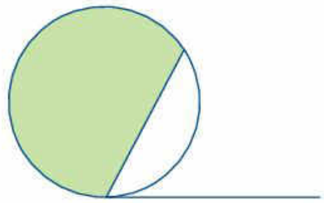


Join O to P, Q, and the point of contact of the tangent with the circle.

Alternate segment theorem

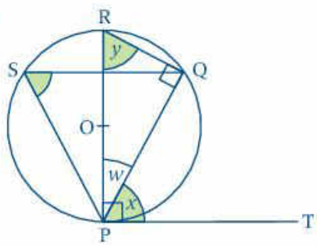
In the diagram, the shaded segment is called the **alternate segment** to the unshaded segment between the chord and the tangent.

There is a relationship between the angle subtended by the chord in the alternate segment and the angle between the chord and the tangent.



The alternate segment means the other segment.

In this diagram, O is the centre of the circle, POR is a diameter, PT is a tangent and S is any point on the major arc PRQ.



In ΔPRQ, $\angle PQR = 90°$ (angle in a semicircle)

$\therefore \quad w + y = 90°$

But $\angle OPT = 90°$ (angle between tangent and radius)

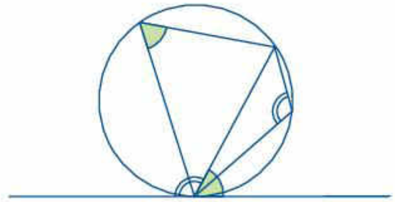
$\therefore \quad w + x = 90°$

So $y = x$.

$\angle PSQ = y$ (both angles subtended at the circumference by minor arc PQ)

So $\angle PSQ = x$, i.e.

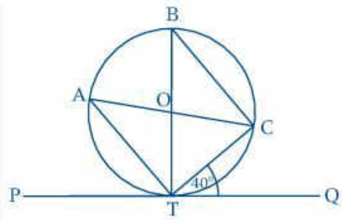
the angle between a tangent and a chord drawn through the point of contact is equal to any angle subtended by the chord in the alternate segment



EXERCISE 10d

Example:

In the diagram, PQ is a tangent to the circle, touching it at T, and O is the centre of the circle. ∠CTQ = 40°.
Find, giving reasons for your answers, the size of
a ∠TAC **b** ∠TBC **c** ∠ACT.



a ∠TAC = 40° (alternate segment theorem)

b ∠TBC = 40° (alternate segment theorem)

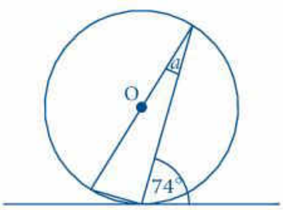
c ∠ATC = 90° (angle in a semicircle)

∴ ∠ACT = 180° − (90° + 40°) = 50° (angle sum of a triangle)

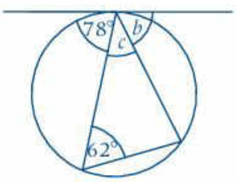
In this exercise give reasons for every statement that you make.

In questions **1** to **3**, find the value of each angle marked with a letter.

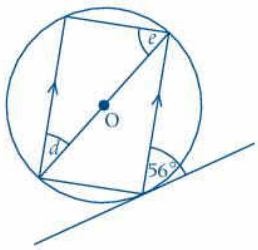
1



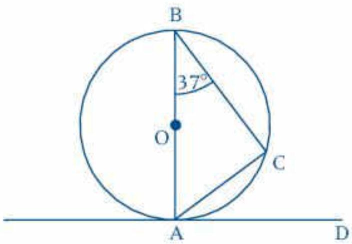
2



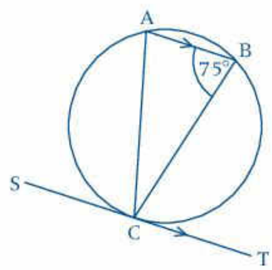
3



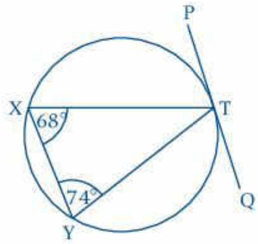
4 AB is a diameter of a circle, centre O. AD is a tangent to the circle at A.
Find
a ∠ACB **b** ∠CAD **c** ∠BAC.



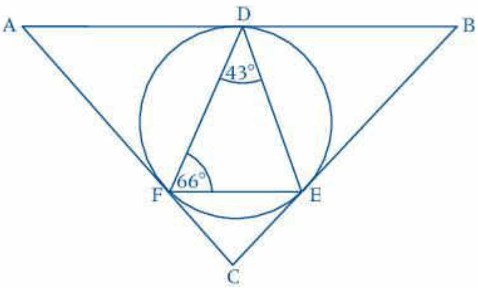
5 ABC is a triangle inscribed in a circle.
The chord AB is parallel to the tangent at C.
∠ABC = 75°.
Find
a ∠BAC **b** ∠ACB.



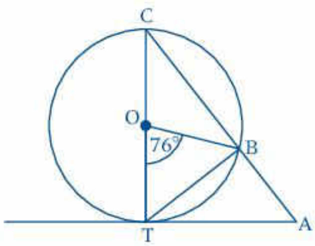
6 PQ is a tangent to a circle at T.
TXY is a triangle such that $\angle TXY = 68°$ and $\angle TYX = 74°$.
Find
a $\angle XTY$ **b** $\angle YTQ$ **c** $\angle XTP$.



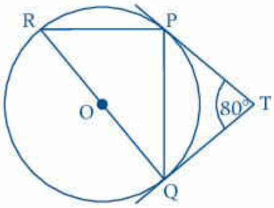
7 AB, AC and BC are tangents to a circle.
DE, EF and DF are chords of the circle.
$\angle EDF = 43°$ and $\angle EFD = 66°$.
Find the size of each angle in triangle ABC.



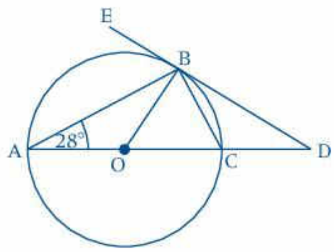
8 In the diagram, AT is a tangent, CT a diameter and ABC a straight line. $\angle TOB = 76°$.
Find
a $\angle OBT$ **b** $\angle CTA$ **c** $\angle TAC$.



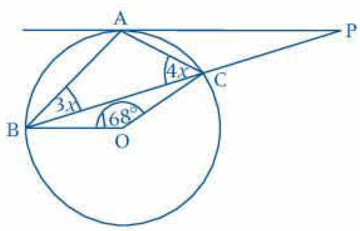
9 PT and QT are tangents to a circle, centre O.
$\angle PTQ = 80°$ and RQ is a diameter.
Find
a $\angle PQR$ **b** $\angle PRQ$.



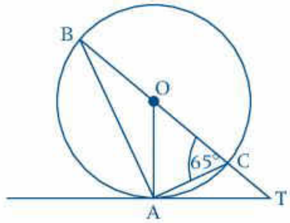
10 EBD is a tangent to a circle, centre O.
The diameter AC when produced meets the tangent at D. $\angle BAC = 28°$.
Find
a $\angle BCO$ **b** $\angle BDA$ **c** $\angle ABE$.



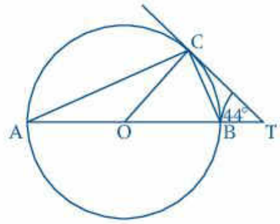
11 When the chord BC is produced it meets the tangent at A to a circle, centre O, at P.
If $\angle BOC = 168°$, $\angle ABC = 3x$ and $\angle BCA = 4x$, find
a the value of x
b the size of angle APC.



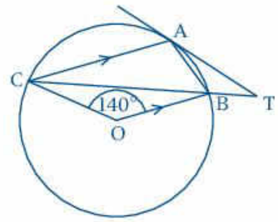
12 The diameter BC of a circle, centre O, is produced to meet the tangent at A in T. ∠OCA = 65°.
Find
a ∠CAT **b** ∠AOC **c** ∠OAB.



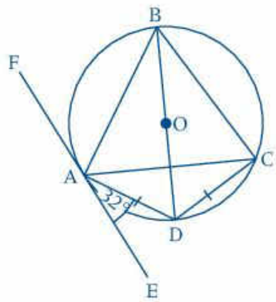
13 AB is a diameter of a circle, centre O.
AB is produced to T and TC is a tangent to the circle.
∠ATC = 44°.
Find
a ∠AOC **b** ∠ABC **c** ∠CAB.



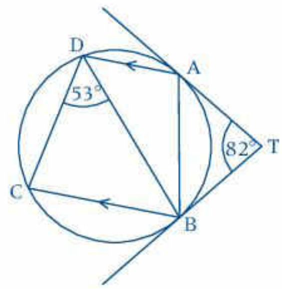
14 A chord CB is produced to meet the tangent to a circle, centre O, from a point A on the circumference, at T.
CA is parallel to OB and ∠COB = 140°.
Find
a ∠CAB **b** ∠CBA **c** ∠ATC.



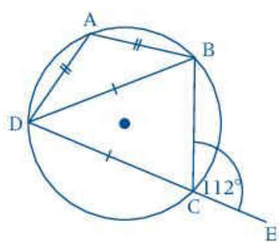
15 ABCD is a cyclic quadrilateral with AD = DC.
The diagonal BD passes through the centre of the circle O, and the tangent FAE at A is such that ∠DAE = 32°.
Find
a ∠ABD **b** ∠CAD
c ∠BAD **d** ∠BAF.



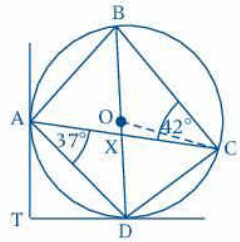
16 ABCD is a cyclic quadrilateral with AD parallel to BC and with the tangents at A and B intersecting at T.
∠ATB = 82° and ∠BDC = 53°.
Calculate the size of
a ∠ADB **b** ∠BCD **c** ∠DBT.



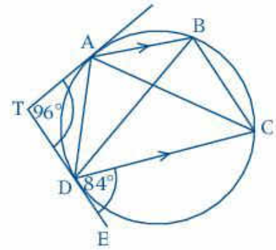
17 ABCD is a cyclic quadrilateral in which AB = AD and DB = DC. DC is produced to E and ∠BCE = 112°.
Find
a ∠DBC **b** ∠BAD **c** ∠ABD.



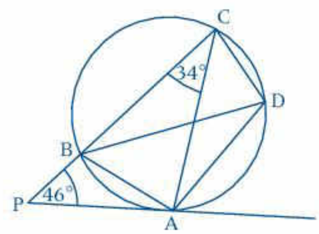
18 ABCD is a cyclic quadrilateral inscribed in a circle, centre O, with the diagonals AC and BOD intersecting at X.
$\angle DAC = 37°$ and $\angle ACB = 42°$.
Find
a $\angle DOC$ **b** $\angle ODC$ **c** $\angle OCB$
d $\angle BAC$ **e** $\angle ABD$ **f** $\angle AXB$.



19 ABCD is a cyclic quadrilateral with AB parallel to DC. The tangents to the circle at A and D intersect at T and TD is produced to a point E.
$\angle ATD = 96°$ and $\angle CDE = 84°$.
Find
a $\angle DAT$ **b** $\angle ACD$ **c** $\angle ABD$
d $\angle CBD$ **e** $\angle ADC$ **f** $\angle BCD$.

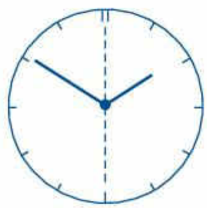


20 PA is a tangent to a circle which circumscribes the quadrilateral ABCD.
$\angle APB = 46°$ and $\angle ACB = 34°$.
Find
a $\angle ABP$
b $\angle ADB$
c $\angle BDC$.



INVESTIGATION

The hands on this clock face make the same angle with the line from 12 o'clock to 6 o'clock. Can you find this time correct to the nearest minute? Investigate other times where the hands have this property. How many such times are there?

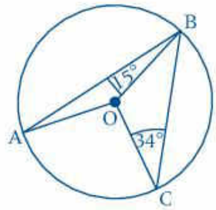


MIXED EXERCISE 10

Several answers are given for these questions.

Write down the letter that corresponds to the correct answer.

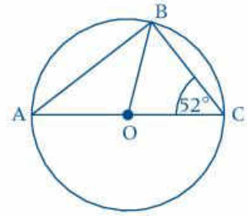
Questions **1** and **2** refer to the diagram alongside where A, B and C are points on the circumference of a circle, centre O.
$\angle OCB = 34°$ and $\angle ABO = 15°$.



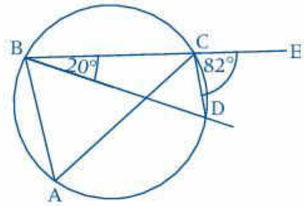
1 $\angle ABC =$
A 15° **B** 34° **C** 49° **D** 98°

2 $\angle AOC =$
A 49° **B** 98° **C** 131° **D** 150°

3 AOC is a diameter and ∠OCB = 52°.
∠AOB =
A 52° **B** 76° **C** 94° **D** 104°



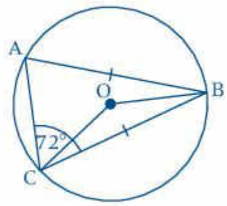
Questions **4** and **5** refer to the diagram alongside where the points A, B, C and D lie on a circle and BC is produced to E.
∠CBD = 20° and ∠DCE = 82°.



4 ∠BAC =
A 20° **B** 62° **C** 70° **D** 82°

5 ∠BCD =
A 20° **B** 82° **C** 90° **D** 98°

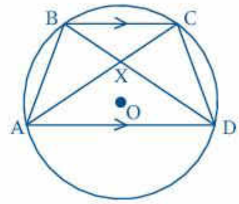
Questions **6** and **7** refer to the diagram alongside where the points A, B and C lie on a circle, centre O. AB = BC and ∠ACB = 72°.



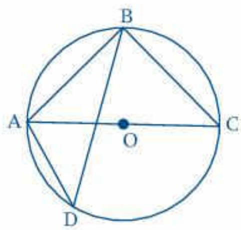
6 ∠CAB =
A 18° **B** 28° **C** 36° **D** 72°

7 ∠OBC =
A 18° **B** 28° **C** 36° **D** 54°

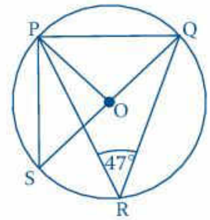
8 A, B, C and D lie on a circle, centre O.
BC is parallel to AD; AC and BD intersect at X.
Which of these statements is false?
A ∠BCA = ∠BDA
B ∠CXD = 2∠CBD
C ∠DBC = ∠DAC
D ∠BCA = ∠DAC



9 The points A, B, C and D lie on a circle, centre O.
AC is a diameter.
Which of these statements is false?
A ∠BAC + ∠BCA = 90°
B ∠BCA = $\frac{1}{2}$∠BAC
C ∠ABC = 90°
D ∠ADB = ∠ACB



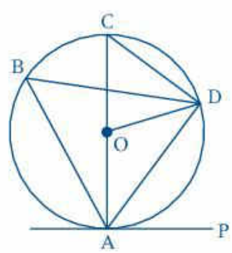
Questions **10** and **11** refer to the diagram alongside where the points P, Q, R and S lie on a circle, centre O.
SQ is a diameter and ∠PRQ = 47°.



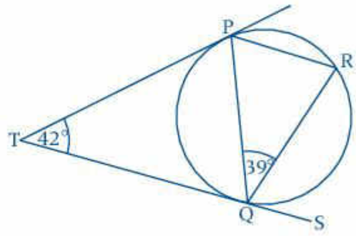
10 ∠PSQ =
A 23.5° **B** 47° **C** 94° **D** 133°

11 ∠OPQ =
A 43° **B** 47° **C** 57° **D** 60°

12 A, B, C and D are points on a circle, centre O.
AC is a diameter and AP a tangent.
Which of these statements is false?
A ∠PAD = ∠ABD **B** ∠ACD = $\frac{1}{2}$∠AOD
C 2∠COD = ∠CAD **D** ∠DAP = ∠ACD



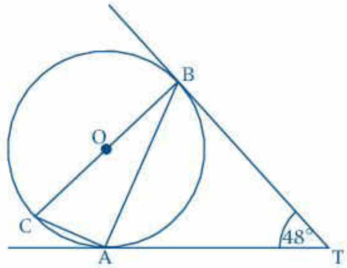
Questions **13** and **14** refer to the diagram alongside where TP and QT are tangents to a circle which inscribes triangle PQR. ∠PTQ = 42° and ∠PQR = 39°.



13 ∠QPR =
A 42° **B** 69° **C** 72° **D** 81°

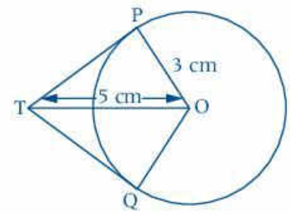
14 ∠RQS =
A 39° **B** 42° **C** 69° **D** 72°

15 TA and TB are tangents to a circle, centre O, which inscribes triangle ABC. BC is a diameter and ∠BTA = 48°. ∠ABC =
A 24° **B** 34°C **C** 48° **D** 66°



16 TP and TQ are tangents to the circle with centre O.
TO = 5 cm, PO = 3 cm.

TQ =
A 3 cm **B** 4 cm **C** 5 cm **D** $\sqrt{34}$ cm



IN THIS CHAPTER YOU HAVE SEEN THAT...

- the angle at the centre of a circle is twice the size of any angle at the circumference subtended by the same arc
- any two angles subtended by the same arc at the circumference of a circle are equal in size
- the angle in a semicircle is a right angle
- the opposite angles of a cyclic quadrilateral are supplementary. Conversely, when the opposite angles of a quadrilateral are supplementary, the quadrilateral is cyclic
- an exterior angle of a cyclic quadrilateral is equal to the interior opposite angle of the quadrilateral
- a tangent is perpendicular to the radius of the circle drawn through the point of contact
- the two tangents drawn from an external point to a circle are equal in length
- the angle between a tangent and a chord drawn through the point of contact is equal to any angle subtended by the chord in the alternate segment.

MATHS IS OUT THERE

Old Euclid drew a circle
On a sand-beach long ago.
He bounded and enclosed it
With angles thus and so.
His set of solemn greybeards
Nodded and argued much
Of arc and of circumference,
Diameter and such.

A silent child stood by them
From morning until noon
Because they drew such charming
Round pictures of the moon

Vachel Lindsay
(1879–1931)

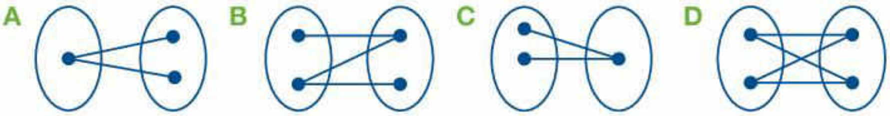
Multiple choice questions

Several possible answers are given.
Write down the letter corresponding to the correct answer.

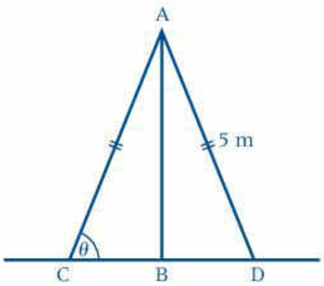
1 Which ordered pair belongs to the relation $\{(x, y), x \in \mathbb{R} \text{ and } x^2 + y^2 = 5\}$?
A (2, 1) **B** (2, 3) **C** (4, 1) **D** (3, 4)

2 The equation of a line is $2y + x = 1$. The equation of a perpendicular line is
A $2y - x = 1$ **B** $y - 2x = 5$ **C** $y + 2x = 1$ **D** $2y + x = 5$

3 Which of these relations is a function?



Questions **4** to **6** refer to this diagram which shows a vertical flag pole, AB, standing on level ground. The top of the flagpole, A, is held in position by two guy ropes AC and AD, each 5 m long, fixed to the ground at C and D.



4 The angle of elevation of A from C is
A ∠ABC **B** ∠CAB **C** ∠ACB **D** ∠DBA

5 AB =
A $5 \tan\theta$ **B** $5 \sin\theta$ **C** $5 \cos\theta$ **D** $\frac{5}{\sin\theta}$

6 CD =
A $5 \cos\theta$ **B** $10 \cos\theta$ **C** $10 \tan\theta$ **D** $5 \tan\theta$

7 The bearing of a ship from a lighthouse is 055°. The bearing of the lighthouse from the ship is
A 035° **B** 090° **C** 235° **D** 305°

8 A regular polygon has sides 8 cm long and an interior angle is 135°. The perimeter of the polygon, in cm, is
A 40 **B** 48 **C** 64 **D** 80

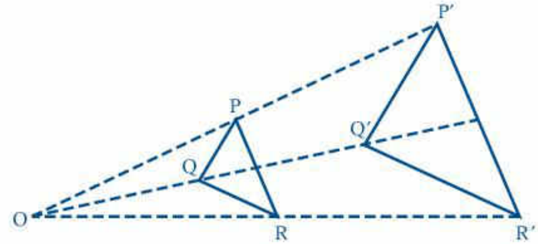
9 An interior angle of a regular polygon is twice the size of an exterior angle. The polygon is
A an equilateral triangle **B** a square
C a pentagon **D** a hexagon

10 If $f(x) = ax + 2$ and $f(2) = 4$, then $f(3) =$
A 1 **B** 3 **C** 4 **D** 5

11 The straight line $\frac{y}{2} - \frac{x}{3} + 1 = 0$ has gradient
A $-\frac{2}{3}$ **B** $-\frac{1}{3}$ **C** $\frac{1}{3}$ **D** $\frac{2}{3}$

12 Triangle PQR of area $9\,cm^2$, is mapped onto triangle P′Q′R′, of area $36\,cm^2$, by an enlargement, centre O and scale factor *k*. The value of *k* is

A 2 **B** 3
C 4 **D** 6



13 If $a * b = \frac{a^b}{b^a}$ then $1 * 2 =$

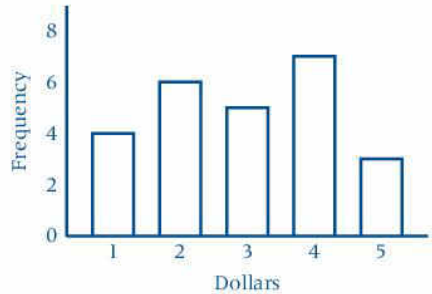
A $\frac{12}{21}$ **B** $\frac{1}{2}$
C 1 **D** 2

14 $(a-b)^2 - 1$ can be expressed as

A $(a-b-1)(a-b+1)$ **B** $(a-b-1)(a-b-1)$
C $(a-b+1)(a-b+1)$ **D** $(a-b-1)(a+b+1)$

15 The diagram shows the daily allowance, in dollars, of students in a class.
The number of students is

A 5 **B** 15 **C** 25 **D** 30



16 The mean of 10 numbers is 46. If one of the numbers is 19, what is the mean of the other nine numbers?

A 36 **B** 38 **C** 49 **D** 52

17 The mean of 8 numbers is 18. If one of the numbers is 25, what is the mean of the other seven numbers?

A 15 **B** 17 **C** 19 **D** 21

Items 18–21 refer to the table below which shows the distribution of the ages of 24 children when they began to play the piano.

Age	7	8	9	10	11	12
No. of children	3	6	5	5	3	2

18 What is the probability that a child chosen at random began piano lessons at the age of 8?

A $\frac{1}{3}$ **B** $\frac{1}{4}$ **C** $\frac{1}{5}$ **D** $\frac{1}{6}$

19 What is the probability that a child was at least 9 years old when they began having piano lessons?

A $\frac{3}{8}$ **B** $\frac{7}{12}$ **C** $\frac{5}{12}$ **D** $\frac{5}{8}$

20 What is the probability that a child was younger than 9 when they began learning to play the piano?

A $\frac{1}{8}$ **B** $\frac{1}{4}$ **C** $\frac{3}{8}$ **D** $\frac{5}{8}$

21 What is the mode of this distribution?

A 2 **B** 5 **C** 6 **D** 8

22 The equation of the line which passes through the point (0, 4) and has a gradient of $\frac{1}{2}$ is

A $y = 2x$ **B** $y = 2x + 4$ **C** $y = \frac{1}{2}x$ **D** $y = \frac{1}{2}x + 4$

23 The gradient of the straight line through the points (2, −4) and (−3, 4) is

A $-\frac{8}{5}$ **B** $\frac{8}{5}$ **C** −8 **D** 8

24 If $y = 5x - 2$, which of the following coordinates satisfies the equation?

A (1, 3) **B** (2, 6) **C** (−1, 7) **D** (−3, 13)

25 If the equation of a line is $y = 3 - 2x$, which of the following points lies on the line?

A (−2, 6) **B** (−4, 11) **C** (5, −4) **D** (3, 9)

26 If $y = 4x + 3$, which of the following points does not lie on the line?

A (−3, −9) **B** (−5, −17) **C** (2, 11) **D** (−1, −1)

27 Given that $p \circ q = p^2 - 2pq + q^2$ then

A $p \circ q = (p - q)^2$ **B** $p \circ (p \circ r) = (p \circ q) \circ r$
C $p \circ q$ is not commutative **D** $(2 \circ -1) \circ 3 = 27$

28 If $f(x) = 5x^2 - x + 1$, $f(-2) =$

A −21 **B** 20 **C** 21 **D** 23

29 If $a \square b = \frac{a^2 - b^2}{a - b}$, $4 \square -2 =$

A 2 **B** 6 **C** 12 **D** 16

30 $(3x - 2)^2 + 4 =$

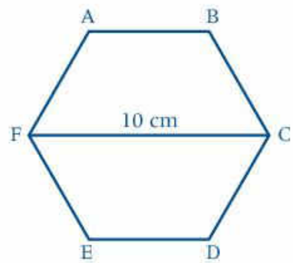
A $9x^2 - 12x$ **B** $9x^2 - 6x$ **C** $9x^2 - 12x + 8$ **D** $9x^2$

General proficiency questions

1 An aircraft has 320 seats. Passengers of average weight 78 kg occupy all the seats and each passenger is allowed a maximum of 20 kg of luggage. Calculate the total maximum weight of the passengers plus their luggage.

2 A 54-seater bus leaves for a package holiday with every seat taken. The average mass of the passengers is 82 kg and each passenger has 25 kg of luggage. Find the total mass of the passengers plus their luggage.

3 A diagonal of this regular hexagon is 10 cm. Find its perimeter.



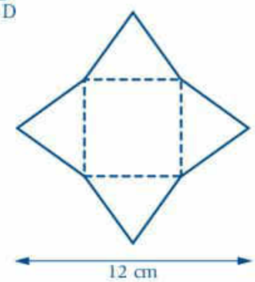
4 $a \wedge b$ means the difference between a^2 and b^2.

a Find **i** $4 \wedge 5$ **ii** $5 \wedge 4$ **iii** $(4 \wedge 5) \wedge 2$

b Explain whether this operation is

i commutative **ii** associative.

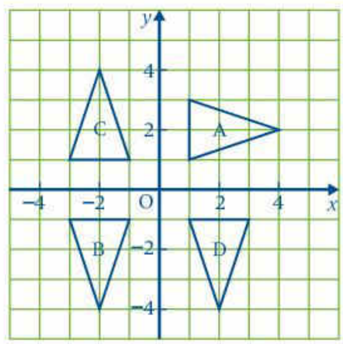
5 This four-pointed star shows four identical equilateral triangles surrounding a square. The distance between opposite points is 12 cm. Find the area of the star.



6 a Draw x and y axes for values of both x and y from -6 to 8. Use 1 cm to represent 1 unit on both axes.
b Plot the points A(4, -2), B(6, 6) and C(-2, 4).
c i Mark the point D such that ABCD is a parallelogram.
ii Write down the coordinates of D.
iii Draw the diagonals AC and BD. Hence write down the coordinates of E, the point of intersection of AC and BD.
d i How many axes of symmetry does ABCD have?
ii What special name is given to this parallelogram?
e Find the gradient of i AB ii CB.
f Find the length of i AB ii BC.

7 A(2, 1), B(3, 6) and C(7, 4) are vertices of a triangle ABC. The triangle is reflected about line that passes through the points E(2, 2) and F(6, 6) to give triangle A′B′C′.
a Write down the coordinates of A′, B′ and C′.
b State the transformation that maps triangle ABC onto triangle A′B′C′.

8 a Describe the transformation that maps
i triangle A onto triangle B
ii triangle B onto triangle C
iii triangle C onto triangle D.
b Is there a single transformation that maps A to D? Justify your answer.



9 The number of people sitting in the lounge of a residential home was counted at hourly intervals one Friday and the results recorded on a line graph. A copy of this graph is given opposite.
a How many people were sitting in the lounge at
i 12 noon ii 5 p.m.?
b Can you say how many there were at
i 11.30 a.m. ii 3.35 p.m.?
c Give an explanation of the pattern.
d Explain, with reasons, whether you would expect to find a similar pattern
i in this lounge on the following day
ii for a lounge in an hotel that has a similar number of beds to the residential home.

10 In an isosceles triangle the equal sides are each x cm. The base is 5 cm less than the sum of the equal sides. If the perimeter is 23 cm, find the length of one of the equal sides.

11 $a \bowtie b$ means 'the square of the difference between a and b'.
a Write down an algebraic expression for $a \bowtie b$.
b Find i $4 \bowtie 7$ ii $7 \bowtie 4$ iii $9 \bowtie 2$ iv $(3 \bowtie 1) \bowtie 5$

12 Expand

a $(x - 3y)(w + 5z)$ **b** $(ax + b)(cx - d)$

c $(m + 3)(m - 9)$ **d** $(7s - 4)(5s - 11)$

13 Factorise

a $x^2 + x - 110$ **b** $x^2 - 10x + 39$ **c** $63 + x^2 + 16x$

d $x^2 - x - 12$ **e** $12x^2 + 28x + 15$ **f** $54 + x^2 + 15x$

14 Simplify

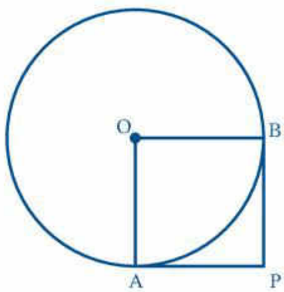
a $\dfrac{10x^2 + x - 2}{5x - 2}$ **b** $\dfrac{x^2 - 3x - 70}{x^2 - 7x - 30}$

15 A square lawn is bounded on three sides by a path 1 m wide. The area of the path is three-quarters the area of the lawn. Find the length of a side of the lawn.

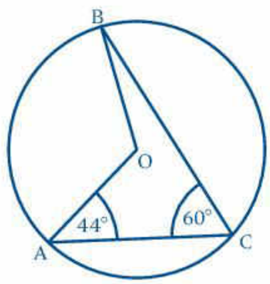
16 Make the letter in the bracket the subject of the formula.

a $T = mg\dfrac{(2a - x)}{a}$ (a) **b** $p = \sqrt{(q^2 - 4p^2)}$ (p)

17 PA and PB are the tangents from P to a circle centre, O, radius 6 cm. ∠AOB = 90°.
Prove that OAPB is a square and write down the length of a side of this square.



18 Find the size of **a** ∠AOB **b** ∠OBC.



19 The coordinates of A and B are, respectively, A(3, −1), B(2, 5).
Find

a the gradient of AB

b the equation of the line through A and B

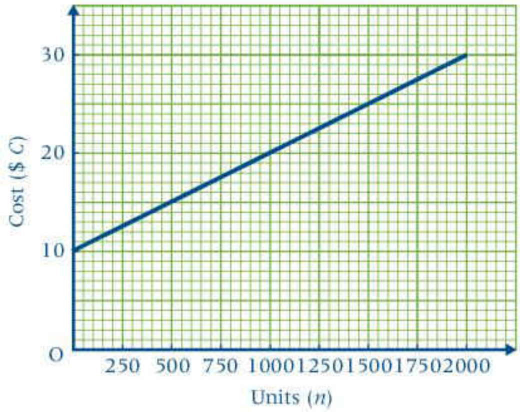
c the equation of the line through (4, −2) that is perpendicular to AB.

20 $f(x) = 3x^2 - 7$ and $g(x) = \dfrac{1}{x}$.

a Find **i** f(3) **ii** f(−3) **iii** $g\left(\frac{1}{2}\right)$

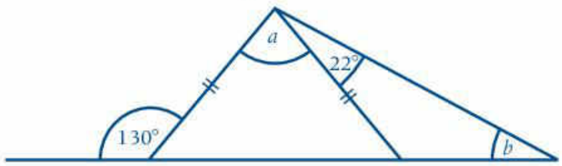
b Explain why g(0) does not exist.

21 The cost of a month's supply of electricity is made up of a fixed charge together with a charge for each unit used.
The graph shows the cost ($*C*) of one month's supply for *n* units electricity for up to 2000 units.
- **a** Write down the fixed charge.
- **b** Calculate the cost of 1 unit of electricity.
- **c** Find the equation of the line.
- **d** Use your equation to find the bill for a month in which 6500 units of electricity are used.

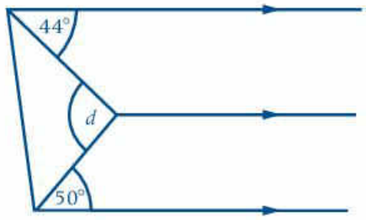


22 Find the size of the angles marked with letters.

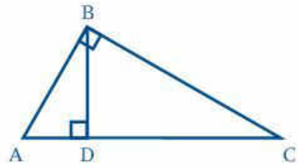
a



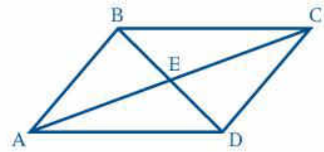
b



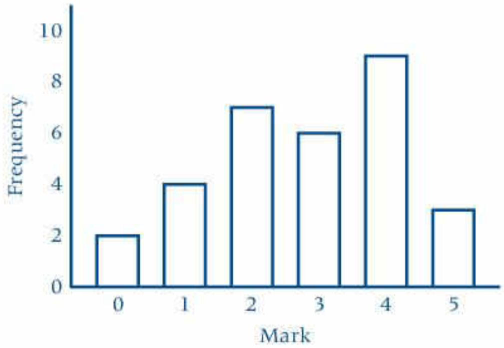
23 In the diagram, triangle ABC has a right angle at B and BD is perpendicular to AC.
Prove that triangles ABC and ABD are similar.



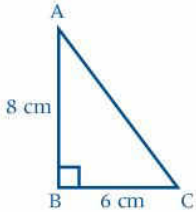
24 ABCD is a parallelogram.
- **a** Prove that triangles ABE and CDE are congruent.
- **b** Hence prove that the diagonals of a parallelogram bisect each other.



25 The bar chart shows the test scores of a group of students.
Find
- **a** the mode
- **b** the median
- **c** the mean
- **d** the probability that one of these students, chosen at random, scored 3 on the test.



26 In triangle ABC, $\angle ABC = 90°$, AB = 8 cm and BC = 6 cm. Calculate
- **a** the length of AC
- **b** the size of angle ACB.



27 The mean daily rainfall in June and July was 1.5 mm.
The mean daily rainfall in June, July and August was 1.8 mm.
Calculate the mean daily rainfall in August.

28 The cost of producing a stove delivered to a retailer is broken down as follows.
Materials: 24%, labour: 48%, transport: 8%, overheads: 20%

a Draw a pie chart to show this information.
b Explain why a bar chart is not an appropriate way of illustrating the information.

29 The numbers 1 to 20 are printed singly on a set of 20 cards. One card is drawn at random.
What is the probability that the number on the card is

a even **b** prime **c** a multiple of 3?

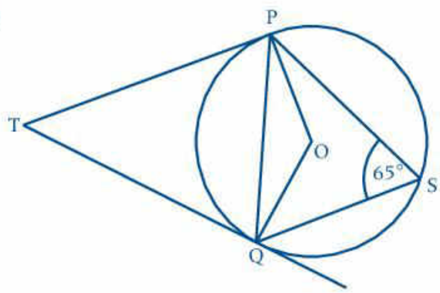
30 Solve the quadratic equations

a $2x^2 - 15x + 27 = 0$ **b** $3x^2 - 2x - 4 = 0$

31 Express $5x^2 + 7x + 1$ in the form $a(x + h)^2 + k$.
Hence solve the equation $5x^2 + 7x + 1 = 0$

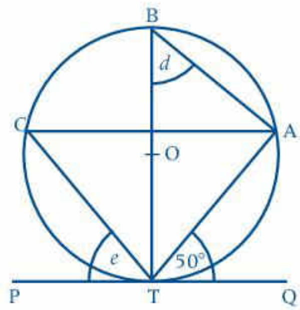
32 In the diagram, TP and TQ are tangents to the circle, centre O.
S is a point on the circumference such that $\angle PSQ = 65°$.

a Write down the size of angle POQ.
b Prove that TPOQ is a cyclic quadrilateral.
c The radius of the circle is 5 cm. Calculate the length of the tangent TP.



33 Draw a diagram to show this information.
Ship A is 10 km east of port B.
Port C is on a bearing of 020° from A.
The distance between B and C is 20 km.

34 In the diagram, O is the centre of the circle and PTQ is a tangent.
Find the sizes of the angles marked with letters.



35 Simplify

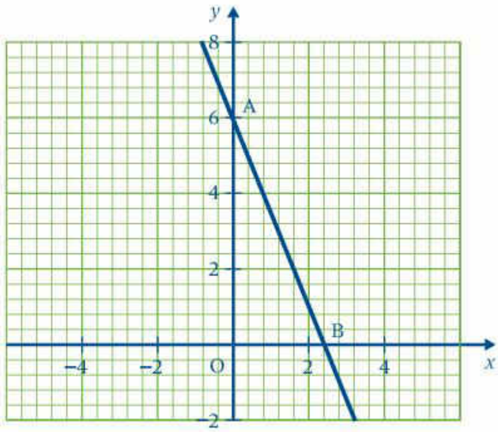
a $\dfrac{2x - 3x^2}{x^3}$ **b** $\dfrac{x^2 + 2x}{x^2 - 4}$

36 $f(x) = x^2 - 5x$.

a Calculate $f(-4)$ **b** the value of x for which $f(x) = 14$

37 The graph shows a straight line which crosses the y-axis at A and the x-axis at B.

a Calculate the gradient of the line AB.
b Find the equation of the line through A and B.
c Find the coordinates of the midpoint of AB.
d Find the angle the line makes with the positive direction of the x-axis.



38 Factorise

a $(x + y)^2 - 1$ **b** $(x + 2)^2 - x - 2$

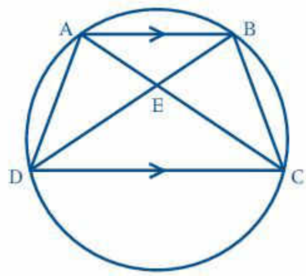
39 These are the marks that 20 students obtained in a test:

6 4 8 7 5 6 6 8 9 3 4 6 5 8 5 6 7 6 9 2

a Calculate
i the mode **ii** the median **iii** the mean mark.
b One of these students is chosen at random. What is the probability that this student scored more than 6 in the test?

40 In triangle ABC, AB = 8 cm, AC = 4 cm and BC = 6 cm. State, with a reason, whether triangle ABC is right-angled.

41 In the diagram, A, B, C and D are points on a circle. AB is parallel to DC. Prove that triangles AED and BEC are congruent. Give reasons for each statement that you make.



42 If $a * b = \dfrac{a^2}{a + b}$ and $2 * c = 1$, find the value of c.

43 Simplify

a $\dfrac{6x^2 - x - 1}{4x^2 - 1}$ **b** $\dfrac{6x - 4}{9x^2 - 12x + 4}$

44 In triangle ABC, D is a point on AB and E is a point on AC such that DE is parallel to BC. DE = 3 cm, BC = 5 cm and AB = 7.5 cm. Calculate the length of AD.

45 Mark obtained an average of 68% in eight examinations. What is the minimum average percentage he needs to score in his next two examinations to give an average of 70% in the ten examinations?

11 MATRICES

AT THE END OF THIS CHAPTER YOU SHOULD BE ABLE TO...

1. Understand what a matrix is.
2. State the size of a matrix.
3. Add and subtract matrices.
4. Multiply a matrix by a constant.
5. Multiply two matrices.
6. Identify a square matrix, a unit matrix and a zero matrix.
7. Calculate the determinant of a square 2×2 matrix.
8. Calculate the inverse of a 2×2 matrix.
9. Write a pair of simultaneous linear equations in matrix form and solve them.
10. Solve simple matrix equations.
11. Use matrices to solve everyday problems.

MATHS IS OUT THERE

The beginning of matrices and determinants can be traced back to the second century BC. In modern times the most important originators and developers were Arthur Cayley (1821–95) and his close friend James Sylvester (1814–95).

BEFORE YOU START you need to know:

✓ how to work with negative numbers
✓ the meaning of commutative and associative.

KEY WORDS

adjoint, column matrix, determinant, element, entry, identity matrix, leading diagonal, matrix, row matrix, singular matrix, square matrix, unit matrix, zero matrix

Definitions

- A **matrix** is a rectangular array of numbers or letters held together in brackets and is denoted by a capital letter in bold, e.g. **A**.
- Each number is called an **entry** or **element**.
- A matrix can have any number of rows and columns.
- The size of a matrix is described by the number of rows followed by the number of columns and is called the order of the matrix. A matrix with n rows and m columns is called an $n \times m$ matrix.

$\begin{pmatrix} 2 & -1 & 0 & 7 \\ 10 & \frac{1}{2} & 5 & 0 \end{pmatrix}$ is a matrix. It has two rows and four columns. This is a 2×4 matrix.

- A **square matrix** has the same number of rows and columns.
- The **leading diagonal** in a square matrix comprises the elements in the diagonal going from top left to bottom right.

$\begin{pmatrix} 2 & 4 \\ 0 & 1 \end{pmatrix}$ is a 2×2 square matrix.

- A **unit matrix** is a square matrix where the elements in the leading diagonal are all 1 and the other elements are zero. A unit matrix is also called an **identity matrix** and is usually denoted by **I**.
- A **zero matrix** is any matrix whose elements are all zero. A zero matrix is also called a null matrix.
- A **column matrix** has just one column, but more than one row.
- A **row matrix** has just one row but more than one column.
- Two matrices can be added provided they are the same size, by adding their corresponding elements. For example, $\begin{pmatrix} 2 & 0 & -1 \\ 4 & 7 & 0 \end{pmatrix} + \begin{pmatrix} 1 & -2 & 4 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & -2 & 3 \\ 4 & 8 & 1 \end{pmatrix}$
- A matrix can be subtracted from another matrix of the same size by subtracting the corresponding elements. For example, $\begin{pmatrix} 2 & 0 & -1 \\ 4 & 7 & 0 \end{pmatrix} - \begin{pmatrix} 1 & -2 & 4 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & -5 \\ 4 & 6 & -1 \end{pmatrix}$
- A matrix can be multiplied by a number by multiplying each element by that number. For example, $2\begin{pmatrix} 3 & 2 \\ 4 & -1 \end{pmatrix} = \begin{pmatrix} 6 & 4 \\ 8 & -2 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is a 2×2 unit matrix and $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ is a 2×2 zero matrix.

$\begin{pmatrix} 2 & 0 & -1 \\ 4 & 7 & 0 \end{pmatrix} + \begin{pmatrix} 2 & 0 \\ 2 & 1 \end{pmatrix}$ cannot be added because they are not the same size.

Matrix multiplication

Multiplication of a matrix with one row by a matrix with one column, provided they have the same number of elements, is defined as follows.

$$(a \quad b \quad c)\begin{pmatrix} x \\ y \\ z \end{pmatrix} = ax + by + cz$$

For example, $(2 \quad 1 \quad -1)\begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix}$
$= 2 \times 3 + 1 \times 0 + (-1) \times (-2)$
$= 6 + 0 + 2 = 8$

i.e. the elements are multiplied in corresponding pairs and then added to give a single element.

We can extend this by adding rows to the first matrix and columns to the second matrix. The product **AX**, where $\mathbf{A} = \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}$ and $\mathbf{X} = \begin{pmatrix} x & p \\ y & q \\ z & r \end{pmatrix}$ is defined as

$$\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix}\begin{pmatrix} x & p \\ y & q \\ z & r \end{pmatrix} = \begin{pmatrix} \text{1st row} \times \text{1st column} & \text{1st row} \times \text{2nd column} \\ \text{2nd row} \times \text{1st column} & \text{2nd row} \times \text{2nd column} \end{pmatrix}$$

Matrix multiplication is sometimes called row-column multiplication.

From this it follows that

- matrices can be multiplied only if the number of elements in the rows of the first matrix is equal to the number of elements in the columns of the second matrix, i.e. **AB** exists only if the number of elements in the rows of **A** equals the number of elements in the columns of **B**
- the order of multiplication matters because in general, $\mathbf{AB} \neq \mathbf{BA}$, i.e. matrix multiplication is not commutative
- $\mathbf{A}^2$ means $\mathbf{A} \times \mathbf{A}$ and this exists only if **A** has the same number of rows and columns, i.e. **A** must be a square matrix.

This is different from multiplication of real numbers, where $ab = ba$.

EXERCISE 11a

Example:

$\mathbf{A} = \begin{pmatrix} 1 & 4 \\ -1 & 0 \end{pmatrix}$, $\mathbf{B} = \begin{pmatrix} 1 & -1 \\ 4 & 0 \end{pmatrix}$, $\mathbf{C} = (-2 \quad 0)$ and $\mathbf{D} = \begin{pmatrix} 3 & -2 & 4 \\ 2 & 0 & -1 \end{pmatrix}$

a Find **i** $3\mathbf{D}$ **ii** $\mathbf{A}^2$ **iii** $\mathbf{CA}$

b Explain why **DB** does not exist.

a i $3D = 3\begin{pmatrix} 3 & -2 & 4 \\ 2 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 9 & -6 & 12 \\ 6 & 0 & -3 \end{pmatrix}$

Multiply every element by 3.

ii $A^2 = \begin{pmatrix} 1 & 4 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} (1 \quad 4)\begin{pmatrix} 1 \\ -1 \end{pmatrix} & (1 \quad 4)\begin{pmatrix} 4 \\ 0 \end{pmatrix} \\ (-1 \quad 0)\begin{pmatrix} 1 \\ -1 \end{pmatrix} & (-1 \quad 0)\begin{pmatrix} 4 \\ 0 \end{pmatrix} \end{pmatrix}$

You can miss out some of these steps.

$= \begin{pmatrix} 1-4 & 4+0 \\ -1+0 & -4+0 \end{pmatrix} = \begin{pmatrix} -3 & 4 \\ -1 & -4 \end{pmatrix}$

iii $CA = (-2 \quad 0)\begin{pmatrix} 1 & 4 \\ -1 & 0 \end{pmatrix} = \left((-2 \quad 0)\begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad (-2 \quad 0)\begin{pmatrix} 4 \\ 0 \end{pmatrix}\right)$

$= (-2 \quad -8)$

b **D** has three elements in each row but **B** has only two elements in each column so **DB** is not possible.

However **BD** is possible because **B** has two elements in each row and **D** has two elements in each column.

Find, where possible,

1 $\begin{pmatrix} 3 \\ 4 \end{pmatrix} + \begin{pmatrix} 9 \\ -7 \end{pmatrix}$

2 $\begin{pmatrix} 4 & 7 \\ 3 & 1 \end{pmatrix} + \begin{pmatrix} 2 & -3 \\ -4 & 2 \end{pmatrix}$

3 $\begin{pmatrix} 3 & 6 \\ 4 & -2 \end{pmatrix} + \begin{pmatrix} -4 & 3 \\ 3 & -5 \end{pmatrix}$

4 $\begin{pmatrix} 5 & 4 \\ 2 & 1 \end{pmatrix} - \begin{pmatrix} 3 & 6 \\ 4 & 1 \end{pmatrix}$

5 $\begin{pmatrix} 4 & 2 \\ 9 & 3 \end{pmatrix} - \begin{pmatrix} -3 & 2 \\ -4 & 3 \end{pmatrix}$

6 $(1 \quad 5 \quad 6) + (2 \quad -2)$

7 $\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} + \begin{pmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{pmatrix}$

8 $\begin{pmatrix} 2 & 4 & 6 \\ 1 & 2 & 3 \end{pmatrix} - \begin{pmatrix} -2 & 1 & -4 \\ 3 & -2 & 5 \end{pmatrix}$

9 $\begin{pmatrix} 2 & 4 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ 5 \end{pmatrix}$

10 $\begin{pmatrix} 3 & -1 \\ -4 & 2 \end{pmatrix}\begin{pmatrix} 4 \\ 2 \end{pmatrix}$

11 $\begin{pmatrix} -5 & 2 \\ 4 & 3 \end{pmatrix}\begin{pmatrix} -2 \\ -1 \end{pmatrix}$

12 $\begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}$

13 $\begin{pmatrix} 3 & 2 \\ 5 & 4 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 2 & 7 \end{pmatrix}$

14 $\begin{pmatrix} 9 & 0 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 2 \\ 4 \end{pmatrix}$

15 $\begin{pmatrix} 3 & 1 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}$

16 $\begin{pmatrix} -2 & -3 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -5 & 2 \end{pmatrix}$

17 $\begin{pmatrix} 4 & -2 \\ 3 & 6 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ -3 & 5 \end{pmatrix}$

18 $\begin{pmatrix} 0 & -2 \\ 6 & -1 \end{pmatrix}\begin{pmatrix} 4 & 0 \\ -5 & 6 \end{pmatrix}$

19 $\begin{pmatrix} -3 & 0 \\ 3 & 5 \end{pmatrix}\begin{pmatrix} 7 & -4 \\ 0 & 0 \end{pmatrix}$

20 $\begin{pmatrix} -10 & -1 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}$

21 $\begin{pmatrix} 0 & -10 \\ 12 & 0 \end{pmatrix}\begin{pmatrix} 3 & 0 \\ 2 & 8 \end{pmatrix}$

22 $\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix} + \begin{pmatrix} 5 & 2 \\ 4 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 5 \\ 3 & 2 \end{pmatrix}$

23 $\begin{pmatrix} 2 & 4 \\ 6 & -8 \end{pmatrix} + \begin{pmatrix} -2 & 1 \\ 4 & 3 \end{pmatrix} + \begin{pmatrix} 3 & 1 \\ -4 & 2 \end{pmatrix}$

24 $\begin{pmatrix} 1 & 2 \\ 5 & 4 \end{pmatrix} - \begin{pmatrix} 3 & 1 \\ 2 & -7 \end{pmatrix} + \begin{pmatrix} -2 & 1 \\ 4 & -5 \end{pmatrix}$

25 $\begin{pmatrix} 8 & 4 \\ -5 & 3 \end{pmatrix} - \begin{pmatrix} -2 & 1 \\ -3 & 6 \end{pmatrix} - \begin{pmatrix} 0 & 2 \\ 8 & -3 \end{pmatrix}$

26 $\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 4 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 5 & 7 \\ 4 & 6 \end{pmatrix}$

27 $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} 1 & 4 \\ 3 & -1 \end{pmatrix}\begin{pmatrix} 3 & -5 \\ -2 & 2 \end{pmatrix}$

28 $\begin{pmatrix} 1 & -2 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$

29 $\begin{pmatrix} 1 & 2 & 5 \end{pmatrix}\begin{pmatrix} 2 & 0 & 1 \\ 4 & -2 & 0 \\ 3 & 1 & 1 \end{pmatrix}\begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}$

Multiply the left-hand pair first then multiply the result by the right-hand matrix. Remember that you must not change the order of the matrices.

Example:

$\begin{pmatrix} -2 & a \\ b & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -4 & -2 \end{pmatrix} = -8\mathbf{I}$ Find the values of a and b.

$\begin{pmatrix} -2 & a \\ b & 2 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -4 & -2 \end{pmatrix} = \begin{pmatrix} -4-4a & 2-2a \\ 2b-8 & -b-4 \end{pmatrix}$

$-8\mathbf{I} = -8\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -8 & 0 \\ 0 & -8 \end{pmatrix}$

$\therefore\ -4-4a = -8 \quad \Rightarrow a = 1 \quad$ (check: when $a = 1, 2-2a = 0$)
and $\ 2b-8 = 0 \quad \Rightarrow b = 4 \quad$ (check: when $b = 4, -b-4 = -8$)

Multiply the matrices then compare the elements in the product with those in $-8\mathbf{I}$. Comparing two elements gives values for a and b. Use the other two elements to check that they are correct.

In questions **30** to **42**, find the values of a and b.

30 $\begin{pmatrix} a & 1 \\ -3 & b \end{pmatrix}\begin{pmatrix} 2 & -1 \\ 3 & -2 \end{pmatrix} = -\mathbf{I}$

31 $\begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} a & 3 \\ -3 & b \end{pmatrix} = \mathbf{I}$

32 $\begin{pmatrix} 4 & a \\ b & 2 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ -2 & 4 \end{pmatrix} = 2\mathbf{I}$

33 $\begin{pmatrix} a & b \\ 5 & 3 \end{pmatrix}\begin{pmatrix} -3 & 2 \\ 5 & -4 \end{pmatrix} = -2\mathbf{I}$

34 $\begin{pmatrix} 2 & a \\ 3 & b \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -3 & 2 \end{pmatrix} = 5\mathbf{I}$

35 $\begin{pmatrix} 5 & 4 \\ 3 & 3 \end{pmatrix}\begin{pmatrix} 3 & b \\ a & 5 \end{pmatrix} = 3\mathbf{I}$

36 $\begin{pmatrix} a & 3 \\ 11 & b \end{pmatrix}\begin{pmatrix} 7 & -3 \\ -11 & 5 \end{pmatrix} = 2\mathbf{I}$

37 $\begin{pmatrix} 4 & a \\ b & -2 \end{pmatrix}\begin{pmatrix} -2 & 3 \\ -2 & 4 \end{pmatrix} = -2\mathbf{I}$

38 $\begin{pmatrix} 8 & 5 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 2 & -5 \\ a & b \end{pmatrix} = \mathbf{I}$

39 $\begin{pmatrix} 2 & -3 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} a & 3 \\ -1 & b \end{pmatrix} = 5\mathbf{I}$

40 $\begin{pmatrix} 4 & a \\ 6 & b \end{pmatrix}\begin{pmatrix} -9 & 5 \\ 6 & -4 \end{pmatrix} = -6\mathbf{I}$

41 $\begin{pmatrix} a & 2 \\ 5 & b \end{pmatrix}\begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ 6 \end{pmatrix}$

42 $\begin{pmatrix} 2 & -1 \end{pmatrix}\begin{pmatrix} a & 4 \\ 2b & b \end{pmatrix} = \begin{pmatrix} 4 & 5 \end{pmatrix}$

Determinants

There is a number associated with a square matrix called the **determinant**.

The determinant of a 2×2 matrix is the product of the numbers in the leading diagonal minus the product of the numbers in the other diagonal, i.e. the determinant of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the number $ad - bc$.

The determinant of a matrix **A** is written as det **A** or as $|\mathbf{A}|$.

A **singular matrix** has a determinant equal to zero.

$\text{Det}\begin{pmatrix} 3 & 5 \\ 2 & 6 \end{pmatrix} = 3 \times 6 - 5 \times 2$
$= 18 - 10 = 8$

The inverse of a matrix

When we can find a matrix **B** such that **AB** = **I**, **B** is called the inverse of **A**.
The inverse is written as $\mathbf{A}^{-1}$.

This is similar to the multiplicative inverse of a real number a: $a \times a^{-1} = a \times \frac{1}{a} = 1$. The difference is that every real number (except 0) has a multiplicative inverse but only square matrices have inverses and not all of them do.

For a 2 × 2 matrix, $\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the inverse is $\frac{1}{|\mathbf{A}|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ provided that $|\mathbf{A}| \neq 0$.

$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ is called the **adjoint** of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

So to find the inverse of a 2 × 2 matrix, interchange the elements in the leading diagonal, change the sign of the elements in the other diagonal then divide by the determinant.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix}$$

Notice that $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$

EXERCISE 11b

Example:

Given that $\mathbf{M} = \begin{pmatrix} 2 & 6 \\ -1 & -4 \end{pmatrix}$

a Show that **M** is not a singular matrix.
b Find $\mathbf{M}^{-1}$.

a $|\mathbf{M}| = (2) \times (-4) - (6) \times (-1) = -8 + 6 = -2$

b $\mathbf{M}^{-1} = \frac{1}{|\mathbf{M}|}\begin{pmatrix} -4 & -6 \\ 1 & 2 \end{pmatrix} = \frac{1}{-2}\begin{pmatrix} -4 & -6 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ -\frac{1}{2} & -1 \end{pmatrix}$

Interchange 2 and −4, change the signs of 6 and −1, then divide each element by −2.

For each matrix find, if possible, the inverse matrix. When you cannot find the inverse matrix state why it does not exist.

1 $\mathbf{A} = \begin{pmatrix} 5 & 4 \\ 2 & 2 \end{pmatrix}$
2 $\mathbf{B} = \begin{pmatrix} 7 & 9 \\ 3 & 4 \end{pmatrix}$
3 $\mathbf{C} = \begin{pmatrix} 5 & 8 \\ 2 & 3 \end{pmatrix}$
4 $\mathbf{D} = \begin{pmatrix} 8 & 4 \\ 4 & 2 \end{pmatrix}$
5 $\mathbf{P} = \begin{pmatrix} 4 & 3 \\ -3 & -2 \end{pmatrix}$
6 $\mathbf{Q} = \begin{pmatrix} -11 & -5 \\ 4 & 2 \end{pmatrix}$
7 $\mathbf{M} = \begin{pmatrix} 7 & 4 \\ -4 & -2 \end{pmatrix}$
8 $\mathbf{N} = \begin{pmatrix} 5 & 6 \\ 7 & 9 \end{pmatrix}$
9 $\mathbf{A} = \begin{pmatrix} -2 & -4 \\ -3 & 4 \end{pmatrix}$
10 $\mathbf{B} = \begin{pmatrix} 5 & 4 \\ 4 & 2 \end{pmatrix}$
11 $\mathbf{F} = \begin{pmatrix} 2 & 0 \\ -1 & -3 \end{pmatrix}$
12 $\mathbf{E} = \begin{pmatrix} -7 & 3 \\ 5 & -2 \end{pmatrix}$

Example:

Given that $\begin{pmatrix} 2x-1 & x \\ 2x & x-2 \end{pmatrix}$ is a singular matrix, find the value of x.

If the matrix is singular, the determinant is zero.

$$\therefore (2x-1)(x-2) - 2x^2 = 0$$
$$2x^2 - 5x + 2 - 2x^2 = 0$$
$$-5x + 2 = 0$$
so $$x = \frac{2}{5}$$

Each matrix in questions **13** to **18** is a singular matrix. Find the value of x.

13 $\begin{pmatrix} x+2 & x-2 \\ x+8 & x-4 \end{pmatrix}$ **14** $\begin{pmatrix} 3x-1 & x \\ 3x+4 & x+2 \end{pmatrix}$ **15** $\begin{pmatrix} 2x+1 & 4x+4 \\ 2x & 4x+1 \end{pmatrix}$

16 $\begin{pmatrix} x+2 & x-2 \\ x+8 & x \end{pmatrix}$ **17** $\begin{pmatrix} x+2 & x-1 \\ 3x+1 & x+1 \end{pmatrix}$ **18** $\begin{pmatrix} 2x & x-1 \\ 3x+3 & 2x-2 \end{pmatrix}$

Applications of matrices

Matrices are ideal for solving problems involved with multiplying items together and adding them. This is because the matrix processes can be mechanised and programmed into software for computers. We will look at two applications here and at a third application in Chapter 19.

Simultaneous equations

Matrices can be used to solve simultaneous linear equations.

Consider the equations
$$2x - 3y = 10$$
$$3x + 2y = 14$$

We can write these using matrices as $\begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10 \\ 14 \end{pmatrix}$

$\begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x-3y \\ 3x+2y \end{pmatrix}$ so $\begin{pmatrix} 2x-3y \\ 3x+2y \end{pmatrix} = \begin{pmatrix} 10 \\ 14 \end{pmatrix}$

We can then solve the equations by multiplying the left-hand sides by the inverse of the 2×2 matrix.

The inverse of $\begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}$ is $\frac{1}{13}\begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}$

so $\frac{1}{13}\begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{13}\begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 10 \\ 14 \end{pmatrix}$

Multiply both sides by 13

$\Rightarrow \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ -3 & 2 \end{pmatrix}\begin{pmatrix} 10 \\ 14 \end{pmatrix}$

Find the matrix product of the two square matrices and the two matrices on the other side of the equals sign.

$\Rightarrow \begin{pmatrix} 13 & 0 \\ 0 & 13 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 62 \\ -2 \end{pmatrix}$, i.e. $\begin{pmatrix} 13x \\ 13y \end{pmatrix} = \begin{pmatrix} 62 \\ -2 \end{pmatrix}$

Find the remaining matrix product, then compare elements.

so $13x = 62$ and $13y = -2$

i.e. $x = \frac{62}{13}$ and $y = -\frac{2}{13}$.

EXERCISE 11c

Use matrices to solve these equations.

1. $3x + 4y = 26$, $4x - 3y = 18$
2. $5p + q = 22$, $2p - 5q = 25$
3. $3m - 4n = 30$, $4m + 5n = 9$
4. $6x - 5y = 11$, $4x + 3y = 1$
5. $10x + 3y = -24$, $5x - 4y = -23$
6. $2x + 3y = 2$, $8x - 9y = 1$
7. $4a - 6b = 13$, $a + b = 2$
8. $12p - 3q = 1$, $4p + 6q = 5$
9. $7x - 5y = 16$, $4x + y = -14$
10. $x - 2y = 2$, $3x - y = -2$
11. $2x - 3y = 4$, $2x + 3y = -10$
12. $5p - 5q = 1$, $p + q = 3$
13. $2x + 3y = -1$, $2x - 2y = 5$
14. $2x + 3y = 4$, $x - y + 4 = 0$
15. $3x + 2y = 16$, $2x + 3y = 29$
16. $3x + y = 6$, $2x - y = 0$
17. $6a - 5b = -7$, $3a + 4b = 16$
18. $3x - 4y = 1$, $6x - 6y = 5$
19. $5x + y = 15$, $7x - 2y = 4$
20. $4x - y = 14$, $3x + 2y = 5$
21. $2x + y = 7$, $3x + 2y = 12$
22. $5x - y = 17$, $2x + 3y = 0$
23. $4x - y = 10$, $3x + 5y = 19$
24. $5x - 3y = 11$, $4x + y = 19$
25. $5x + 7y = 4$, $x + 2y = 2$
26. $3x + 5y = -14$, $3x - 2y = 14$
27. $7x + 5y = 6$, $3x - 4y = 21$

Numerical applications

Calculations that involve multiplying pairs of numbers, then adding them, can be done using matrices.

These two tables show the purchases and costs of items bought by a school over a three-month period.

	May	June	July
Ink cartridges	6	10	2
Reams of paper	10	15	5
Packets of CD-R discs	5	2	5

	1 ink cartridge	1 ream of paper	1 packet CD-R discs
Cost ($)	25	5	10

We can write the information in the top table as a 3 × 3 matrix:

	My	Jn	Jly
Ink cartridges	6	10	2
Reams of paper	10	15	5
Packets of CD-R discs	5	2	5

$$\begin{pmatrix} 6 & 10 & 2 \\ 10 & 15 & 5 \\ 5 & 2 & 5 \end{pmatrix}$$

and we can write the information in the lower table as a 1 × 3 matrix:

	1 ink cartridge	1 ream of paper	1 packet CD-R discs
Cost ($)	(25	5	10)

To find the total amount spent in May, we need to multiply the cost of each item by the number of each item in May then add these sums up.

We need to do a similar calculation to find the total amount spent in June and again in July.

Multiplying the one-row matrix by the square matrix gives these amounts:

$$\begin{pmatrix} 25 & 5 & 10 \end{pmatrix}\begin{pmatrix} 6 & 10 & 2 \\ 10 & 15 & 5 \\ 5 & 2 & 5 \end{pmatrix} = \begin{pmatrix} 250 \\ 345 \\ 125 \end{pmatrix}$$

Notice that the numbers of each item are in the columns of the square matrix and the cost of each item is in the one-row matrix.

INVESTIGATION

Find a matrix that can be multiplied by the column matrix above to give the total amount spent over the three months.

EXERCISE 11d

Example:

In a multiple choice test of 40 questions, marks are awarded as follows:
1 for a correct answer, 0 for no answer and –1 for a wrong answer
Three students A, B and C answered the questions as follows:

A: 25 correct, 6 not answered and 9 wrong

B: 30 correct, 2 not answered and 8 wrong

C: 24 correct, 14 not answered and 2 wrong

a Make a column matrix, **A**, to represent the marks awarded for a correct answer, no answer and a wrong answer.

b Make a 3 × 3 matrix, **B**, to represent the numbers of questions correct, not answered and wrong by the three students.

c Construct a matrix product that gives the total mark awarded to each student. Write down these marks.

a $A = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

b $B = \begin{pmatrix} 25 & 6 & 9 \\ 30 & 2 & 8 \\ 24 & 14 & 2 \end{pmatrix}$

The marks awarded for each type of answer go down the column in **A**, so we want the number of each answer to go across the rows in **B**.

c $\begin{pmatrix} 25 & 6 & 9 \\ 30 & 2 & 8 \\ 24 & 14 & 2 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 16 \\ 22 \\ 22 \end{pmatrix}$

A got 16 marks, B got 22 marks and C got 22 marks.

AB is not possible, but **BA** is. Check that the square matrix is correct by checking that the type of answer is multiplied by the mark for that answer.

1 Last week Keith bought twelve 50c stamps and eight 25c stamps.This week he bought eight 50c stamps and fifteen 25c stamps.Show this information in two matrices and multiply them together to show how much Keith spent on stamps
a last week **b** this week.

2 Nikki bought 2 bars of chocolate and 3 packets of crisps while Joe bought 3 bars of chocolate and 1 packet of crisps.
a Show this information in a matrix **A**.
b Bars of chocolate cost 130c and packets of crisps 80c. Show this in a matrix **B**.
c Construct a matrix product to show how much each person paid for their purchases.

3 After ten games the results of the four local teams in a cricket league were as given in the table.

	Wins	Draws	Losses
Abbott	7	1	2
Berwick	4	1	5
Curtleigh	5	4	1
Denster	6	2	2

The points system is 3 for a win, 1 for a draw and 0 for a loss.
a Write this information as two matrices and multiply them together.
b Explain the meaning of the numbers in the resulting matrices.
c Which team has scored **i** the most points **ii** the least points?

4 Three contestants took part in a quiz. Each contestant was asked twenty questions. Their replies showed that
Anne got 13 correct, 5 incorrect and the rest were unanswered
Ben got 11 correct, 2 incorrect and the rest were unanswered
Colin got 14 correct, 4 incorrect and the rest were unanswered.
The scoring was: 3 points for a correct answer, −1 for an incorrect answer and 0 for an unanswered question.
a Make a 3×3 matrix, **P**, for the numbers of questions answered correctly, incorrectly and unanswered by the three contestants and a column matrix, **Q**, for the points system.
b Construct the matrix product **PQ** to give the score for each contestant. Who won?

5 This table shows the coins two ladies had in their purses.

	50 cent coins	25 cent coins	10 cent coins
Mrs Cappola	3	8	5
Mrs Raman	6	4	2

Write this information in a matrix, **A,** and a column matrix, **B**, so that when the matrices are multiplied together they show the total value of these coins in each purse. What is the difference in the amounts?

6 Mrs Rohan went shopping for three items A, B and C. She priced these items in three different shops. These prices are given in the table.

	Shop X	Shop Y	Shop Z
A	10 cents	11 cents	12 cents
B	21 cents	20 cents	18 cents
C	14 cents	16 cents	15 cents

a Show this information in a suitable matrix.
b Mrs Rohan decides that she wants to buy 3 of Item A, 5 of Item B and 2 of Item C.
Show this information in a column matrix.
c Use the product of two matrices to find the total costs of these items in each of the three shops. In which shop should she make her purchase?

7 Given below are the time sheets for the four employees in a small business. They work different numbers of agreed hours and often do overtime.

	Agreed hours	Overtime in hours
Mr Singh	16	2
Miss Walcott	24	5
Mrs Elcott	30	3
Mr Smith	20	4

a Show this information in a matrix.
b The rates of pay are \$12 per hour for agreed hours and \$15 per hour for overtime.
Show this as a column matrix.
c Multiply the matrices together and give a meaning to each of the elements.
d How much more was the greatest income than the least?

8 The number of litres of fuel sold from three pumps at a service station during the first five days of one week are shown below.

	Mon	Tues	Wed	Thurs	Fri
Pump 1	240	140	230	200	310
Pump 2	180	180	200	190	260
Pump 3	260	150	140	190	280

a Show this information in a matrix.
b The cost per litre, in dollars, at the three pumps is:
Pump 1: \$8   Pump 2: \$7   Pump 3: \$6.50
Show this as a column matrix.
c Use these matrices to work out a column matrix that shows the total cash taken each day. How much more was taken on Friday than on Tuesday?

9 On Friday Mrs Quinllan posts 5 letters under 100 g, 3 letters under 250 g and 2 letters under 500 g. On Saturday she posts 10 letters under 100 g, 4 under 250 g and 3 under 500 g.
The postage rates are, respectively, 90 c, 120 c and 200 c for first class mail and 65 c, 110 c and 165 c for second class mail.
Show this information in suitable matrices and by matrix multiplication show Mrs Quinllan's postage costs on each day for first class mail and for second class mail.

10 Three textbooks, A, B and C, are available in hardback and paperback. For book A the hardback costs \$35 and the paperback \$20. For book B the prices are \$30 and \$15, and for book C \$40 and \$25.
A school decides to order small numbers of these books for three groups of students.
For Group 1 they order 4 of A, 7 of B and 3 of C,
for Group 2 the numbers are 5, 6 and 4
and for Group 3 they are 4, 6 and 3 respectively.
Use matrices to show the costs for each group if they purchase
a all hardback books   **b** all paperback books.

ABCD MIXED EXERCISE 11

Several answers are given for these questions.
Write down the letter that corresponds to the correct answer.

1 $(2\ 1)\begin{pmatrix}-1\\2\end{pmatrix} =$

A 3   **B** $\begin{pmatrix}-2\\2\end{pmatrix}$ **C** $(-2\ \ 2)$ **D** 0

2 $\mathbf{A} = \begin{pmatrix}1 & -1\\2 & 0\end{pmatrix}$, det $\mathbf{A}$ =

A −1   **B** −2   **C** 1   **D** 2

3 $(1\ -1)\begin{pmatrix}-1\\1\end{pmatrix} =$

A (−1)   **B** (−2)   **C** $\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix}$ **D** $\begin{pmatrix}-1 & 1\\1 & -1\end{pmatrix}$

4 $\mathbf{M} = \begin{pmatrix}2 & 4\\3 & x\end{pmatrix}$. If $\mathbf{M}$ is singular then x =

A −12   **B** −6   **C** 6   **D** 12

5 $\mathbf{M}$ is a non-singular 2 × 2 matrix. $\mathbf{MM}^{-1}$ =

A $\begin{pmatrix}0 & 1\\1 & 0\end{pmatrix}$ **B** $\begin{pmatrix}1 & 1\\1 & 1\end{pmatrix}$ **C** $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$ **D** $\begin{pmatrix}1 & 1\\0 & 0\end{pmatrix}$

6 If $\begin{pmatrix} 2 \\ -3 \end{pmatrix} + 3\begin{pmatrix} 5 \\ y \end{pmatrix} = \begin{pmatrix} 17 \\ y \end{pmatrix}$, then $y =$

A $\frac{2}{3}$ **B** $\frac{3}{4}$ **C** $1\frac{1}{2}$ **D** 3

7 If **M** and **N** are both 2×2 matrices and $\mathbf{M} + \mathbf{N} = \mathbf{M}$, then **N** is

A $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ **B** $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ **C** $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ **D** $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

8 $\mathbf{M} = \begin{pmatrix} 2x & 4 \\ -1 & 1 \end{pmatrix}$ is a non-singular matrix. x cannot be

A -4 **B** -2 **C** $-\frac{1}{2}$ **D** $\frac{1}{2}$

9 $\mathbf{M} = \begin{pmatrix} x & 3 \\ 2 & 6 \end{pmatrix}$ is a singular matrix, $x =$

A -1 **B** 0 **C** 1 **D** $\frac{1}{2}$

10 $\mathbf{M} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, $\mathbf{M}^{-1} =$

A $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ **B** $\begin{pmatrix} -1 & 1 \\ 1 & -2 \end{pmatrix}$ **C** $\begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$ **D** $\begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$

PUZZLE

If there are five Sundays, five Mondays and five Tuesdays in July on which day of the week will the first of August fall?

MATHS IS OUT THERE

Matrices and determinants have their origin in the study of systems of linear equations. Their beginnings go back to the second century BC.

IN THIS CHAPTER YOU HAVE SEEN THAT...

- a matrix is a rectangular array of numbers
- matrices can be added or subtracted, provided they are the same size, by adding or subtracting their corresponding elements
- matrices can be multiplied provided that the number of elements in the rows of the first matrix is equal to the number of elements in the columns of the second matrix. The element in the nth row and mth column of the product is the sum of the products of the corresponding elements in the nth row of the first matrix and the mth column of the second matrix,

  i.e. $\begin{pmatrix} a & b \\ . & . \end{pmatrix} \begin{pmatrix} : & p \\ : & q \end{pmatrix} = \begin{pmatrix} * & ap + bq \\ * & * \end{pmatrix}$

- the determinant of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is the number $ad - bc$
- the inverse of the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\frac{1}{|\mathbf{A}|}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$ where $|\mathbf{A}|$ is the determinant of matrix **A**.

12 SEQUENCES

AT THE END OF THIS CHAPTER YOU SHOULD BE ABLE TO...

1. Understand what a sequence is.
2. Recognise some types of number patterns.
3. Continue a sequence.
4. Find the rule for continuing a sequence.

BEFORE YOU START

you need to know:
✓ how to work with numbers
✓ how to form an algebraic expression.

KEY WORDS

number pattern, sequence

MATHS IS OUT THERE

The great logician Betrand Russell once claimed that he could prove anything if given that 1 + 1 = 1. So one day, some smarty-pants asked him, 'OK. Prove that you're the Pope.' He thought for a while and proclaimed, 'I am one. The Pope is one. Therefore, the Pope and I are one.'

Sequences

A **sequence** is an ordered list of objects. The objects are called the terms, so there is a first term, a second term, and so on.

The terms may be numbers or geometric shapes or algebraic expressions. When the terms are, for example, 1×2, 2×3, 3×4, ..., the sequence may be called a **number pattern**.

These are sequences:
2, 4, 6, 8, ...
, ...
$x, x^2, x^3, \ldots$

When the terms are geometric shapes, a number sequence may be formed from the perimeters or areas of those shapes (or some other property). For example, this table shows a sequence of shapes formed from equilateral triangles with sides of length 1 cm and the sequence of numbers formed from the perimeters of the shapes.

Shape				...
Perimeter, cm	3	4	5	...

There is usually a pattern that can be used to continue a sequence. We can usually spot the pattern in simple cases such as

numbers increasing by the same amount: 1, 5, 9, 13, 17, ...

squares of numbers: 1, 4, 9, 16, 25, ...

powers of a number: 3, 9, 27, 81, ...

When we have spotted the pattern, it is easy to continue the sequence. To find, say, the 50th term in a sequence, we can list all the terms up to this term, but that is time-consuming. If we can find an expression for the nth term in terms of n, we can then substitute a value for n to find a particular term. The nth term is sometimes denoted by u_n.

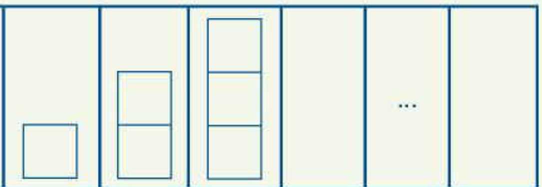
If we know that the nth term of a sequence is $n^2 - n$ then the 4th term is the value of $n^2 - n$ when $n = 4$, i.e. $16 - 4 = 12$

It helps to construct a table listing the term numbers and the terms. Then look for numbers increasing by the same amount or squares of numbers or powers of a number. The worked examples illustrate how to proceed.

EXERCISE 12a

Example:

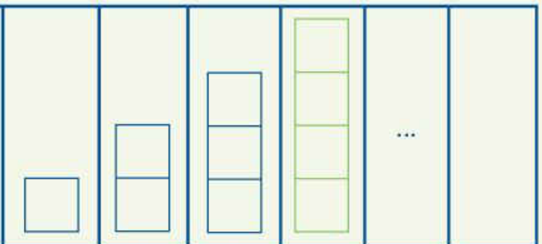
The table shows a sequence of shapes made from squares with sides of length 1 unit.



Shape number, *n*	1	2	3	4		
Perimeter in units	4	6	8			

a Draw the fourth shape in the sequence.
b A shape in the sequence has a perimeter of 50.
What is the area of the shape?

a



Pattern number, *n*	1	2	3	4	...	n
Perimeter in units	4	6	8	10	...	
$2n$	2	4	6	8		$2n$

The shapes increase by one square each time so the fourth shape has four squares.

b

First find an expression for the perimeter of the nth shape. The perimeters form a sequence of numbers that increase by 2 each time. Start by multiplying the shape numbers by 2 and add a row to the table to show this sequence.

The numbers in the sequence formed by the perimeters are each 2 more than the numbers in the sequence formed by the values of $2n$.

The perimeter of the nth shape $= 2n + 2$

When the perimeter is 50, $2n + 2 = 50$

$2n = 48$

so $n = 24$

The area of the shape number 24 is 24 square units.

Check this by substituting a value of n to see that it gives the correct perimeter: $n = 3 \Rightarrow 2n + 2 = 8$

Each shape has the same number of squares as the pattern number. The area of 1 square is 1 square unit.

1 The table shows a sequence of shapes made from squares with side 1 cm.

Shape	**Shape number**	**Area of shape (cm^2)**	**Perimeter (cm)**
	1	1	4
	2	2	8
	3	3	12

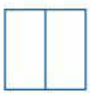
a Draw the next two shapes to continue the sequence.

b For each shape drawn in part **a**, complete the table by stating
 i the area of the shape **ii** the perimeter of the shape.

c A shape in this sequence has an area of $10\,m^2$. What is the perimeter of the shape?

d A shape in this sequence has a perimeter of 60 cm. What is the area of this shape?

2 The table shows a sequence of shapes made from rectangles measuring 2 cm by 1 cm.

Shape	**Shape number**	**Area of shape (cm^2)**	**Perimeter (cm)**
	1	2	6
	2	4	8
	3	6	10

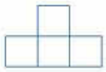
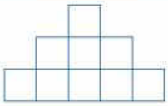
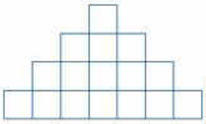
a Draw the next two shapes to continue the sequence.

b For each shape drawn in part **a**, complete the table by stating
 i the area of the shape **ii** the perimeter of the shape.

c A shape in this sequence has an area of $36\,m^2$. What is the perimeter of the shape?

d A shape in this sequence has a perimeter of 48 cm. What is the area of this shape?

3 The table shows a sequence of shapes made from squares with sides of 1 unit.

Shape	Shape number	Area of shape in square units	Perimeter of shape in units
	1	4	10
	2	9	16
	3	16	22

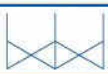
a Draw the next two shapes to continue the sequence.
b For each shape drawn in part **a**, complete the table by stating
  i the area of the shape **ii** the perimeter of the shape.
c A shape in the sequence has an area of 49 square units. What is the perimeter of the shape?
d A shape in this sequence has a perimeter of 58 units. What is the area of this shape?

4 The table shows a sequence of shapes made from sticks of various lengths.

Shape	Area of shape	Number of sticks used
	1 square unit	5
	2 square units	9
	3 square units	13

a Draw the next two shapes to continue the sequence.
b For each shape drawn in part **a**, complete the table by stating
  i the area of the shape **ii** the number of sticks used in each shape.
c A shape in this sequence has an area of 7 square units. How many sticks are used to form it?
d A shape in this sequence uses 25 sticks. What is the area of this shape?

5 The table shows a sequence of shapes made from rods of equal lengths. The number of rods in each shape depends on its position in the sequence.

Shape	Position number	Number of rods in shape
	1	4
	2	7
	3	10

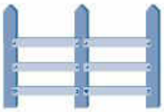
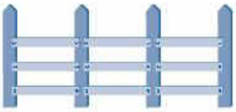
a Draw the next two shapes to continue the sequence.

b For each shape drawn in part **a**, complete the table by stating
i the position number **ii** the number of rods used.

c A shape in this sequence is in 9th position. How many rods are used?

d A shape in this sequence uses 37 rods. What is the position number?

6 The table shows a sequence of fences made from posts and rails.

Shape			
Shape number	1	2	3
Number of posts	2	3	4
Number of rails	3	6	9

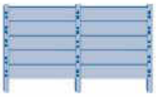
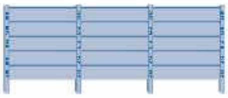
a Draw the next two fences to continue the sequence.

b For each new fence drawn in part **a**, complete the table by stating
i the number of posts supporting the fence
ii the number of rails in the fence.

c A fence in this sequence has 10 posts. How many rails does it have?

d A fence in this sequence has 36 rails. How many posts are needed?

7 The table shows a sequence of fence panels with horizontal slats tied together by vertical lengths.

Shape			
Shape number	1	2	3
Number of vertical lengths	2	3	4
Number of slats	5	10	15
Total number of pieces of wood used	7	13	19

a Draw the next two fence panels to continue the sequence.

b For each shape drawn in part **a**, complete the table by stating
i the number of vertical lengths in the shape
ii the number of slats in the shape
iii the total number of pieces of wood in each shape.

c A fence panel in this sequence has 40 slats.
How many pieces of wood have been used?

d A fence panel in this sequence uses 43 pieces of wood. How many vertical lengths are there?

e A fence panel in this sequence uses 37 pieces of wood. How many slats are there?

8 The table shows a sequence of Us made from small sticks.

Shape			
Position in sequence	1	2	3
Number of sticks used	3	7	11

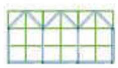
a Draw the next two Us to continue the sequence.

b For each new U drawn in part **a**, complete the table by stating
  i the position of the U in the sequence
  ii the number of sticks needed to make the U.

c A U in this sequence uses 27 sticks. What is the position of this U in the sequence?

d How many sticks are needed to make the twelfth U in the sequence?

9 The table shows a sequence of shapes made from long and short sticks to form pentagons which are laid on a square grid of side 1 unit.

Shape			
Shape number	1	2	3
Area of shape in square units	5	10	15
Number of short sticks used	2	4	6
Number of long sticks used	3	5	7
Total number of sticks used	5	9	13

a Draw the next two shapes to continue the sequence.

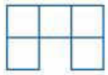
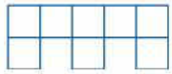
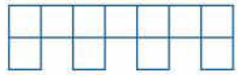
b For each shape drawn in part **a**, complete the table by stating
  i the number of short sticks in the shape
  ii the number of long sticks in the shape
  iii the total number of sticks in each shape.

c A shape in this sequence has an area of 30 square units. How many short sticks are in it?

d A shape in this sequence uses 13 long sticks. What is its area?

e A shape in this sequence has an area of 65 square units. How many sticks have been used to make it?

10 The table shows a sequence of shapes made from squares of side 1 unit.

Shape			
Shape number	1	2	3
Area of shape in square units	5	8	11
Perimeter in units	12	18	24

- **a** Draw the next two shapes to continue the sequence.
- **b** For each shape drawn in part **a**, complete the table by stating
  - **i** the area of the shape **ii** the perimeter of the shape.
- **c** A shape in this sequence has an area of 26 square units. What is its perimeter?
- **d** The perimeter of a shape in this sequence is 66 units. What is its area?

11 Rohan made these patterns with used matchsticks.

- **a** Write down the number of matchsticks required for each of the first five patterns in the sequence.
- **b** How many matchsticks are needed for
  - **i** the nth term in the sequence **ii** the 30th term in the sequence?
- **c** Rohan has a box of 200 matchsticks. How many patterns can he make in this sequence? How many matchsticks are left over?

Example:

These patterns are formed by drawing a dark triangle in each light triangle as shown.

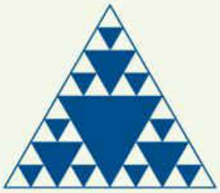
Pattern number	1	2	3	4	…	n
Number of light triangles	1	3	9	27		

- **a i** Draw pattern number 4.
  - **ii** Write down the number of light triangles it contains.
- **b** Determine the number of light triangles in pattern number 7.
- **c** Calculate the pattern number that contains 59 049 light triangles.
- **d** Find an expression for the number of light triangles in pattern number n.

MATHS IS OUT THERE

When continued, the pattern in this example forms Sierpinski's Triangle (or gasket), named after the Polish mathematician Waclaw Sierpinski who described it in 1916.

Time for a library (or Internet) search: Find out more about Sierpinski's Triangle.

a i  **ii** 27

b The number of white triangles is $3^6 = 729$.

The sequence of numbers of white triangles are powers of 3: writing them as powers of 3: $3^0, 3^1, 3^2, 3^3, \ldots$ shows that the power is 1 less than the pattern number. So the number of light triangles in pattern number 7 is 3^6.

c 59 049 is a power of 3.
$59\,049 = 3^{10}$
The pattern number is 11.

Try different powers of 3 on a calculator, or express 59 049 as a product of prime factors.

d The number of white triangles in pattern number n is 3^{n-1}.

12 This array of numbers is called Pascal's triangle.

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

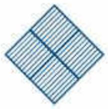
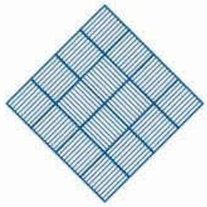
a Copy this pattern and write down the next three rows.
b Add the numbers in each row to form a sequence. The first term is 1, the second $1 + 1 = 2$, the third $1 + 2 + 1 = 4$ and so on.
c Find an expression for the sum of the numbers in the nth row of Pascal's triangle.

13 The nth term of a sequence is defined by $u_n = \frac{1}{2}n(n + 1)$.
Find
a the first five terms of the sequence
b the 20th term
c an expression for the term before u_n (i.e. u_{n-1})
d an expression for $u_n - u_{n-1}$
e the values of $u_2 - u_1, u_3 - u_2, u_4 - u_3, u_5 - u_4$
f the 20th term of the sequence in part **e**.

14 The nth term of a sequence is defined by $u_n = \frac{n}{6}(n + 1)(2n + 1)$.
Find
a the first five terms of the sequence
b the 20th term
c an expression for the term before u_n (i.e. u_{n-1})
d an expression for $u_n - u_{n-1}$
e the values of $u_2 - u_1, u_3 - u_2, u_4 - u_3, u_5 - u_4$
f the 20th term of the sequence in part **e**.

15 This is a sequence of pairs of numbers: (1, 2), (2, 5), (3, 10), (4, 17) ...
Find
a the next pair in the sequence
b the 10th pair in the sequence
c the nth pair in the sequence.

16 The table shows a sequence of shapes made from squares of side 1 unit.

Shape			
Area of shape in square units	4	9	16
Perimeter in units	8	12	16

a Draw the next two shapes to continue the sequence.

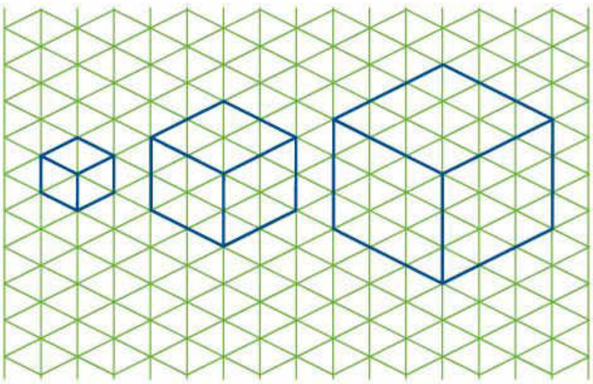
b For each shape drawn in part **a**, complete the table by stating

i the area of the shape **ii** the perimeter of the shape.

c A shape in this sequence has an area of 64 square units. What is its perimeter?

d The perimeter of a shape in this sequence is 28 units. What is its area?

17 Starting with a unit cube the length of each edge is increased by 1 to give the next cube in the sequence.



Pattern number	1	2	3	4	...	n
Volume (cubic units)	1	8	27			
Surface area (square units)	6	24	54			

a Sketch pattern number 4.

b Write down **i** its volume **ii** its surface area.

c For pattern number 8, find **i** the volume **ii** the surface area.

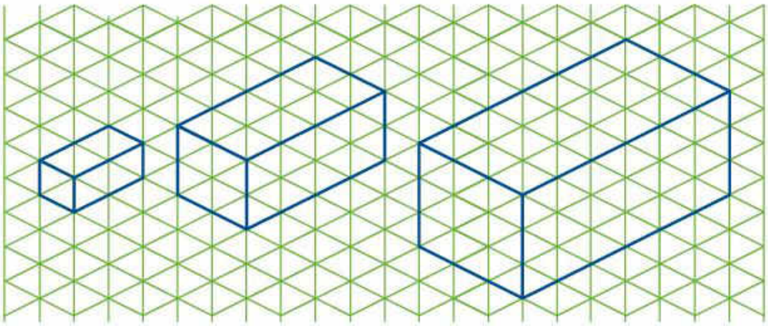
d Calculate the pattern number that has

i a volume of 3375 cubic units

ii a surface area of 600 square units.

e For pattern number n, find an expression for

i the volume **ii** the surface area.

18 

Starting with a cuboid measuring 2 units by 1 unit by 1 unit the edges of each square end are increased by 1 unit and the length by 2 units to give the next cuboid in the sequence.

Pattern number	1	2	3	4	...	*n*
Volume (cubic units)	2	16	54			
Total length of the edges (units)	16	32	48			

a Sketch pattern number 4.
b Write down **i** its volume **ii** the total length of its edges.
c For pattern number 7, find
 i the volume **ii** the total length of the edges.
d Calculate the pattern number that has
 i a volume of 432 cubic units.
 ii edges that total 144 units in length.
e For pattern number *n*, find an expression for
 i the volume **ii** the total length of the edges.

19 This sequence shows a number of triangles of increasing size made with dots.
The number of dots gives the sequence of triangular numbers.

							
Pattern number	1	2	3	4	5	6	7
Triangular number	1	3	6	10			
Sum of consecutive triangular numbers		4	9	16			

a Draw the next triangle in the sequence. How many dots does it have?
b Copy the table and fill in the blanks.
c The triangular number for pattern number n is $\frac{nx}{2}$.
 Find x in terms of n.
d Use your answer to part **c** to find the 15th triangular number.
e The sum of a triangular number and the one immediately preceding it is 484. What is the number?

20 This sequence shows the maximum number of regions a circle can be divided into as the number of chords increases.

				
Number of chords	1	2	3	4
Number of regions	2	4	7	11

a Draw the next diagram in this sequence.

b What is the greatest number of regions a circle can be divided into by 5 chords?

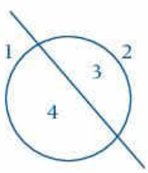
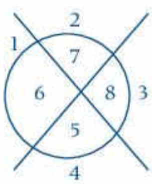
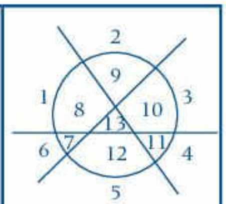
c Seven chords are drawn in a circle.
What is the greatest number of regions that result?

d When there are n chords the greatest possible number of regions is $\frac{a(n + 1)}{2} + 1$.
Use the number pattern to find a in terms of n.

e Chords are drawn in a circle to give 56 regions. Use your answer to part **d** to find how many chords have been drawn.

21 This sequence shows the maximum number of regions that can be drawn, inside and outside a circle, as the number of secants increases.

A secant is a line that cuts a circle in two points.

			
Number of secants	1	2	3
Number of regions	4	8	13

a Draw the next diagram in this sequence.

b What is the greatest number of regions possible if 4 secants are drawn? Illustrate this on a diagram.

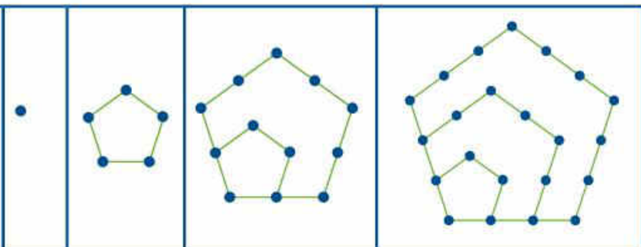
c Use the pattern to determine the greatest number of regions 5 secants will divide the area inside and outside a circle into.

d Eight secants are drawn through a circle. What is the greatest number of regions that result?

e When there are n secants the greatest possible number of regions is $\frac{(n + 2)(n + 3)}{2} - a$.
Use the number pattern to find a.

f Secants are drawn through a circle to give 89 regions. Use your answer to part **e** to find how many secants have been drawn.

22 This sequence shows an ever increasing number of regular pentagons with dots marked on the sides always the same distance apart. The number of dots gives the sequence of pentagonal numbers.



Pattern number	1	2	3	4	5	6
Pentagonal number	1	5	12	22		

a Draw the next two pentagons in the sequence.
b Copy the table and fill in the blanks.
c The pentagonal number for pattern number n is $x(3n - 1)$. Find x in terms of n.
d Use your answer to part **c** to find the 10th pentagonal number.

23

1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16
17	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32
33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48

In this number grid, the shaded cross is called X_{20} because 20 is in the centre of the cross.
a Draw X_n and fill in the squares with expressions in terms of n.
b Find the nth term of the sequence 2, 11, 20, 29, ...

24 Study the number patterns below.

$1 = 1$ $\qquad 1^3 = 1$
$1 + 2 = 3$ $\qquad 1^3 + 2^3 = 9$
$1 + 2 + 3 = 6$ $\qquad 1^3 + 2^3 + 3^3 = 36$
$1 + 2 + 3 + 4 = 10$ $\qquad 1^3 + 2^3 + 3^3 + 4^3 = 100$

Use these patterns to find the value of each number denoted by a letter.
a $1 + 2 + 3 + 4 + 5 + 6 = a$ $\qquad 1^3 + 2^3 + 3^3 + 4^3 + 5^3 + 6^3 = b$
b $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 = 36$ $\qquad 1^3 + 2^3 + 3^3 + 4^3 + ... + 7^3 + 8^3 = c$
c $1 + 2 + 3 + ... + 9 + 10 = 55$ $\qquad 1^3 + 2^3 + 3^3 + 4^3 + ... + 9^3 + 10^3 = d$
d $1 + 2 + 3 + ... + 10 + 11 + 12 = e$ $\qquad 1^3 + 2^3 + 3^3 + ... + 10^3 + 11^3 + 12^3 = 6084$

25 Study the number pattern below.

$3^2 - 2^2 = 9 - 4 = 5 = 3 + 2$
$4^2 - 3^2 = 16 - 9 = 7 = 4 + 3$
$5^2 - 4^2 = 25 - 16 = 9 = 5 + 4$

a Use the pattern to complete each of the following lines
i $6^2 - 5^2 =$ **ii** $8^2 - 7^2 =$ **iii** $12^2 - 11^2 =$
b One line in the pattern ends ... $= 10 + 9$. What comes in front of this?
c Write down the twelfth line in the pattern.
d Complete the line in the pattern $n^2 - (n - 1)^2 =$

26 Study this number pattern.

$3^2 = 9 \qquad 2 \times 4 = 8$

$4^2 = 16 \qquad 3 \times 5 = 15$

$5^2 = 25 \qquad 4 \times 6 = 24$

$6^2 = 36 \qquad 5 \times 7 = 35$

a Write down the next three lines in the pattern.

b Write down and complete the following line of the pattern:

i $14^2 = 196 \qquad 13 \times ... = ...$ **ii** $17^2 = ... \qquad ... \times 18 = ...$

c If $43^2 = 1849$, what number multiplied by 44 gives 1848?

d Given that $56^2 = 3136$ which two consecutive odd numbers multiply together to give 3135?

27 Study this number pattern.

$$1 = 1^2$$
$$1 + 3 = 2^2$$
$$1 + 3 + 5 = 3^2$$
$$1 + 3 + 5 + 7 = 4^2$$

a Use the pattern to find the sum of the first 10 odd numbers.

b The sum of a certain number of odd numbers, starting with 1, is 64. How many odd numbers have been added together?

c What is the sum of

i the first 90 odd numbers

ii the first 50 odd numbers

iii the odd numbers from 51 to 90 inclusive?

28 Using your calculator to help you, write down the first eight digits in the decimal value of the fractions $\frac{1}{7}, \frac{2}{7}, \frac{3}{7}, ... \frac{6}{7}$.

a What do you notice about the first six digits in each decimal?

b Without using a calculator, write down the decimal value of $\frac{8}{7}$ and $\frac{13}{7}$

c Use the pattern you have observed to write down the value of $\frac{4}{7}$ to 15 decimal places.

d Write the decimal 2.857 142 857 142 ... as an improper fraction.

29 Study the number pattern below.

$\frac{1}{11} = 0.090\,909\,09...$

$\frac{2}{11} = 0.181\,818\,18...$

$\frac{3}{11} = 0.272\,727\,27...$

$\frac{4}{11} = 0.363\,636\,36...$

a Use this pattern to write down the decimal value of $\frac{3}{11}$ to 12 decimal places.

b Use the pattern to write down, to 8 decimal places,

i $\frac{5}{11}$ **ii** $\frac{6}{11}$

30 Study this number pattern.

$\frac{1}{2} - \frac{1}{3} = \frac{1}{6}$, $\frac{1}{3} - \frac{1}{4} = \frac{1}{12}$, $\frac{1}{4} - \frac{1}{5} = \frac{1}{20}$, ...

Use the pattern to find

a **i** $\frac{1}{8} - \frac{1}{9}$ **ii** $\frac{1}{20} - \frac{1}{21}$

b the value of x if $\frac{1}{12} - \frac{1}{x} = \frac{1}{156}$

c the fraction denoted by y if $y - \frac{1}{11} = \frac{1}{110}$

Example:

This is a sequence of expressions.

$\{(x - 0) + 1\}^2 - 2x$, $\{2(x - 1) + 2\}^2 - 3x$, $\{3(x - 2) + 3\}^2 - 4x$, ...

a Copy and complete this term: $\{\square(x - \square) + 6\}^2 - \square x$.

b Write down an expression for the nth term of this sequence.

a

Term number	1	2	3	...
Term	$\{(x - 0) + 1\}^2 - 2x$	$\{2(x - 1) + 2\}^2 - 3x$	$\{3(x - 2) + 3\}^2 - 4x$	

Constructing a table makes it easier to spot the relationships between the term numbers and the numbers in the expressions.

$\{6(x - 5)^2 + 6\} - 7x$

b $\{n(x - (n - 1)) + n\}^2 - (n + 1)x$

The only number given is 6 at the end of the curly bracket. From the table this is the same as the term number. So this is term number 6.

The number multiplied by the round bracket is also the same as the term number. The number subtracted from x in the curly bracket is 1 less than the term number. The number of xs at the end is one more than the term number.

31 This is a sequence of equations.

$3x - 1 = 0$, $6x - 2 = 0$, $12x - 4 = 0$, $24x - 8 = 0$, ...

a This equation is in the sequence: $\square x - 64 = 0$. Write down the value of the coefficient of x.

b Write down the nth equation in this sequence.

c Write down the solutions of the equations in this sequence.

d State, with a reason, whether the equation $50x - 150 = 0$ is a term in this sequence.

32 This is a sequence of expressions.

$3(x + 7)$, $4(x + 3)$, $5(x - 1)$, $6(x - 5)$, ...

a Copy and complete this term in the sequence: $10(x - \square)$

b Explain why $20(x - 64)$ is not an expression in this sequence.

c Write down an expression for the nth term in this sequence.

33 Study this sequence of expressions:

$x^2 + 2x$, $2x^2 + 3x$, $3x^2 + 4x$, $4x^2 + 5x$, ...

a Write down the next two terms.

b Find an expression for the nth term.

c Find **i** the 10th term **ii** the 20th term.

34 This is a sequence of expressions:

$\frac{1}{6}(x + 1)(2x - 1), \frac{1}{3}(2x + 1)(4x - 1), \frac{1}{2}(3x + 1)(6x - 1), \frac{2}{3}(4x + 1)(8x - 1), \ldots$

a Write down the next two terms.
b Find an expression for the *n*th term in the sequence.
c Find **i** the 12th term **ii** the 20th term.

> Write the fractions as equivalent fractions with a common denominator.

35 This is a sequence of expressions:

$x - 2x^2, 2x + 3x^3, 3x - 4x^4, 4x + 5x^5, \ldots$

a Write down the next two terms.
b Complete 10th term: $10x +$
c Explain why $12x - 13x^{13}$ is not an expression in the sequence.
d Find an expression for the *n*th term.
e Find **i** the 15th term **ii** the 23rd term.

> The signs between the terms alternate. They are negative when *n* is odd and positive when *n* is even. The sign in the *n*th expression will be correct if the second term in that expression is multiplied by $(-1)^n$.

36 This is a sequence of expressions:

$\frac{x^2}{2}, \frac{-2x^3}{5}, \frac{3x^4}{10}, \frac{-4x^5}{17}, \ldots$

a Write down the next two terms.
b Find an expression for the *n*th term.
c Explain why $\frac{10x^{10}}{101}$ is not an expression in this sequence.
d Write down the 12th term.

37 Study this sequence of expressions:

$(x + 1)^2 - (2x - 1)^2, (2x + 1)^2 - (4x - 1)^2, (3x + 1)^2 - (6x - 1)^2, (4x + 1)^2 - (8x - 1)^2$

a Write down the next two terms.
b Find an expression for the *n*th term.
c Explain why $(7x + 1)^2 - 14(x - 1)^2$ is not a term in this sequence.
d Explain why $(10x + 1)^2 - (1 - 20x)^2$ does belong to this sequence.

38 A boy is given a large bar of chocolate and decides to make it last by eating half of what is left each day. Thus he eats half of the bar on the first day; he eats half of half the bar, i.e. quarter of the bar, on the second day; he eats half of quarter of the bar on the third day, and so on.
Write down the sequence giving the fraction of the bar left at the end of the first, second, third, fourth and fifth days. *In theory*, how long will the bar of chocolate last?

INVESTIGATION

A Fibonacci sequence is formed by starting with any two numbers, then adding the previous two numbers to get the next term.

For example, 3, 7, 10, 17, 27, 44, ...

a Write down the next 5 terms of this sequence.
b Another sequence is formed by dividing a term in the Fibonacci sequence by the one before it. The first two terms are $7 \div 3 = 2.333...$, $10 \div 7 = 1.428...$
Work out 10 terms of this sequence.
Comment on what appears to be happening.
c Investigate what happens to the ratio of successive terms for different start numbers of the Fibonacci sequence.

PUZZLE

You will need to do some lateral thinking to solve these.

1 Each line uses the same pattern

2, 4, 2, 8
5, 7, 5, 17
9, 11, 9, 29
4,

Complete the last line.

2 Write down the next two letters in this sequence.
J, F, M, A, M, ...

3 Write down the next two numbers in this sequence.
1, 2, 4, 5, 10, 12, 16, 18, 22, ...

4 Write down the next number in this sequence.
1, 11, 101, 111011, 11110101, ...

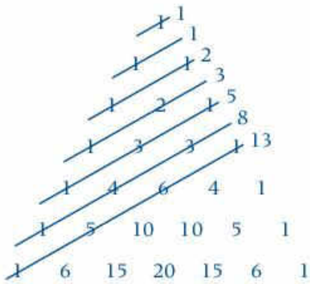
5 Write down the next four numbers in this sequence.
5, 7, 10, 4, 6, 9, 3, 5, 8, 2, 4, 7, 1, 3, 6, 0, 2, 5, –1, ...

6 What is the next term in this sequence?
1, 11, 21, 1211, 1231, 131221, ...

Describe the figure in each term in words.

MATHS IS OUT THERE

Did you know that if you add the numbers in the diagonals in Pascal's triangle, as shown in the diagram, you get a Fibonacci sequence.



Time for another library (or Internet) search: Find out what other connections there are with a Fibonacci sequence.

IN THIS CHAPTER YOU HAVE SEEN THAT...

- a sequence (or number pattern) is an ordered list of terms with a rule for continuing the sequence
- to spot a pattern, start by looking for numbers that increase by the same amount, or squares or cubes of numbers, or increasing powers of the same number.

13 QUADRATIC FUNCTIONS

AT THE END OF THIS CHAPTER YOU SHOULD BE ABLE TO...

1. Draw the graph of a quadratic function.
2. Solve quadratic equations graphically.
3. Solve quadratic inequalities.
4. Find the maximum or minimum of a quadratic function.
5. Solve graphically and algebraically simultaneous equations, where one is linear and the other is quadratic.

BEFORE YOU START

you need to know:

- ✓ how to draw and scale a set of x and y axes
- ✓ what a function is and the meaning of domain and range
- ✓ that the equation of a straight line is $y = mx + c$ where m is the gradient and c is the intercept on the y-axis
- ✓ how to draw the graph of a straight line from its equation
- ✓ how to find the gradient of a straight line from its graph
- ✓ how to solve a pair of linear simultaneous equations
- ✓ how to factorise a quadratic expression
- ✓ how to solve a quadratic equation by factorisation and by completing the square.

KEY WORDS

parabola, quadratic function

MATHS IS OUT THERE

Did you know that 'For thousands of years, Africa was in the mainstream of mathematics history? This history began with the first written numerals of ancient Egypt, a culture whose African origin has been reaffirmed by the most recent discoveries of archaeology.' Beatrice Lumpkin, *Blacks in Science*.

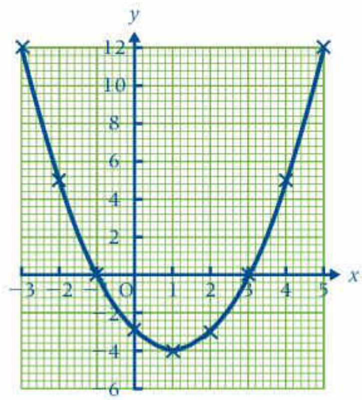
The graph of a quadratic function

The general **quadratic function** has the form $f(x) = ax^2 + bx + c$ where a, b and c are constants and $a \neq 0$.

If we let $y = f(x)$ then we can make a table of values to show some of the ordered pairs of the function which we can plot as points on the xy-plane to give a graphical representation of the function.

For example, for $y = x^2 - 2x - 3,\ x \in \mathbb{R}$

x	−3	−2	−1	0	1	2	3	4	5
y	12	5	0	−3	−4	−3	0	5	12



These points all lie on a curve. Because x can take any real value, all the possible ordered pairs are represented by the points between and beyond the points we have plotted.

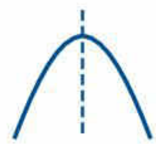
This means that the curve goes on for ever in both directions.

The graphs of all quadratic functions have a shape like this curve. They are called **parabolas**.

Every parabola has an axis of symmetry which goes through the vertex, i.e. the point where the curve turns back upon itself.

If the coefficient of x^2 is positive, i.e. $a > 0$, then $f(x)$ has a least value, and the parabola looks like this.

If the coefficient of x^2 is negative, i.e. $a < 0$, then $f(x)$ has a greatest value and the parabola is this way up.



The axis of symmetry of the graph shown is the line $x = 1$ and y has a minimum value of -4, i.e. $f(x) = x^2 - 2x - 3$ has a minimum value of -4.

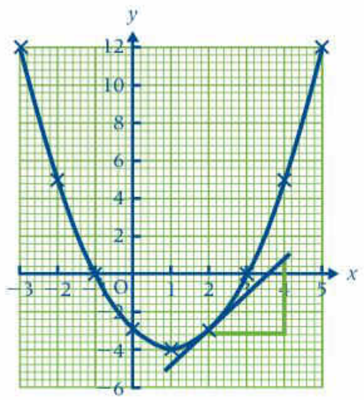
This means that the range of $f(x)$ is $f(x) \geqslant -4$

Estimation of the gradient at a point

The gradient of a curve varies continuously.
The gradient at a point on a curve is defined as the gradient of the tangent to the curve at that point.

A tangent to a curve is a straight line that touches the curve at a point.

We can estimate the gradient of this curve at the point on the curve where $x = 2$ by drawing the tangent at that point.



Use your eyes to position your ruler carefully before you draw the tangent.

From the diagram, the gradient of the tangent is $\frac{3}{2} = 1.5$

So at the point on the curve where $x = 2$, gradient is 1.5.

Remember to read the lengths of the lines from the scales on the axes.

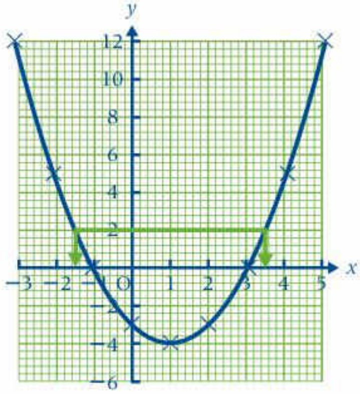
Graphical solution of quadratic equations

We can use the graph of $y = x^2 - 2x - 3$ to solve the equation $x^2 - 2x - 3 = 0$.

The solutions of the equation $x^2 - 2x - 3 = 0$ are the values of x for which $y = 0$ when $y = x^2 - 2x - 3$.

We can read these values from the graph: $x = -1$ and $x = 3$.

$y = 0$ where the curve cuts the x-axis. These values of x are exact because they are in the table.

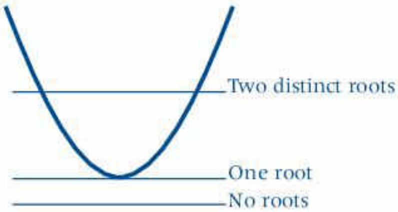


Similarly the solution of the equation $x^2 - 2x - 5 = 0$ can be read from the graph. First rearrange the equation so that the left-hand side is $x^2 - 2x - 3$. Adding 2 to both sides gives $x^2 - 2x - 3 = 2$, so the solutions are the values of x where $y = 2$.

We can estimate these by drawing the line $y = 2$ on the graph and reading the values of x where this line cuts the curve: $x = -1.5$ and $x = 3.5$

These values are estimates. Their accuracy depends on the scales used and the accuracy of the drawing.

In general, the solutions of the equation $ax^2 + bx + c = d$ are the values of x where the line $y = d$ cuts the curve $y = ax^2 + bx + c$.

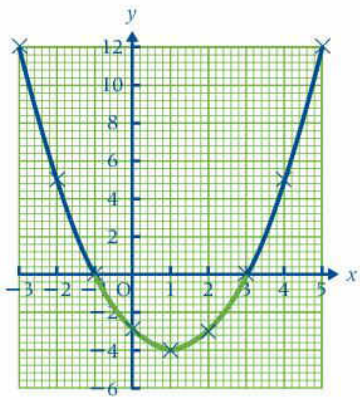


The line may cut the curve in two distinct places; this gives two separate solutions.

The line may just touch the curve (in which case it is called a tangent); this gives only one solution.

The line may miss the curve; this gives no real solutions.

Graphical solution of quadratic inequalities



$x^2 - 2x - 3 < 0$ is an example of a quadratic inequality.

From the graph we can see that y $(= x^2 - 2x - 3) < 0$ for values of x between -1 and 3.

So $x^2 - 2x - 3 < 0$ for $-1 < x < 3$

EXERCISE 13a

Example:

Given that $f(x) = -x^2 + 3x + 2$

a Copy and complete this table of values for $f(x)$.

x	−2	−1	0	1	2	3	4	5
$f(x)$	−8		2					−8

b Draw the graph of $f(x) = -x^2 + 3x + 2$ for $-2 \leqslant x \leqslant 5$ using a scale of 1 cm to represent one unit on the x-axis and 2 cm to represent 5 units on the $f(x)$-axis.

c i Draw the axis of symmetry of the curve and give its equation.

ii Calculate the maximum value of f(x).

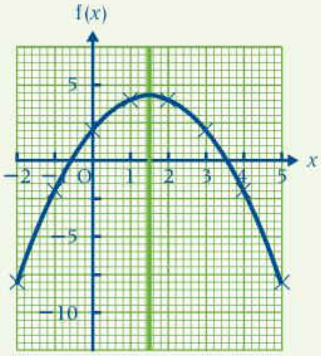
d Use the graph to find the values of x for which

i $f(x) = 0$ **ii** $x^2 - 3x - 2 = -3$ **iii** $f(x) < 2$.

a

x	-2	-1	0	1	2	3	4	5
f(x)	-8	-2	2	4	4	2	-2	-8

b



c i $x = 1.5$

The curve turns halfway between $x = 1$ and $x = 2$ and this is where f(x) has its maximum value.

ii When $x = 1.5$,

$$f(x) = -(1.5)^2 + 3(1.5) + 2$$
$$= -2.25 + 4.5 + 2 = 4.25$$

d i $f(x) = 0$ where $x = -0.6$ and $x = 3.6$

$f(x) = 0$ where the curve crosses the x-axis.

ii $x^2 - 3x - 2 = -3 \Rightarrow -x^2 + 3x + 2 = 3$

i.e. $f(x) = 3$

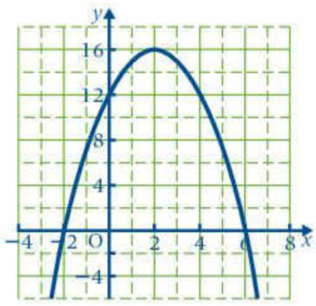
This occurs where $x = 0.4$ and $x = 2.6$

Rearrange the equation so that the left-hand side is equal to f(x). We can do this by multiplying both sides by -1.

iii $f(x) < 2$ for $x < 0$ and $x > 3$

$f(x) < 2$ for the parts of the curve below the line $f(x) = 2$.

1 The diagram shows the graph of $y = 12 + 4x - x^2$.

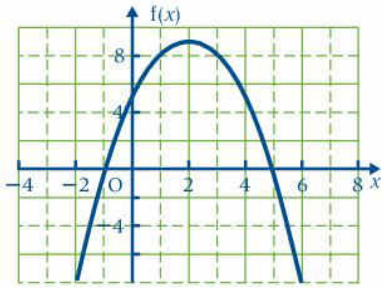


a i Write down the values of x where the graph crosses the x-axis.

ii Write down the equation whose roots are the values in part **i**.

b Use the graph to find the maximum value of $12 + 4x - x^2$ and the value of x for which it occurs.

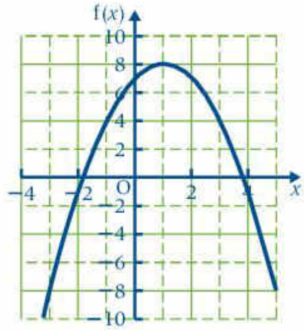
2 The diagram shows the graph of $f(x) = 5 + 4x - x^2$.



a Write down the values of x where the graph crosses the x-axis.
Write down the equation which has these values as its roots.

b Use the graph to find the maximum value of f(x) and the value of x for which it occurs.

3 The diagram shows the graph of $y = 7 + 2x - x^2$.



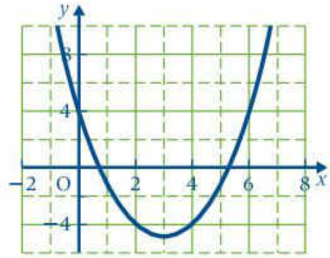
a Write down the value of y when x is
 i -1 **ii** 3

b Write down the values of x where the graph crosses the x-axis.
Write down the equation which has these values as its roots.

c Use the graph to find the maximum value of $7 + 2x - x^2$ and the value of x for which it occurs.

d Write down the values of x for which $y > 0$.

4 The diagram shows the graph of the function $y = x^2 - 6x + 4$.

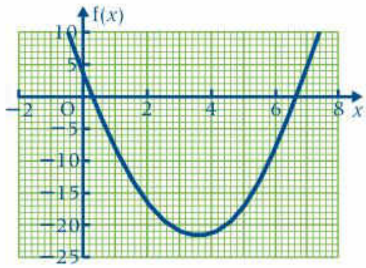


a Estimate the values of x where the graph crosses the x-axis. Write down the equation that has these values as roots.

b Estimate the minimum value of y and the value of x for which this occurs.

c Give estimates for the values of x where
 i $y = 2$ **ii** $y = -3$.

5 The diagram shows the graph of the function $f(x) = 2x^2 - 14x + 3$.



Use the graph to determine

a the value of $f(x)$ when **i** $x = 3$ **ii** $x = -1.5$

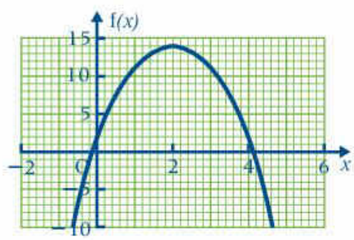
b the values of x for which $f(x) = 0$

c the minimum value of $f(x)$

d the value of x at which $f(x)$ is a minimum

e the interval on the domain for which $f(x)$ is less than -2.

6 The diagram shows the graph of the function $f(x) = 2 + 12x - 3x^2$.



Use the graph to determine

a the value of $f(x)$ when **i** $x = -0.5$ **ii** $x = 2.5$

b the values of x for which $f(x) = 0$

c the maximum value of $f(x)$

d the value of x at which $f(x)$ is a maximum

e the values of x for which $f(x) = 10$

f the equation which has these two values as its roots

g the interval on the domain for which $f(x)$ is greater than 5.

7 Copy and complete the following table which gives the values of $2x^2 + 3x - 5$ for values of x from -4 to $+3$.

x	-4	-3	-2	-1	$-\frac{1}{2}$	0	1	2	$2\frac{1}{2}$
$2x^2$	32	18	8	2	$\frac{1}{2}$	0		8	$12\frac{1}{2}$
$+3x$	-12	-9	-6	-3	$-1\frac{1}{2}$	0		6	$7\frac{1}{2}$
-5	-5	-5	-5	-5	-5	-5	-5	-5	-5
$2x^2 + 3x - 5$	15	4		-6	-6	-5		9	15

Taking 2 cm as 1 unit for x and 1 cm as 1 unit for y, draw the graph of $y = 2x^2 + 3x - 5$.

a **i** Write down the values of x where your graph crosses the x-axis.
 ii Write down the equation for which these x values are the roots.

b From your graph estimate the minimum value of $2x^2 + 3x - 5$.

c From your graph estimate the value of $2x^2 + 3x - 5$ when $x = 1.7$.

d Estimate the gradient of the curve at the point where $x = -1$.

8 Copy and complete the following table which gives values of $12 - 2x - 3x^2$ for values of x between -4 and $+4$.

x	-4	-3	-2	-1	0	1	2	3	$3\frac{1}{2}$	4
12	12	12	12	12	12	12	12	12	12	12
$-2x$	8	6		2		-2	-4	-6	-7	-8
$-3x^2$	-48	-27	-12	-3		-3		-27	$-36\frac{3}{4}$	-48
$12 - 2x - 3x^2$	-28	-9		11		7		-21	$-31\frac{1}{4}$	-44

Taking 2 cm to represent 1 unit on the x-axis and 2 cm to represent 5 units on the y-axis, plot the graph of $y = 12 - 2x - 3x^2$.

Use your graph to

a find the values of x which make $12 - 2x - 3x^2 = 0$

b estimate the maximum value of $12 - 2x - 3x^2$ and the value of x for which this maximum occurs.

c estimate the gradient of the curve at the point where $x = -2$.

9 **a** Copy and complete the following table for $y = 3x^2 + 2$ for $-3 \leqslant x \leqslant 3$.

x	-3	-2	-1	0	1	2	3
$y = 3x^2 + 2$	29		5		5	14	

b Draw the graph of $y = 3x^2 + 2$ for $-3 \leqslant x \leqslant 3$ using 1 cm to represent 1 unit on the x-axis and 1 cm to represent 5 units on the y-axis.

c Write down the equation of the line of symmetry.

d What is the minimum value of y and the value of x for which it occurs?

e Use your graph to solve the equation $3x^2 - 5 = 0$.

10 Calculate the missing values in the following table which gives the value of $x^2 - 5x + 4$ for given values of x in the range 0 to $4\frac{1}{2}$.

x	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4	$4\frac{1}{2}$
$x^2 - 5x + 4$	4	$1\frac{3}{4}$		$-1\frac{1}{4}$	-2	$-2\frac{1}{4}$		$-1\frac{1}{4}$	0	$1\frac{3}{4}$

Use these values to plot the graph of $f(x) = x^2 - 5x + 4$ for values of x from 0 to $4\frac{1}{2}$.

Take 4 cm as the unit on each axis. From your graph find

a the values of x for which $x^2 - 5x + 4 = 1$

b the value of $x^2 - 5x + 4$ when $x = 3.2$.

c an estimate for the gradient of the curve where $x = \frac{1}{2}$.

11 The table gives values of $5x - x^2$ for given values of x.

x	$-\frac{1}{2}$	0	1	2	3	4	$4\frac{1}{2}$	5	$5\frac{1}{2}$
$5x - x^2$	$-2\frac{3}{4}$	0		6	6		$+2\frac{1}{4}$		$-2\frac{3}{4}$

Copy the table and fill in the missing values. Hence draw the graph of $f(x) = 5x - x^2$ for values of x from $-\frac{1}{2}$ to $5\frac{1}{2}$. Take 2 cm as the unit on each axis.

Use your graph to find

a the value of $5x - x^2$ when $x = 1.3$

b the values of x when $5x - x^2 = 3$

c the solutions of the equation $5x - x^2 + 2 = 0$.

12 **a** Draw the graph of $f(x) = 2x^2 + 3x - 7$ for $-4 \leqslant x \leqslant 2$ using 2 cm to represent 1 unit on the x-axis and 1 cm to represent 2 units on the $f(x)$ axis.

b **i** Draw the axis of symmetry of the curve and give its equation.

ii What is the minimum value of $f(x)$?

c Estimate the values of x for which **i** $f(x) = 0$ **ii** $f(x) = 5$

13 **a** Draw the graph of $y = 2x^2 - 4x - 5$ for $-2 \leqslant x \leqslant 4$ using 2 cm to represent 1 unit on the x-axis and 1 cm to represent 2 units on the y-axis.

b Use the graph to estimate the roots of the equation

i $2x^2 - 4x - 5 = 0$ **ii** $2x^2 - 4x - 1 = 0$

14 **a** Draw the graph of $y = 4x - x^2$ for $-1 \leqslant x \leqslant 5$ using 2 cm to represent 1 unit on the x-axis and 1 cm to represent 2 units on the y-axis.

b Draw the axis of symmetry of the curve and write down its equation.

c Find the maximum value of y and the value of x for which it occurs.

d Use your graph to estimate the values of x for which

i $y = 2$ **ii** $x^2 - 4x + 1 = 0$

15 **a** Draw the graph of $f(x) = 5 + 2x - x^2$ for $-2 \leqslant x \leqslant 4$. Use 2 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the $f(x)$-axis.

b Estimate the maximum value of $f(x)$ and the value of x for which it occurs.

d Use your graph to estimate the roots of the equation

i $5 + 2x - x^2 = 0$ **ii** $2 + 2x - x^2 = 0$ **iii** $x^2 - 2x - 6 = 0$

16 **a** Draw the graph of $f(x) = x^2 + 2x - 5$ for $-4 \leqslant x \leqslant 2$. Choose your own scales.

b Draw the line of symmetry of the curve and give its equation.

c Use your graph to solve the equation

i $x^2 + 2x - 5 = 0$ **ii** $x^2 + 2x - 2 = 0$

17 Draw the graph of $f(x) = 9 - x^2$ for values of x between -3 and $+4$. Use 2 cm to represent one unit on the x-axis and two units on the $f(x)$-axis.
Find the two values of x which correspond to $f(x) = 6$.

18 Draw the graph of $f(x) = x^2 - 3x - 4$ for values of x from -3 to $+5$. Use 2 cm to represent one unit on the x-axis and two units on the $f(x)$-axis.
Find the values of x which make $f(x)$ unity.

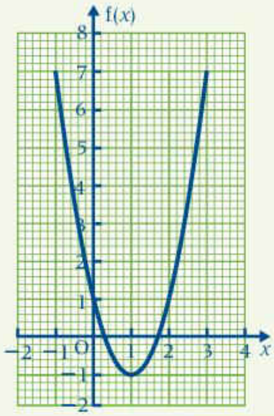
19 Draw the graph of $f(x) = 2x(x - 5)$ for values of x from -1 to $+6$. Take 2 cm as the unit on the x-axis and as two units on the $f(x)$-axis. Use your graph to find the minimum value of $f(x)$ and the value of x for which it occurs.

Example:

The graph illustrates the function $f(x) = ax^2 + bx + c$.

a Find the values of a, b and c.

b State the domain and the range $f(x)$.



a When $x = 0$, $f(x) = 1 \Rightarrow c = 1$

When $x = 2$, $f(x) = 1$

$\Rightarrow 1 = 4a + 2b + 1$

$4a + 2b = 0$

$2a + b = 0$ [1]

When $x = 1$, $f(x) = -1$

$\Rightarrow -1 = a + b + 1$

$a + b = -2$ [2]

$[1] - [2] \Rightarrow a = 2$

From [2] $b = -4$

b The domain is $-1 \leq x \leq 3$.

The range of the function is $-1 \leq f(x) \leq 7$

Reading from the graph, we can find corresponding values of $f(x)$ and x which we can substitute into $f(x) = ax^2 + bx + c$ to give equations in a, b and c. Start with $x = 0$ as this eliminates a and b. Then choose values of x that correspond to exact values of $f(x)$.

Equations [1] and [2] can be solved simultaneously to give the values of a and b.

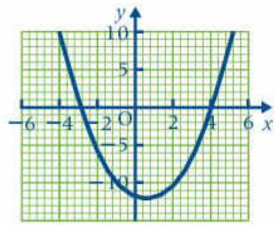
The domain of f: $x \rightarrow f(x)$ is the set of values of x. This curve starts where $x = -1$ and stops where $x = 3$. The range of $f(x)$ is the set of values of $f(x)$.

20 The diagram shows the graph of $y = ax^2 + bx + c$.

a Write down the values of x, and the corresponding values of y, where the graph crosses the x-axis.

b Find the values of a, b and c.

c Find the minimum value of y and the value of x for which this occurs.



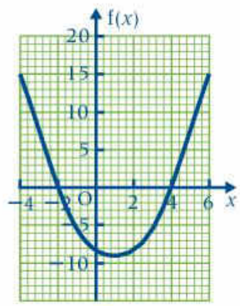
21 The diagram shows the graph of the function $f(x) = ax^2 + bx + c$.

a Find the values of a, b and c.

b Use the graph to determine the minimum value of $f(x)$ and the value of x at which it occurs.

c Write down the equation of the line of symmetry.

d Write down the domain and range of $f(x)$.



22 The diagram shows the graph of the function $f(x) = ax^2 + bx + c$.

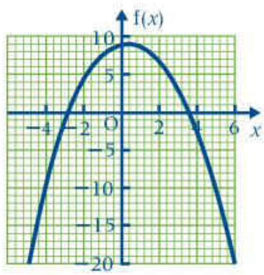
a Find the values of a, b and c.

b State the domain and range for $f(x)$.

c Use the graph to determine the maximum value of $f(x)$ and the value of x at which it occurs.

d Use the graph to determine the value of $f(x)$ when

i $x = -1$ **ii** $x = 3.5$.



23 **a** Draw, on the same axes, the graphs of $y = 5 - \frac{1}{2}x^2$ and $y = \frac{1}{2}x + 2$ for $-4 \leqslant x \leqslant 4$. Use 2 cm to represent 1 unit on Both axes.

b **i** Estimate the values of x where the two graphs intersect.

ii Find the equation for which these values are the roots.

> The graphs intersect where the values of y are the same on both graphs, i.e. where $5 - \frac{1}{2}x^2 = \frac{1}{2}x + 2$

24 **a** Draw the graph of $f(x) = 3x^2 - 2x - 4$.

b Use your graph to solve the equation

i $3x^2 - 2x - 4 = 0$ **ii** $3x^2 - 2x - 2 = 0$.

c Draw, on the same axes, the graph of $f(x) = x - 2$ and write down the coordinates of the points where the two graphs cross. For what equation are these values the roots?

25 **a** Draw the graph of $y = x^2$ for $-4 \leqslant x \leqslant 4$. Use 2 cm to represent 1 unit on the x-axis and 1 cm to represent 1 unit on the y-axis.

b Use your graph to estimate

i 1.7^2 **ii** -2.4^2 **iii** $\sqrt{12}$ **iv** $\sqrt{7}$

c **i** On the same axes, draw the graph of $y = x + 3$.

ii Estimate the values of x where the two graphs cross.

iii Find the equation for which these values are the roots.

d By drawing a suitable straight line graph, solve the equation $x^2 + x - 4 = 0$.

26 **a** Draw the graph of $y = 6 - x - x^2$ for $-4 \leqslant x \leqslant 3$.

b Draw the line of symmetry and write down its equation.

c Find the range of y for the given domain of x.

d On the same axes, draw the graph of $y = \frac{1}{2}x + 3$ and write down the values of x where the two graphs intersect. For what equation are these x-values the roots?

27 **a** Draw the graph of $y = 2x^2 + 2x + 1$ for $-3 \leqslant x \leqslant 2$.

b The graph does not cross the x-axis. What meaning can you attach to this fact?

c Draw, on the same axes, the graph of $y = 3x + 1$ and write down the coordinates of the points where the graphs cut. Find the equation for which these x-values are the roots.

Finding the maximum or minimum value of a quadratic function algebraically

$f(x) = ax^2 + bx + c$ can be written in the form $f(x) = a(x - h)^2 + K$.

> This is based on completing the square on $ax^2 + bx$.

This shows that, for all values of x,

$$f(x) = a(x - h)^2 + K \geqslant K \text{ if } a > 0$$

and

$$f(x) = a(x - h)^2 + K \leqslant K \text{ if } a < 0$$

> The square of any number, positive or negative, is positive so $(x - h)^2 = 0$ when $x = h$ and $(x - h)^2 \geqslant 0$ for all other values of x.

Therefore $f(x)$ has a minimum value of K when $a > 0$ and a maximum value of K when $a < 0$ and these both occur when $x = h$.

$x = h$ is also the equation of the axis of symmetry of the graph since it goes through the minimum or maximum point.

We can find the value of h in terms of a, b and c by completing the square on $ax^2 + bx$ and simplifying.

This gives $f(x) = a\left(x + \frac{b}{2a}\right)^2 + \left(\frac{4ac - b^2}{4a^2}\right)$

Hence $f(x)$ has a maximum or minimum value when $x = -\frac{b}{2a}$

You can see where this comes from in Chapter 9 where we derived the formula for solving a quadratic equation.

EXERCISE 13b

Example:

Find the maximum or minimum value of $f(x) = 2x^2 + 6x - 5$

$$2x^2 + 6x - 5 = a(x - h)^2 + K$$
$$= a(x^2 - 2hx + h^2) + K$$
$$= ax^2 - 2ahx + ah^2 + K$$

$a = 2$,

$-2ah = 6$ so $-4h = 6 \Rightarrow h = -1\frac{1}{2}$

$ah^2 + K = -5$ so $(2)\left(-1\frac{1}{2}\right)^2 + K = -5 \Rightarrow K = -9\frac{1}{2}$

$\therefore 2x^2 + 6x - 5 = 2\left(x + 1\frac{1}{2}\right)^2 - 9\frac{1}{2}$

so $f(x)$ has a minimum value of $-9\frac{1}{2}$

Alternatively, $f(x)$ has a minimum value $(a > 0)$ where $x = -\frac{b}{2a} = -\frac{3}{2}$

Hence the minimum value of $f(x) = 2\left(-\frac{3}{2}\right)^2 + 6\left(-\frac{3}{2}\right) - 5 = -9\frac{1}{2}$

Start by expressing the function in the form $a(x - h)^2 + K$. Expand the right-hand side then compare the coefficients of x^2 and of x and compare the constants.

The minimum value occurs when $x = 1\frac{1}{2}$

Express each quadratic expression in the form $a(x - h)^2 + K$. Hence, or otherwise. Find the maximum or minimum values for the following expressions.

1 $x^2 + 8x + 3$

2 $x^2 - 3x - 10$

3 $-x^2 + 6x - 5$

4 $2x^2 + 6x - 5$

5 $-6x^2 + 3x - 2$

6 $4 - 5x - 10x^2$

7 $x^2 + 10x - 3$

8 $x^2 - 8x + 24$

9 $x^2 + 12x + 7$

10 $x^2 - 7x + 16$

11 $x^2 - x + 5$

12 $-x^2 + 10x - 3$

13 $-x^2 + 20x + 14$

14 $-x^2 - 6x + 5$

15 $9x^2 + 18x + 13$

16 $2x^2 + 8x - 7$

17 $-4x^2 + 4x - 3$

18 $-4x^2 - 12x + 5$

19 $12 - 3x - 5x^2$

20 $5 - 4x - 6x^2$

21 $3 - 2x - 2x^2$

Example:

In a triangle PQR, the sum of the length of the side PR and the altitude from Q to PR is 12 cm. Find the length of PR and the altitude from Q so that the triangle has maximum area.

Let the length of PR be x cm and the altitude be y cm.

$\therefore \quad x + y = 12$

Area, $A = \frac{1}{2}xy$

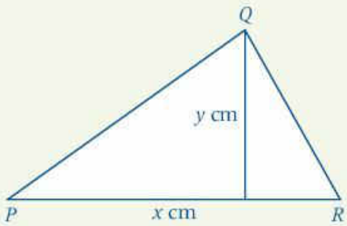
$= \frac{1}{2}x(12 - x)$

$= 6x - \frac{1}{2}x^2$

$= -\frac{1}{2}(x^2 - 12x)$

$= -\frac{1}{2}(x^2 - 12x + 36 - 36)$

$= -\frac{1}{2}(x - 6)^2 + 18$



Start by drawing a diagram. Then form an equation and use it to give the area as a function of one variable.

Complete the square on $x^2 - 12x$ by adding $\left(\frac{1}{2} \text{ of } 12\right)^2$

To keep the expression equal to A, subtract $\left(\frac{1}{2} \text{ of } 12\right)^2$ as well.

A is maximum when $(x - 6)^2 = 0$

i.e. $\quad x = 6 \quad \therefore \quad y = 6$

Therefore PR = 6 cm and altitude = 6 cm

22 A fence of length 300 m is to be used to enclose a rectangular area. Calculate

a the maximum area of the enclosure

b the dimensions of the rectangle of maximum area.

23 The sum of two real numbers is 18.
Find their value if their product is a maximum.

24 A farmer needs to enclose a rectangular field which has a stream on one side and therefore needs no fence. Calculate the dimensions of the field of maximum area that can be enclosed with 150 m of fence.

25 A bullet is shot directly upwards with a velocity of 500 metres per second. Its distance above the ground, y, at the time, t, in seconds is given by the formula $y = 500t - 5t^2$. Calculate the maximum height of the bullet and after how many seconds it reaches this height.

26 A thin length of wire, 42 cm long, is bent into the shape of a rectangle ABCD so that the side CD occurs twice.
Find the maximum area of the rectangle.

27 The length of the base of an isosceles triangle is x cm and its height is y cm. If the sum of its height and the length of its base is 5 m, form an expression for its area in terms of x.
For what value of x is the area a maximum? Calculate this maximum area.

Sketching the graphs of quadratic functions

To sketch the graph of a quadratic function, we can use these properties.

$$f(x) = ax^2 + bx + c$$

- has a shape like  or 
- has a maximum value when $a < 0$
- has a minimum value when $a > 0$
- is symmetrical about the line $x = -\frac{b}{2a}$
- has a maximum or minimum value of $f\left(-\frac{b}{2a}\right)$
- intercepts the y-axis at $y = f(0)$, i.e. where $y = c$
- when the equation is a perfect square (e.g. $y = (x + 1)^2$) the x-axis is a tangent to the curve at its turning point.

A sketch is not an accurate plot. A sketch should show the shape of the curve and place the curve in the correct position on a set of axes.

We can use a quick sketch of a quadratic function to solve quadratic inequalities. The second worked example shows how to do this.

EXERCISE 13c

Example:

a Find the greatest or least value of the function given by $f(x) = 2x^2 - 7x - 4$ and hence sketch the graph of $f(x) = 2x^2 - 7x - 4$.

b State the range of $f(x)$.

a $f(x) = 2x^2 - 7x - 4 \quad \Rightarrow \quad a = 2, b = -7$ and $c = -4$

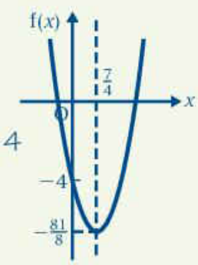
As $a > 0$, $f(x)$ has a least value

and this occurs when $x = -\frac{b}{2a} = \frac{7}{4}$

$\therefore$ the least value of $f(x)$ is $f\left(\frac{7}{4}\right) = 2\left(\frac{7}{4}\right)^2 - 7\left(\frac{7}{4}\right) - 4$

$= -\frac{81}{8}$

$f(0) = -4$



b $f(x) \geqslant -\frac{81}{8}$

We now have one point on the graph of f(x) and we know that the curve is symmetrical about this value of x. However to locate the curve more accurately we need another point and we use f(0) as it is easy to find.

1 Find the greatest or least value of $f(x)$ and the range of $f(x)$ where $f(x)$ is

a $x^2 - 3x + 5$ **b** $2x^2 - 4x + 5$ **c** $3 - 2x - x^2$
d $7 + x - x^2$ **e** $x^2 - 2$ **f** $2x - x^2$

2 Sketch the graph of each of the following quadratic functions, showing the greatest or least value and the value of x at which it occurs.

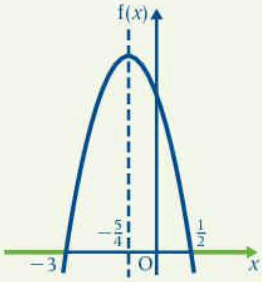
a $x^2 - 2x + 5$ **b** $x^2 + 4x - 8$ **c** $2x^2 - 6x + 3$
d $4 - 7x - x^2$ **e** $x^2 - 10$ **f** $2 - 5x - 3x^2$

3 a Express $x^2 - 3x - 1$ in the form $(x - h)^2 + K$. Hence
  i sketch the graph of $y = x^2 - 3x - 1$
  ii state the number of roots of the equation $x^2 - 3x - 1 = 0$
 b Express $5 - x - x^2$ in the form $K - (x - h)^2$. Hence
  i sketch the graph of $y = 5 - x - x^2$
  ii state the number of roots of the equation $5 - x - x^2 = 0$
 c Express $x^2 + 2x + 5$ in the form $(x - h)^2 + K$. Hence
  i sketch the graph of $y = x^2 + 2x + 5$
  ii state the number of roots of the equation $x^2 + 2x + 5 = 0$

Example:

Solve the inequality $(1 - 2x)(x + 3) \leqslant 0$

Draw a quick sketch of the graph of $f(x) = (1 - 2x)(x + 3)$. The coefficient of x^2 is negative, so $f(x)$ has a greatest value. The curve cuts the x-axis when $f(x) = 0$.
When $f(x) = 0$, $(1 - 2x)(x + 3) = 0 \Rightarrow x = \frac{1}{2}$ or -3
The average of these values is $-\frac{5}{4}$ so the curve is symmetrical about $x = -\frac{5}{4}$.



$\therefore \quad (1 - 2x)(x + 3) \leqslant 0 \quad$ for $\quad x \leqslant -3$ and $x \geqslant \frac{1}{2}$

Now we can see where $f(x) < 0$ (i.e. below the x-axis)

Note that this method is suitable only when the quadratic factorises.

Solve the following inequalities.

4 a $(x - 1)(x - 3) < 0$	b $(x + 2)(x - 4) \geqslant 0$	c $(2x - 1)(x - 3) \leqslant 0$
5 a $(1 + x)(2 - x) \geqslant 0$	b $(x^2 - 9) < 0$	c $3x^2 + 5x - 2 > 0$
6 a $3x^2 - x - 2 < 0$	b $2x^2 - x - 1 \geqslant 0$	c $3x^2 + x - 10 < 0$
7 a $3x^2 - 5x - 2 \geqslant 0$	b $2x^2 - 11x + 15 < 0$	c $4x^2 - 7x - 2 \leqslant 0$
8 a $2x^2 - 5x - 3 > 0$	b $2x^2 - 5x - 12 < 0$	c $6x^2 + 11x - 10 \geqslant 0$
9 a $5x^2 - 3x < 2$	b $4x^2 + 5x \geqslant 21$	c $3x^2 - 16x > 12$
10 a $2x^2 + 3x \leqslant 20$	b $2x^2 - x < 6$	c $3x^2 - x > 2$

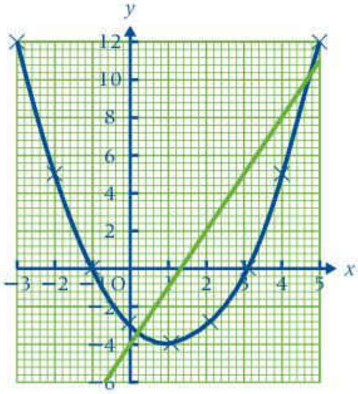
First rearrange the inequality so that the right-hand side is 0. Remember that you can add or subtract the same term on both sides of an inequality without changing it.

Simultaneous equations – one linear and the other quadratic

Graphical solution

When the quadratic equation has the form $y = ax^2 + bx + c$, we can draw the graph. A linear equation can always be arranged as $y = mx + c$ so we can add the straight line to the curve.

For example, to solve the equations $y = x^2 - 2x - 3$ and $y = 3x - 4$ graphically we draw the graphs representing these equations on the same set of axes.



The solutions are the coordinates of the points where the curve and the line intersect, i.e. $x = 0.2$, $y = -3.4$ and $x = 4.8$, $y = 10.4$

Note that these values are estimates.

A graphical solution is quick and easy if we can use a graphical calculator or a curve drawing program on a computer. Without these aids, it is a slow method and it is not very accurate.

INVESTIGATIONS

You need a graph drawing program.
It is possible to plot the graph of any equation on a graphics calculator, but the equation must be entered in the form $y = f(x)$.

1 To solve the equations $x^2 + y^2 = 25$ and $y = -1$, you will also need to add the graph of the line. Then you can use the Trace function together with the Zoom function to find the coordinates of the points of intersection.

For example, to plot the equation $x^2 + y^2 = 25$

first rearrange the equation as $y^2 = 25 - x^2$

which gives $y = \pm\sqrt{25 - x^2}$

So you need to enter two equations,
i.e. $y = \sqrt{25 - x^2}$ and $y = -\sqrt{25 - x^2}$

2 **a** Investigate the position on the axes of the graph of $y = ax^2$ for different values of a.

b Investigate the position on the axes of the graph of $y = ax^2 + bx$ for different values of a and b.

c Investigate the position on the axes of the graph of $y = ax^2 + c$ for different values of a and c.

d Investigate the position on the axes of the graph of $y = ax^2 + bx + c$ for different values of a, b and c.

Algebraic solution

To solve the equations $\begin{matrix} x^2 + y^2 = 17 \\ x + y = 3 \end{matrix}$, we use the method of substitution to reduce the two equations to one equation containing one unknown.

We do this by using the linear equation to find one letter in terms of the other and then make a substitution in the more complicated equation.

EXERCISE 13d

Example:

Solve the pair of equations $x^2 + y^2 = 17$, $x + y = 3$

$x^2 + y^2 = 17$ [1]

$x + y = 3$ [2]

From [2] $x = 3 - y$ [3]

so $x^2 = (3 - y)(3 - y)$

$= 9 - 6y + y^2$

Substitute for x^2 in [1]

$9 - 6y + y^2 + y^2 = 17$

$2y^2 - 6y + 9 = 17$

$2y^2 - 6y - 8 = 0$

$y^2 - 3y - 4 = 0$

$(y - 4)(y + 1) = 0$

From [3], when $y = 4$, $x = -1$

When $y = -1$, $x = 4$

The solution is $x = 4$, $y = -1$ or $x = -1$, $y = 4$

Using equation [2] we can express either x or y in terms of the other variable and substitute this value in equation [1]. Here x and y are equally convenient to use.

Check in [1]: When $x = 4$ and $y = -1$, $x^2 + y^2 = 16 + 1 = 17$

When $x = -1$ and $y = 4$, $x^2 + y^2 = 1 + 16 = 17$

Solve the following pairs of equations.

1 $x^2 + y^2 = 20$, $y = x + 2$

2 $x^2 + y^2 = 25$, $y = x + 1$

3 $x^2 + y^2 = 13$, $x + y = 5$

4 $x^2 + y^2 = 5$, $x + y = 3$

5 $x^2 + y^2 = 10$, $x - y = 4$

6 $x^2 + y^2 = 34$, $y - x = 2$

7 $x^2 + y^2 - 25 = 0$, $x - y = 1$

8 $x^2 + y^2 - 29 = 0$, $y = 7 - x$

Write $x + y = 5$ as $x = 5 - y$.

Example:

Solve the pair of equations

$x - y = 7$ [1]

$x^2 - xy - y^2 = 19$ [2]

From [1] $x = 7 + y$ i.e. $x = y + 7$

Substituting into [2] $(y + 7)^2 - y(y + 7) - y^2 = 19$

$y^2 + 14y + 49 - y^2 - 7y - y^2 = 19$

$-y^2 + 7y + 30 = 0$

$y^2 - 7y - 30 = 0$

$(y - 10)(y + 3) = 0$

Multiplying every term by -1

Either	$y = 10$	or	$y = -3$
i.e.	$x = 10 + 7$	or	$x = -3 + 7$
i.e.	$x = 17$	or	$x = 4$

This solution can be written as (17, 10) and (4, −3).

Solve algebraically each of the following pairs of equations.

Remember that xy is the same as yx.

Multiply both sides by xy to get rid of the fractions.

9 $x^2 + y^2 = 25$
$y = 2x + 5$

10 $x = 2y + 4$
$x^2 + y^2 = 5$

11 $x = 3y - 5$
$x^2 + 2y^2 = 6$

12 $y = 10 - 3x$
$2x^2 + y^2 = 19$

13 $y = 3x - 1$
$4x^2 - xy = 2$

14 $3x - y = 3$
$9x^2 - y^2 = 45$

15 $x + 2y = 1$
$x + 12y + 3y^2 = -6$

16 $5x - 2y = 2$
$25x^2 - 4y^2 = 36$

17 $3x - 4y = 2$
$9x^2 + 16y^2 = 52$

18 $2x - 3y = 12$
$x^2 - xy = 15$

19 $x + y = 10$
$\frac{12}{y} - \frac{12}{x} = 5$

20 $2x + y = 3$
$\frac{1}{x} + \frac{1}{y} + \frac{1}{2} = 0$

21 $2xy + y = 10$
$x + y = 4$

22 $x^2 - 2xy = 32$
$y = 2 - x$

23 $9 - x^2 = y^2$
$x + y = 3$

24 $xy - 12$
$x + y = 7$

25 $x^2 - y^2 = 0$
$3x + 2y = 5$

26 $xy = 6$
$4x - 3y = 6$

27 $x(x - y) = 6$
$x + y = 4$

28 $x^2 + xy + y^2 = 7$
$2x + y = 4$

29 $xy + x = -3$
$2x + 5y = 8$

30 $x^2 = y + 3$
$2x - 3y = 8$

31 $\frac{1}{x} - \frac{1}{y} - \frac{1}{12} = 0$
$2x - y = 2$

32 $x(2x - y) - 8 = 0$
$x - y = 7$

33 $xy = 8$
$3x - 2y = 2$

34 $4x^2 - xy = 2$
$6x + y = 1$

35 $x^2 + y^2 = 20$
$y = x + 2$

36 $x^2 + y^2 = 25$
$x - y + 1 = 0$

37 $x^2 - 2y - 3 = 0$
$x + y + 2 = 0$

38 $x^2 + y^2 = 29$
$x = y + 3$

39 $x^2 + y^2 - 13x - 7 = 0$
$y = 3x + 2$

40 $6x^2 + y^2 - x - 6y + 6 = 0$
$y = 2x + 3$

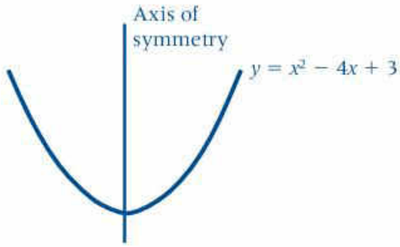
41 $2x^2 + 16y^2 - 10x - 8y + 9 = 0$
$x - 4y + 1 = 0$

MIXED EXERCISE 13

Several answers are given for these questions.
Write down the letter that corresponds to the correct answer.

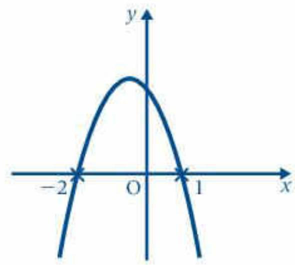
1 The axis of symmetry of the quadratic graph $y = x^2 - 4x + 3$, shown in the diagram, is

A $x = -4$ **B** $x = -2$ **C** $x = 2$ **D** $x = 4$

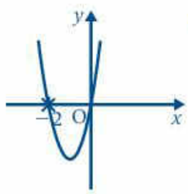


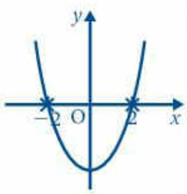
2 The equation of the quadratic graph shown in the diagram is

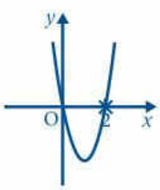
A $y = 2 - x - x^2$ **B** $y = x^2 + x - 2$
C $y = x^2 - x - 2$ **D** $y = 2 + x - x^2$

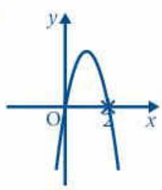


3 Which of the following curves represents the graph with the equation $y = x^2 - 2x$?

A


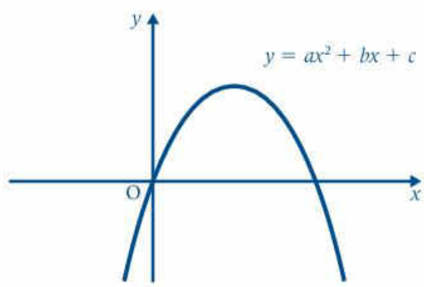
B


C


D


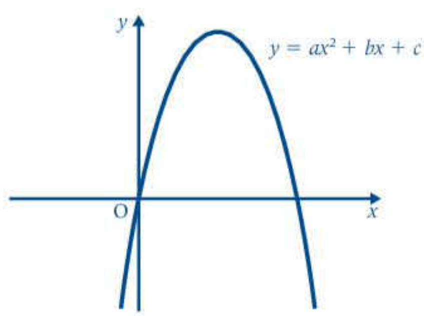
4 The diagram illustrates the curve with equation $y = ax^2 + bx + c$ where a, b and $c \in \mathbb{R}$. The value of c is

A a positive integer **B** a negative integer
C zero **D** a fraction

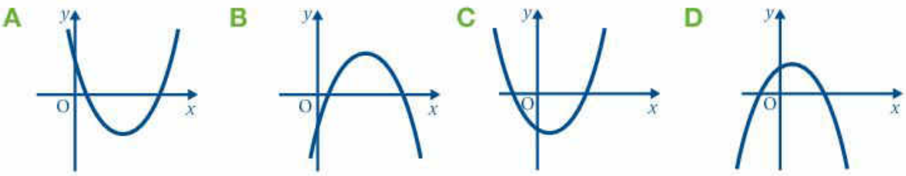


5 The diagram illustrates the graph of the quadratic $y = ax^2 + bx + c$. Hence

A a is positive and $c = 0$.
B a is negative and c is positive.
C a is positive and c is negative.
D a is negative and c is zero.



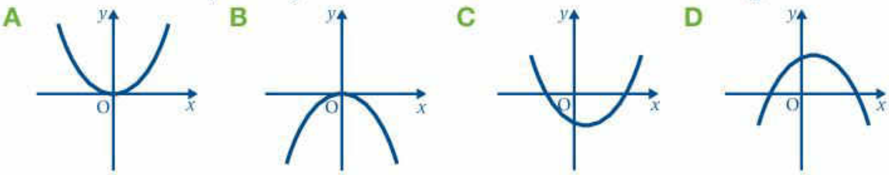
6 Which of the diagrams shown below is a likely illustration of the curve $y = x^2 - 3x + 2$?



7 The equation $x^2 - 6x + 9 = 0$ has
A two real and distinct roots
B only one real root
C no real roots
D more than two roots

8 The graph with equation $y = x^2 - 6x + 1$ has an axis of symmetry with equation
A $x = -6$ **B** $x = -3$ **C** $x = 3$ **D** $x = 6$

9 The curve with equation $y = 2 + x - x^2$ is best illustrated in diagram



10 When $3x^2 - 6x + 1$ is written in the form $a(x - h)^2 + K$, $h =$
A 1 **B** 2 **C** 3 **D** 9

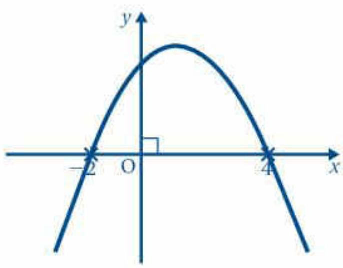
11 $y^2 + 2y + x = 0$ and $x - y = 1$ so
A $y^2 + y - 1 = 0$ **C** $y^2 + 3y + 1 = 0$
B $y^2 + 2y + 1 = 0$ **D** $y^2 + 2y - 1 = 0$

12 The graph of $y = x^2 - 2x + 6$ is drawn.
To solve the equation $x^2 - 2x + 4 = 0$, the value of x is required when $y =$
A -2 **B** 2 **C** 4 **D** 6

13 The minimum value of $f(x) = (x - 1)(x + 7)$ occurs when $x =$
A -3 **B** 1 **C** 3 **D** 7

14 The graph of $y = f(x)$, where $f(x) = 3x^2 - 7x - 4$ intercepts the y-axis where
A $f(x) = -7$ **B** $f(x) = -4$ **C** $f(x) = 0$ **D** $f(x) = 3$

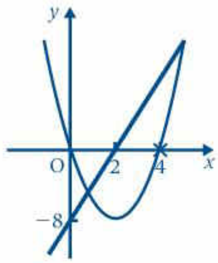
Questions **15** and **16** refer to the quadratic graph alongside.



15 The equation representing the graph is
A $y = 8 - 2x + x^2$ **B** $y = 8 + 2x - x^2$
C $y = -8 - 2x + x^2$ **D** $y = 8 - 2x - x^2$

16 The range of values of x for which $y \geqslant 0$ is
A $-2 < x < 4$ **B** $-2 \leqslant x \leqslant 4$
C $-2 < x \leqslant 4$ **D** $-2 \leqslant x < 4$

Questions **17** to **20** refer to the quadratic graph alongside.



17 The equation of the line of symmetry of the curve is

A $y = 2$ **B** $y = 0$ **C** $x = 2$ **D** $x = 0$

18 $y = 0$ for

A $x = 4$ only
B $x \leqslant 0$
C $0 < x < 4$
D $x = 0$ and $x = 4$

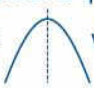
19 The equation of the curve is

A $y = x(x - 4)$
B $y = 4x(x - 4)$
C $y = x(4 - x)$
D $y = x^2 - 8$

20 The values of x where the line and the curve intersect are the solutions of the equation

A $x^2 = -8$
B $4x - 8 = 0$
C $4x(x - 4) = 0$
D $x(x - 4) = 4x - 8$

IN THIS CHAPTER YOU HAVE SEEN THAT...

- a quadratic function has the form $f(x) = ax^2 + bx + c$
- the graph of a quadratic function looks like  when $a > 0$ and like  when $a < 0$
- you can find the maximum or minimum of a quadratic function by expressing $ax^2 + bx + c$ in the form $a(x - h)^2 + K$
- the graph of $f(x) = ax^2 + bx + c$ is symmetric about the line $x = -\frac{b}{2a}$
- one linear and one quadratic equation can be solved using algebra. You do this by using the linear equation to give one unknown in terms of the other then substitute into the quadratic equation.

MATHS IS OUT THERE

The derivation of the word 'quadratic' comes from the Latin word 'quadratus' meaning 'squared'.

14 MORE FUNCTIONS

AT THE END OF THIS CHAPTER YOU SHOULD BE ABLE TO...

1 Represent cubic, exponential and trigonometric functions, graphically.
2 Recognise simple functions which have inverses.
3 Find the inverse $f^{-1}(x)$ of a function $f(x)$.
4 Find composite function $fg(x)$.

MATHS IS OUT THERE

Look up the word 'function' in the dictionary. How many meanings can you find? How do you think the mathematical meaning of 'function' was derived?

BEFORE YOU START

you need to know:
✓ how to use set notation
✓ how to plot points in a Cartesian plane
✓ how to draw graphs
✓ how to describe transformations.

KEY WORDS

composite function, cosine, cubic function, domain, exponent, exponential, function, hyperbola, inverse function, mapping, range, reciprocal curve, reciprocal function, sine, self inverse, tangent

Cubic functions

When a function contains x^3 as its highest power (and possibly terms involving x^2, x or a number), it is called a **cubic function**.

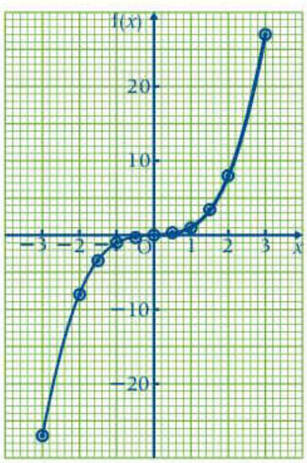
These are cubic functions:
$f(x) = x^3 + x$, $f(x) = 2x^3 - 5$, $f(x) = x^3 - 2x^2 + 6$

We start by plotting the simplest cubic function, $f(x) = x^3$.

The table gives values of x^3 for some values of x from -3 to 3.

x	−3	−2	−1.5	−1	−0.5	0	0.5	1	1.5	2	3
x^3	−27	−8	−3.4	−1	−0.1	0	0.1	1	3.4	8	27

Plotting these points and joining them with a smooth curve gives the curve below.



EXERCISE 14a

1 Copy and complete the following table which gives values of $\frac{1}{5}x^3$, correct to one decimal place, for values of x from -3 to $+3$.

x	-3	-2.5	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2	2.5	3
x^3	-27	-15.6	-8	-3.4		-0.13		0.13	1				27
$\frac{1}{5}x^3$	-5.4	-3.1	-1.6	-0.7		-0.03		0.03	0.2				5.4

Hence draw the graph of $y = \frac{1}{5}x^3$ for values of x from -3 to $+3$.
Take 2 cm as unit on each axis.
Use your graph to solve the equations

a $\frac{1}{5}x^3 = 4$ **b** $x^3 = -15$

2 Make your own copy of the graph of $f(x) = x^3$.

a On the same axes draw the line $f(x) = x + 6$

b Give the values of x at the point of intersection of the curve and the straight line.

c Write down the equation for which the values of x in part **b** are the roots.

Example:

The graph of $f(x) = (x - 1)(x + 1)(x - 3)$ is given alongside.
Use the graph to find

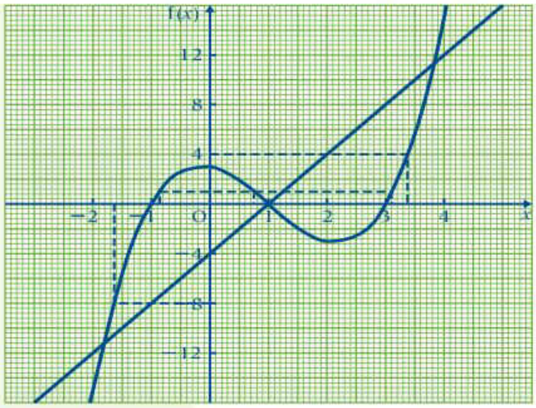
a the values of x when **i** $f(x) = 1$ **ii** $f(x) = -8$

b the solutions to the equations

i $(x - 1)(x + 1)(x - 3) = 0$ **ii** $(x - 1)(x + 1)(x - 3) = 4$

c **i** Add the line $f(x) = 4x - 4$ to the graph and write down the values of x at the point of intersection of the line and the graph.

ii Show that these values of x are the roots of the equation $x^3 - 3x^2 - 5x + 7 = 0$.



a **i** $f(x) = 1$ where $x = -0.9, 0.75, 3.1$

Drawing the line $f(x) = 1$, we see that there are three values of x where $f(x) = 1$.

ii $f(x) = -8$ where $x = -1.6$

The line $f(x) = -8$ cuts the curve once, so there is one value of x where $f(x) = -8$.

b **i** $(x - 1)(x + 1)(x - 3) = 0$ where $f(x) = 0$, i.e. where the curve crosses the x-axis, which is when $x = -1$, 1 and 3.

ii $(x - 1)(x + 1)(x - 3) = 4$ where the curve cuts the line $f(x) = 4$, i.e. when $x = 3.4$.

c **i** The line and curve intersect where $x = -1.8, 1, 3.8$

ii The values of x are those for which

Expand the brackets.

$(x - 1)(x + 1)(x + 3) = 4x - 4$

i.e. $(x^2 - 1)(x - 3) = 4x - 4$

$x^3 - 3x^2 - x + 3 = 4x - 4$

$x^3 - 3x^2 - 5x + 7 = 0$

3 Copy and complete the table which gives the values of y when $y = x(x-2)(x-4)$ for values of x from 0 to 4.

x	0	0.5	1	1.5	2	2.5	3	3.5	4
$x-2$		-1.5						1.5	
$x-4$		-3.5						-0.5	
y		2.63						-2.63	

Values of y are given correct to 2 d.p. It is not practical to give them to a greater degree of accuracy as they are then difficult to plot.

Hence draw the graph of $y = x(x-2)(x-4)$, using 4 cm for 1 unit on each axis.

a Use your graph to find
 i the lowest value **ii** the highest value
 of $x(x-2)(x-4)$ within the given range of values for x.

b Find the solutions of the equation $x(x-2)(x-4) = 2$ within the given range.

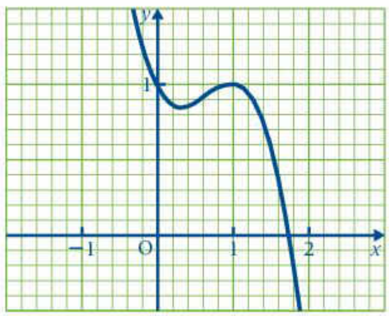
c If the values of x were extended, do you think there may be another solution to the equation in part **b**? Explain your answer.

4 Copy and complete the table which gives the value, correct to 2 decimal places, of $\frac{1}{3}x^3 - 2x + 3$ for values of x from -2 to 2.

x	-2	-1.5	-1	-0.5	0	0.5	1	1.5	2
$\frac{1}{3}x^3$	-2.67	-1.13	-0.33		0		0.33	1.13	
$-2x$	4	3	2		0		-2	-3	
-3	3	3	3		3		3	3	
$\frac{1}{3}x^3 - 2x + 3$	4.33	4.87	4.67		3		1.33	1.13	

Hence draw the graph of $y = \frac{1}{3}x^3 - 2x + 3$ using 4 cm for 1 unit on each axis. Estimate the value(s) of x where the graph crosses the x-axis.

5 The graph of $y = 1 - x + 2x^2 - x^3$ is given alongside.



a Write down the value of x where the curve crosses the x-axis. What can you deduce about the number of solutions to the equation $1 - x + 2x^2 - x^3 = 0$?

b How many solutions are there to the equation $1 - x + 2x^2 - x^3 = -22$? Explain your answer.

c Give a value of c if the equation $1 - x + 2x^2 - x^3 = c$
 i has only one solution **ii** has three solutions.

6 **a** Without drawing the curve, or working out a table of values, explain how you can state the values of x where the curve $f(x) = (x-2)(x-3)(x-4)$ crosses the x-axis. *Sketch* the curve.

b Hence state the values of x for which $(x-2)(x-3)(x-4) > 0$.

7 **a** Given that $y = x^3$, copy and complete the table below.

x	-2	-1	-0.5	0	0.5	1	2
y	-8		-0.125			1	

b Draw the graph of $y = x^3$ for values of x from -2 to 2.

c Use the graph to estimate
 i the roots of the equation $x^3 = 4$
 ii the values of x for which $x^3 = x$.

8 **a** Draw the graph of $y = \frac{1}{4}x^3$ for values of x from -2 to 2 using a scale of 2 cm to represent 1 unit on both axes.

b Use the graph to estimate the values of x for which $\frac{1}{4}x^3 < 1$.

c **i** Add the graph of $y = \frac{1}{2}x + 1$ on the same axes.

ii Write down the equation in x whose solution is given by the intersection of the two graphs.

9 **a** Draw the graph of $y = x^2(x - 1)$ for values of x using a scale of 1 cm for 1 unit on the x-axis and 1 cm for 1 unit on the y-axis.

b Use the graph to write down the values of x for which $x^2(x - 1) < 0$.

c **i** On the same axes, add the graph of $y = x - 1$.

ii Write down the equation in x whose solution is given by the intersection of the two graphs.

The shape of a cubic curve

From the last exercise we see that a cubic curve looks like

 or  and  or 

when the x^3 term is positive when the x^3 term is negative.

Reciprocal functions

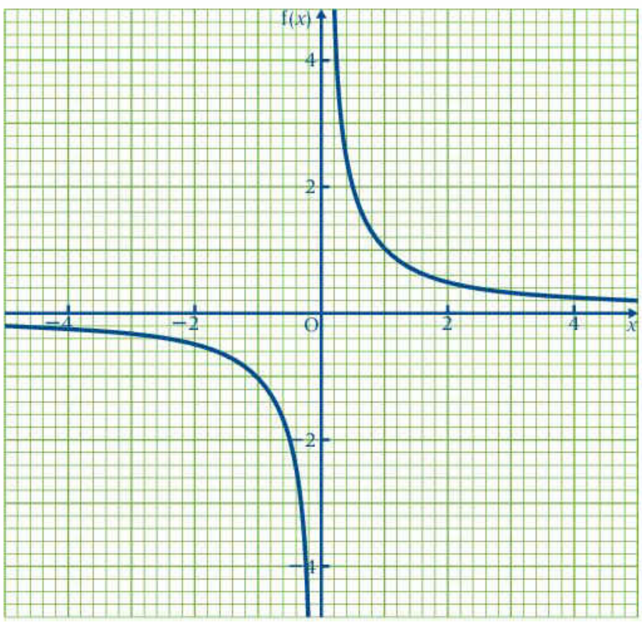
The function $f(x) = \frac{a}{x}$ where a is a number, is called a **reciprocal function**.

The simplest reciprocal function is $f(x) = \frac{1}{x}$.

Making a table showing values of $f(x)$ for some values of x from -4 to $-\frac{1}{4}$ and from $\frac{1}{4}$ to 4 gives

x	-4	-3	-2	-1	$-\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	3	4
$f(x)$	-0.25	-0.33	-0.5	-1	-2	-4	4	2	1	0.5	0.33	0.25

Plotting these points on a graph and joining them with a smooth curve gives this distinctive two-part shape.



Notice that $x = 0$ is not used in the table.

This is because $\frac{1}{0}$ is meaningless.

We say that $\frac{1}{x}$ is undefined when $x = 0$.

EXERCISE 14b

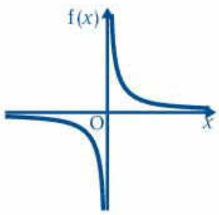
1 Use the graph of $f(x) = \frac{1}{x}$, drawn above, to answer these questions.
   a Give the value of $f(x)$ when $x = 2.5$.
   b What is the value of x when $f(x) = 2.5$?
   c What is the value of $f(x)$ when $x = 0.2$?
   d What happens to the value of $f(x)$ when x gets even smaller than 0.2?
   e Why is there no point on the curve shown when $x = 0$?
   f How many forms of symmetry does the graph have?

2 Draw the graph of $f(x) = \frac{2}{x}$ for values of x from -4 to $-\frac{1}{2}$ and from $\frac{1}{2}$ to 4. Use 2 cm for 1 unit on both axes.
   a Why is there no point on the graph where $x = 0$?
   b Give the value of $f(x)$ when
      i $x = 2.6$ ii $x = -1.8$.

3 *Sketch* the graph of $f(x) = \frac{1}{x}$ for values of x from -10 to $-\frac{1}{10}$ and from $\frac{1}{10}$ to 10.
   a What happens to $f(x)$ as the value of x increases beyond 10?
   b Is there a value of x for which $f(x) = 0$? Explain your answer.
   c Is there a value of $f(x)$ for which $x = 0$? Explain your answer.

4 Draw the graph of $f(x) = \frac{1}{x^2}$ for the domain $-3 < x < -\frac{1}{3}$ and $\frac{1}{3} < x < 3$.
   a Why is $-3 < x < 3$ not a suitable domain?
   b Find the value of $f(x)$ when $x = 2$.
   c Describe the range of the function.

Reciprocal curves

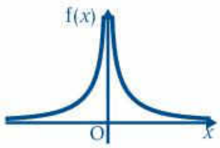


A function of the form $f(x) = \frac{a}{x}$, where a is a positive constant (i.e. a positive number), gives a distinctive two-part curve called a **reciprocal curve**.

The shape is also called a **hyperbola**.

(If a is negative the curve is the same shape but reflected in the x-axis.)

The shape of the graph of the function $f(x) = \frac{a}{x^2}$ looks like this.



Recognising curves

We are now in a position to look at a graph and recognise that its equation could be

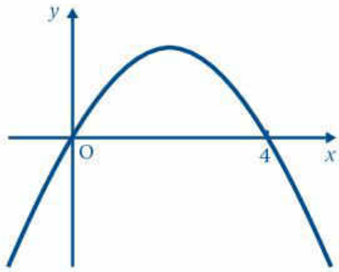
- $y = mx + c$
- $y = ax^2 + bx + c$
- $y = ax^3 + bx^2 + cx + d$
- $y = \frac{a}{x}$
- $y = \frac{1}{x^2}$
- none of these.

EXERCISE 14c

For questions **1** to **4**, write down the letter that corresponds to the correct answer.

1 The equation of this curve could be

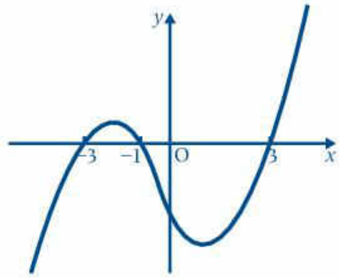
A $y = x^2$ **B** $y = x^3$ **C** $y = \frac{1}{x}$ **D** $y = 4x - x^2$



2 The equation of this curve could be

A $y = x^2 + x - 9$ **B** $y = (x - 3)(x + 3)(x + 1)$

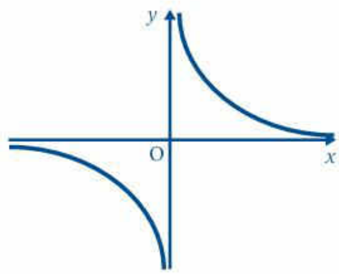
C $y = \frac{9}{x}$ **D** $y = x^3$



3 The equation of this curve could be

A $y = \frac{12}{x}$ **B** $y = x^2 - 9$

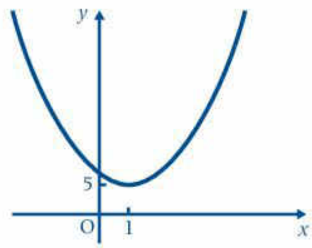
C $y = 9 - x^2$ **D** $y = x^3$



4 The equation of this curve could be

A $y = x^2$ **B** $y = 4 - x^2$

C $y = x^2 - 2x + 6$ **D** $y = x^3 - 4x^2 + 3$



5 In questions **1** to **4**, explain why you cannot be certain that the equation chosen is the equation of the graph.

Trigonometric functions

We first met **sines**, **cosines** and **tangents** when dealing with angles in right-angled triangles and so tend to think of these trigonometric ratios in relation to acute angles only. However, sin $x°$ has a value for any value of x and so has cos $x°$. Use your calculator to find, say, sin 240° and cos 515°.

EXERCISE 14d

1 Copy and complete the following table, using a calculator to find sin $x°$ correct to 2 d.p. for each value of x from 0 to 360 at intervals of 15.

x	0	15	30	45	60	75	90	105	120	135	150	165	180	195	210	225	240	255	270	285	...	345	360
sin $x°$																							

Use the values in your table to draw the graph of $y = \sin x°$ for $0 \leqslant x \leqslant 360$ and $-1 \leqslant y \leqslant 1$. Use a scale of 2 cm for one unit on the y-axis and 1 cm for 60 units on the x-axis.

2 Make another table, using the same values of x as given in question **1**, for values of cos $x°$ correct to 2 d.p., i.e.

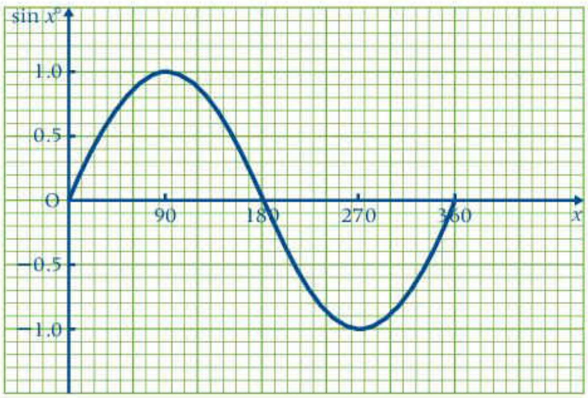
x	0	15	30	...	345	360
cos $x°$	1	0.97				

Use the values in your table to draw the graph of $y = \cos x°$. Use the same ranges and scales as in question **1**.

The graph of f(x) = sin x°

The graph drawn for question **1** in the last exercise should look like this. It is the graph of the function $f(x) = \sin x°$ for values of x between 0 and 360.

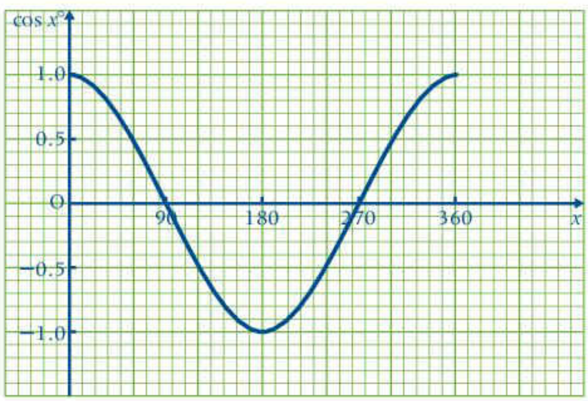
The curve has a distinctive shape: it is called a sine wave.



The graph of f(x) = cos x°

The graph drawn for question **2** in the last exercise should look like this. It is the graph of $f(x) = \cos x°$ for values of x between 0 and 360. Notice that the cosine curve looks quite different from the sine curve for values of x in the range $0 \leqslant x \leqslant 360$.

However, if both curves are drawn for a larger range of values of x, we find that there is a relationship between the curves. Questions **4** and **5** in the next exercise explore this relationship further.



EXERCISE 14e

1 a Draw a sketch of the curve $f(x) = \sin x°$ for $0 \leqslant x \leqslant 360$.
  b From your sketch, find the values of x for which $f(x) = 0$.
  c On the same axes draw a line to show how the value(s) of x can be found for which $f(x) = 0.4$.

2 a Draw a sketch of the curve $f(x) = \cos x°$ for $0 \leqslant x \leqslant 360$.
  b For what values of x is $f(x) = 0$?
  c On the same axes draw a line to show how the values of x can be found for which
    i $f(x) = 0.5$ ii $f(x) = -0.8$.

3 For $0 \leqslant x \leqslant 360$, give the range of values of the functions
  a $f(x) = \sin x°$ b $f(x) = \cos x°$.

4 Draw the graph of $f(x) = \sin x°$ for $0 \leqslant x \leqslant 720$ using the following steps.
  a Make a table of values of $\sin x°$ for values of x from 0 to 720 at intervals of 30 units. Use a calculator and give values of $\sin x°$ correct to 2 d.p.
  b Draw the vertical axis on the left-hand side of a sheet of graph paper and scale it from −1 to 1 using 2 cm to 0.5 units.
    Draw the x-axis and scale it from 0 to 720 using 1 cm to 60 units.
  c Plot the points given in the table made for **a** and draw a smooth curve through them.

5 Draw the graph of $y = \cos x°$ for $0 \leqslant x \leqslant 720$ using the same sequence of steps as in question **6**.
  Compare the graphs of $y = \sin x°$ and $y = \cos x°$. What do you notice?

You will need a graphics calculator or a computer with graph drawing software for questions **6** to **8**. If you can print out the results, do so and keep them.

6 Set the range to $-360 \leqslant x \leqslant 360$ and $-1 \leqslant y \leqslant 1$.
  a Draw the graph of $y = \sin x°$ and then superimpose the graph of $y = \sin 2x°$.
  b Describe the transformation that transforms the first curve to the second.
  c Clear the screen and repeat **a** and **b** for $y = \sin x°$ and $y = \sin 3x°$.

7 a Set the range to $0 \leqslant x \leqslant 1080$ and draw the graph of $f(x) = \sin x°$.
  b What is the range of $f(x)$?

8 Repeat question **7** for $f(x) = \cos x°$.

The sine and cosine functions

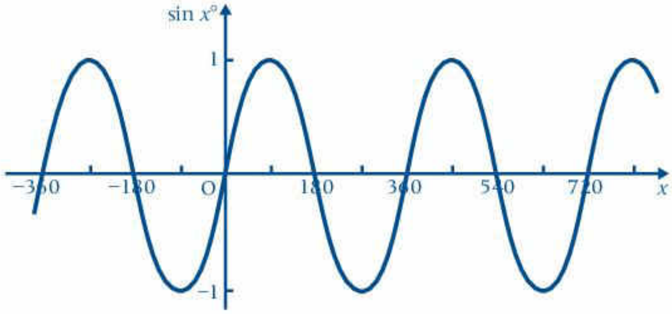
The graphs drawn for the last exercise demonstrate that the sine of an angle is never greater than 1 and never less than −1,

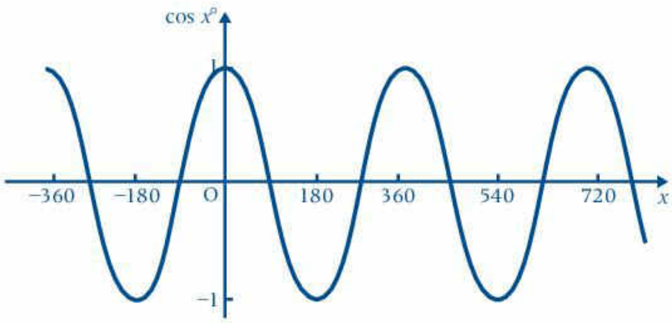
i.e. for all values of x, $-1 \leqslant \sin x° \leqslant 1$.

Note also that the curve repeats at intervals of 360 units.

The function $f(x) = \cos x°$ has the same properties. In fact, if the curve $y = \sin x°$ is translated 90 units to the left, we get the curve $y = \cos x°$.

From this we deduce that $\sin (x + 90)° = \cos x°$.

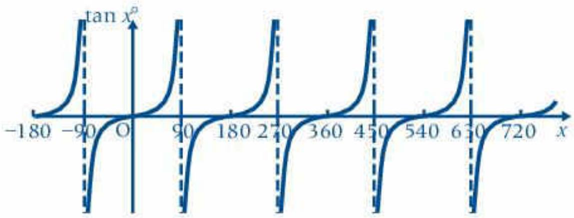




The tangent function

The graph that results from plotting values of $\tan x°$ against x in the range $-180 \leqslant x \leqslant 720$ is given below.

Find out what the broken lines are called. Are there any similar lines in the graphs of $y = \frac{1}{x}$ and $y = \frac{1}{x^2}$?



This shows that the function given by $y = \tan x°$ has properties that are quite different from those of the sine and cosine functions.

The main differences are:

- $\tan x°$ has no greatest or least value;
- the graph of $y = \tan x°$ has a repeating pattern but it repeats at intervals of 180 units;
- the graph has 'breaks' in it which occur when $x = 90$ and after every interval of 180 units from there along the x-axis.

(If you find tan 89°, tan 89.9°, tan 89.99°, etc. on your calculator you get larger and larger values. When you find tan 90° the calculator display reads 'error'. Why do you think this is so?)

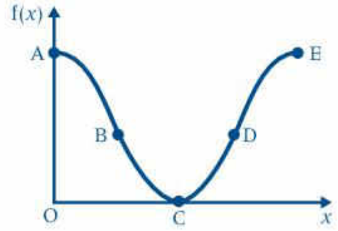
EXERCISE 14f

1 Sketch the graph of $y = \tan x°$ for $0 \leqslant x \leqslant 360$.

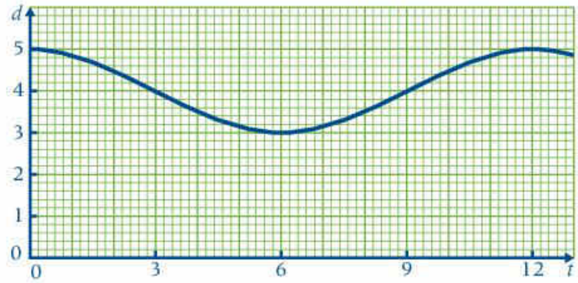
2 Sketch the curves $y = \sin x°$ and $y = \tan x°$ for $0 \leqslant x \leqslant 360$.
Hence give the number of values of x from 0 to 360 inclusive for which $\sin x° = \tan x°$.
If you have access to a computer or graphics calculator, find these values, each correct to the nearest degree.

3 Repeat question **2**, using $y = \cos x°$ instead of $y = \sin x°$.

4 The sketch shows part of the graph of $f(x) = 1 + \cos x°$.
Write down the coordinates of the points marked on the curve.



5 The depth of water, d metres, in a harbour, t hours after noon, is given by $d = 4 + \cos 30t°$ and is illustrated in the sketch below.



a Estimate the times when the depth of water in the harbour is less than 3.5 m.

b A ship, which has a draught of 3.2 m (i.e. it requires water to a depth of 3.2 m to be able to float), ties up in the harbour at noon. What is the longest time it can remain in the harbour without becoming stranded and having to wait for the next tide?

6 Use sketch graphs for values of x from 0 to 360 to find the *exact* values of x for which

a $\sin x° = -\sin x°$ b $\cos x° = -\cos x°$.

7 Sketch the graphs of $y = \sin x°$ and $y = 2 - \frac{x}{30}$ for $0 \leqslant x \leqslant 90$.

Hence show that there is one solution of the equation
$60 - x = 30 \sin x°$ between $x = 30$ and $x = 50$.

8 Sketch the graphs of $y = \cos x°$ and $y = \frac{x}{6}$ for $0 \leqslant x \leqslant 90$. Hence show that there is one solution of the equation $x = 6 \cos x°$ between 0 and 20.
If the values of x are not restricted to those between 0 and 90, how many solutions does the equation have?

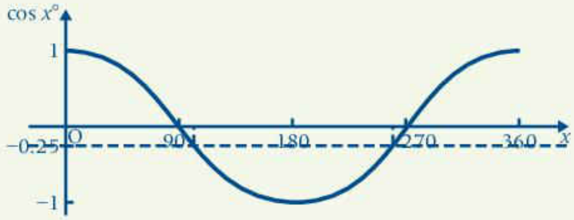
9 a What graphs need to be drawn to solve graphically the equation $50 \cos x° = x - 20$?

b What graphs could you draw to solve graphically the equation $15 \sin x° = 10 - x$?

c The x-coordinates of the points of intersection of the graphs $y = 14 \tan x°$ and $y = (x - 1)^2$ are found.
Write down the equation that has these values of x as roots.

Example:

Sketch the graph of $y = \cos x°$ for $0 \leqslant x \leqslant 360$ and use the sketch to find the values of x for which $\cos x° = -0.25$, giving values of x to the nearest whole number.



$\cos x° = -0.25$ so $x = 104.4...$ or $360 - 104.4...$

i.e. $x = 104$ or 256 to the nearest whole number.

From the graph we can see that there are two values of x for which $\cos x° = -0.25$. Using a calculator. (SHIFT $\cos^{-1}$ -0.25), gives $x = 104.4...$; this is the smaller value of x. As the graph is symmetrical about $x = 180$, the other value of x is $360 - 104.4...$

10 Use the sketch in the worked example to help find the values of x between 0 and 360 for which
a $\cos x° = 0.4$ **b** $\cos x° = -0.7$ **c** $\cos x° = -0.1$.
Give your answers correct to the nearest whole number.

11 Sketch the graph of $y = \sin x°$ for $0 \leqslant x \leqslant 360$ and use it to help find the values of x in this range for which
a $\sin x° = 0.2$ **b** $\sin x° = -0.2$ **c** $\sin x° = -0.8$.

12 Copy and complete the table to give values of $\sin x° + \cos x°$ for values of x from 0 to 360. Give each value correct to 2 d.p.

x	0	15	30	45	...	330	345	360
$\sin x°$	0		0.5				−0.26	0
$\cos x°$	1		0.87				0.97	1
$\sin x° + \cos x°$	0		1.37				0.71	1

Hence draw the graph of $f(x) = \sin x° + \cos x°$ for values of x from 0 to 360.
a Use your graph to find the least value and the greatest value of $f(x)$ within the given range.
b Describe a transformation which maps $y = \sin x°$ to $y = \sin x° + \cos x°$.

Growth curves

There are situations where quantities grow by a constant factor at regular intervals.

In the next exercise we investigate the relationship between the size of the quantity and the time such growth takes place.

For example, when cells divide every day,
1 cell becomes 2 cells after 1 day,
those 2 cells become 4 cells after another day,
the 4 cells become 8 cells after a further day, and so on,
i.e. the number of cells doubles every day.

EXERCISE 14g

1 Under ideal conditions, bacteria reproduce by dividing themselves every hour (i.e. each hour 1 bacterium becomes 2 bacteria). One bacterium is placed on a Petri dish and then incubated.

a Copy and complete the table which shows the number of bacteria present *t* hours after the dish is placed in the incubator.

Number of hours, t	0	1	2	3	4	5	6
Number of bacteria, n	1	2	4	8			

b Plot the points representing corresponding values of n and t on graph paper using a scale of 2 cm for one hour on the horizontal axis and 1 cm for 5 bacteria on the vertical axis. Draw a smooth curve through them.

c Add another row to the table as shown and complete it to give the values of n as powers of 2.

Number of hours, t	0	1	2	3	4	5	6
Number of bacteria, n	1	2	4	8			
				2^3			

By comparing the corresponding values of t and n, deduce a formula for n in terms of t.

d If 10 bacteria were placed on the Petri dish at the start, what would the relationship between n and t become?

2 Peter invests \$1000 in a savings bond which grows at 10% per annum (i.e. the compound interest is 10% p.a.).

a Copy and complete the table to show the value of the bond n years later. Give the value of P in the first row exactly in index form and in the second row give the value of P to the nearest \$100.

n years after investment	0	1	2	3	10	20	30	40
Value of the bond, \$$P$	1000	1000×1.1^1	1000×1.1^2					

Use the first two rows of the table to deduce the relationship between n and P.

b Plot the points representing these values using a scale of 1 cm for 5 years on the horizontal axis and 1 cm for \$2000 on the vertical axis. Draw a smooth curve through them.

c Use your graph to find after how many years the initial value of the bond has doubled. After how many further years does the value double again?

d If Peter invests \$5000 in this bond, what is the relationship between P and n?

Decay curves

In other situations quantities decay (shrink in size) by a constant factor at regular intervals.

The relationship between the value of a car and its age is investigated in the following exercise.

For example if, in one year, the value of a car depreciates by $\frac{1}{3}$,
then a \$9000 car is worth $\frac{2}{3} \times$ \$9000 (=\$6000) a year later
and another year on it is worth $\frac{2}{3} \times$ \$6000 (=\$4000), and so on.

EXERCISE 14h

1 A car is bought for \$9000 and then depreciates each year to $\frac{2}{3}$ of its value at the start of that year.

a Copy and complete the table showing the value, \$*P*, of the car *t* years after it is bought. In the first row for values of *P*, give the answers in index form and in the second row give the answers correct to the nearest \$10.

Number of years, t	0	1	2	3	4	5	6
Value of car, \$$P$	9000	$9000 \times \frac{2}{3}$	$9000 \times \left(\frac{2}{3}\right)^2$				
	9000	6000	4000				

b Plot the points representing corresponding values of *t* and *P* on a graph using scales of 2 cm for 1 year on the horizontal axis and 1 cm for \$500 on the vertical axis. Draw a smooth curve through the points.

c How long after buying the car does its value halve? After how much longer does it halve again? Does the value of the car ever drop to zero? Explain your answer.

2 A substance loses half its mass through radioactive decay every hour.

a Starting with 5 grams of this substance, copy and complete the table to give the mass, *m* grams, of the substance *t* hours later, giving values of *m* in the second row correct to 3 significant figures.

Number of hours, t	0	1	2	3	4	5	6
Mass of substance, m grams	5	$5 \times \frac{1}{2}$	$5 \times \left(\frac{1}{2}\right)^2$	$5 \times \left(\frac{1}{2}\right)^3$			
	5	2.5	1.25	0.625			

b Plot the points representing corresponding values of *t* and *m* on a graph using scales of 2 cm for 1 hour on the horizontal axis and 1 cm for 0.2 grams on the vertical axis. Draw a smooth curve through the points.

c By writing $\frac{1}{2}$ as 2^{-1}, show that $m = 5 \times 2^{-t}$.

d What happens to *m* as *t* gets very large?

Exponential growth and decay

Growth

Question **1** in Exercise 14g demonstrates that when a quantity grows by a factor of 2 each unit of time, then the relationship between the size of the quantity and time involves a power of 2. In this case, $n = 2^t$.

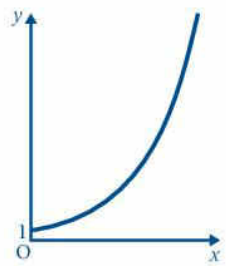
Exponent is another word for power (so *t* is the exponent of 2) and we describe the way in which *n* increases as **exponential growth**.

The function that changes values of *t* into values of *n* is called an **exponential function**.

In general, a function, f, that gives exponential growth is such that

$$f(x) = a^x$$

where a is a number greater than 1.



Question **2** in Exercise 14g is another example of exponential growth; in this case $P = 1000 \times 1.1^n$ when the initial investment is \$1000. For both questions, the shape of the curve is similar. It is called an *exponential growth curve*.

Decay

In Exercise 14h, the questions involve quantities decaying by the same factor in each unit of time.

In question **1**, the relationship is $P = 9000 \times \left(\frac{2}{3}\right)^t$.

As $\frac{2}{3} = \left(\frac{3}{2}\right)^{-1} = 1.5^{-1}$, we can write

$$P = 9000 \times 1.5^{-t} \qquad [1]$$

In question **2**, the relationship between the quantities is $m = 5 \times \left(\frac{1}{2}\right)^t$

which, since $\frac{1}{2} = 2^{-1}$, can be written as

$$m = 5 \times 2^{-t} \qquad [2]$$

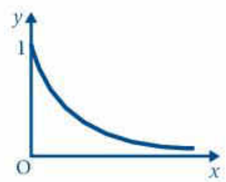
Again, the rule governing the decreasing size of each quantity involves a power; we describe the way in which the quantities decrease as **exponential decay**.

There is clearly a connection between exponential growth and exponential decay. The relationships in [1] and [2] show that when we have the power of a number greater than 1, the exponent is negative.

In general, a function, f, that gives exponential decay is such that

$$f(x) = a^{-x}$$

where a is a number greater than 1.



The shape of an exponential decay curve is typically like the one shown in the diagram.

The relationship between the functions for exponential growth and exponential decay is explored further in the next exercise.

Investigate what happens to f(x) as x gets very large.

EXERCISE 14i

1 Copy and complete the following table.

x	−3	−2	−1	0	1	2	3	4	5	6
2^x	0.125			1			8			

Use the values in your table to draw the graph of $y = 2^x$ using scales of 1 cm for 1 unit on the x-axis and 1 cm for 5 units on the y-axis.

2 Copy and complete the following table.

x	-5	-4	-3	-2	-1	0	1	2	3
2^{-x}	32			4					0.125

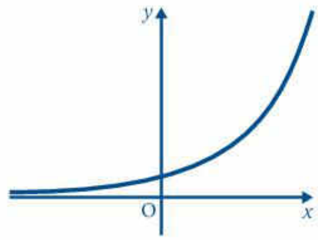
Use the values in your table to draw the graph of $y = 2^{-x}$ using the same scales as in question **1**.

3 What is the relationship between the curves drawn for questions **1** and **2**? Is this the relationship you would expect by considering the equations of the two curves? Explain your answer.

4 The diagram shows a sketch of the curve $y = 3^x$.

a Give the coordinates of the point where the curve cuts the y-axis.

b Does the curve cross the x-axis? Justify your answer.



5 Alongside is a sketch of the curve $y = 1.5^x$.

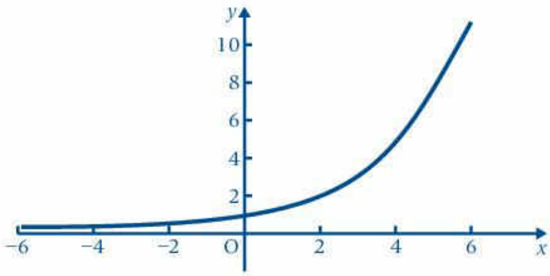
a Explain why this curve cuts the y-axis at the same point as the curve $y = 3^x$. Where does the curve $y = a^x$ cut the y-axis ($a \neq 0$)? Explain your answer.

b Copy this sketch and add a sketch of the curve $y = 5 \times 1.1^x$.

c Where does the curve $y = 500 \times 1.1^x$ cut the y-axis?

d Repeat part **c** for the curves

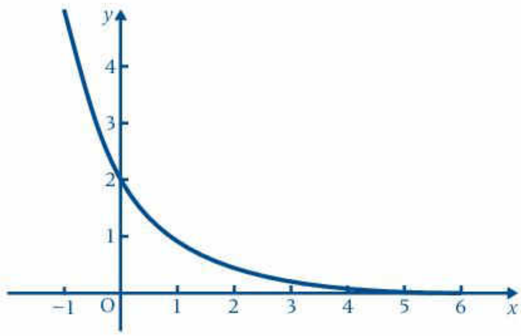
i $y = 10 \times 5^x$ **ii** $y = 10 \times 5^{-x}$.



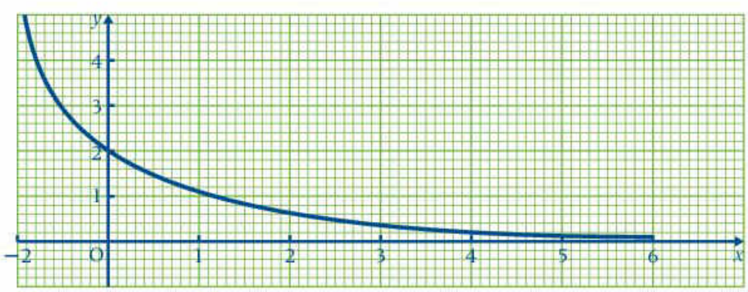
6 The equation of this curve is $y = a \times b^x$.

a Write down the value of a.

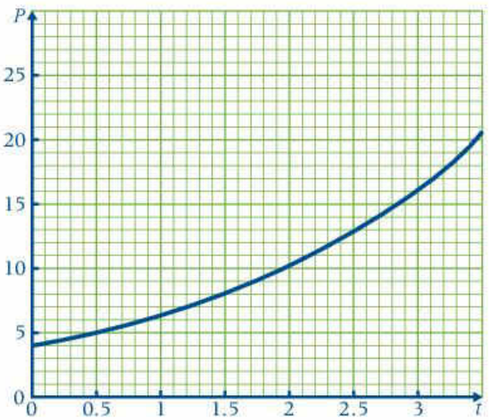
b From the graph estimate the value of y when $x = 1$. Hence estimate the value of b.



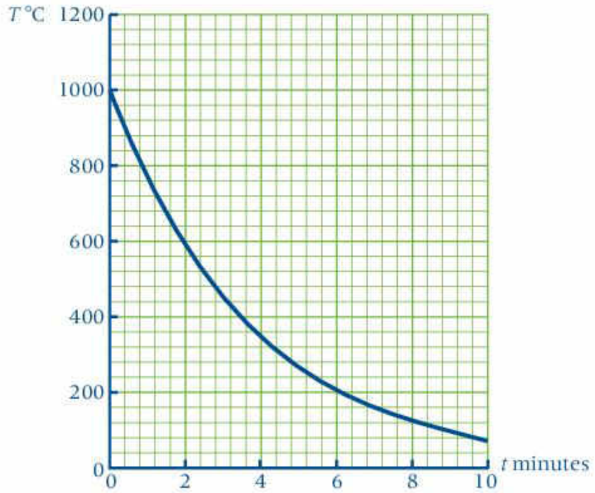
7 The equation of this curve is $y = ab^{-x}$.
Estimate the values of a and b and hence write down the equation of the curve.



8 The diagram shows a graph plotted from experimental results in a science lesson.
P and t are related by the equation $P = ab^t$.
Use the graph to estimate values of a and b.



9 This graph illustrates the temperature, T °C of a metal ingot t minutes after it is removed from a furnace.
The relationship between T and t is $T = ka^{-t}$.
Use the graph to estimate the values of k and a.



10 A sum of money, \$$P$, is invested at compound interest of r% p.a.
After T years, the value of the investment is \$$A$.
a Copy and complete the table.

T years	0	1	2	3	4
\$$A$	P	$P \times \left(1 + \frac{r}{100}\right)$	$P \times \left(1 + \frac{r}{100}\right)^2$		

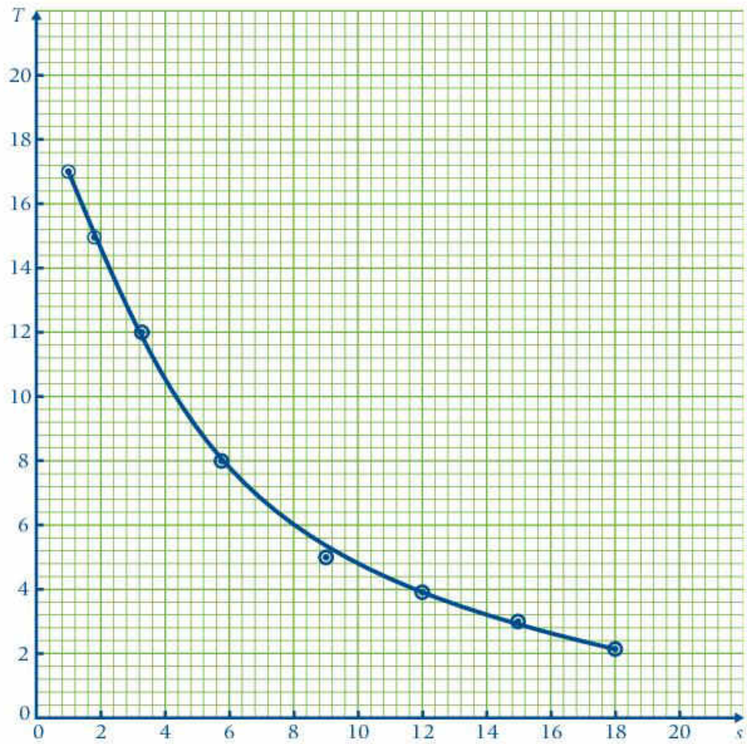
b Find a formula for A in terms of P, r and T.
c Sketch a graph showing how A changes as T increases.

11 When a particular performance-enhancing drug is taken, its concentration in the bloodstream reduces by 20% each day.
a If 100 mg is taken into the bloodstream on Monday,
i how many milligrams will be in the bloodstream on Tuesday?
ii on which day will the concentration be halved?
b Sketch a graph showing the quantity in the bloodstream from Monday to the following Saturday.

12 A population of fruit flies grows to 8 times the initial number in 18 days. Assuming that the growth is exponential, how many days does it take for the population to double?

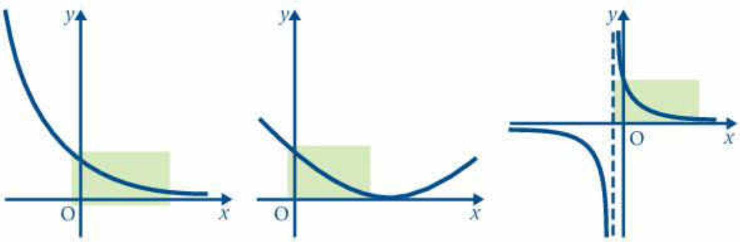
INVESTIGATION

Cheryl took some readings from an experiment and then plotted the results on a graph. She hoped that she would be able to discover the relationship between *T* and *s* from the shape of the curve through her points.



This is Cheryl's graph.

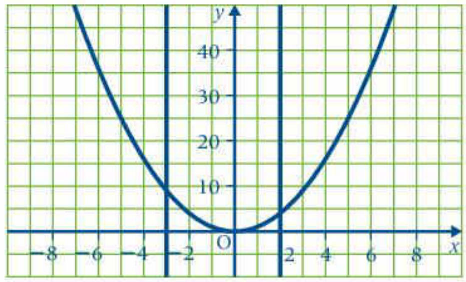
When she looked at it, she realised that her graph could be a section from several curves, e.g.



Can you help Cheryl?

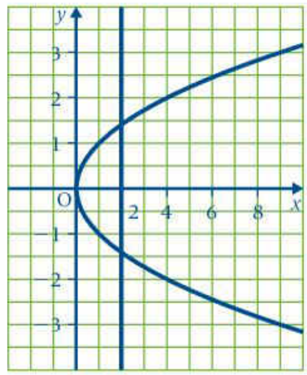
Vertical line test for a function

A function is a one–one mapping or a many–one mapping. A relation that is a one–many mapping is not a function. You can tell whether or not a relation is a function from its graph.



For example, the graph of the relation given by $y = x^2$ for $x \in \mathbb{R}$ shows that wherever we choose to draw a line vertically through the x-axis, that line cuts the graph once only.

This confirms that the relation is a one–one mapping and therefore is a function.



The graph of the relation $y = \pm\sqrt{x}$ for $x \geq 0$ and $x \in \mathbb{R}$ shows that a line drawn vertically through the x-axis cuts the graph twice. Therefore the relation is a one–many mapping and therefore is not a function.

When a vertical line can be drawn anywhere on a graph and only cuts the graph once, the relation it represents is a function.

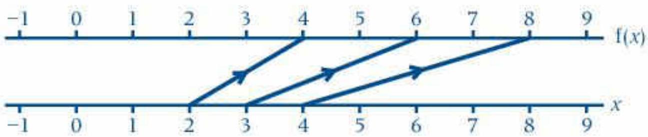
However when a vertical line drawn somewhere on a graph (it does not have to be everywhere) cuts the graph more than once, the relation it represents is not a function.

This is called the **vertical line test**.

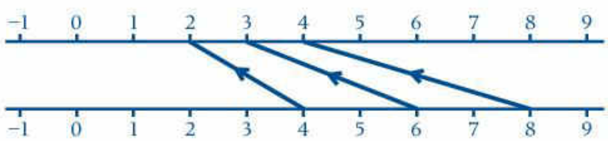
Inverse functions

Consider the function f where $f(x) = 2x$ for $x \in \{2, 3, 4\}$.

Under this function, the **domain** $\{2, 3, 4\}$ maps to the image-set $\{4, 6, 8\}$ and this is illustrated by the arrow diagram.



It is possible to reverse this **mapping**, i.e. we can map each member of the image-set back to the corresponding member of the domain by halving each member of the image-set.



This procedure can be expressed algebraically, i.e. for $x \in \{4, 6, 8\}$, $x \to \frac{1}{2}x$ maps 4 to 2, 6 to 3 and 8 to 4.

This reverse mapping is a function in its own right and it is called the **inverse function** of f where $f(x) = 2x$.

Therefore, for $f(x) = 2x$, $f^{-1}(x) = \frac{1}{2}x$ is such that f^{-1} reverses f for all real values of x, i.e. f^{-1} maps the output (range) of f to the input (domain) of f.

Denoting this inverse function by f^{-1} we can write $f^{-1}(x) = \frac{1}{2}x$. In fact, $f(x) = 2x$ can be reversed for all real values of x and the procedure for doing this is a function.

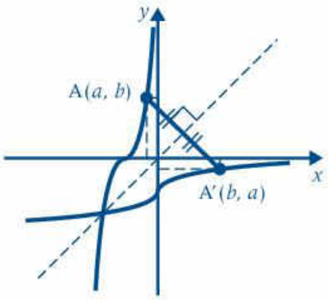
In general, for any function f,

if there exists a function, g, that maps the output (range) of f back to its input (domain),

i.e. g: f(x) $\to x$, then g is called the inverse of f and it is denoted by f^{-1}.

The graph of a function and its inverse

Consider the curve that is obtained by reflecting $y = f(x)$ in the line $y = x$. The reflection of a point A(a, b) on the curve $y = f(x)$, is the point A′ whose coordinates are (b, a), i.e. interchanging the x- and y-coordinates of A gives the coordinates of A′.

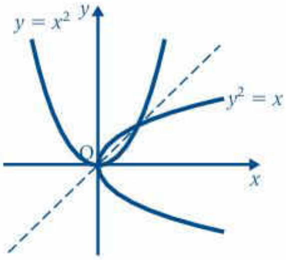


We can therefore obtain the equation of the reflected curve by interchanging x and y in the equation $y = f(x)$.

Now the coordinates of A on $y = f(x)$ can be written as $[a, f(a)]$. Therefore the coordinates of A′ on the reflected curve are $[f(a), a]$, i.e. the equation of the reflected curve is such that the output of f is mapped to the input of f.

Hence if the equation of the reflected curve can be written in the form $y = g(x)$, then g is the inverse of f, i.e. $g = f^{-1}$.

Any curve whose equation can be written in the form $y = f(x)$ can be reflected in the line $y = x$. However this reflected curve may not have an equation that can be written in the form $y = f^{-1}(x)$.



Consider the curve $y = x^2$ and its reflection in the line $y = x$.

The equation of the image curve is $x = y^2 \Rightarrow y = \pm\sqrt{x}$ and $x \to \pm\sqrt{x}$ is not a function.

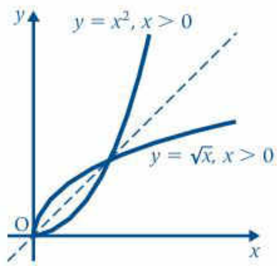
(We can see this from the diagram as, on the reflected curve, one value of x maps to two values of y. So in this case y cannot be written as a function of x.)

Therefore the function f: $x \to x^2$ does not have an inverse, i.e. not every function has an inverse.

A function has an inverse if and only if the function is a one-to-one mapping.

If we change the definition of f to f: $x \to x^2$ for $x \in \mathbb{R}^+$ then the inverse mapping is

$x \to \sqrt{x}$ for $x \in \mathbb{R}^+$ and this is a function,
$f^{-1}(x) = \sqrt{x}$ for $x \in \mathbb{R}^+$



To summarise:

The inverse of a function undoes the function, i.e. it maps the output (range) of a function back to its input (domain).

The inverse of the function f is written f^{-1}.

Not all functions have an inverse.

When the curve whose equation is $y = \text{f}(x)$ is reflected in the line $y = x$, the equation of the reflected curve is $x = \text{f}(y)$. If this equation can be written in the form $y = \text{g}(x)$ then g is the inverse of f, i.e. $\text{g}(x) = \text{f}^{-1}(x)$.

Consider the compound function $\text{ff}^{-1}(x)$.

When $\text{f}(x) = 3x - 2$, $\text{f}^{-1}(x) = \dfrac{x+2}{3}$ so $\text{ff}^{-1}(x) = 3\left(\dfrac{x+2}{3}\right) - 2 = x + 2 - 2 = x$

Also $\text{f}^{-1}\text{f}(x) = \dfrac{(3x-2)+2}{3} = x$

so $$\text{ff}^{-1}(x) = \text{f}^{-1}\text{f}(x) = x.$$

This is true for any function that has an inverse.

EXERCISE 14j

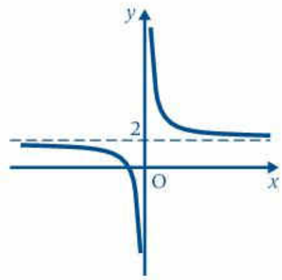
Example:

Determine whether there is an inverse of the function f given by $\text{f}(x) = 2 + \frac{1}{x}$.
If f^{-1} exists, express it as a function of x.

From the sketch of $\text{f}(x) = 2 + \frac{1}{x}$, we see that one value of f(x) maps to one value of x, therefore the reverse mapping is a function. The equation of the reflection of $y = 2 + \frac{1}{x}$ can be written as

$$x = 2 + \frac{1}{y} \Rightarrow y = \frac{1}{x-2} \text{ (interchanging } x \text{ and } y)$$

$\therefore$ when $\text{f}(x) = 2 + \frac{1}{x}$, $\text{f}^{-1}(x) = \frac{1}{x-2}$, provided that $x \neq 2$



Example:

Find $\text{f}^{-1}(4)$ when $\text{f}(x) = 5x - 1$.

If $y = \text{f}(x)$, i.e. $y = 5x - 1$
then for the reflected curve $x = 5y - 1 \quad \Rightarrow \quad y = \frac{1}{5}(x + 1)$

i.e. $\quad \text{f}^{-1}(x) = \frac{1}{5}(x + 1)$

$\therefore \quad \text{f}^{-1}(4) = \frac{1}{5}(4 + 1) = 1$

1 Sketch the graphs of f(x) and $\text{f}^{-1}(x)$ on the same axes.

a $\text{f}(x) = 3x - 1$ **b** $\text{f}(x) = 2^{-x}$ **c** $\text{f}(x) = (x - 1)^3$

d $\text{f}(x) = 2 - x$ **e** $\text{f}(x) = \dfrac{1}{x-3}$ **f** $\text{f}(x) = \dfrac{1}{x}$

2 Which of the functions given in Question **1** are their own inverses?

3 Determine whether f has an inverse function, and if it does, find it when

a $\text{f}(x) = x + 1$ **b** $\text{f}(x) = x^2 + 1$

c $\text{f}(x) = x^3 + 1$ **d** $\text{f}(x) = x^2 - 4, x \geq 0$

e $\text{f}(x) = (x + 1)^4, x \geq -1$.

4 The function f is given by $f(x) = 1 - \frac{1}{x}$. Find
 a $f^{-1}(4)$
 b the value of x for which $f^{-1}(x) = 2$
 c any values of x for which $f^{-1}(x) = x$.

5 If $f(x) = 3^x$, find
 a $f(2)$ b $f^{-1}(9)$ c $f^{-1}\left(\frac{1}{3}\right)$

Compound functions

Consider the two functions f and g given by

$$f(x) = x^2 \quad \text{and} \quad g(x) = \frac{1}{x}$$

These two functions can be combined in several ways.

1 They can be added or subtracted,

i.e. $f(x) + g(x) = x^2 + \frac{1}{x}$ and $f(x) - g(x) = x^2 - \frac{1}{x}$.

2 They can be multiplied or divided,

i.e. $f(x)\,g(x) = (x^2)\times\left(\frac{1}{x}\right) = x$ and $\frac{f(x)}{g(x)} = \frac{x^2}{\frac{1}{x}} = x^3$.

3 The output of f can be made the input of g,

i.e. $x \xrightarrow{f} x^2 \xrightarrow{g} \frac{1}{x^2}$ or $g[f(x)] = g(x^2) = \frac{1}{x^2}$

Therefore the function $x \to \frac{1}{x^2}$ is obtained by taking the function g of the function f.

Function of a function

A compound function formed in the way described in **3** above is known as a **function of a function** or **composite function** and it can be denoted by gf (or by $g \circ f$).

For example, if $f(x) = 3^x$ and $g(x) = 1 - x$ then $gf(x)$ means the function g of the function $f(x)$,

i.e. $gf(x) = g(3^x) = 1 - 3^x$

Similarly $fg(x) = f(1 - x) = 3^{(1-x)}$

Note that $gf(x)$ is not the same as $fg(x)$.

Also $gg(x) = g(1 - x) = 1 - (1 - x) = x$. Note that $gg(x)$ is also written as $g^2(x)$ and $f^3(x)$ means $fff(x)$.

EXERCISE 14k

1 If f, g and h are functions defined by $f(x) = x^2$, $g(x) = \frac{1}{x}$, $h(x) = 1 - x$ find as a function of x
 a fg b fh c hg d ff e gf f hf

2 If $f(x) = 2x - 1$ and $g(x) = x^3$ find the value of
 a $g \circ f(3)$ b $f \circ g(2)$ c $f \circ g(0)$ d $g \circ f(0)$

3 Given that $f(x) = 2x$, $g(x) = 1 + x$ and $h(x) = x^2$, find as a function of x
 a hg b fhg c ghf

4 The function $f(x) = (2 - x)^2$ can be expressed as a function of a function. Find g and h as functions of x such that $gh(x) = f(x)$.

5 Repeat Question **4** when $f(x) = (x + 1)^4$.

6 Express the function $f(x)$ as a combination of functions $g(x)$ and $h(x)$, and define $g(x)$ and $h(x)$, where $f(x)$ is

a $10^{(x+1)}$ b $\frac{1}{(3x-2)^2}$ c $2^x + x^2$

d $\frac{(2x+1)}{x}$ e $(5x - 6)^4$ f $(x - 1)(x^2 - 2)$

Example:

The functions f and g are defined as

$$f(x) = \frac{x+3}{2x-1} \quad \text{and} \quad g(x) = x + 4$$

a Calculate i $g(-1)$ ii $f(5)$

b Write expressions for i $g^{-1}(x)$ ii $fg(x)$ iii $f^{-1}(x)$

c Show that $(fg)^{-1}(x) = g^{-1}f^{-1}(x)$

a i $g(-1) = -1 + 4 = 3$ ii $f(5) = \frac{5+3}{10-1} = \frac{8}{9}$

b i Let $y = x + 4$.

For the inverse function, $x = y + 4 \Rightarrow y = x - 4$

$\therefore g^{-1}(x) = x - 4$

ii $fg(x) = \frac{(x+4)+3}{2(x+4)-1} = \frac{x+7}{2x+7}$

iii Let $y = \frac{x+3}{2x-1}$ so for the inverse function,

$$x = \frac{y+3}{2y-1}$$

so $2xy - x = y + 3 \Rightarrow 2xy - y = x + 3$

$\Rightarrow \quad y = \frac{x+3}{2x-1}$

$\therefore \quad f^{-1}(x) = \frac{x+3}{2x-1}$

$f^{-1}(x) = f(x)$ so $f(x)$ is its own inverse or **self inverse**.

c If $y = fg(x)$, then for $(fg)^{-1}(x)$,

$$x = \frac{y+7}{2y+7} \Rightarrow 2yx + 7x = y + 7$$

so $2yx - y = 7 - 7x \Rightarrow y = \frac{7-7x}{2x-1}$

i.e. $(fg)^{-1}(x) = \frac{7-7x}{2x-1}$

$$f^{-1}(x) = \frac{x+3}{2x-1} \quad \text{and} \quad g^{-1}(x) = x - 4$$

so $g^{-1}f^{-1}(x) = \left(\frac{x+3}{2x-1}\right) - 4 = \frac{x+3-4(2x-1)}{2x-1}$

$$= \frac{x+3-8x+4}{2x-1} = \frac{7-7x}{2x-1}$$

$\therefore \quad (fg)^{-1}(x) = g^{-1}f^{-1}(x)$

The result proved in part **c** is true for any two functions, i.e.

if f and g are functions and if f^{-1} and g^{-1} exist, then $(fg)^{-1}(x) = g^{-1}f^{-1}(x)$

7 The functions f and g are defined by $f(x) = 7x - 1$ and $g(x) = 3 - 4x$
- **a** Evaluate **i** $f(-1)$ **ii** $g(-1)$ **iii** $f(0)$ **iv** $g(-4)$
- **b** Find **i** $fg\left(\frac{1}{2}\right)$ **ii** $gf(2)$
- **c** **i** If $f(x) = 34$ find the value of x.
  ii If $g(x) = 15$ find the value of x.

8 The functions f and g are defined by $f: x \to 5x - 4$ and $g: x \to 2x + 1$
- **a** Evaluate **i** $f(-3)$ **ii** $g(-1)$ **iii** $fg(3)$ **iv** $gf(3)$
- **b** Find the value of x if **i** $f(x) = 31$ **ii** $g(x) = 9$

9 The functions f and g are defined by $f: x \to 12 - 3x$ and $g: x \to 2x + 3$
- **a** Evaluate **i** $f(4)$ **ii** $g\left(-\frac{1}{2}\right)$ **iii** $fg(-1)$
- **b** Find an expression for **i** $fg(x)$ **ii** $gf(x)$

10 The functions f and g are defined by $f(x) = 2x - 5$ and $g(x) = 5 - 3x$
- **a** Evaluate **i** $f(-1)$ **ii** $g(-1)$ **iii** $fg(-1)$ **iv** $gf(-1)$
- **b** Find an expression for **i** $f^{-1}(x)$ **ii** $g^{-1}(x)$ **iii** $f^{-1}g^{-1}(x)$

11 The functions f and g are defined by $f(x) = \frac{x + 2}{2x + 3}$ and $g(x) = \frac{x - 1}{2 - x}$
- **a** Evaluate **i** $f(1)$ **ii** $g(1)$
- **b** Find an expression for $fg(x)$
- **c** Hence show that $fg(2) = \frac{1}{2}$

12 The functions f and g are defined by $f(x) = \frac{2 - x}{3x + 4}$ and $g(x) = 1 - 2x$
- **a** Evaluate **i** $f(3)$ **ii** $g(-3)$
- **b** Find an expression for $fg(x)$
- **c** Hence show that $fg(-3) = -\frac{1}{5}$
- **d** Find $gf(-3)$.

13 The functions f, g and h are defined by $f(x) = x + 3$, $g(x) = 3x - 4$ and $h(x) = 1 - 2x$
- **a** Find **i** $fgh(x)$ **ii** $ghf(x)$ **iii** $hfg(x)$
- **b** Evaluate **i** $fgh(3)$ **ii** $ghf(2)$ **iii** $hfg(2)$

14 Given that $f(x) = \frac{2 + x}{3 - x}$ find

a $f(1)$ **b** $f\left(\frac{1}{2}\right)$ **c** $f^{-1}(x)$ **d** $f^{-1}(2)$ **e** $f^{-1}\left(\frac{1}{2}\right)$

15 Given that $f: x \to \frac{3}{x + 2}$ and $g: x \to x + 3$ find
- **a** **i** $fg(x)$ **ii** $gf(x)$ **iii** $fg(2)$ **iv** $gf(3)$
- **b** **i** $f^{-1}(x)$ **ii** $g^{-1}(x)$ **iii** $f^{-1}g^{-1}(x)$ **iv** $g^{-1}f^{-1}(x)$

16 The functions f and g are defined by $f(x) = \frac{x - 2}{x + 4}$ and $g(x) = x - 3$
- **a** Calculate **i** $f(4)$ **ii** $g(5)$
- **b** Find an expression for **i** $f^{-1}(x)$ **ii** $fg(x)$ **iii** $g^{-1}(x)$ **iv** $gf(x)$

17 The functions f and g are defined by $f(x) = \frac{x - 2}{3x - 2}$ and $g(x) = \frac{2x + 1}{x + 4}$
- **a** Evaluate **i** $f(-1)$ **ii** $g(1)$
- **b** Find **i** $fg(1)$ **ii** $gf(-1)$
- **c** Find an expression for $fg(x)$

18 The functions f and g are defined by $f(x) = \frac{x}{x+3}$ and $g(x) = 4 + 2x$
a Calculate **i** f(3) **ii** g(3)
b Find an expression for **i** $f^{-1}(x)$ **ii** fg(x) **iii** $g^{-1}(x)$ **iv** gf(x)

19 The functions f and g are defined by $f(x) = \frac{3x+3}{2x+1}$ and $g(x) = \frac{2x-3}{3x+2}$
a Calculate **i** f(4) **ii** g(−2)
b State the real number that cannot be in the domain of
i f(x) **ii** g(x)
c Find an expression for **i** $f^{-1}(x)$ **ii** $g^{-1}(x)$

20 The functions g and h are defined by $g(x) = \frac{x^2}{2}$ and $h(x) = \frac{1}{2x}$
Find **a** g(3) + g(23) **b** h(3) − h(−3) **c** gh(2) **d** fg(2)

21 The functions f and g are defined by $f(x) = \frac{1}{3}x + 4$ and $g(x) = x^3$
a Evaluate **i** g(2) − g(−2) **ii** $f^{-1}(5)$ **iii** $f^{-1}(4)$ **iv** fg(2)
b Find an expression for **i** $f^{-1}(x)$ **ii** fg(x) **iii** $g^{-1}(x)$ **iv** gf(x)

22 The function $f(x) = \frac{x+2}{x-1}$
a Calculate $f\left(\frac{1}{2}\right)$
b Given that $f^2(x) = ff(x)$ show that $f^2(x) = x$
c If $f^3(x) = f(f^2(x))$ find $f^3(x)$
d Suggest an expression for $f^{21}(x)$.

23 The functions f and g are defined by $f(x) = 2x + 3$ and $g(x) = \frac{3x-1}{x+2}$
a Find **i** $f\left(\frac{3}{2}\right)$ **ii** g(1)
b Show that $gf(-1) = \frac{2}{3}$
c Find fg(1)

24 Given that $f: x \to x^2$ and $g: x \to 3 - x$ find
a fg(x) **b** gf(x) **c** fg(3) **d** gf(3)
e $(fg)^{-1}(x)$ **f** $(gf)^{-1}(x)$ **g** $f^{-1}g^{-1}(x)$ **h** $g^{-1}f^{-1}(x)$

25 The functions f and g are defined by $f(x) = \frac{x+3}{x+5}$ and $g(x) = \frac{x}{x+1}$
a Calculate **i** f(2) **ii** f(0) **iii** f(−3) **iv** g(3) **v** g(−4)
b Find an expression for **i** $f^{-1}(x)$ **ii** $g^{-1}(x)$ **iii** fg(x) **iv** gf(x)

26 Given that $f: x \to x + 2$, $g: x \to x^2$ and $h: x \to x - 3$ find
a fgh(x) **b** fgh(2) **c** $fgh\left(-\frac{1}{2}\right)$

27 Given that $f(x) = 2x + 1$, $g(x) = 3x$ and $h(x) = x - 3$
find an expression for
a fgh(x) **b** ghf(x) **c** fgh(1) **d** $ghf\left(\frac{1}{2}\right)$

28 Given that $f(x) = x^2$, $g(x) = x^3$ and $h(x) = 2x$
find an expression for
a fgh(x) **b** ghf(x) **c** hfg(x) **d** fgh(1) **e** ghf(2) **f** hfg(2)

29 Given that f and g are defined by $f: x \to 2x$, $g: x \to 3 - x$
a Find an expression for
i fg(x) **ii** gf(x) **iii** $(fg)^{-1}(x)$ **iv** $g^{-1}f^{-1}(x)$
b Show that $(fg)^{-1}(4) = g^{-1}f^{-1}(4)$

30 Given that f and g are defined by $f: x \to x + 2$, $g: x \to 3x$
a Find $(fg)^{-1}(x)$ and $g^{-1}f^{-1}(x)$
b Evaluate $(fg)^{-1}(5)$ and $g^{-1}f^{-1}(5)$

31 The functions f and g are defined by $f(x) = \frac{x+2}{5-x}$ and $g(x) = \frac{2x-1}{x+1}$

a Evaluate **i** f(3) **ii** g(−2)

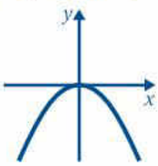
b Show that $fg\left(-\frac{1}{2}\right) = \frac{2}{9}$

c Find gf(3)

ABCD MIXED EXERCISE 14

Several answers are given for these questions.
Write down the letter that corresponds to the correct answer.

1 The graph representing $y = \frac{3}{x}$ could be

A


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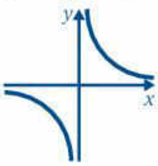

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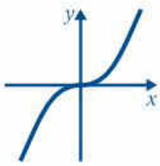

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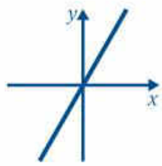
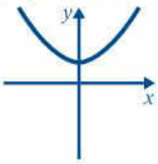
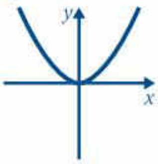
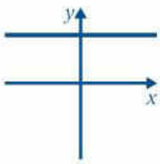
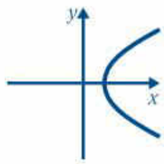


2 The graph representing $y = 10x^2$ could be

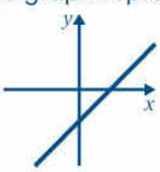
A


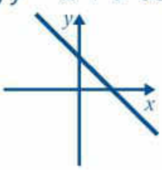
B


C


D


3 The graph representing $y = x + 2$ could be

A


B


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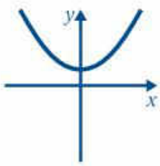

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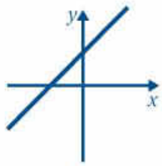
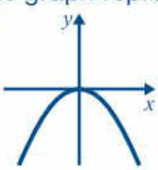


4 The graph representing $y = -x^2$ could be

A


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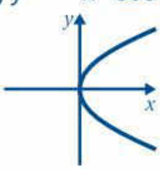

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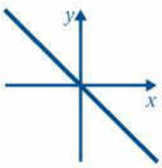
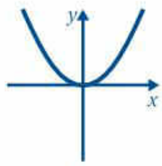
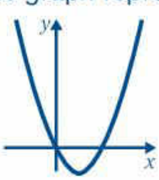
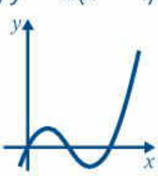
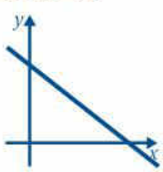


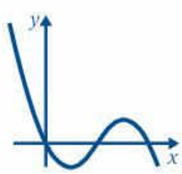
D


5 The graph representing $y = x(x-1)(x-2)$ could be

A


B


C


D


6 $f(x) = 3x - 1$, $f^{-1}(x) =$

A $3x - 1$ **B** $\frac{1}{3}(x - 1)$ **C** $\frac{1}{3}(x + 1)$ **D** $\frac{1}{3}x + 1$

7 $f(x) = 5x + 2$, $g(x) = 1 - 3x$, $fg(x) =$

A $-15x - 5$ **B** $2x + 3$ **C** $7 - 15x$ **D** $2 - x - 15x^2$

8 The values of x where the graphs of $y = x^3$ and $y = 2 - x$ intersect are the solutions of the equation

A $2 - x = 0$ **B** $x^3 = 2$ **C** $x^3 = x - 2$ **D** $x^3 + x - 2 = 0$

9 $\cos x° = 0$ when $x =$

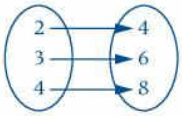
A 0 **B** 90 **C** 180 **D** 360

10 The diagram shows a function f.

f^{-1} is described by the mapping

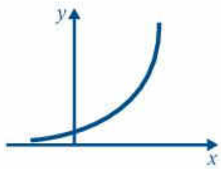
A $x: x \to 2x$ **B** $x: x \to \frac{1}{2}x$

C $x: x \to x + 2$ **D** $x: x \to 2 - x$

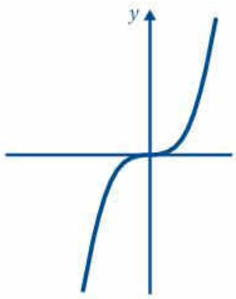


11 The equation of the graph shown could be

A $y = x^2$ **B** $y = x^3$ **C** $y = \frac{1}{x}$ **D** $y = 2^x$



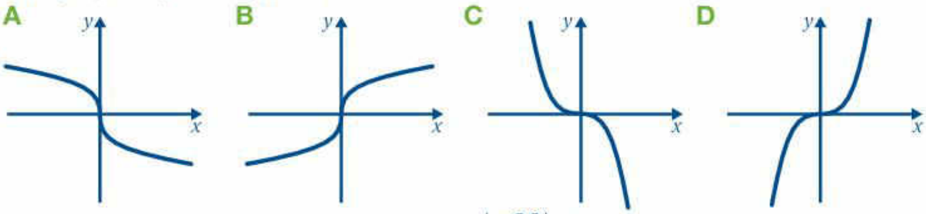
Questions **12** and **13** refer to this diagram which shows the graph of $y = f(x)$.



12 $f(x)$ could be

A x^2 **B** x^3 **C** $\sin x°$ **D** 2^x

13 The graph of $y = f^{-1}(x)$ could be



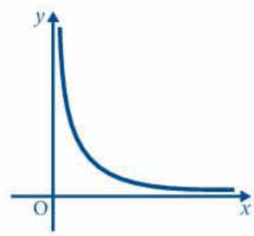
14 The graph of $y = \sin x°$ is translated by $\begin{pmatrix} -90 \\ 0 \end{pmatrix}$. The equation of the translated curve is

A $y = \sin x° - 90$ **B** $y = 90 + \sin x°$

C $y = \cos x°$ **D** $y = 90 + \cos x°$

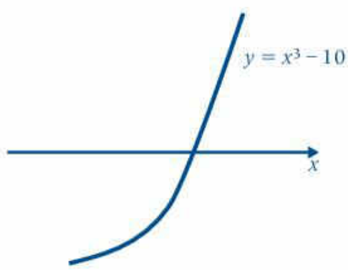
15 The diagram on the right is most likely to illustrate the graph with equation

A $y = x^2$ **B** $y = x + 1$ **C** $y = \frac{3}{x}$ **D** $y = 2x^2$



16 The curve with equation $y = x^3 - 10$ cuts the x-axis between the integers

A -3 and -2 **B** -2 and -1 **C** 1 and 2 **D** 2 and 3



17 Given that $\sin x°$ and $\cos x°$ are both negative and that $0 \leqslant x \leqslant 360$ then

A $0 < x < 90$ **B** $90 < x < 180$

C $180 < x < 270$ **D** $270 < x < 360$

18 Which of the following functions is its own inverse?

A $f(x) = 2x$ B $f(x) = \frac{1}{x}$ C $f(x) = x + 1$ D $f(x) = x^2$

19 $\sin 90° + \cos 90° =$

A -1 B 0 C 1 D 2

20 If $f(x) = 2x + 1$ then $f^{-1}(5) =$

A $\frac{1}{11}$ B $\frac{1}{5}$ C 2 D 11

MATHS IS OUT THERE

Did you know that there were mathematical cover-ups? For example: while on a sea voyage, Hippasus of Metapontum was thrown overboard from the ship by a crowd of Pythagoreans for revealing the existence of irrational numbers (numbers which cannot be written as fractions). The Pythagoreans believed that only whole numbers and their ratios could describe anything geometric.

IN THIS CHAPTER YOU HAVE SEEN THAT...

- for all real values of x, $-1 \leqslant \sin x° \leqslant 1$ and $-1 \leqslant \cos x° \leqslant 1$
- the range of the tangent function, $f(x) = \tan x$, is all real numbers but the domain cannot include odd multiples of 90
- the function $f(x) = a^x$ is called an exponential function. When $a > 1$, the graph of this function is called an exponential growth curve. The domain of the function is all real numbers. The range is all numbers greater than zero. For any value of a, the curve $y = a^x$ crosses the y-axis where $y = 1$
- the graph of the exponential function $f(x) = a^{-x}$, $a > 1$, is called an exponential decay curve. The curve $y = a^{-x}$ crosses the y-axis where $y = 1$ for any value of a
- the inverse of a function maps the range of a function to its domain
- a function has an inverse if and only if the function is a one-to-one mapping
- you can find the inverse mapping for $y = f(x)$ by interchanging x and y then solving the resulting equation for y. If this new mapping is a function, it is called the inverse of f and is denoted by f^{-1}
- the graph of an inverse function can be found by reflecting the graph of the original function in the line $y = x$.

15 TRAVEL GRAPHS

AT THE END OF THIS CHAPTER YOU SHOULD BE ABLE TO...

1. Interpret the gradient of a curve at a point on the curve.
2. Draw a distance–time, *s*–*t*, graph, given pairs of values for *s* and *t*.
3. Use a distance–time graph to find the speed of an object.
4. Use a distance–time graph to estimate the average velocity between two times.
5. Use a velocity–time, *v*–*t*, graph to estimate the acceleration of an object at a given time.
6. Use a velocity–time graph to find the distance travelled by an object.

BEFORE YOU START you need to know:
- ✓ the relationship between distance, speed and time
- ✓ how to find the gradient of a straight line
- ✓ how to plot points and draw graphs
- ✓ how to find the area of a trapezium.

KEY WORDS
acceleration, chord, distance, distance–time graph, gradient, speed, tangent, time, velocity, velocity–time graph

MATHS IS OUT THERE

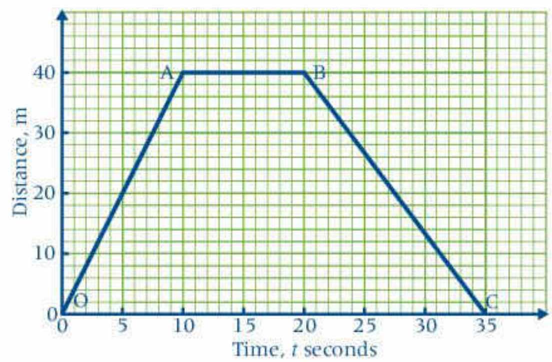
We know that a *rolling stone gathers no moss*. What about *a rolling curve*?

As a circle rolls along a straight line, the curve generated by a point on it is called a *cycloid*. Galileo is responsible for naming this curve.

Straight line distance–time graphs

When we travel, we cover **distance** and take **time** to do it.
A **distance–time graph** is a plot of the distance covered against the time taken.

This graph shows the journey of a runner practising sprints.



The section OA shows that, in the first 10 seconds, the runner covered 40 m.
So the **speed** for these 10 seconds is $\frac{40}{10}$ m/s = 4 m/s.

The section AB shows that the runner did not cover any distance, i.e. for these 10 seconds, the runner was resting.

The section BC shows that the runner took 15 seconds to run back to his starting point. His speed for this section was $\frac{40}{15}$ m/s = 2.67 m/s (2 d.p.)

The average speed for the whole journey

$= \frac{\text{total distance}}{\text{total time}} = \frac{80}{35}$ m/s = 2.29 m/s (2 d.p.)

Total distance = 40 m out + 40 m back

EXERCISE 15a

Example:

A car leaves home and travels at 60 km/h for 1 hour. It stops for 30 minutes then returns home at constant speed, arriving 3 hours after it left.

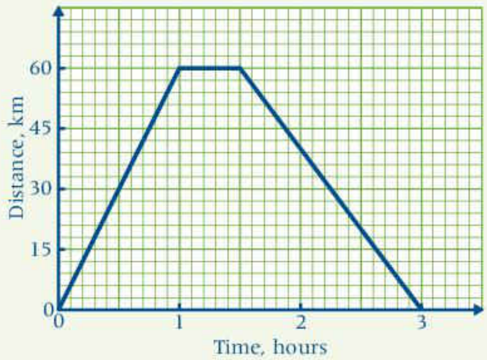
a Draw a distance–time graph to represent this journey.

b Calculate

  i the speed on the return journey

  ii the average speed of the car for the whole journey.

a



At 60 km/h, the car travels 60 km in one hour.
In the next half-hour, the car does not move, so the graph is flat.

b i *Speed on the return journey* $= \frac{60}{1.5}$ km/h $= 40$ km/h

  ii *Average speed for the whole journey* $= \frac{120}{3}$ km/h $= 40$ km/h

1 Connie leaves home on her bicycle to visit her grandparents. The graph shows her journey.

  a How long does she stay with her grandparents?

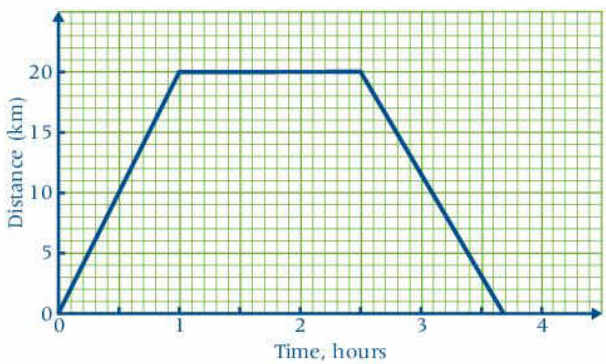
  b Which journey is the faster – the outward journey or the homeward one?

  c What was her average speed

    i on the outward journey

    ii on the homeward journey

    iii for the total time she was cycling?



2 The graph shows the journeys of two trains from Axford to Beardsley.

  a How far is it from Axford to Beardsley?

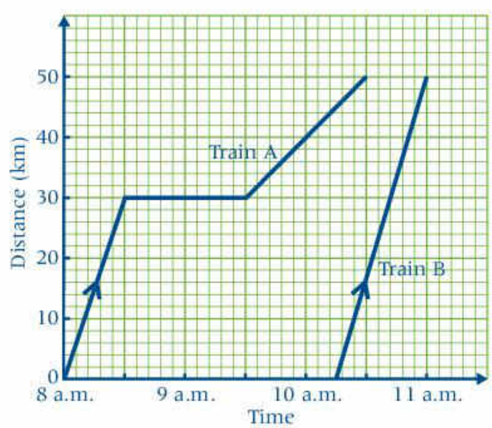
  b i How far does train A travel before it stops?

    ii For how long is it stationary?

    iii Find its average speed for the second part of its journey.

  c i Which train arrives at Beardsley first?

    ii How long is it before the other train arrives?



3 A motorist travelling at a steady speed along a main road passed a point A at 12.00 noon. At 2.00 p.m. he stopped at a café B, which is 160 km from A, and at 3.00 p.m. he continued his journey at the same speed to C which is 30 km from B. He then continued at a steady speed of 100 km/h.

Taking 2 cm to 1 hour on one axis and 2 cm to 50 km on the other, draw the graph representing the two stages of the motorist's journey on the main road.
Hence, or otherwise, find the time at which the motorist arrived at a point D, a distance of 350 km from A measured along the main road.

4 Winston leaves Bridgetown at noon to cycle to St John, a town 20 km away. He pedals at a steady speed of 15 km/h but after 17 km he breaks down. After spending 15 min trying to repair his cycle, he decides to push it the rest of the way. This he does at a constant speed of 4 km/h. Draw a travel graph to illustrate the journey and use it to find

a the time of arrival of at St John
b the distance that Winston is from Bridgetown one hour after he left.

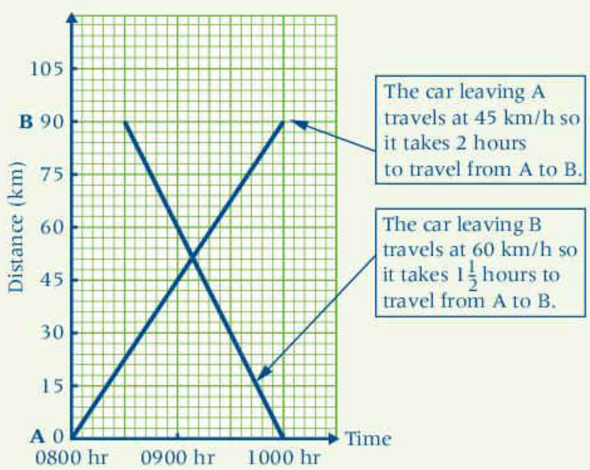
Use 6 cm ≡ 5 km and 2 cm ≡ 15 min.

Example:

A car leaves village A at 0800 hr and travels at 45 km/h to village B. Another car leaves village B at 0830 hr and travels at 60 km/h to village A. The distance between the villages is 90 km.

a Draw a distance–time graph showing the journeys of the two cars.
b Use the graph to find
  i the time when the cars meet
  ii the distance from A where the cars meet.

a

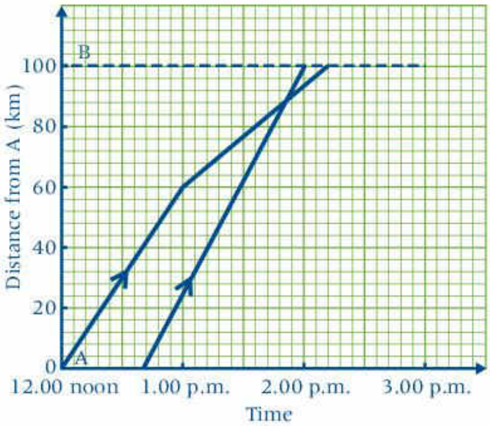


Choose scales so that each subdivision represents a whole number.
For the scales used here, each subdivision on the time axis represents 6 minutes.
Each subdivision on the vertical axis represents 3 km.

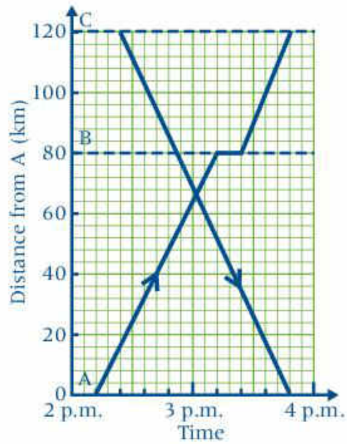
This diagram can be used to find the relationship between distance, speed and time:

Cover up the one you want to find.

b i *The cars meet at 0909 hr.*
  ii *The cars meet 51 km from A.*



5 This travel graph shows the journeys of two buses from town A to town B. The first bus leaves A at noon and the other bus follows later.
Use the graph to find
   a the speed of the first bus during the first hour
   b the average speed of the first bus for the second part of the journey
   c the average speed of the first bus for the whole of the journey
   d the time at which the second bus leaves A and the time it arrives at B
   e the average speed of the second bus
   f the time and distance from A when the second bus passes the first.



6 This travel graph shows the journeys of two cars travelling between three service stations of a highway. The first car travels from A to B and stops at B for a short while before proceeding to C. The second car travels from C to A without stopping.
Use the graph to find
   a the average speed of the first car on the first stage of the journey
   b how long it stops at B
   c the average speed of the first car for the whole of the journey
   d the time that the second car leaves C and the time it arrives at A
   e the average speed of the second car
   f when and where the two cars pass.

7 Dick leaves the village of Axeter at 12.06 p.m. to walk to the neighbouring village of Botlow $9\frac{1}{2}$ kilometres away. He walks at a steady 4 km/h but after walking for 5 km he gets a lift in a friend's car which takes him the remainder of the journey at a steady speed of 30 km/h. At 12.16 p.m. his brother George leaves Axeter and jogs to Botlow at a steady 6 km/h. Draw travel graphs for the two journeys taking 2 cm to represent 1 km and 6 cm to represent 1 h. Use your graphs to find
   a which of the brothers arrives at Botlow first, and how long he must wait for the other brother to arrive
   b when and where they pass
   c for how long George is ahead of Dick.

8 A bus leaves a town A at 10 a.m. and travels at a steady speed to a town B, which is 40 km from A, arriving at 11.00 a.m. It remains at B for half an hour, and then continues to a third town C, which is 60 km further on from B, arriving at 12.45 p.m. A second bus leaves C at 10.45 a.m., travels at a constant speed, arriving at A at 1.45 p.m. Using 2 cm ≡ 10 km on the vertical axis, and 2 cm ≡ 30 minutes on the horizontal axis, draw the travel graphs for these journeys, and use them to find
   a when and where the buses pass
   b the distance between the buses at 12.15 p.m.
   c the constant speed of the second bus.

9 Antley, Bexeter and Cridgley are three towns (in that order) on a straight road, Antley being 60 km south of Bexeter and Cridgley 75 km north of Bexeter. A car leaves Antley at noon and travels at a constant speed of 60 km/h to Bexeter where it remains for 20 minutes before travelling on to Cridgley, arriving at 2.40 p.m. A second car leaves Antley at 12.36 p.m. and travels to Cridgley without a stop, arriving at 2.28 p.m.

Using 1 cm ≡ 10 km and 2 cm ≡ 20 minutes, draw suitable travel graphs and use them to find

a when and where the cars pass

b the speed of the first car for the second part of the journey

c when the second car is exactly 10 km ahead of the first.

10 A and B are two garages 120 km apart. One car leaves A at 11.08 a.m. and travels to B, without stopping, at a constant speed, arriving at 1.32 p.m. A second car leaves B for A, travels at a constant 60 km/h, and arrives at 1.36 p.m. Draw travel graphs for these journeys and from them find

a the time of departure from B of the second car

b the average speed of the first car

c when and where they pass

d their distance apart at 1 p.m.

Use 2 cm ≡ 10 km and 2 cm ≡ 20 minutes.

11 A lorry leaves A at noon to travel to B, a town 95 km away. It travels the first 40 km at an average speed of 40 km/h and, without stopping, completes the journey to B arriving there at 2.30 p.m. A second lorry leaves B and travels at 60 km/h, arriving at A at 2.44 p.m. Taking 2 cm to represent 10 km on one axis and 20 minutes on the other, draw travel graphs for the two journeys and use them to determine

a the starting time for the second lorry

b when and where they pass

c the average speed of the first lorry for the whole journey.

12 Two hikers leave a mountain centre A at 8 a.m. and set out on a 24-kilometre walk to B. They walk at a steady 8 km/h to a check point which is at the halfway stage. After a 15-minute rest they continue at the same speed but 4 km from the end of the walk one of them injures his foot and cannot continue. After resting with him for half an hour the second hiker decides to press on to B to seek help, walking at 10 km/h. At noon the rescue party leaves B and walks towards the injured hiker at a steady 8 km/h. The rescue party spends 15 minutes with him, then leaves with him to return to A arriving there at 5 p.m.
Draw a distance–time graph to show these journeys and from it find

a the time the hiker who was seeking help arrived at B

b the time the rescue party reached the injured hiker

c the average speed at which he was transported to A.

13 A, B and C are three stations on a straight railway line such that B is 50 km from A and C a further 70 km on from B. A train leaves A at 2.30 p.m. and travels to B at a steady speed of 70 km/h where it stops for 6 minutes before proceeding to C at a steady speed of 50 km/h. A second train leaves C at 3.12 p.m. and travels directly to A arriving there at 4.21 p.m. Draw a travel graph for these two journeys taking 2 cm ≡ 10 km and 8 cm ≡ 1 hour. Use your graphs to find

a the time of arrival of the first train at C

b the average speed of the second train

c when and where they pass.

14 A and B are two service stations on a highway 120 kilometres apart. Mrs Brown leaves A at noon to travel to B. She covers the first 40 km in 1 h, then rests for 45 minutes before proceeding to B at a steady 70 km/h. At 12.27 p.m. Mrs White leaves B at a steady 60 km/h. After 1 h she takes a 15-minute rest before proceeding to A at a steady 50 km/h. Draw travel graphs to represent these journeys taking 4 cm ≡ 1 hour and 2 cm ≡ 10 km.

Use your graph to determine

a when and where they pass

b which woman arrives at her destination first

c their distance apart at 1.31 p.m.

15 John sets out at noon from his home village of Atley, to call at Bentham which is 16 km away, before going on to Cottle, a village which is 9 km beyond Bentham. He walks for 1 h at 6 km/h, then rests for 20 minutes before running the remaining distance at 10 km/h to his cousin's home at Bentham. Here he chats with his cousin for 4 minutes before leaving on a borrowed bicycle, cycling to Cottle at an average speed of 16 km/h.

Draw a travel graph to represent John's journey taking 6 cm ≡ 1 hour and 4 cm ≡ 5 km.

Use your graph to find his time of arrival at Cottle.

In the meantime his friend Tim leaves Cottle at 12.40 p.m. and cycles leisurely to Atley at a steady 12 km/h. Show this on the same graph and hence find

a when and where the two friends pass

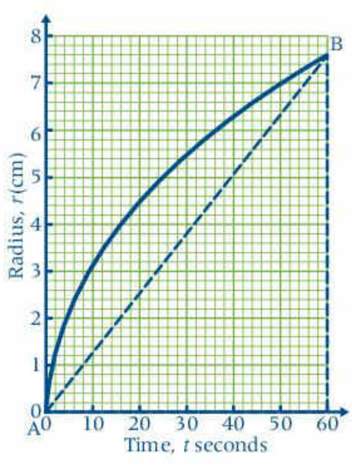
b Tim's time of arrival at Atley

c their distance apart at 1.34 p.m.

Rates of change

An outside tap is left dripping and the water from it forms a circular puddle on the ground.

The graph shows how the radius of the puddle changes as time passes.



The graph shows that the radius of the puddle continues to increase as the tap drips. Looking at the shape of the curve, we see that the curve is steepest at the start and continues to get flatter as time passes. This means that the radius is increasing most rapidly when the tap starts to drip and the rate of increase of the radius slows down over the 60-second time span.

- Reading from the graph, we can see that the radius has increased from 0 to 7.7 cm in the time span from 0 to 60 seconds, so it has increased at an average rate of $\frac{7.7}{60}$ cm per second over this time span. This corresponds to the gradient of the line joining A and B.
- We can also find the average rate at which the radius has increased for any other time span on this graph. It is not, however, immediately obvious how to find the rate at which the radius is changing at any particular instant, for example, exactly 5 seconds after the tap started dripping. The next exercise investigates how this can be done.

EXERCISE **15b**

1 Copy the graph on the previous page and use your copy to answer the questions.

a Find the points on the curve where $t = 5$ and where $t = 50$ and label them C and D respectively. Draw the line *through* C and D (extend it beyond both C and D). Find

i the increase in r over this time span

ii the average rate of increase in r over this time span.

Explain the relationship between the answer to part **ii** and the gradient of the line through C and D.

b Mark the point E on the curve where $t = 30$ and repeat part **a** for the points C and E.

c Mark the point F on the curve where $t = 10$ and repeat part **a** for the points C and F.

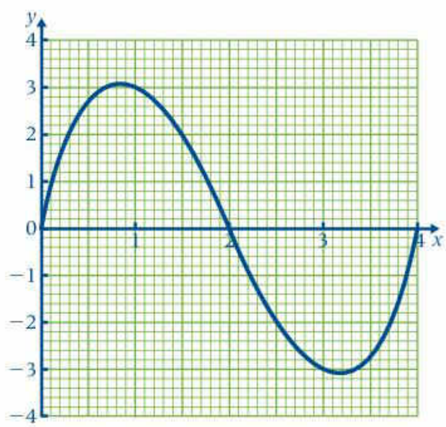
d Mark the point G on the curve where $t = 6$ and repeat part **a** for the points C and G.

e Use your results from parts **a** to **d** to give an estimate of the rate at which the radius is increasing when $t = 5$. Discuss the relationship between your estimate and the gradients of the lines.

f Discuss how an improved estimate for the rate of increase of r when $t = 5$ can be found.

2 This is the graph of the curve whose equation is

$$y = x(x - 2)(x - 4)$$



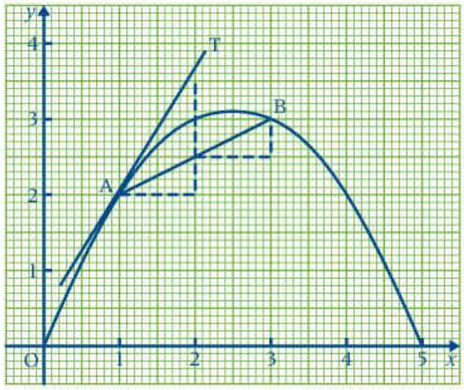
a Discuss how y changes as x increases from $x = 0$ to $x = 3$.

b Find the gradient of the line joining the points (1, 3) and (2, 0) on this curve and discuss the meaning of the gradient in relation to how y is changing with respect to values of x from $x = 1$ to $x = 2$.

c Discuss how you could estimate the change in y with respect to x at the point on the curve where $x = 2$.

Tangents to curves

This graph shows the curve whose equation is $y = \frac{1}{2}x(5 - x)$.



A line joining two points on a curve is a **chord**.

The line AB is a chord of the curve shown.

A line which touches a curve at a point is a **tangent** to the curve at the point. In the diagram, AT is the tangent to the curve at A.

When moving along a curve, the **gradient** changes continuously. Imagine moving along the curve in the diagram, starting from O. When you get to A imagine that you stop following the curve and move on in a constant direction: you will move along the tangent AT.

Therefore the gradient of a curve at a point is defined as the gradient of the tangent to the curve at this point.

In the diagram, the gradient of the tangent AT is $\frac{1.5}{1} = \frac{3}{2}$, therefore the gradient of the curve at A is $\frac{3}{2}$.

Discussion from Exercise 15b shows that, at a particular point on a curve, we can estimate the rate at which values of y are increasing as values of x increase by finding the gradient of a chord from that point. This estimate is improved by making the chord as short as possible.

Finding a gradient by drawing and measurement means that the curve must be accurately drawn and then the tangent must be positioned carefully. Using a transparent ruler helps and, as a rough guide, the tangent should be approximately at the same 'angle' to the curve on each side of the point of contact.

From this we deduce that

the gradient of the tangent to the curve at a point gives the rate at which values of y are increasing as values of x increase at this point.

We also saw that the gradient of a chord gives the average rate of increase of y-values for the values of x between the ends of the chord.

EXERCISE 15c

1 Draw x- and y-axes from -6 to $+6$ using a scale of 1 cm for 1 unit on both axes. Plot the points A(1, 1), B(3, 2), C(-4, -1), D(-5, 1), E(1, 6) and F(4, -5). Find the gradient of

a AB **b** BC **c** DE **d** EF **e** AD

Explain why it is not possible to give a value for the gradient of AE.

2 Copy and complete the table for $y = \frac{x^2}{10}$

x	0	1	2	3	4	5	6	7	8
y	0	0.1	0.4						

Use a scale of 2 cm for 1 unit on both axes and draw the curve.

a P is the point on the curve where $x = 2$ and Q is the point on the curve where $x = 4$. Draw the chord PQ and find its gradient.

b Draw, as accurately as possible, the tangents to the curve at the points where $x = 1$, $x = 4$ and $x = 6$.

c Find the gradients of the tangents to the curve at the points where $x = 1$, $x = 4$ and $x = 6$.

3 Copy and complete the table for $y = \frac{10}{x}$ giving values of y correct to 1 decimal place.

x	1	1.5	2	3	4	5	6	7	8
y	10	6.7	5	3.3	2.5				

Use a scale of 2 cm for 1 unit on both axes and draw the curve for values of x from 1 to 8.

a A is the point on the curve where $x = 1$ and B is the point on the curve where $x = 4$. Find the gradient of the chord AB.

b Find the gradients of the tangents to the curve at A and B.

c Find the gradient of the tangent to the curve at the point where $x = 3$.

4 Draw the graph of $y = x^3$ for values of x from 0 to 3.5 using a scale of 2 cm for 1 unit on both axes. Find the gradient of the curve at the points where

a $x = 1$ **b** $x = 3$

In the many practical applications of graphs it is rarely possible to have the same scales on both axes.

When the scales are not the same, care must be taken to read vertical measurements from the scale on the vertical axis and horizontal measurements from the scale on the horizontal axis.

Example:

This table shows the girth, w cm, of a pumpkin, n days after being fed with fertiliser.

n	1	2	3	4	5	6	7
w	15	17	20	24	29	35	41

Use a scale of 2 cm for 1 day and 2 cm for 10 cm of girth to draw the graph illustrating this information.

a Find the gradient of the chord joining the points where $n = 3$ and $n = 7$ and interpret the result.

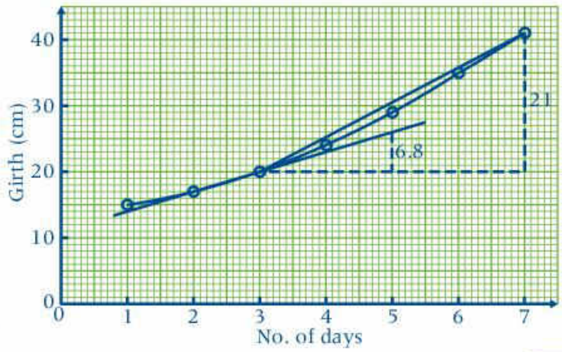
b Find the gradient of the tangent at the point where $n = 3$ and interpret the result.

a Gradient of chord $= \frac{21}{4} = 5.25$

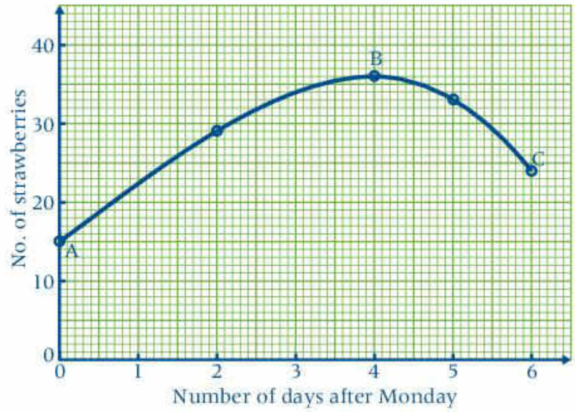
This shows that the girth of the pumpkin is increasing by an average of 5.25 cm per day over the four-day period.

b Gradient of tangent $\approx \frac{6.8}{2} = 3.4$

This shows that the girth of the pumpkin is increasing by 3.4 cm per day at the time of measurement on the third day.



5 The number of ripe strawberries on a particular strawberry plant were counted on Monday, Wednesday, Friday, Saturday and Sunday during one week. The results were recorded and plotted to give the following graph.



a How many ripe strawberries were there on
i Wednesday **ii** Saturday?

b How many strawberries were probably ripe on Thursday?

c Find the gradient of the chord joining the points A and B and interpret the result.

d Find the gradient of the chord joining B and C and interpret the result.

6 The table shows the population of an island at 10-year intervals from 1900 to 1980.

Date, D	1900	1910	1920	1930	1940	1950	1960	1970	1980
No. of people, N	500	375	280	210	160	120	90	65	50

Using a scale of 2 cm ≡ 10 years and 2 cm ≡ 50 people, draw the graph illustrating this information.

a Find the gradient of the chord joining the points on the curve where $D = 1910$ and $D = 1940$. Interpret your result.

b Find the gradient of the tangent to the curve where $D = 1910$ and interpret the result.

7 The table shows the sales of 'Jampot' jam for 5 months following an advertising campaign.

Month, M	1	2	3	4	5
Sales (number of jars)	2000	2500	3500	5000	7000

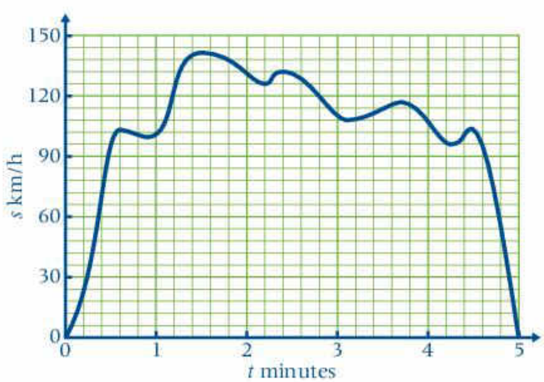
Using a scale of 2 cm ≡ 1 month and 2 cm ≡ 1000 jars draw the graph illustrating this information.

a Find the gradient of the tangent to the curve where $M = 2$ and interpret the result.

b Find the gradient of the tangent to the curve where $M = 4$ and interpret the result.

Travel graphs

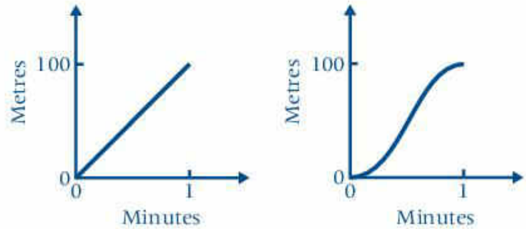
The graph shows the speed of a police car plotted against time during a chase.



- We can read quite a lot of information about the behaviour of the car from this graph; it started from rest, its speed increased most rapidly during the first 0.6 minutes, it reached a maximum speed of about 143 km/h, and so on.
- We cannot, however, read how far the car travelled during the chase. Nor can we tell whether the car travelled just one way down a road – it may have made a U-turn at some point. In this chapter we develop the work on travel graphs so that they can be used to find such quantities.

EXERCISE 15d

1 These two graphs each illustrate the journey of a car between two sets of traffic lights.

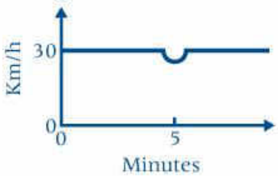


a Which of these graphs is a more realistic representation of the car's journey? Explain your answer.

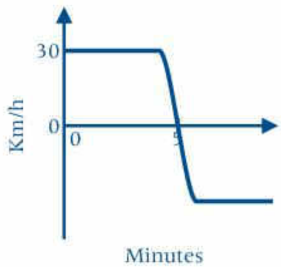
b Find the average speed, in m/s, of the car between the two sets of lights. Is it possible to use either graph to work this out? Explain your answer.

2 A car driver takes the wrong exit from a roundabout so he continues to the next roundabout where he turns through an angle of 180° and comes back along the same road. The car travels at a steady 30 km/h in both directions. These two graphs are attempts to illustrate the journey of the car along this road.

A



B



a Discuss whether, from graph **A**, you can tell in which direction the car is travelling along the road. What do you think the dip in the line in graph **A** attempts to show?

b Discuss what graph **B** attempts to show. Include in your discussion the speed shown after 5 minutes.

Curved distance–time graphs

When an object moves with constant speed, the distance–time graph representing its motion is a straight line. However, when an object moves so that its speed is constantly changing (for example, a car) then the distance–time graph representing its motion is a curved line. To draw such a graph, we need a relation between the time and the distance travelled.

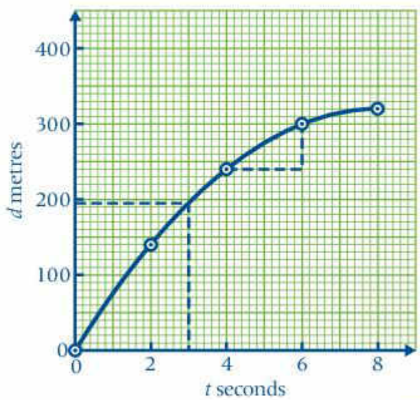
Consider, for example, a rocket fired from the ground so that its distance, d metres, from the launching pad after t seconds is given by

$$d = 80t - 5t^2$$

Taking values of t at 2-second intervals and calculating the corresponding values of d gives a set of points which we can plot.

Drawing a smooth curve through the points gives this distance–time graph.

From this graph we can see that 3 seconds after launching, the rocket is 195 m above the launch pad.



We can also find the average speed of the rocket over any interval of time. Consider, for example, the motion during the fifth and sixth seconds (from $t = 4$ to $t = 6$).

The rocket moves (300 – 240) m, i.e. 60 m, in these 2 seconds.

Therefore the average speed for the interval from $t = 4$ to $t = 6$ is $\frac{60}{2}$ m/s = 30 m/s.

Notice that the chord joining the points on the curve where $t = 4$ and $t = 6$, has a gradient of $\frac{60}{2} = 30$.

This confirms what we expect; we saw earlier that the gradient of a chord gives the average rate of increase of the quantity on the vertical axis with respect to the quantity on the horizontal axis. In this case, the gradient of a chord gives the rate of increase of distance with respect to time, that is, metres per second.

EXERCISE **15e**

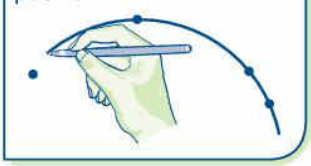
1 A car moves away from a set of traffic lights. The table shows the distance, d metres, of the car from the lights after t seconds.

t	0	1	2	3	4	5
d	0	2	8	18	32	50

Draw the distance–time graph using scales of 1 cm ≡ $\frac{1}{2}$ second and 1 cm ≡ 5 m.
From your graph find

a the distance of the car from the lights after $2\frac{1}{2}$ seconds
b the average speed of the car during the 2nd second
c the average speed of the car during the first 5 seconds.

Draw the curve in one continuous movement. Many people find it easier to draw a curve with their hand inside the curve. Turn the paper round to get a comfortable position.



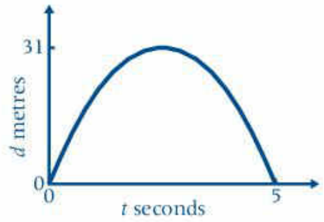
2 A rocket is launched and the table shows the distance travelled, d metres, after a time t seconds from lift-off.

t	0	1	2	3	4	5
d	0	5	40	135	320	625

Draw the distance–time graph using scales of 2 cm ≡ 1 second and 1 cm ≡ 100 m.
Use your graph to find

a the distance of the rocket from the launch pad $4\frac{1}{2}$ seconds after lift-off
b the average speed of the rocket for the first 4 seconds of its journey
c the average speed of the rocket during the fourth second.
(Keep this graph; it is needed for question **1**, **Exercise 15g**.)

3 This distance–time graph illustrates the motion of a ball thrown upwards from the ground.

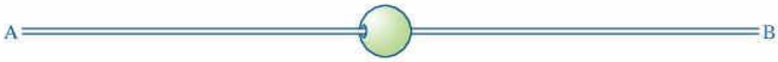


a How far does the ball go above the ground?
b What is the average speed of the ball on its way up?
c What is the average speed of the ball on its way down?
d Copy the diagram and draw the chord whose gradient represents
  i the answer to part **b**
  ii the answer to part **c**.
e **i** How far does the ball travel from when it leaves the ground until it returns?
  ii How long is it in the air?
f What is the average speed for the whole of the motion shown?
g What is the gradient of the chord joining the points representing the start of the motion and the end of the motion?
h Discuss the meaning of your answers to parts **f** and **g**.
i Explain why this situation did not arise in questions **1** and **2**.

Velocity

Discussion from **Exercise 15d** shows that when an object is moving along a straight line, the speed of the object gives no indication of the direction of its movement.

Consider a bead threaded on a straight wire AB.



If we are told that the bead is moving along the wire at 5 m/s, we know something about the motion of the bead but we do not know which way the bead is moving.

If we are told that the bead is moving from A to B at 5 m/s we know *both* the direction of motion *and* the speed of the bead.

> **Velocity** is the name given to the quantity that includes *both* the speed *and* the direction of motion.

When an object moves along a straight line, like the bead, there are only two possible directions of motion. In this case a positive sign is used to indicate motion in one direction and a negative sign is used to indicate motion in the opposite direction.

Taking the direction A to B as positive, we can illustrate velocities of +5 m/s and −5 m/s on a diagram.

Velocity = +5 m/s

Velocity = −5 m/s

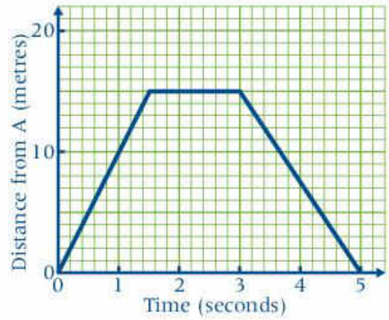
EXERCISE 15f

1 For each of the following statements state whether it is the velocity or the speed of the object that is given.
- a A boat travels between Roseau and Scotts Head in Dominica at 20 km/h.
- b A boat travels from Roseau to Scotts Head in Dominica at 20 km/h.
- c A ball rolls down a hill at 5 m/s.
- d A ball rolls along a horizontal groove at 3 m/s.
- e A lift moves between floors at 2 m/s.
- f A lift moves up from the ground floor at 2 m/s.

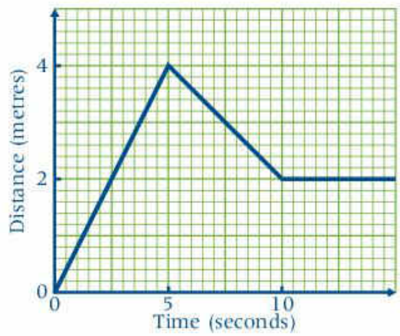
2 A bead moves along a horizontal wire AB. Taking the direction from A to B as positive, draw a diagram to illustrate the motion of the bead if its velocity is

a −2 m/s b 4 m/s c −10 m/s d 0

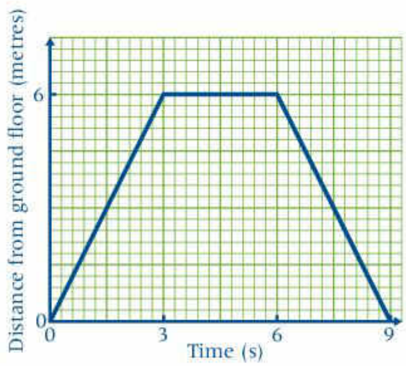
3 The graph illustrates the motion of a bead along a straight wire AB as the bead moves from A to B, stops at B, and moves back to A again.
Taking the direction from A to B as a positive, find
- a the speed of the bead as it moves from A to B
- b the velocity of the bead as it moves from A to B
- c the speed of the bead as it moves from B to A
- d the velocity of the bead as it moves from B to A
- e the average speed for the whole motion
- f the length of time for which the velocity is zero.



4 The graph illustrates the motion of a ball rolling in a straight line along horizontal ground. Taking the direction of the first part of the motion as positive, describe the motion of the ball, giving its velocity for each section of the motion.



5 The graph illustrates the motion of a lift which travels from the ground floor to the first floor and then returns to the ground floor.
Taking the upward direction as positive, state which of the following statements *must* be true.
- a The velocity of the lift is the same on both the upward and downward journeys.
- b On the downward journey the speed is 2 m/s.
- c The average speed of the lift between leaving the ground floor and returning to it, is zero.
- d On the upward journey the velocity of the lift is 2 m/s.
- e The velocity of the lift is zero for three seconds.



PUZZLE

Find three numbers x, y and z, such that

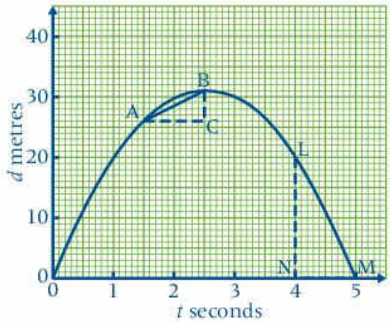
$$x^3 + y^4 = z^5$$

You will need your calculator

$2(2^{24}) = 2^{25} = (2^5)^5$.

Finding velocity from a distance–time graph

This distance–time graph illustrates the motion of a ball thrown upwards from the ground.



Taking the upward direction as positive we see that, up to the point where $t = 2.5$, the distance of the ball from the ground is increasing, i.e. the ball has a positive velocity (the ball is going up).

From $t = 2.5$ to $t = 5$, the distance of the ball from the ground is decreasing, i.e. the ball has a negative velocity (the ball is going down).

Average velocity is defined as $\dfrac{\text{increase in distance}}{\text{time taken}}$

Therefore the average velocity of the ball in the interval from $t = 1.5$ to $t = 2.5$ is

$$\frac{\text{increase in distance from } t = 1.5 \text{ to } t = 2.5}{2.5 - 1.5} \text{ m/s} = \frac{5}{1} \text{ m/s} = 5 \text{ m/s}$$

On the graph this is represented by the gradient of the chord AB.

Similarly the average velocity of the ball from $t = 4$ to $t = 5$ is

$$\frac{\text{increase in distance in this time}}{5 - 4} \text{ m/s} = -20 \text{ m/s}$$

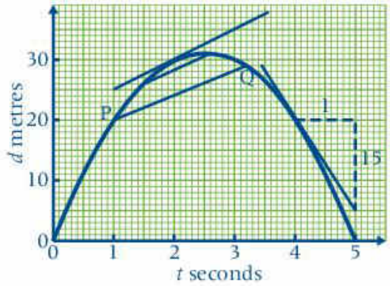
On the graph this is represented by the gradient of the chord LM.

In a distance–time graph, the gradient of a chord gives the average velocity over the time interval spanned by the chord.

The average velocity during any time interval can now be found using this fact, as the following example shows.

From $t = 1.5$ to $t = 4$ the average velocity is given by the gradient of the chord AL, i.e. $\frac{-6}{2.5}$ m/s $= -2.4$ m/s

Velocity at an instant



This is the distance–time graph first shown on the previous page.

Suppose that we want to find the velocity of the ball at the instant when $t = 2$.

We can get an approximate value for this velocity by finding the average velocity from, say, $t = 1$ to $t = 3.2$, i.e. by finding the gradient of the chord PQ.

A better approximation is obtained by taking a smaller interval of time, say from $t = 1.5$ to $t = 2.6$, that is by making the ends of the chord closer together.

The best answer is obtained when the interval of time is as small as possible, that is when the ends of the chord coincide. When this happens the chord becomes a tangent to the curve at the point where $t = 2$.

Therefore the velocity of the ball at the instant when $t = 2$ is given by the gradient of the tangent to the curve at the point where $t = 2$.

By drawing and measurement, the velocity is 5 m/s.

Similarly, at the point where $t = 4$, the gradient of the tangent is found to be -15. Therefore the velocity when $t = 4$ is -15 m/s. (Both results are estimates because they were obtained from a graph.)

This confirms what we expect; we saw earlier that the gradient of a tangent to a curve gives the rate of increase of the quantity on the vertical axis with respect to the quantity on the horizontal axis at that point on the curve. In this case, the gradient of a tangent gives the rate of increase of distance with respect to time, that is, the velocity.

In a distance–time graph, the gradient of a tangent to the curve gives the velocity at that instant.

EXERCISE 15g

1 Use the graph drawn for question **2** in **Exercise 15e** to estimate

- **a** the average velocity of the rocket during the time from $t = 2$ to $t = 4$.
- **b** the average velocity of the rocket over the interval $t = 2$ to $t = 3$
- **c** the velocity of the rocket when **i** $t = 2$ **ii** $t = 2.5$.

2 The table shows the distance, d metres, of a ball from its starting position, t seconds after being thrown into the air.

t	0	1	2	3	4	5	6
d	0	25	40	45	40	25	0

Using scales of 2 cm ≡ 1 second and 1 cm ≡ 5 m, draw the graph of d against t. From your graph estimate

- **a** when the ball returns to the starting point
- **b** the average velocity of the ball from $t = 1$ to $t = 2$
- **c** the average velocity of the ball from $t = 1$ to $t = 1.5$
- **d** the velocity of the ball when **i** $t = 1$ **ii** $t = 4$ **iii** $t = 5$.

3 Use the graph on the previous page to estimate
- a the velocity when $t = 1$
- b the average velocity during the first second
- c the average velocity during the first four seconds
- d the greatest height of the ball
- e the average velocity for the time between $t = 3$ and $t = 5$.

4 A particle moves in a straight line so that t seconds after leaving a fixed point O on the line, its distance, d metres, from O is given by

$$d = 8t - 2t^2$$

- a Copy and complete the following table.

t	0	1	2	3	4
d	0	6			

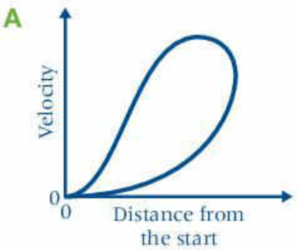
- b Use scales of 2 cm ≡ 1 second and 1 cm ≡ 2 m and draw the distance–time graph.
- c From your graph estimate the velocity of the particle when $t = 2$ and when $t = 3$.
- d What is the greatest distance of the particle from its starting point?

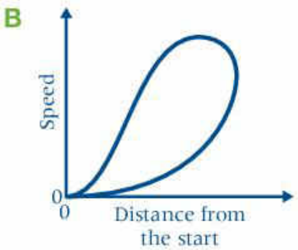
PUZZLE

A car was tested on a race track.

Graph **A** purports to show the velocity of the car plotted against the distance (as the crow flies) between the car and its starting point.

Graph **B** purports to show the speed of the car plotted against its distance (as the crow flies) from the starting point.





Both graphs look impossible, but one of them is possible.

Which one and why?

Acceleration

When the velocity of a moving object is changing we say that the object is accelerating.

Velocity includes speed and direction; we are concerned here with objects that move in just one direction so the changes in velocity involve only changes in speed.

If a train moves away from a station A and accelerates from rest so that its speed increases by 2 m/s each second then

1 second after leaving A the train has a speed of 2 m/s
2 seconds after leaving A the train has a speed of 4 m/s
3 seconds after leaving A the train has a speed of 6 m/s.

The train is said to have an **acceleration** of 2 m/s per second, i.e. 2 m/s/s and this is written as $2\,m/s^2$ or $m\,s^{-2}$.

If the speed of the train decreases it is said to be decelerating.

Suppose that the speed of the train decreases by 1 m/s each second, then we say that the deceleration is 1 m/s per second or $1\,m/s^2$.

We can also say that the train has an acceleration of $-1\,m/s^2$, i.e. a deceleration is a negative acceleration.

Consider a car that accelerates from rest at $5\,m/s^2$ for 10 seconds and then decelerates at $2\,m/s^2$ back to rest.

An acceleration of $5\,m/s^2$ means that the speed of the car increases by 5 m/s each second. Therefore after 10 seconds its speed is 50 m/s. A deceleration of $2\,m/s^2$ means that the speed of the car reduces by 2 m/s each second. Therefore, the speed of 50 m/s is reduced by 2 m/s each second, and this means that the car takes 25 seconds to stop.

EXERCISE 15h

1 A train accelerates from rest at $1\,m/s^2$ for 30 seconds. How fast is the train moving at the end of the 30 seconds?
If the train now decelerates back to rest at $0.5\,m/s^2$ how long does it take for the train to stop?

2 A train accelerates from rest at $0.2\,m/s^2$. How fast is the train moving after
a 2 seconds **b** 30 seconds **c** 1 minute?

3 A bus moves away from rest at a bus stop with an acceleration of $0.4\,m/s^2$ for 5 seconds; it then has to decelerate to rest at $0.2\,m/s^2$. How long after leaving the bus stop is the bus again stationary?

4 The speed of a lift increases from 6 m/s to 20 m/s in 7 seconds. Find the acceleration.

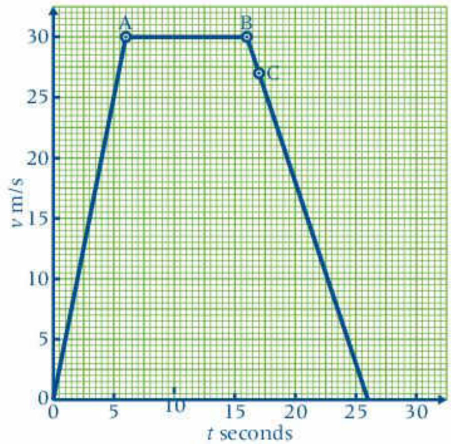
5 A train accelerates from rest at $0.5\,km/minute^2$.
How fast (in km/h) is the train moving after
a 3 minutes **b** 10 minutes **c** 45 seconds?

6 The speed of a car increases from 10 km/h to 80 km/h in 5 seconds. Find the acceleration.

7 Find, in m/s^2, the acceleration of a motorbike when its speed increases from 10 km/h to 50 km/h in 4 seconds.

Velocity–time graphs

A car accelerates from rest at $5\,m/s^2$ for 6 seconds and then travels at a constant speed for 10 seconds after which it decelerates to rest at $3\,m/s^2$. This information can be shown on a graph by plotting velocity against time. This is called a **velocity–time graph**.



After 6 seconds, the car is moving at 30 m/s, so we draw a straight line from the start to the point A, where $t = 6$ and $v = 30$.

Notice that the gradient of this line represents the acceleration (i.e. the rate of increase of velocity with respect to time).

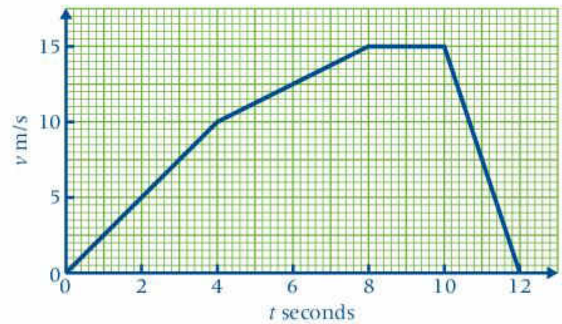
The line AB (zero gradient) represents the car moving at constant speed.

The last section of the journey is represented by the line drawn from B through C to the time axis, where C is 1 unit along the time axis and 3 units down the velocity axis from B. Notice that the gradient of BC is -3 and this represents the deceleration of $3\,m/s^2$.

> Acceleration is represented by the gradient of the velocity–time graph.

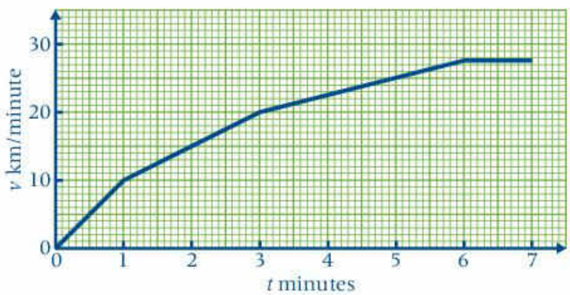
EXERCISE 15i

1 This velocity–time graph illustrates the journey of a car between a set of traffic lights and a zebra crossing.



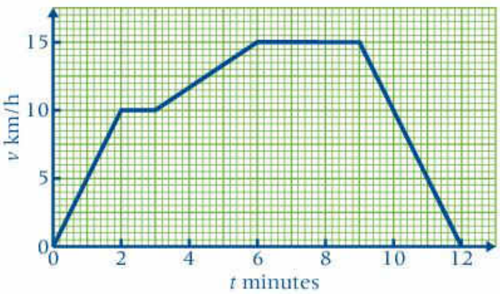
   a What is the car's acceleration for the first 4 seconds?
   b What happens when $t = 4$?
   c For how long is the car moving at a constant speed?
   d For how long is the car braking?
   e What is the deceleration of the car?
   f For how long is the car moving?

2 This velocity–time graph illustrates the first 7 minutes of the flight of a rocket.



   a What is the initial acceleration of the rocket?
   b What is the speed of the rocket 2 minutes after its launch?
   c What steady speed is attained by the rocket?
   d What is the acceleration of the rocket during the fourth minute of its flight?

3 This velocity–time graph illustrates the journey of a train between two stations.
   a What is the acceleration for the first 2 minutes?
   b What is the greatest speed of the train?
   c For how long is the train travelling at constant speed?
   d What is the acceleration during the third minute?
   e What is the deceleration of the train?



Draw a velocity–time graph to illustrate the following journeys. Use scales of 1 cm ≡ 2 seconds and 1 cm ≡ 5 m/s.

4 A car accelerates steadily from rest reaching a speed of 12 m/s in 10 seconds.

5 A train accelerates from rest reaching a speed of 8 m/s in 5 seconds and then immediately decelerates to rest in 4 seconds.

6 A motorbike accelerates from rest to a speed of 20 m/s in 4 seconds, maintains this steady speed for 8 seconds and then decelerates to rest in 5 seconds.

Sketch a velocity–time graph for each of the following journeys.

7 A bullet is fired into a block of wood at 100 m/s and comes to rest 3 seconds later.

8 A car accelerates from rest at 2 m/s^2 for 5 seconds, then moves with constant speed for 15 seconds before decelerating back to rest at 4 m/s^2.

9 A train accelerates from rest at 1 m/s^2 for 3 seconds, 2 m/s^2 for 3 seconds and then 5 m/s^2 for 5 seconds.

10 A car accelerates from rest at 10 m/s^2 for 2 seconds, 5 m/s^2 for 5 seconds and then 2 m/s^2 for 3 seconds. The car then travels at constant speed for 10 seconds before decelerating at 8 m/s^2 back to rest.

11 A bullet is fired at 50 m/s into sand which retards the bullet at 30 m/s^2.

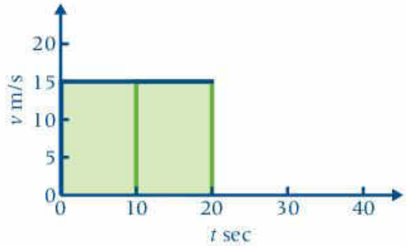
12 A car travels at 30 m/s for 5 seconds, then decelerates at 4 m/s^2 for 3 seconds and travels at constant speed for another 10 seconds.

13 A block of wood is thrown straight down into the sea. The wood enters the water at 50 m/s and sinks for 6 seconds.

14 A train accelerates from rest at 5 km/h per minute for 5 minutes, then at 15 km/h per minute for 5 minutes and then maintains its speed for 10 minutes.

Finding the distance from a velocity–time graph

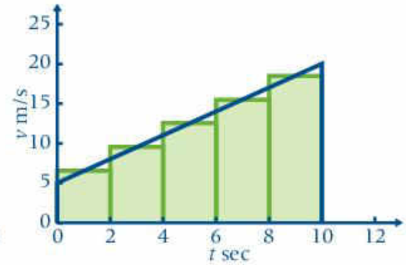
This graph shows a car travelling at a steady 15 m/s for 20 seconds.



The distance travelled by the car in this time is given by speed × time, i.e. distance travelled = 15 × 20 metres.

Now the area of the shaded rectangle is also 15 × 20 square units, so the distance travelled is represented by the area under the velocity–time graph.

Now consider this velocity–time graph which represents a car accelerating at a constant rate from a speed of 5 m/s to a speed of 20 m/s in 10 seconds.



The changing speed can be approximated by considering a car moving at a constant speed of 6.5 m/s for 2 seconds, 9.5 m/s for 2 seconds, and so on, as shown by the green lines on the graph.

The distance travelled by the car is then represented by the sum of the areas of the rectangles shown on the graph.

Now, because the parts of the rectangles above the line are equal in area to the white parts under the blue line, the sum of the areas of these rectangles is equal to the area under the blue line.

Hence

> the distance travelled is equal to the area under the velocity–time graph.

EXERCISE **15j**

Example:

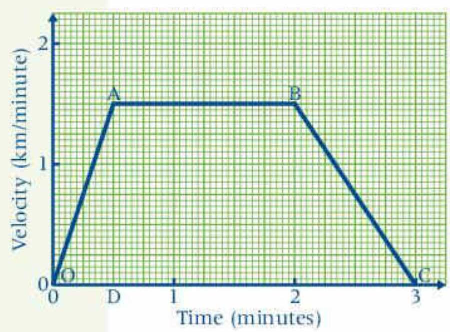
The velocity–time graph illustrates a train journey between two stations.

a What is the maximum speed of the train in km/h?

b What is the train's acceleration in the first half-minute?

c How far does the train travel in the first 30 seconds?

d What is the distance between the stations?



a From the graph, the maximum speed is 1.5 km/min

$= 1.5 \times 60$ km/h

$= 90$ km/h

b The acceleration is given by the gradient of OA, which is $\frac{1.5}{0.5} = 3$

Therefore the acceleration is 3 km/minute^2.

c Area $\triangle OAD = \frac{1}{2}(OD) \times (AD)$

$= 0.5 \times (0.5) \times (1.5) = 0.375$

Therefore the train travels 0.375 km in the first 30 seconds.

d Area $OABC = \frac{1}{2}(OC + AB) \times AD$

$= 0.5\,(3 + 1.5) \times 1.5$

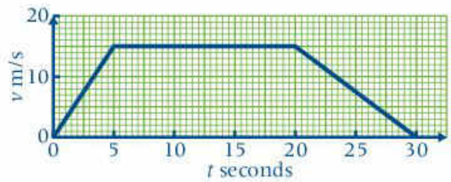
$= 3.375$

Therefore the distance between the stations is 3.375 km.

The distance travelled in the first 30 seconds is given by the area of $\triangle OAD$.

The distance between the stations is represented by the area of trapezium OABC.

1 The velocity–time graph represents a car journey between two sets of traffic lights.



a What is

i the acceleration **ii** the deceleration of the car?

b For how long does the car accelerate?

c How many metres does the car travel while braking?

d Find the distance between the two sets of lights.

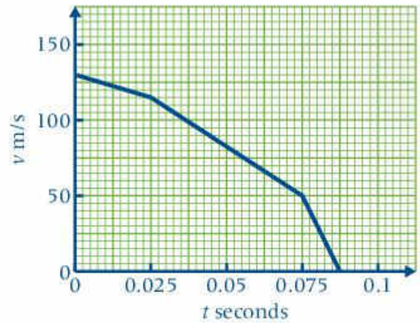
2 Use the graphs for questions **1**, **2** and **3** of **Exercise 15i** to find

a the distance covered by the car in question **1**

b the distance travelled by the rocket in the first 3 minutes in question **2**

c the distance travelled by the train in the first 2 minutes in question **3**. (Be careful with the units.)

3 The velocity–time graph represents a missile fired into a 'wall' made up of a layer of sand followed by a layer of wood and then a layer of brick.



a Find the deceleration of the missile as it passes through the layer of sand.

b Find the depth of the sand.

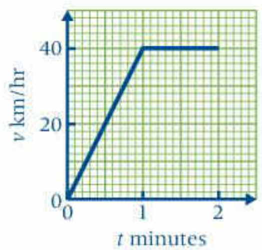
c Find the deceleration of the missile as it passes through the layer of wood.

d Find the depth of the layer of wood.

e What retardation of the missile does the layer of brick cause?

f Find the depth to which the missile penetrates the brick.

4 The graph represents a two-minute section of a car journey. Find



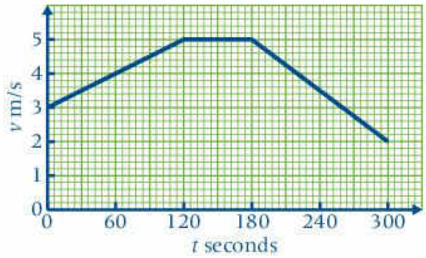
a the acceleration, in m/s^2, of the car during the first minute

b the distance, in metres, travelled by the car during the two minutes.

5 A cross-country runner covers three sections of the course in succession. The first is a downhill sweep, then there is a level section followed by a hill climb. The graph shows the speed, v m/s, and the time, t s, of the runner over the three sections.

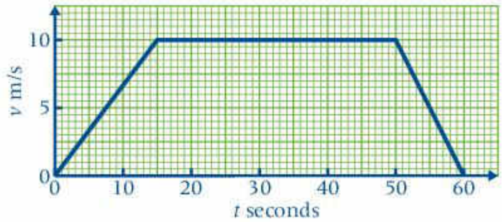
Find

a the acceleration of the runner on the downhill section
b the constant speed over the level section
c the deceleration of the runner during the hill climb
d the distance covered on the level section
e the distance covered on the hill climb.



6 The graph represents the journey made by a bus between two bus stops.

a What is the acceleration of the bus?
b What is the deceleration of the bus?
c What distance does the bus cover while accelerating?
d What distance does the bus cover while decelerating?
e What is the distance between the two bus stops?



7 A motorist starts from rest and accelerates uniformly to 30 m/s in 5 seconds. This speed is maintained for 15 seconds until the brakes are applied, decelerating the car to rest in 4 seconds.

a Draw a velocity–time graph which shows this journey.
b Use your graph to determine
   i the acceleration, in m/s^2, during the first 5 seconds
   ii the deceleration, in m/s^2, during the last 4 seconds
   iii the total distance travelled, in metres
   iv the average speed of the car for the whole journey.

8 A car, starting from rest, attains a speed of 60 km/h in 30 seconds. It continues at this speed for 4 minutes before decelerating to rest at a constant rate. The total time for the journey is 5 minutes.
Illustrate this journey on a velocity–time graph and use your graph to find

a the acceleration of the car in m/s^2 during the first part of the journey
b the deceleration of the car in m/s^2
c the total distance covered.

> Make sure that you are using consistent units.

9 A motorist starts from rest and accelerates uniformly to 20 m/s in 4 seconds. This speed is maintained for 20 seconds until the brakes are applied, decelerating the car to rest in 3 seconds.

a Draw a velocity–time graph which shows this journey.
b Use your graph to determine
   i the acceleration, in m/s^2, during the first 4 seconds
   ii the deceleration, in m/s^2, during the last 3 seconds
   iii the total distance travelled, in metres
   iv the average speed of the car for the whole journey.

10 A motorcyclist, starting from rest, accelerates uniformly to 90 m/s in 6 seconds. This speed is maintained for 16 seconds until the brakes are applied, decelerating the motorcycle to rest in 3 seconds.

a Draw a velocity–time graph which shows this journey.

b Use your graph to determine

i the acceleration, in m/s^2, during the first 6 seconds

ii the deceleration, in m/s^2, during the last 3 seconds

iii the total distance travelled, in metres

iv the average speed of the motorcyclist for the whole journey.

11 A high performance car, starting from rest, travels with a uniform acceleration of $12 m/s^2$ for 8 seconds. The speed it attains is then maintained for 16 seconds before the brakes are applied, decelerating the car to rest in 3 seconds.

a Draw a velocity–time graph which shows this journey.

b Use your graph to determine

i the distance travelled in the first 8 seconds

ii the average speed for the whole journey.

12 A heavy lorry starts from rest and accelerates uniformly to a speed of 60 km/h in 4 minutes. This speed is maintained for another 8 minutes until the brakes are applied, decelerating the lorry to rest uniformly in 10 seconds.

a Draw a velocity–time graph which shows this journey.

b Use your graph to determine

i the acceleration, in km/h^2, during the first 4 minutes

ii the deceleration, in km/h^2, during the last 10 seconds

iii the total distance travelled, in metres

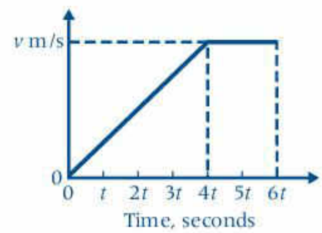
iv the average speed of the lorry for the whole journey.

Example:

The velocity–time graph shows the movement of a marble down a slope. The marble starts from rest, then accelerates for $4t$ seconds to a speed of v m/s. It then maintains this speed for $2t$ seconds before hitting a wall. The average speed for the whole journey is 10 m/s.

a Find the value of v.

b The marble travels a distance of 12 m before it is stopped by the wall. Find the value of t.



a Total distance $= \frac{1}{2}(6t + 2t) \times v = 4vt$

Total time $= 6t$

Average speed $= \frac{4vt}{6t} = \frac{2v}{3}$

$\therefore \quad \frac{2v}{3} = 10$ so $v = 15$

b Distance $= 4vt = 10 \times 6t = 60t$

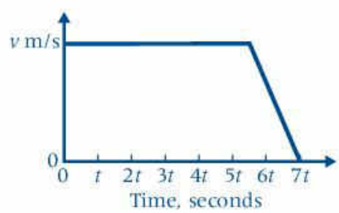
$\therefore \quad 60t = 12$ so $t = \frac{1}{4}$

Use the graph to find an expression for the average speed. The total distance is the area under the graph which is a trapezium. Then form an equation.

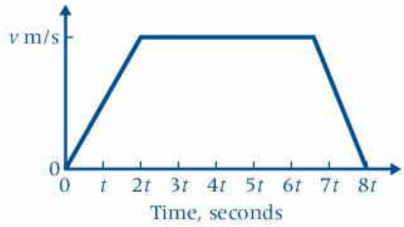
13 The velocity–time graph shows an object moving for $5.5t$ seconds at a constant speed of v m/s and then decelerating to 0 after a further $1.5t$ seconds. The average speed for the whole movement is 25 m/s.

a Find the value of v.

b If the total distance travelled is 35 metres, find the value of t.

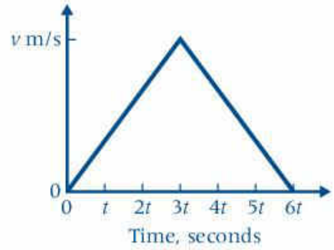


14 This velocity–time graph shows an object moving for $8t$ seconds. For the first $2t$ seconds it moves with uniform acceleration until it acquires a speed of v m/s. Then it travels at a constant speed of v m/s for $4.5t$ seconds before a uniform retardation brings it to rest in a further $1.5t$ seconds. The average speed for the whole journey is 25 m/s.



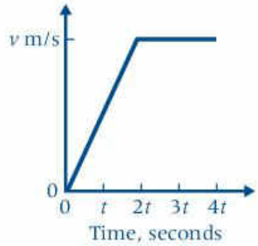
a Find the value of v.
b If the total distance travelled is 50 metres, find the value of t.

15 The velocity–time graph shows the journey of a car. From a standing start it accelerates uniformly until it reaches a speed of v m/s. Then it immediately decelerates uniformly to rest after a total journey time of $6t$ seconds. The average speed for the whole movement is 16 m/s.



a Find the value of v.
b If the total distance travelled is 32 metres find the value of t.

16 The velocity–time graph shows the movement of an object for the first $4t$ seconds of a journey. Starting from rest it reaches a speed of v m/s by accelerating uniformly for $\frac{5t}{3}$ seconds. It continues to move at v m/s. The average speed of the object for the first $4t$ seconds is 19 m/s.



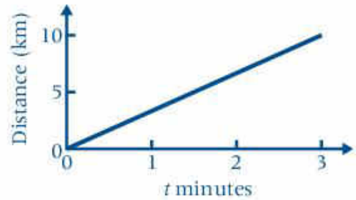
a Find the value of v.
b If the object travels 8 metres in $4t$ seconds, find the value of t.

ABCD MIXED EXERCISE 15

Each question is followed by several alternative answers.
Write down the letter that corresponds to the correct answer.

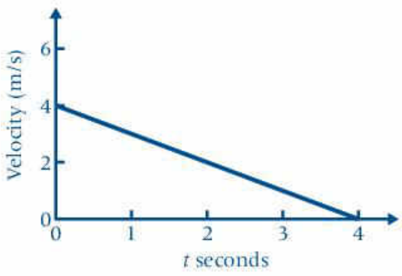
1 The graph shows an object moving at

A $\frac{3}{10}$ km/min **B** $3\frac{1}{3}$ km/min **C** 10 km/min **D** 30 km/min



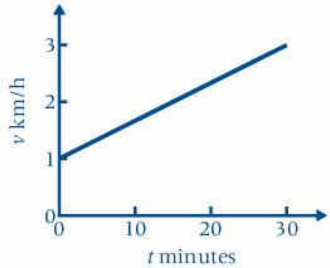
2 The graph shows an object which in 4 seconds covers a distance of

A −8 m **B** 1 m **C** 8 m **D** 16 m

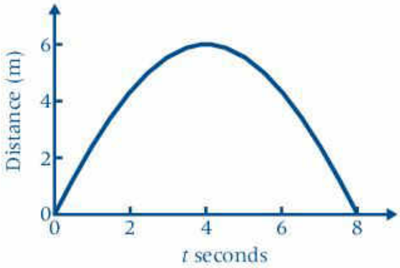


3 The acceleration of the object whose motion is given in question **2** is
A $-1\,\text{m/s}^2$ B $1\,\text{m/s}^2$ C $4\,\text{m/s}^2$ D $16\,\text{m/s}^2$

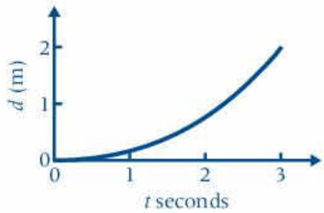
4 From the graph the distance covered in the 30 minutes is
A 1 km B 2 km C 4 km D 60 km



5 The graph shows an object
A whose velocity is constant
B whose speed when $t = 0$ is zero
C which continues to move away from its initial position
D which returns to its initial position after 8 seconds.

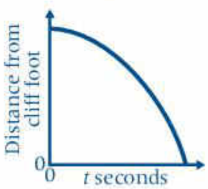


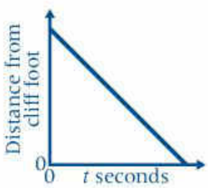
6 The average velocity from $t = 0$ to $t = 3$ is
A $\frac{2}{3}$ m/s B 1 m/s C $1\frac{1}{2}$ m/s D 2 m/s



7 Deducing from the shape of the graph in question **6**, the acceleration when $t = 2$ could be
A $-1\,\text{m/s}^2$ B zero C $1\,\text{m/s}^2$ D $20\,\text{m/s}^2$

8 The distance–time graph representing the motion of a stone dropped from a cliff top could be

A


B


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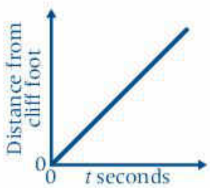
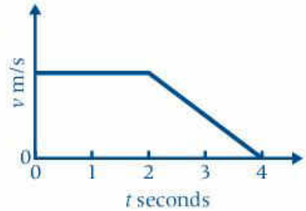


9 A bead moves along a straight wire with a constant speed for 2 seconds and then its speed decreases at a constant rate to zero. The velocity–time graph illustrating this could be

A



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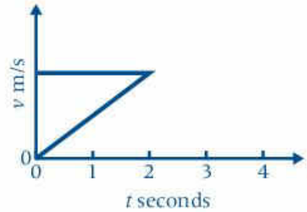

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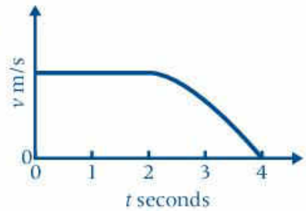

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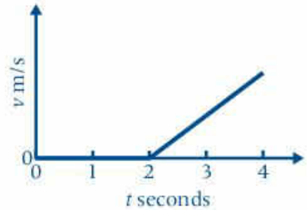
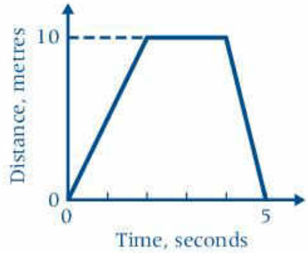


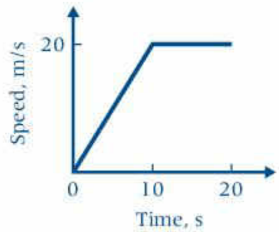
Questions **10** and **11** refer to this diagram which shows a journey lasting 5 seconds.



10 The distance travelled is
A 5 m **B** 10 m **C** 15 m **D** 20 m

11 The average speed is
A 4 m/s **B** 5 m/s **C** 10 m/s **D** 20 m/s

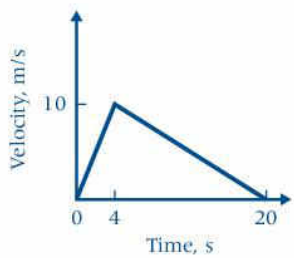
Questions **12** and **13** refer to this diagram which shows a speed–time graph for a car, which accelerates uniformly from rest for 10 s reaching a speed of 20 m/s and then travels at this speed for 10 s.



12 The distance covered in the first 20 seconds is
A 200 m **B** 300 m **C** 400 m **D** 600 m

13 The average speed of the car in the first 20 seconds is
A 10 m/s **B** 15 m/s **C** 20 m/s **D** 30 m/s

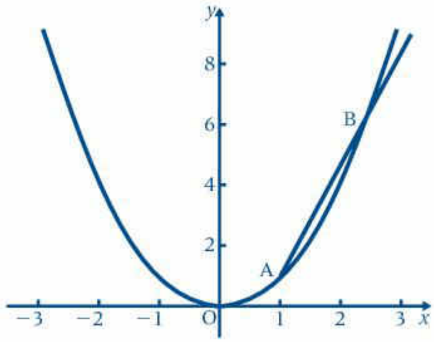
14 This velocity–time graph, illustrates the motion of a car that accelerates uniformly from rest to 10 m/s in 4 s, and then decelerates uniformly to rest in 16 s. The distance covered in this time, is
A 200 m **B** 160 m **C** 100 m **D** 40 m



15 In the velocity–time graph shown in question **14**, the acceleration in the first 4 seconds is
A $0.2\,m/s^2$ **B** $0.4\,m/s$ **C** $2\,m/s^2$ **D** $2.5\,m/s^2$

INVESTIGATION

The graph shows the curve whose equation is $y = x^2$.



a A is the point on the curve where $x = 1$. B is the point on the curve where $x = 1 + h$. Find the gradient of AB in terms of h.

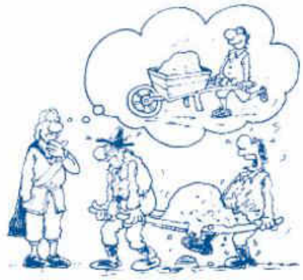
b Find the gradient of AB when $h = 2$, $h = 1$, $h = 0.5$, $h = 0.25$, $h = 0.1$. Investigate the sequence of values obtained for the gradient of AB as h gets smaller. What value do these appear to approach? Justify your conclusion. What does this value represent in relation to the curve?

c Repeat parts **a** and **b** when A is the point on the curve where $x = 2$. Try finding the gradient of the curve at the point where $x = -3$. Find a relationship between the gradient of the curve at a point and the x-coordinate of the point.

d Try a similar method for finding the relationship between the gradient at a point on the curve $y = x^3$ and its x-coordinate.

MATHS IS OUT THERE

Did you know that Pascal has been credited (inaccurately) with the invention of the present day's one-wheeled wheelbarrow? Medieval illustrations show this not to have been the case.



IN THIS CHAPTER YOU HAVE SEEN THAT...

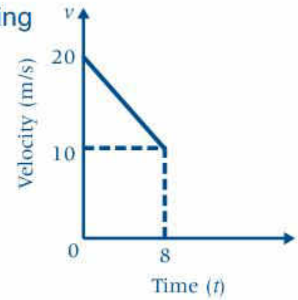
- the gradient of a curve at a point tells you how the quantity on the vertical axis is changing in relation to the corresponding value on the horizontal axis
- the gradient of a curve at a point is the gradient of the tangent to the curve at that point
- velocity gives speed and the direction of motion
- the gradient of a chord on a distance–time graph gives the average velocity over that time interval, whereas the gradient of a tangent to the curve gives the velocity at that instant
- when velocity is plotted against time, the gradient gives the acceleration and the area under the graph gives the distance travelled.

REVIEW TEST 3 Chapters 11–15

Multiple choice questions

Several possible answers are given.
Write down the letter that corresponds to the correct answer.

1 The diagram shows the velocity–time graph of a particle decelerating uniformly over a period of 8 seconds.
The distance covered in this time is
A 160 m **B** 120 m **C** 80 m **D** 40 m



2 $(2 \quad -1)\begin{pmatrix} 2 \\ -1 \end{pmatrix} =$

A $(4 \quad 1)$ **B** 5 **C** 3 **D** $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$

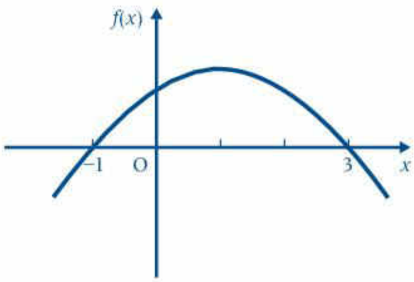
3 A car travelling on a straight road passes a point P at a steady speed of 10 m/s. Two seconds later a second car, travelling on the same road, passes P at a steady speed of 15 m/s. The second car overtakes the first car after
A 2 s **B** 4 s **C** 5 s **D** 10 s

4 Given that $f(x) = \frac{3}{x}$, then $f^{-1}(x) =$

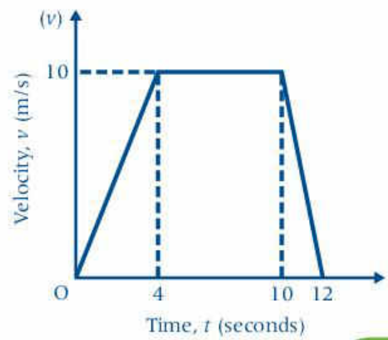
A $\frac{x}{3}$ **B** $\frac{-3}{x}$ **C** $3x$ **D** $\frac{3}{x}$

5 A car travels at 72 km/h for $1\frac{1}{2}$ h and then at 28 km/h for a further $1\frac{1}{2}$ h. The average speed for the entire journey is
A 100 km/h **B** 50 km/h **C** 48 km/h **D** $33\frac{1}{3}$ km/h

6 Which of the following equations is best illustrated by the graph given in the diagram?
A $f(x) = 3 + 2x - x^2$
B $f(x) = x^2 + 2x - 3$
C $f(x) = 3 - 2x - x^2$
D $f(x) = x^2 - 2x - 3$



7 This (t, v) graph illustrates the speed of a cyclist over 12 s.
The average speed during this interval is
A 5 m/s **B** $7\frac{1}{2}$ m/s
C 10 m/s **D** 11 m/s



8 The next term in the sequence 2, 5, 10, 17, is
A 6 **B** 18 **C** 24 **D** 26

9 The value of x for which the matrix $\begin{pmatrix} 3 & x \\ -6 & 12 \end{pmatrix}$ is singular is
A −6 **B** 6 **C** 12 **D** 36

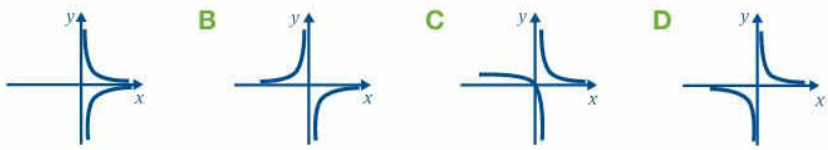
10 The equation of the axis of symmetry of the graph of $y = (x - 2)(x + 4)$ is
A $x = -1$ **B** $x = 2$ **C** $x = 3$ **D** $x = 4$

11 $f(x) = 5x$ and $g(x) = 1 - x$, $fg(x) =$
A $1 - 5x$ **B** $1 + 4x$ **C** $5 - 5x$ **D** $1 - 6x$

12 To solve the equation $x^2 - 5x + 2 = 0$ using the graph of $y = x^2 + 1$, the line that needs to be added is
A $y = -5x + 1$ **B** $y = 5x - 2$ **C** $y = 5x - 1$ **D** $y = 5x + 1$

13 $\mathbf{M} = \begin{pmatrix} 3 & 1 \\ 2 & 1 \end{pmatrix}$, $\mathbf{M}^{-1} =$
A $\begin{pmatrix} 1 & -1 \\ -2 & 3 \end{pmatrix}$ **B** $\begin{pmatrix} -3 & 2 \\ 1 & -1 \end{pmatrix}$ **C** $\begin{pmatrix} 1 & 2 \\ 1 & 3 \end{pmatrix}$ **D** $\begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}$

14 The graph of $y = \frac{1}{x}$ could be



15 The nth term of the sequence 2, 4, 6, 8, 10, ... is
A n **B** $n + 1$ **C** $2n$ **D** $n + 2$

16 If $f(x) = 3x + 4$ and $g(x) = 2x + 3$, then fg (−2) is
A 25 **B** 10 **C** 1 **D** −1

17 If $f(x) = 2x - 5$ and $g(x) = 3 - 2x$, then fg(−3) is
A −23 **B** −11 **C** 7 **D** 13

18 If $g(x) = 3x + 4$ and $h(x) = 2x - 3$, then gh(1) is
A 25 **B** 10 **C** 1 **D** −1

19 If $f(x) = \frac{2}{x - 3}$, $x \neq -3$, then $f^{-1}(x) =$
A $\frac{2 - 3x}{x}$ **B** $\frac{x}{2 + 3x}$
C $\frac{2 + 3x}{x}$ **D** f^{-1} does not exist.

20 The nth term, u_n, of a sequence is given by $u_n = 3n - n^2$. The fourth term of the sequence is
A 2 **B** 0 **C** −1 **D** −4

21 When y is eliminated from the equations $\begin{cases} x^2 + y = 0 \\ x + y = 4 \end{cases}$ the resulting equation could be

A $x^2 = 4$ **B** $x^2 - x + 4 = 0$ **C** $x^2 + x + 4 = 0$ **D** $x^2 + x = 4$

22 If $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$, then $\mathbf{AB} =$

A $\begin{pmatrix} 1 & 2 \\ 0 & 4 \end{pmatrix}$ **B** $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ **C** $\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$ **D** $\begin{pmatrix} 2 & 1 \\ 0 & 4 \end{pmatrix}$

23 A car travels at a uniform speed of 13 m/s for 5 seconds and then at a uniform speed of 10 m/s for another 10 seconds. The average speed of the car in this interval of time is

A 16.5 m/s **B** 11.5 m/s **C** 11 m/s **D** 2.2 m/s

24 The matrix equation $\begin{pmatrix} 1 & 4 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$ represents the equations

A $4y = x + 2$, $3y = 2x + 1$

B $4y + x - 2 = 0$, $3y - 2x + 1 = 0$

C $y + 4x = 2$, $-2y + 3x = -1$

D $x - 2y = 2$, $4x + 3y = -1$

25 The inverse of the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is

A $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ **B** $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ **C** $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ **D** $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

26 Given that $f(x) = \sin x$ for $0° \leq x \leq 360°$, which of the following statements are true?

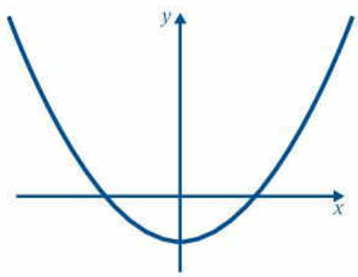
A $f(x) = 0$ when $x = 90°$ **B** $f(x) < 0$ when $x > 90°$

C $f(180°) = 1$ **D** $f(90°) = 1$

27 The diagram alongside could illustrate the curve

A $y = x^2 - 1$ **B** $y = x^2$

C $y = x^2 + 1$ **D** $y = 1 - x^2$



28 Here is a sequence of expressions: $4x + 2$, $5x$, $6x - 2$. The next term is

A $7x + 4$ **B** $7x - 3$

C $7x - 4$ **D** $4x - 4$

29 $(1\ 0\ 2)\begin{pmatrix} 3 & 0 & 1 \\ 2 & 1 & -1 \\ 1 & 0 & 0 \end{pmatrix} =$

A 7 **B** $\begin{pmatrix} 6 \\ 0 \\ 1 \end{pmatrix}$

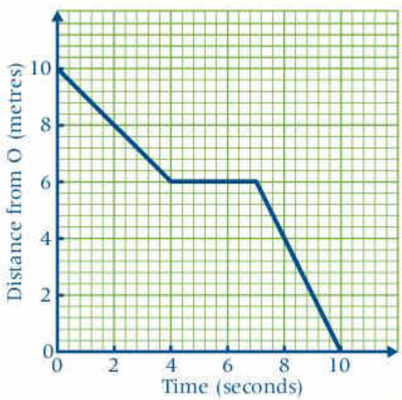
C $\begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix}$ **D** $(5\ 0\ 1)$

30 The distance–time graph shows a journey.

The average speed for this journey is

A 10 m/s **B** 5 m/s

C 1.75 m/s **D** 1 m/s



General proficiency questions

1 Solve for x and y in the following equations.

$$x^2 + y^2 = 2$$
$$x - 2y = -1$$

2 a A car accelerates uniformly from rest attaining a speed of 10 m/s in 4 s. It travels at this speed for 8 s and then decelerates uniformly for 3 s before coming to rest.
   i Draw a (t, v) graph to illustrate the above data.
   ii Find the acceleration of the car at 2 s.
   iii Obtain the total distance covered by the car in the 15 s.
   iv Calculate the deceleration of the car in the last 3 s.
   v Determine the average speed of the car over the 15 s interval.
b A second car travels at a constant speed of 5 m/s and covers the same distance as the first car. Find the time taken by the second car.
c A third car accelerates uniformly from rest for 23 s and travels the same distance as the first and second cars. Find the acceleration of the third car.

3 a Express $2x^2 - 3x + 1$ in the form $a(x + b)^2 + c$, where a, b and c are constants. Hence, solve exactly the equation $2x^2 - 3x + 1 = 0$.
b If $f(x) = a(x + b)^2 + c$, find
   i the minimum value of $f(x)$
   ii the value of x for which $f(x)$ is minimum.
c Sketch the graph of $f(x) = 2x^2 - 3x + 1$ and find x for which $f(x) > 0$. Hence, or otherwise, find the values of x for which $f(x) < 0$.

Each matrix in questions **4** and **5** is a singular matrix. Find the value of x.

4 $\begin{pmatrix} x & 6x + 1 \\ 2x + 1 & 3(3x + 4) \end{pmatrix}$

5 $\begin{pmatrix} 3x + 4 & 2(3x + 1) \\ 9x - 1 & 6x + 1 \end{pmatrix}$

6 a Draw the graph of $y = 2x^2 - 12x + 9$ for $-1 \leqslant x \leqslant 7$.
b Use your graph to solve the equation $2x^2 - 12x + 9 = 0$.
c Draw, on the same axes, the graph of $y = 1 - 3x$ and write down the coordinates of the points where the graphs cut. Find the equation for which these x-values are the roots.

7 Solve for x and y the equations $12x^2 - 3y^2 + 8 = 0$
$y = 2(x + 1)$

8 Given that $f(x) = 10^x$, $g(x) = x^2$ and $h(x) = \frac{1}{x}$
a find fg(2), hg(3) and gf(−1)
b find, in terms of x, hfg(x) and gfh(x)
c if they exist, find $g^{-1}(x)$ and $h^{-1}(x)$
d find the value(s) of x for which gh(x) = 9
e does the function $(gh)^{-1}$ exist?

9 Draw sketches of the following curves
a $y = (x - 2)(x - 3)(x - 4)$ b $y = \frac{1}{x}$ c $y = 2^{4-x}$

10 A function f is defined by $f(x) = \frac{1}{(1 - x)}, x \neq 1$
a Why is 1 excluded from the domain of f?
b Find the value of f(−3).
c Sketch the curve $y = f(x)$.
d Find $f^{-1}(x)$ in terms of x and give the domain of f^{-1}.

11 Find the greatest or least value of each of the following functions, stating the value of x at which they occur.

a $f(x) = x^2 - 3x + 5$

b $f(x) = 2x^2 - 7x + 1$

c $f(x) = (1 - x)(x + 5)$

12 The function f is given by $f(x) = 2^{(3x - 2)}$

a Find $g(x)$ and $h(x)$ such that $f = gh$.

b Evaluate $ff(2)$ and $f^{-1}(2)$.

13 Solve for x and y the equations $xy + x + 21 = 0$

$x + y = 3$

14 **a** Express $2x^2 - 5x + 6$ in the form $2(x + p)^2 + q$

b Hence explain why the equation $2x^2 - 5x + 6 = 0$ has no real solutions.

15 The functions f and g are defined by $f(x) = 2x$ and $g(x) = 3x^2$.

a Sketch the curves $y = g(x - 3)$ and $y = gf^{-1}(x)$.

b Find the value(s) of x for which $f^{-1}(x) = g(x)$.

16 If $f(x) = 3x$, $g(x) = \frac{1}{x}$ and $h(x) = x^2 - 1$, find

a $fg(x)$ **b** $gfh(x)$ **c** $g^{-1}f^{-1}(x)$ **d** $(gf)^{-1}(x)$

17 The function f is given by $f(x) = x^2 - 4$, find

a $f(0)$ **b** $f(-2)$ **c** $f(5)$

18 Sketch the graph of $y = \sin x°$ and use it to find the values of x between 0 and 360 for which $\sin x° = \frac{1}{2}$

19 Sketch the graph of $y = \cos x°$ and use it to find the values of x between 0 and 360 for which $\cos x° = -\frac{1}{2}$

20 The function $k(x) = x^2 - 1$ is defined for all real values of x. Given that $k(x) = fg(x)$ where f and g are functions, find f and g given that f only is a linear function.
Determine f^{-1} and sketch the graphs of f and f^{-1} on the same axes.
Show that $(fg)^{-1} = g^{-1}f^{-1}$

21 Given that $f: x \rightarrow x^2 - 4x + 1$, find $f^{-1}(x)$ for $x \geqslant 2$.
Find the range of $f(x)$ for the given domain.

22 $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 1 & 0 \\ 2 & k \end{pmatrix}$

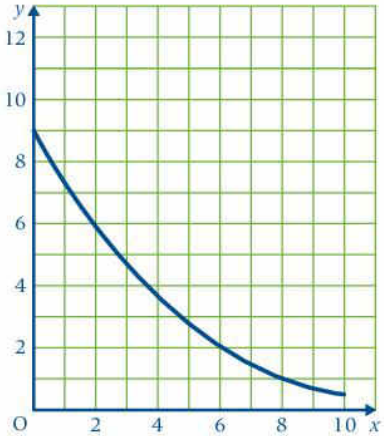
a Find **AB** and **BA**.

b If **AB** = **BA** find k.

c If $k = 0$ why does **B** have no inverse?

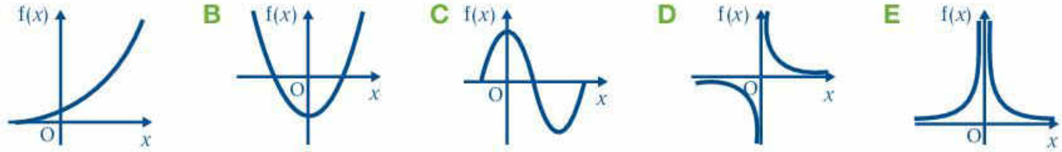
23 If $\begin{pmatrix} 1 & 2 \\ 3 & 8 \end{pmatrix} \mathbf{P} = \begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$ find **P**.

24 This is the graph of $y = ka^{-x}$.
Use the graph to estimate the values of k and a.



25 Which of the following curves could represent the function f where

a $f(x) = \cos x°$ **b** $f(x) = x^2 - 1$ **c** $f(x) = \frac{1}{x^2}$ **d** $f(x) = 3^x$
e $f(x) = \frac{1}{x}$



26 $\mathbf{A} = \begin{pmatrix} 6 & 2 \\ 5 & 2 \end{pmatrix}$ and $\mathbf{B} = \begin{pmatrix} 2 & -3 \\ -1 & 2 \end{pmatrix}$

Give as a single matrix

a $\mathbf{AB}$ **b** $\mathbf{BA}$ **c** $\mathbf{A}^{-1}$ **d** $\mathbf{B}^{-1}$
e $\mathbf{A}^2$ **f** $\mathbf{B}^3$ **g** $2\mathbf{A}$ **h** $3\mathbf{B}$

27 $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Find

a $2\mathbf{A}$ **b** $\mathbf{A}^{-1}$ **c** $\mathbf{A}^2$ **d** $\mathbf{A}^3$
e $\mathbf{A}^4$ **f** $\mathbf{A}^5$ **g** $|\mathbf{A}|$ **h** $|\mathbf{A}^2|$
i $\mathbf{A}^n$ if n is even **j** $\mathbf{A}^n$ if n is odd

28 If $4\begin{pmatrix} 6 & -3 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 2 & -1 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 1 & -2 \\ 3 & 0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ find a, b, c and d.

29 Consider this sequence: 2, 5, 8, 11, 14, ...

a Write down the next four terms.
b Find an expression for the nth term, u_n, in terms of n.
c Hence show that $u_n - u_{n-1} = 3$.

30 Square tiles measuring 50 cm by 50 cm are used to surround a square pool.

a How many tiles are needed to surround a pool of side 2 metres.
b Tiles are supplied in packs of 50.
  i Find the largest pool that can be surrounded by one pack of 50 tiles.
  ii How many packs are needed to surround a pool of side 30 metres?

31 This is a sequence of calculations:

$$4 \times 9 = 36$$
$$44 \times 9 = 396$$
$$444 \times 9 = 3996$$

a Write down the value of 444 444 $\times$ 9.
b State, in words, a general rule for this sequence of calculations.

32 Solve the inequality $5x^2 < 20$.

33 **a** Sketch the graph of $y = x^3$.
b Add to your sketch the line needed to solve the equation $x^3 - 2x + 1 = 0$.

34 This is a sequence of calculations.

$\frac{2}{1} + \frac{1}{2} = \frac{5}{2}$

$\frac{3}{2} + \frac{2}{3} = \frac{11}{6}$

$\frac{4}{3} + \frac{3}{4} = \frac{25}{12}$

a Write down the next calculation in this sequence.
b Write down, in terms of n, the nth calculation in this sequence.

35 The temperature, T °C, of some water t minutes after it was placed over a source of heat is given in the table.

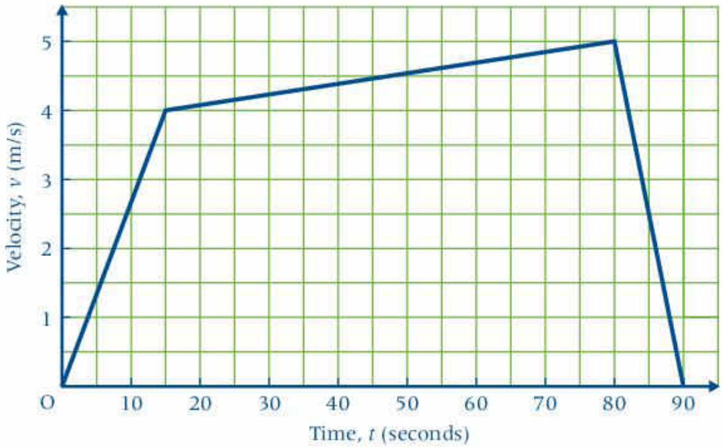
Time, *t* minutes	0	1	2	4	5	7	9	11	13
Temperature (°C)	18	23	30	46	56	78	92	100	100

a On graph paper, using scales of 1 cm to 1 minute and 1 cm to 10 °C, draw a graph of temperature against time.
b Concurrently, in a similar experiment alcohol was heated instead of water. The following values were obtained.

Time, *t* minutes	0	1	2	4	5	7	9	11	13
Temperature (°C)	18	32	48	72	76	78	78	78	78

Plot a graph to represent this information on the same graph paper as you used for part **a**.
c Use your graph to find the rate of increase of the temperature after 3 minutes for
i the water **ii** the alcohol.
d During the first 6 minutes of the experiments, is the temperature of the alcohol always increasing faster than the temperature of the water? Justify your answer.

36



The graph shows the velocity of a cyclist during a short journey.

a How long did the journey take?

b Did the cyclist travel at a constant speed at any time during the journey? Justify your answer.

c What was the cyclist's acceleration for the first 15 seconds?

d What happened from $t = 80$ to the end of the journey?

e What was the length of the journey?

f What was the cyclist's average speed for the whole journey?

37 The height, h metres, of the tide at Limpster is given relative to a particular rung marked A, on a ladder fixed to the sea wall.

On 1 June the height was given by the equation $h = 6 \cos (30t)°$, where t is the time in hours after 6 a.m.

a Draw a graph of $h = 6 \cos (30t)°$ for values of t from 0 to 12 plotting points at 1 hour intervals.
Take 2 cm ≡ 1 hour on the t-axis and 1 cm ≡ 1 m on the h-axis.

b Use your graph to estimate

i when the height of the tide will be 4 m above A

ii the height of the tide with reference to A at 10 a.m.

38 In this question assume all speeds are uniform.

A, B and C are three towns on a straight road such that B is 35 miles from A, and C is 86 miles from A in the same direction.

a A motorist sets out from A at 12 noon and drives to C without stopping, arriving there at 2.18 p.m. Draw a travel graph to represent this journey using 2 cm ≡ 10 miles and 2 cm ≡ 20 minutes. From your graph determine the motorist's average speed.

b A second motorist leaves C at 12.10 p.m. and drives to B at 47 mph. She stops at B until 1.56 p.m. when she continues her journey to A, arriving there at 2.30 p.m. On the same axes draw a travel graph to represent this journey and use your graph to determine

i the second motorist's time of arrival at B

ii the second motorist's average speed for the journey between B and A.

c A third motorist leaves C at 12.40 p.m. and travels directly to A, arriving there at 2.36 p.m.

On the same axes draw a travel graph for this journey, and from your graphs determine
- i the average speed of the third car
- ii when and where the third motorist passes the other two
- iii which motorist is nearest to A at 1.12 p.m.

39 An inter-island vessel leaves a port A at noon carrying cement for delivery at three ports, B, C and D. It travels to B, a distance of 150 nautical miles at a steady speed of 12 knots (nautical miles per hour). At B it spends $3\frac{1}{2}$ hours unloading before continuing the journey to C, a further distance of 150 nautical miles. It arrives at C at 9 a.m. the following morning. At C the unloading takes 4 hours. The vessel then proceeds to D, a distance of 200 nautical miles, arriving there at 9 a.m. the following day. A naval protection ship sails from D to A at a steady speed of 25 knots, arriving there at 4 a.m. on the third day. Draw travel graphs to show these journeys. Take 1 cm to represent $2\frac{1}{2}$ hours on the horizontal axis and choose your own scale on the distance axis.
Use your graphs to find
- a the average speed of the vessel between B and C and between C and D
- b the time the protection ship leaves D
- c when and where they pass.

40 A slow train leaves Bodley at 0200 hr and travels west at a steady 52.5 km/h. A faster train leaves Bodley at 0336 hr and travels at a steady 110 km/h along a parallel line. Draw travel graphs for the two journeys taking 4 cm ≡ 1 h on the time axis and 1 cm ≡ 50 km on the distance axis. From your graph find
- a how far the slow train is from Bodley when the other train starts its journey
- b when and where the two trains pass
- c the distance between the trains at 0530 hr.

41 A motorist leaves Caxton at 1016 hr to travel north to Dewchurch which is 160 km away. He is able to travel at a steady 60 km/h and completes the journey without a stop. A second motorist leaves Eastwood, which is 40 km north of Dewchurch, at 1040 hr to drive to Caxton. He travels to Dewchurch at a steady 60 km/h where he stops for 10 minutes before continuing his journey to Caxton, arriving there at 1240 hr. Taking 6 cm ≡ 1 h and 1 cm ≡ 20 km, draw travel graphs for these two journeys.
Use your graphs to find
- a the time the first motorist arrives at Dewchurch
- b the average speed of the second motorist between Dewchurch and Caxton
- c where and when the two pass
- d the second motorist's average speed for the whole journey.

42 An object, starting from rest, accelerates at $50\,\text{m/s}^2$ for 2 minutes, continues at the speed it has reached for a further 5 minutes, then retards to rest at $100\,\text{m/s}^2$.
- a Show this information on a velocity–time graph.
- b Use your graph to determine
  - i the maximum speed the object attains
  - ii the total distance travelled.

16 INEQUALITIES

AT THE END OF THIS CHAPTER YOU SHOULD BE ABLE TO...

1. Represent inequalities involving two variables in the $x-y$ plane.
2. Draw diagrams to represent regions in the $x-y$ plane defined by a set of inequalities.
3. Find, in a given region, the coordinates of a point where $ax + by$ is greatest or least.
4. Solve problems using linear programming.

BEFORE YOU START you need to know:

✓ the meaning of inequalities
✓ how to represent an inequality involving x or y as a line parallel to an axis in the $x-y$ plane
✓ the equation of a straight line
✓ how to draw a line from its equation
✓ how to find the gradient of a line from its graph
✓ how to find the equation of a line from its graph
✓ how to identify the point of intersection of two lines.

KEY WORDS boundary line, inequality, linear programming, region, vertex (plural vertices)

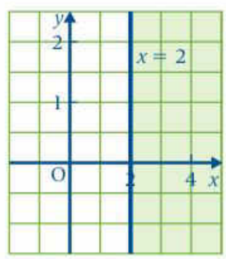
MATHS IS OUT THERE

Did you know that the Ishango bone was an 8000-year-old bone found by Dr de Heinzelin near the fishing village of Ishango, in the Congo? There are markings on it which the discoverer claimed is a numeration system. Dr Marshack who examined the notches on this bone concluded that it was used as a lunar calendar.

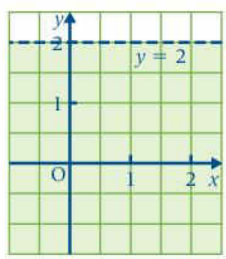
Linear inequalities

The inequality $x \geqslant 2$ can be illustrated on a number line:

This inequality can also be represented on the x–y plane by the set of points whose x-coordinates are greater then 2.
The boundary line represents all the points for which $x = 2$ and the shaded area represents all the points where $x > 2$.



The inequality $y < 2$ can be represented on the x–y plane by the set of points whose y-coordinates are less than 2.
The boundary line is broken because it shows where $y = 2$ and these values are not included in the inequality. The shaded area shows all the points where $y < 2$.



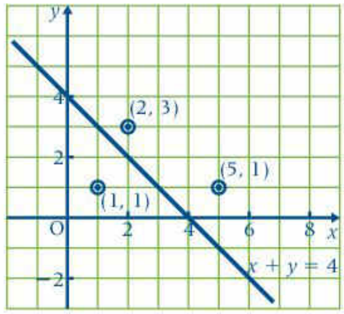
Inequalities involving two variables

The **boundary lines** for **inequalities** are parallel to the x or y axis when the equalities contain either x or y, but not both.

Now we will consider some inequalities involving *both x and y* and we will find that the boundary lines are no longer parallel to an axis.

Consider $x + y \geqslant 4$.

In this case the boundary line is $x + y = 4$; as it is included in the **region** it is drawn as a solid line.



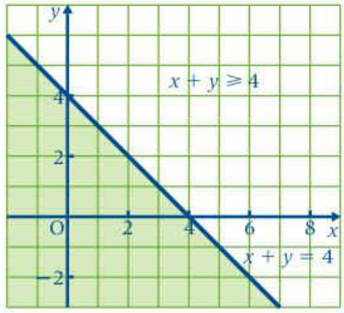
The boundary line divides the space into two regions, one on each side of the line. We need to decide which of the two regions is the one that we want.

Test a point such as (2, 3).

When $x = 2$ and $y = 3$, $x + y = 5$ which is greater than 4, so the point (2, 3) is in the required region.

The point (5, 1) is also in the region, but the point (1, 1) is not.

We can now see which region is required; it is above the line. We will therefore shade the region which is not required, as shown in the diagram.



EXERCISE 16a

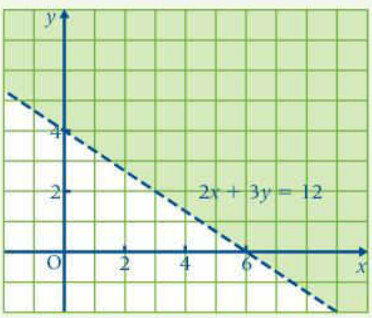
Example:

Leave unshaded the region defined by the inequality $2x + 3y < 12$.

x	0	6	3
y	4	0	2

The boundary line is $2x+3y = 12$ (not included in the inequality).

Plot this line: first find some points on it, then, because the line is not included, draw a *broken* line through them.



When $x = 0$ and $y = 0$, $2x + 3y = 0$ which is less than 12.

Therefore (0, 0) is in the required region. (Shade the region on the other side of the line.)

The unshaded region represents the inequality $2x + 3y < 12$.

To decide which of the two regions is wanted, we test an easy point such as (0, 0).

Find the regions defined by the following inequalities (draw axes for values of x and y from -6 to 6).

1 $x + y \leqslant 3$ **2** $x + 4y \leqslant 8$ **3** $x + y > 1$

4 $x + y \leqslant 2$ **5** $2x + 5y \geqslant -6$ **6** $3x + 4y \geqslant 12$

7 $4x + y < 4$ **8** $2x + 5y > 10$ **9** $2x + y \leqslant 6$

10 $3x + 2y > 5$

Sometimes the x and y terms are not on the same side of the inequality. Find the regions defined by the following inequalities:

11 $y \leqslant x + 1$ **12** $y > 2x - 1$ **13** $y \geqslant \frac{1}{2}x + 1$

14 $y < 2 - 2x$ **15** $y < 4 - x$ **16** $y \geqslant 2x - 2$

17 $y < 2x + 3$ **18** $y < 5 + 3x$ **19** $y > 3 + x$

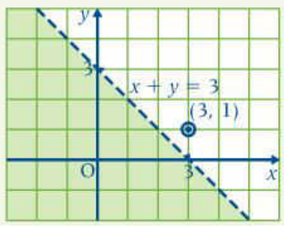
20 $y \leqslant 5 - 2x$ **21** $y \leqslant x - 4$ **22** $y \geqslant 1 - x$

Draw the boundary line then test the point (0, 0)

EXERCISE 16b

Example:

Find the inequality defining the unshaded region.



The boundary line is $x + y = 3$ and is not included.

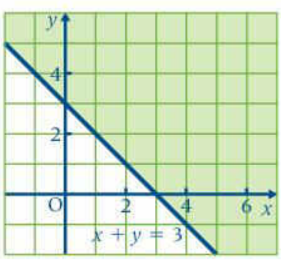
When $x = 3$ and $y = 1$, $x + y = 4$

$4 > 3$, so the inequality is $x + y > 3$

Test a point which is in the required region; we use (3, 1).

Find the inequalities that define the unshaded regions:

1



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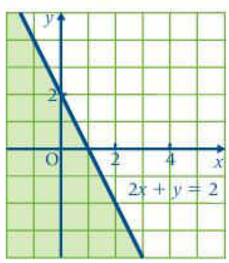

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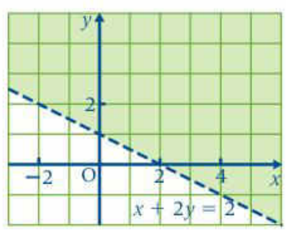

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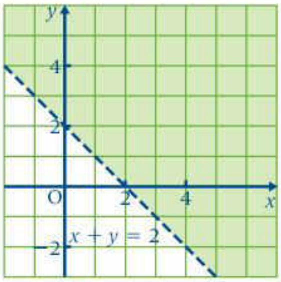

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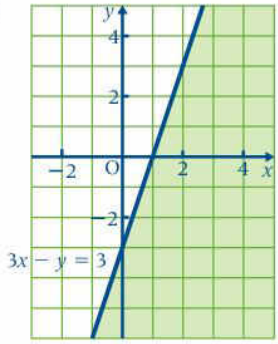

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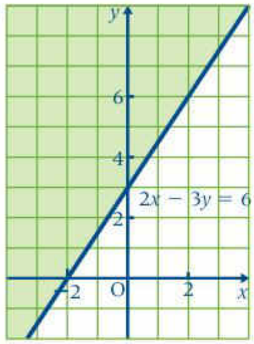
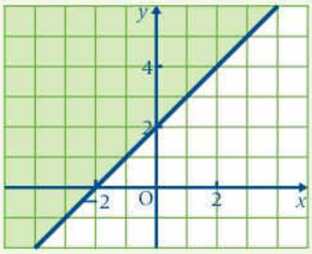


Example:

Find the inequality that defines the unshaded region.



Moving from (−2, 0) to (0, 2) the gradient of the boundary line is

$$\frac{2-0}{0-(-2)} = \frac{2}{2} = 1$$

To find the equation of a line you need to work out its gradient and find where it cuts the y-axis.

The line cuts the y-axis when $y = 2$ so the equation of the boundary line is $y = x + 2$.

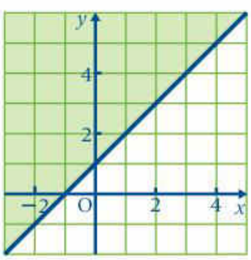
Test the point (0, 0), which is in the required region.

At (0, 0), $y = 0$ and $x + 2 = 2$

$0 < 2$, so the inequality is $y \leqslant x + 2$.

The boundary line is solid so is included in the region.

Find the inequalities that define the unshaded regions:

7


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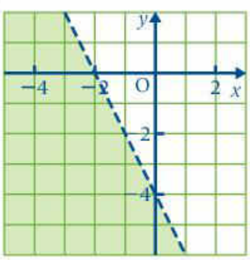

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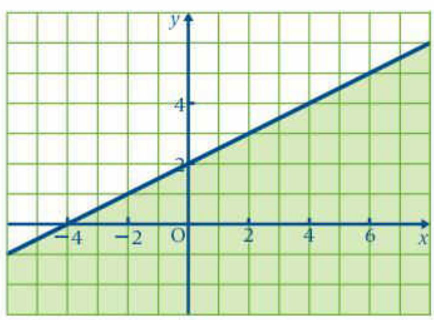

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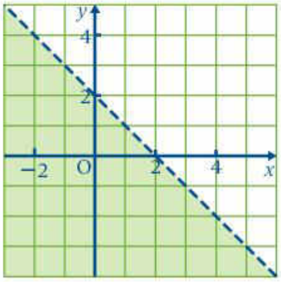

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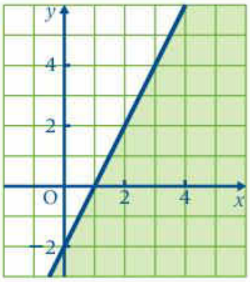

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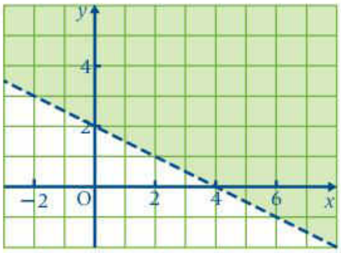

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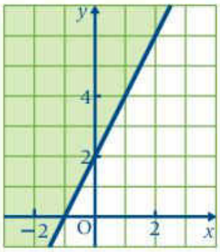


EXERCISE 16c

Example:

Leave unshaded the region defined by the set of inequalities $x + y < 4$, $x \geqslant 0$ and $x + 2y \geqslant 2$.

1st boundary line (not included) is $x + y = 4$

x	4	0	2
y	0	4	2

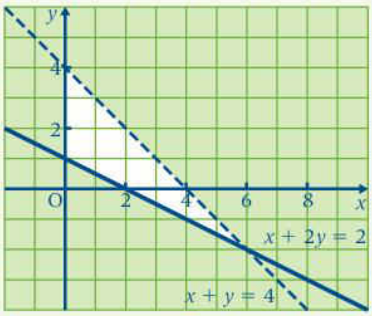
Plot this line. Testing (0, 0) shows that the region below the line is wanted.

2nd boundary line (included) $x = 0$.
This is the y-axis

3rd boundary line (included) $x + 2y = 2$.

x	0	2	4
y	1	0	−1

Plot this line. Test the point (0, 0). This shows that the region above this line is wanted.



The unshaded region is defined by the given inequalities.

Leave unshaded the regions defined by the following sets of inequalities:

1 $x \geqslant -3, y \geqslant -2, x + y \leqslant 3$

2 $x > -1, -2 < y < 4, x + y < 4$

3 $y > 0, y \leqslant \frac{1}{2}x, x + y \geqslant 1, x + y \leqslant 5$

4 $y < 3, 2x + 3y \geqslant 6, y > x - 2$

5 $y \leqslant 0, x \leqslant 0, x + y \geqslant -4$

6 $y > x, y < 4x, x + y < 5$

7 $y \geqslant 1, x \leqslant 0, y \leqslant x + 2$

8 $x + y \leqslant 6, 3x + y \geqslant 3, y \geqslant -1, x \geqslant -1$

9 $x \geqslant 0, y \geqslant x - 1, 2y + x < 4$

10 $x > 0, y \geqslant \frac{1}{2}x, x + y \geqslant 1, x + y \leqslant 5$

11 $y \geqslant 0, x \geqslant 0, x + y \leqslant 1$

12 What can you say about the region defined by $x + y > 4$, $x + y < 1$, $x > 0$ and $y > 0$?

13 Do the regions defined by the following sets of inequalities exist?
a $x + y \geqslant 3, y \leqslant 2, y \geqslant 2x$
b $x + y > 3, y < 2, y > 2x$

MATHS IS OUT THERE

Did you know that Fibonacci (Leonardo Pisano) (1170–1240) got his start in mathematics during his long residence in the North African coastal city where his father was a merchant?

Fibonacci was responsible for introducing the Arabic notation for numerals to Europe.

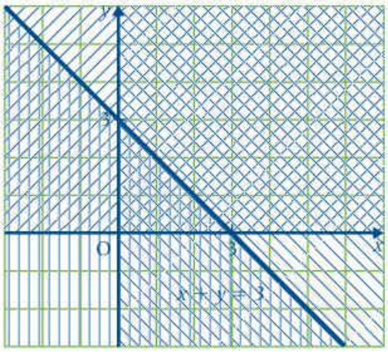
PUZZLE

Which fills the space better, a round peg in a square hole or a square peg in a round hole?

Shading the required region

In some simple cases you might be asked to shade the region defined by the inequality, instead of leaving it unshaded.

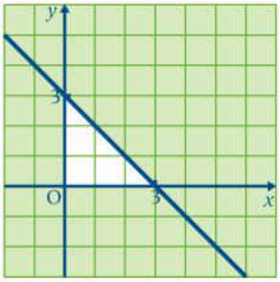
Occasionally, you may be asked to shade the required region when it is defined by several inequalities. If you try to do it by shading the required side of each boundary line, you will find yourself with overlapping shadings, resulting in a confused diagram.



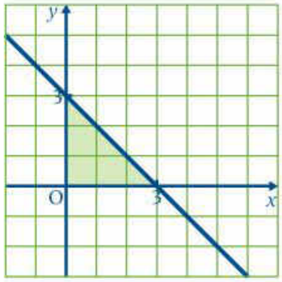
For instance, if $y \geqslant 0$, $x \geqslant 0$ and $x + y \leqslant 3$, the diagram looks like this and the required region disappears in a muddle.

A better method is to do the shading as before so that the required region is left unshaded, then draw a second diagram on which you shade the required area.

1st diagram



2nd diagram
The required region is shaded.



EXERCISE 16d

Shade the regions defined by the sets of inequalities:

> Shade the unwanted regions first then redraw the diagram and shade the required region.

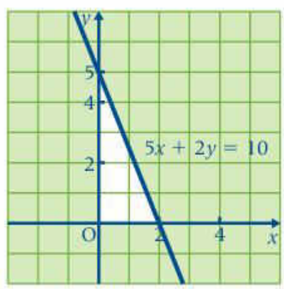
1. $x \geqslant 0, y \geqslant 0, x + y \leqslant 3$
2. $x > -2, y \geqslant 2x, x + y < 4$
3. $x \leqslant 4, y \leqslant 3, x + y \geqslant 0$
4. $2x + y \geqslant 4, y \leqslant 0, x \leqslant 4$
5. $y > \frac{1}{2}x, 0 < x < 2, x + y < 4, y < 3$
6. $\frac{x}{2} + \frac{y}{3} \geqslant 1, \frac{x}{2} - \frac{y}{3} \geqslant 1, x \leqslant 8$
7. $3x + 4y \leqslant 12, y \leqslant 2x + 1, y \geqslant -1$
8. $5x + 2y \leqslant -10, y \geqslant -1, x \geqslant -6$
9. $\frac{x}{5} + \frac{y}{4} \leqslant 1, y \leqslant 3x, y \geqslant \frac{1}{2}x$
10. $3x + 2y \geqslant -6, y \geqslant -3, x \geqslant -1$

If you wish to use diagrams for solving problems, it is best to leave the required regions unshaded.

EXERCISE 16e

Example:

Give the inequalities that define the unshaded region.



> Test the point (1, 1).

Two of the boundary lines are the x-axis and the y-axis.

The unshaded region is above the x-axis: this is $y \geqslant 0$, and to the right of the y-axis: this is $x \geqslant 0$.

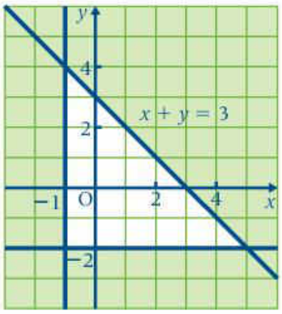
The 3rd boundary line is $5x + 2y = 10$

When $x = 1$ and $y = 1$, $5x + 2y = 7$

$7 < 10$ so the 3rd inequality is $5x + 2y \leqslant 10$

Give the sets of inequalities that define the unshaded regions:

1



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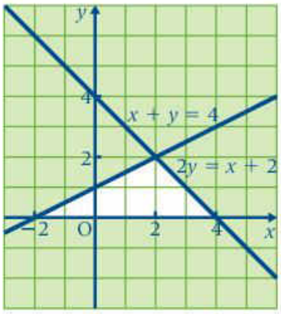

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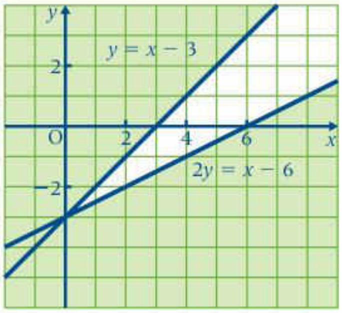

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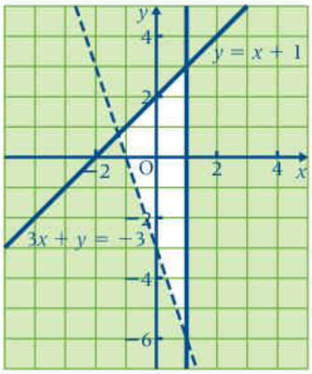
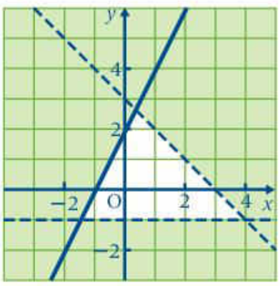


In the next four questions it is necessary to find the equations of the boundary lines first:

5



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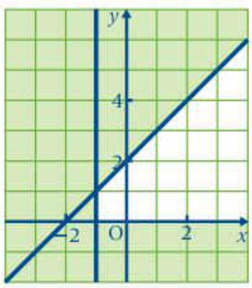

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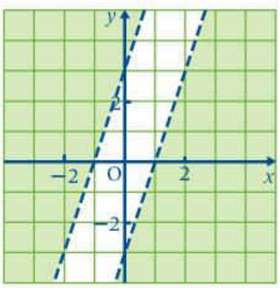

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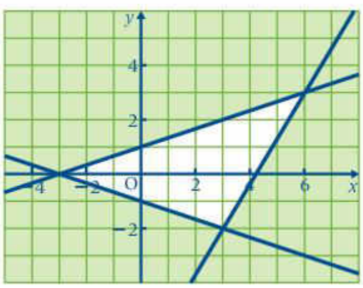
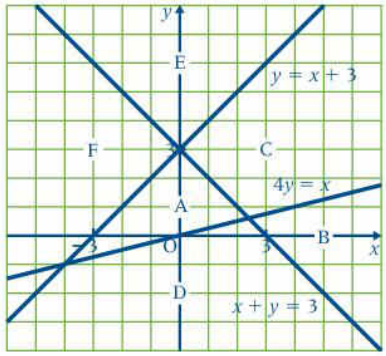


9 Use inequalities to describe the regions:

a A **b** B **c** C **d** D

e E **f** A + F

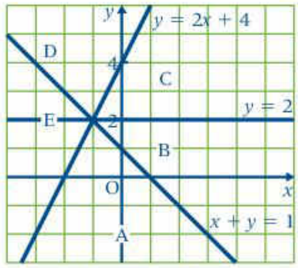


10 Use inequalities to describe the regions:

a A **b** B **c** C

d D **e** E **f** A + B

(The axes are not boundary lines.)

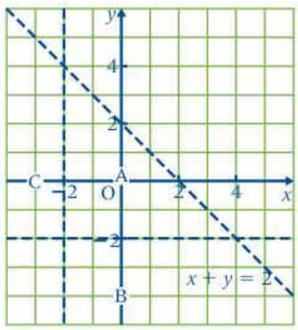


11 In the diagram, which region (A, B or C) does each of the following sets of inequalities refer to?

a $x + y < 2, x < -2, y > -2$

b $x + y < 2, x > -2, y > -2$

c $x + y < 2, x > -2, y < -2$

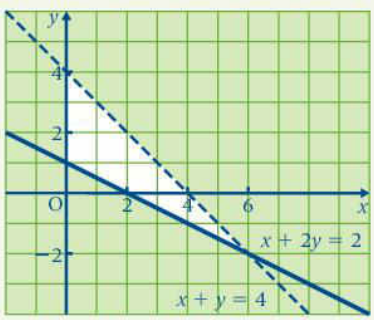


Coordinates of points in a region

EXERCISE 16f

Example:

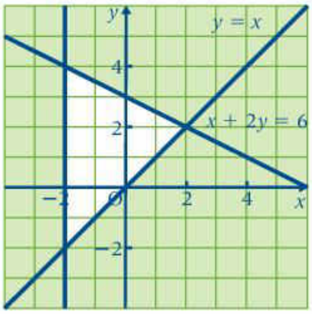
From the given diagram read off the coordinates of the **vertices** of the unshaded region.



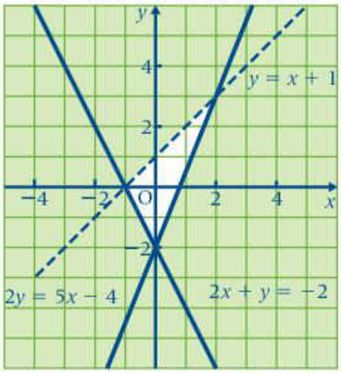
The vertices are (0, 1), (0, 4) and (6, −2).

From the diagrams, give the coordinates of the vertices of the unshaded regions in questions **1** to **4**:

1



2



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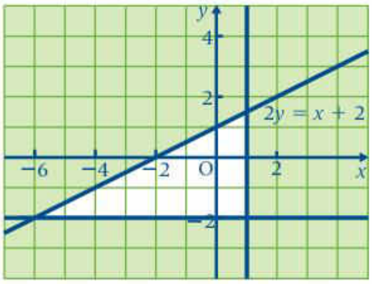

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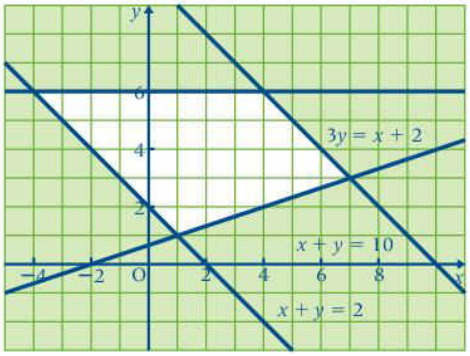
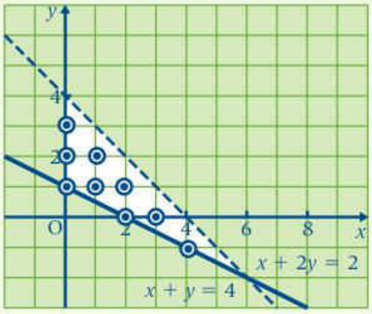


5 Draw a diagram and find the coordinates of the vertices of the region defined by the inequalities $x \geqslant -2$, $y \geqslant 2x$ and $y \leqslant x$.

6 Draw a diagram and give the vertices of the region defined by the inequalities $y \geqslant 1$, $x \leqslant 0$ and $y \leqslant x + 2$.

Example:

Give the points whose coordinates are integers and that lie in the region given in the worked example at the beginning of the exercise.



Notice that points on the broken line are *not* in the region.

Points are (0, 1), (0, 2), (0, 3), (1, 1), (1, 2), (2, 0), (2, 1), (3, 0) and (4, −1).

7 Give the points whose coordinates are integers and that lie in the regions in questions **1** to **3**.

8 Draw a diagram and give the coordinates of the points whose coordinates are integers and that lie in the region defined by the inequalities $y \leqslant 3x + 6$, $y > x - 2$, $x + y > -2$ and $x + y \leqslant 3$.

9 Draw a diagram and give the points whose coordinates are integers and that lie in the region defined by the inequalities $y > 0$, $x > 0$, $3x + 4y < 12$ and $4x + 3y < 12$.

10 Draw a diagram and give the points with coordinates that are integers, on the boundaries of the region defined by the inequalities $x \geqslant 2$, $y \geqslant -1$ and $x + y \leqslant 4$.

Greatest and least values

EXERCISE 16g

Example:

Find the value of $x - 2y$ at the points (4, 1), (3, 0) and (2, −1). At which point is the value greatest?

At (4, 1), $x - 2y = 4 - 2 = 2$

At (3, 0), $x - 2y = 3 - 0 = 3$

At (2, −1), $x - 2y = 2 - (-2) = 4$

$x - 2y$ has its greatest value at (2, −1).

1 Find the value of $x + y$ at the points (2, 3), (4, −2) and (−3, −1).

2 Find the value of $x - y$ at the points (6, 2), (1, 4) and (4, −3).

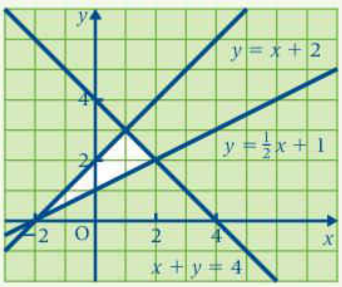
3 Find the value of $2x + y$ at the points (5, 1), (−6, −2) and (−1, 2).

4 Find the value of $3x - 2y$ at the points (5, 5) and (2, −8). At which point is the value greatest?

5 Find the value of $y - 3x$ at the points (−6, −8) and (3, 8). At which point is the value least?

Example:

From the diagram give the coordinates of the vertices of the unshaded region. At which vertex is the value of $x + 2y$ least?



The three vertices are (1, 3), (2, 2) and (−2, 0).

At (1, 3) $x + 2y = 1 + 6 = 7$

At (2, 2) $x + 2y = 2 + 4 = 6$

At (−2, 0) $x + 2y = -2 + 0 = -2$

$\therefore x + 2y$ is least at point (−2, 0).

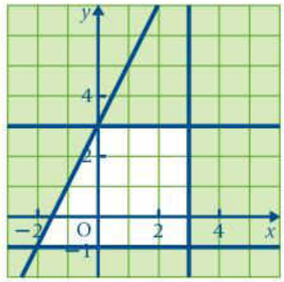
MATHS IS OUT THERE

Did you know that three great periods of African mathematics are

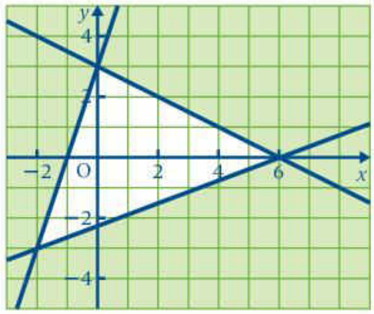
1. the ancient Egyptian mathematics of the pyramids, obelisks and great temples;
2. African participation in classical mathematics of the Hellenistic period;
3. the African participation in Muslim mathematics?

Can you list others?

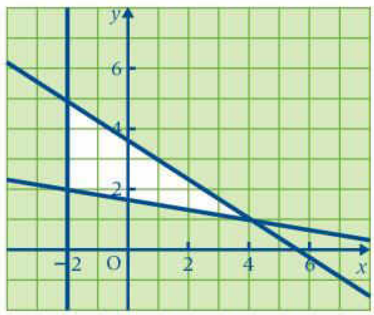
6 a Find from the diagram the coordinates of the vertices of the unshaded region.
b Find the vertex at which the value of $x + y$ is greatest.
c Find the vertex at which the value of $x + y$ is least.
d Mark the other points in the unshaded region whose coordinates are integers. How many are there?
e Is there a point in the region where the value of $x + y$ is greater than its value at the vertex chosen in **b**?



7 a Find from the diagram the coordinates of the vertices of the region.
b Find the vertex at which the value of $2x - y$ is greatest.



8 a From the diagram find the coordinates of the vertices of the unshaded region.
b Find the vertex at which the value of $3x - 2y$ is
i greatest ii least.
c List the other points in the unshaded region whose coordinates are integers.
d Is there a point in the region where the value of $3x - 2y$ is less than its value at the vertex chosen in **b ii**?



9 a Draw axes for the values of x and y from -5 to 5 and leave unshaded the region defined by the set of inequalities $x \geqslant -2$, $y \geqslant -3$ and $x + y \leqslant 2$.
b Find the coordinates of the vertices of the region.
c Mark the other points in the region with integer coordinates. How many are there?
d At which of these points in the region is the value of $x - y$ greatest and at which point is it least?

As we see from the answers to questions **8** and **9**, the greatest and least values of any expression such as $2x + 3y$ occur at or near the vertices of a region and not at points between the vertices or in the middle of the region. Therefore to find the greatest or least values, we need to test points at or near the vertices only.

EXERCISE 16h

Example:

For the region defined by the set of inequalities $x \geqslant -1$, $y \geqslant -2$ and $x + y < 3$, draw a diagram and find the points with coordinates that are integers, where the value of $2x - y$ is **a** greatest **b** least.

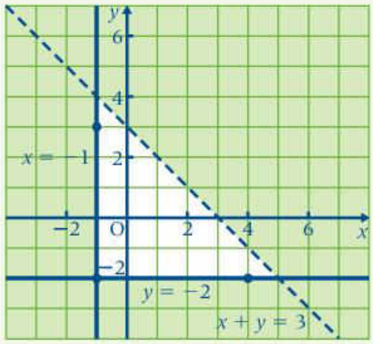
At $(-1, 3)$ $\quad 2x - y = -2 - 3 = -5$

At $(-1, -2)$ $\quad 2x - y = -2 + 2 = 0$

At $(4, -2)$ $\quad 2x - y = 8 + 2 = 10$

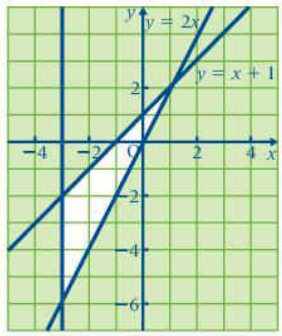
a $(2x - y)$ is greatest at $(4, -2)$.

b $(2x - y)$ is least at $(-1, 3)$.

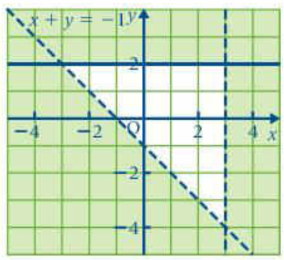


Remember that points on the broken line are not included in the region, When the point at a vertex is not in the region, choose the point that is nearest to the vertex. It helps if you mark the points on the diagram.

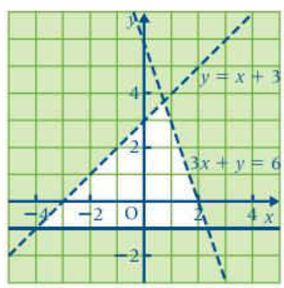
1 In the region defined by $x \geqslant -3$, $y \leqslant x + 1$ and $y \geqslant 2x$, find the point with integer coordinates where the value of $3x - y$ is greatest.



2 In the region defined by $x + y > -1$, $x < 3$ and $y \leqslant 2$, find the point with integer coordinates where the value of $x + 2y$ is least.



3 In the region defined by $y < x + 3$, $3x + y < 6$ and $y \geqslant -1$, find the point with integer coordinates where the value of $3x - 2y$ is greatest.



4 In the region defined by $y \leqslant 4 - 2x$, $y \geqslant -2$ and $x \geqslant 0$, by drawing a diagram, find the points or point with integer coordinates at which the value of $2x + y$ is greatest.

5 In the region defined by $x + y < 4$, $x > -1$ and $y > 1$, by drawing a diagram, find the point with integer coordinates at which the value of $2x - y$ is least.

6 In the region defined by $4x + y \leqslant 4$, $y \leqslant 3$ and $x \leqslant 1$, by drawing a diagram, find the point with integer coordinates at which the value of $x + y$ is greatest. Is there a point at which the value of $x + y$ is least?

Linear programming

A branch of mathematics known as **linear programming** is an essential tool in business planning. It provides a useful application for systems of linear inequations similar to those solved above.

The following exercise will illustrate this.

EXERCISE 16i

Example:

Dana Ceramics produce a new commemorative plate and mug. Russell runs a china shop and has to decide how many of each to order. Plates will cost him \$15 each and mugs will cost him \$8 each. After going through the various options he decides that his order must contain at least 10 plates and 25 mugs; at least twice as many mugs as plates, not exceed 40 pieces, and have a value of at least \$360 to qualify for a 5% discount off his next order.

His profit on a plate is \$5 and his profit on a mug is \$4. On the assumption that he can sell all that he buys, find how many of each he should order to maximise his profit.

First let x be the number of plates and y be the number of mugs that Russell buys.

First allocate letters to the unknown quantities.

At least 10 plates and at least 25 mugs gives
$x \geqslant 10$ and $y \geqslant 25$

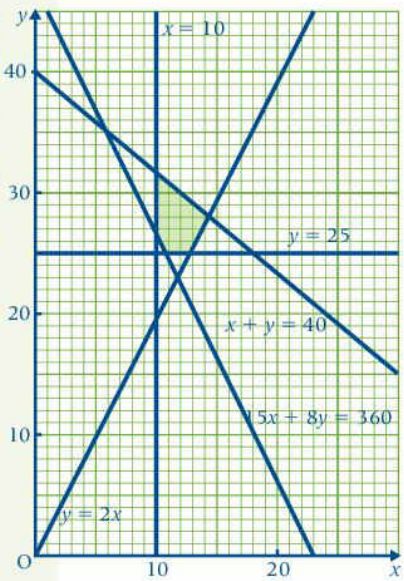
Now use the conditions to set up inequalities in terms of x and y.

At least twice as many mugs as plates gives
$y \geqslant 2x$

Draw a diagram showing the region satisfied by all the inequalities.

Not exceed 40 pieces gives
$x + y \leqslant 40$

Have a value of at least \$360 gives
$15x + 8y \geqslant 360$



His profit is $5x + 4y$ and we need to find the values of x and y that give the maximum value of this.

$5x + 4y$ has its maximum value in the region when $x = 13$ and $y = 27$

So Russell should order 13 plates and 27 mugs.

There are five vertices. Test each of the nearest points with whole-number coordinates.

1 A factory produces two grades of a product, grade A and grade B. Each unit of grade A requires 2 hours work on a machine and each unit of grade B requires 3 hours work on the same machine. The factory can only afford a maximum of 40 working hours per week on the machine. Denoting the number of units of grade A product produced weekly by x and the number of units of grade B product by y, graph the relation which shows the combinations of the two products that the factory is capable of producing weekly.
If grade A units are sold at \$150 per unit and grade B units are sold at \$200 per unit, what is the maximum value of sales which the factory can make in one week?

2 A builder has a $5000\,m^2$ plot of land and outline planning permission to build a mixture of two-bedroomed houses and two-bedroomed bungalows. The planning regulations state that the area of a plot for a bungalow must be at least $300\,m^2$ and the area of a plot for a house must be at least $250\,m^2$. It takes 260 man-days to build a bungalow and 180 man-days to build a house. The builder wants to keep as many of his work force employed as possible so plans to use at least 2000 man-days. He plans to build at least 6 bungalows. What is the greatest number of dwellings that he can build on this plot?

3 A building contractor can employ no more than 5 carpenters; also, he can employ at least 7 masons but no more than 10.
He must have at least 1 carpenter and the total number of carpenters and masons must not exceed 12.
Using x to represent the number of carpenters and y to represent the number of masons, write a system of inequalities to show the permissible combinations he can employ.

4 A recommended animal feed is to be a mixture of two foodstuffs, each unit of which contains protein, fat and carbohydrate in the number of grams shown in the table below.

	Foodstuff	
	1	2
Protein (g)	10	5
Fat (g)	0.1	0.9
Carbohydrate (g)	10	30

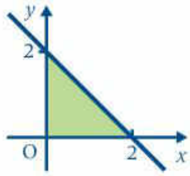
Each bag of the resulting mixture is to contain at least 40 g of protein, 1.8 g of fat and 120 g of carbohydrate. Denoting the number of units of foodstuff 1 in the mixture by x and the number of units of foodstuff 2 by y, write and graph the system of inequalities which shows the mixtures that meet these requirements.

MATHS IS OUT THERE

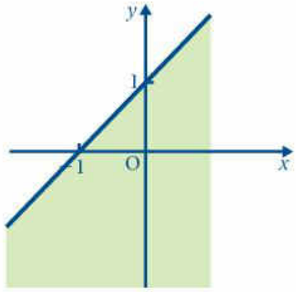
Did you know that Euclid is the best known of the Alexandrian mathematicians? For some time his texts, *The Elements,* dominated the teaching of mathematics.

MIXED EXERCISE 16

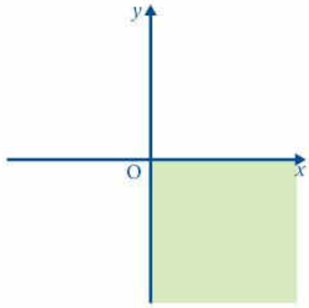
Several answers are given for these questions.
Write down the letter that corresponds to the correct answer.



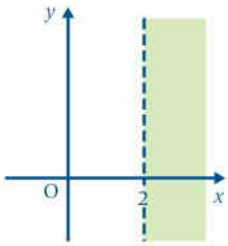
1 The shaded region in the diagram is defined by
A $x \leqslant 2, y \leqslant 2, x + y \leqslant 2$ **B** $x \leqslant 0, y \leqslant 0, x + y \leqslant 2$
C $x \geqslant 0, y \geqslant 0, x + y \leqslant 2$ **D** $x \geqslant 0, y \geqslant 0, x + y \geqslant 2$



2 The shaded region shown in the diagram is represented by the inequation
A $y \leqslant x + 1$ **B** $y \geqslant x + 1$ **C** $y \leqslant x + 1$ **D** $y \geqslant x + 1$

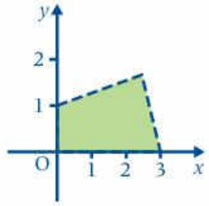


3 The shaded region represents
A $x > 0$ and $y < 0$ **B** $x < 0$ and $y > 0$
C $x > 0$ and $y > 0$ **D** $x < 0$ and $y < 0$



4 The shaded region represents
A $x < 2$
B $x > 2$
C $y > 2$
D None of these

5 The shape of the region represented by the inequalities $x \leqslant 2, y \geqslant 0, y \geqslant x, y \leqslant x + 2$ could be
A a triangle **B** a parallelogram **C** a trapezium **D** a pentagon



6 x and y are integers.
The greatest value of $x + y$ in the region shown in the diagram is
A 1 **B** 2 **C** 3 **D** 4

7 A farm uses a combination of x large and y small trays to pack the eggs it produces.
A large tray holds 12 eggs and a small tray holds 6 eggs.
No more than 80 eggs need to be packed in at most 9 trays.
Which of the following inequalities satisfy these conditions?

A $x + y \leqslant 9$ and $12x + 6y \leqslant 80$
B $x + y \leqslant 9$ and $12y + 6x \geqslant 80$
C $x + y \geqslant 9$ and $12x + 6y \leqslant 80$
D $x + y \leqslant 80$ and $12x + 6y \leqslant 9$

IN THIS CHAPTER YOU HAVE SEEN THAT...

- you can illustrate inequalities in the $x-y$ plane by drawing the boundary line and testing a point to see which side of the boundary line is required. You can then shade out the region not wanted
- several inequalities can be illustrated on one diagram: then all points inside the unshaded region satisfy all those inequalities
- to find the maximum, or minimum value of an expression, substitute the values of the coordinates at or near the vertices of the region.

17 VARIATION

AT THE END OF THIS CHAPTER YOU SHOULD BE ABLE TO...

1 Write a given statement in the form of an equation.
2 Complete a table of values of variables for a given equation.
3 Calculate the value of one variable, given the corresponding value of the other variable and the equation defining the relation.

BEFORE YOU START

you need to know:
✓ how to write a given statement in the form of an equation
✓ how to solve simple equations
✓ how to complete a table of values of variables for a given equation
✓ how to calculate the value of one variable, given its corresponding variable and the equation defining the relation
✓ how to plot the graph for a set of data points
✓ how to change the subject of a given formula.

KEY WORDS

dependent variable, direct proportion, direct variation, independent variable, inverse variation, variable

MATHS IS OUT THERE

Carl Gauss (1777–1855), once said to a friend, 'I cannot understand how a man as intelligent as Archimedes failed to invent a place value system.' At the same time however, he was accused of failing to publish his own fantastic discoveries.

Formulae

The formula $s = 20t$ tells us the distance s, in kilometres, travelled by the cyclist in t hours.

This equation is an example of two quantities that are directly proportional.
As t increases, s also increases.
If t is doubled, s is also doubled.

We say that 's varies directly as t'.

Also, $\frac{s}{t} = 20$.

In this example, 20 is the constant of proportion.

If we draw the graph of $s = 20t$, we get a straight line.

> The graph of two **variables** in direct proportion will always be a straight line.

How fast did you run the 100 metres race at your school's last Sports Day? Was the record for this race broken at your last meet?

You know that the faster you run, the less time it takes you to cover a distance of 100 m, say.

As the speed increases the time decreases.

If we let t represent the time and v represent the speed, then for a 100-metre race, $vt = 100$ or $v = \frac{100}{t}$

This equation is an example of two quantities that are inversely proportional.

If t is doubled, v is halved.

We say that 'v varies inversely as t'.

Try drawing the graph of $v = \frac{100}{t}$

Here are some values for v and t:

t	5	10	20	25	15
v	20	10	5	4	6.3

Plot these points and draw the resulting graph. Did you get a curve?

Any two quantities that are inversely proportional will have a graph with this shape.

You should recognise the shape. We drew graphs with this shape in Chapter 14.

Relationships

Frequently, in everyday life, we come across two quantities that appear to be related to each other in some way. The amount I spend on potatoes, when they cost 60 c a kilogram, depends on the number of kilograms I buy; the distance I travel in a car, at a constant speed, depends on the length of time that I am travelling; the number of blank tapes I can buy for \$30, depends on the price of one such tape. These are examples of quantities that are related by a simple algebraic equation.

On the other hand, there is no simple algebraic relationship between the amount a person earns and how much that person spends on food; between our weight and our height; or between how far we travel to school and the time we get up in the morning.

The first exercise in this chapter helps us to recognise some of the simple relationships that connect sets of varying quantities.

EXERCISE 17a

Example:

Write down the equation connecting the two variables given in the table.

x	2	3	5	10	12
y	6	9	15	30	36

$y = 3x$

We observe that in each case the value of y is three times the value of x.

In each of the following questions write down the equation connecting the variables.

When you think you have found the equation, test it using at least two sets of values of x and y.

1

x	1	2	4	7	10
y	3	6	12	21	30

2

p	0	1	2	3	4
q	0	1	4	9	16

3

x	1	2	3	4	5
V	1	8	27	64	125

4

A	0	4	9	16	25
r	0	2	3	4	5

5

x	2	4	6	24
y	12	6	4	1

6

r	0	2	4	6	10
s	0	0.2	0.4	0.6	1

7

x	−3	−1	0	2	4
y	36	4	0	16	64

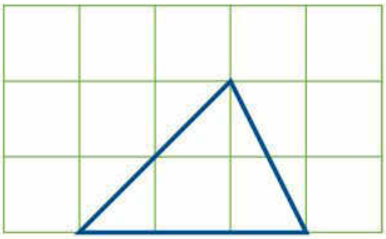
8

p	−9	−6	−3	2	4
q	4	6	12	−18	−9

9 A set of rectangles is such that, in each one, the length is twice the breadth. Use the lengths given in the table and complete the table to give the area of each rectangle. What is the relationship between the area A and the length L?

Length of rectangle, L cm	2	4	5	6	8	10
Area of rectangle, A cm^2						

10 Copy the triangle given on squared paper. Its base is 3 cm and its height is 2 cm.



Draw three similar triangles with bases 6 cm, 9 cm and 12 cm whose heights will be 4 cm, 6 cm and 8 cm respectively. Find the area of each triangle. Use these values to complete the following table.

Base, b cm	3	6	9	12
Area, A cm^2				

What relationship connects A and b?

11 For this question imagine that you have a quantity of identical cubes. Use these cubes to build bigger cubes whose edges are larger than the basic cube by factors of 2, 3, 4 and 5. It will help if you draw diagrams. For each of these cubes find how many times larger its volume is than the volume of the basic cube, i.e. how many of the smallest cubes are required to make each of the larger cubes. Use your results to complete the following table.

Factor, x, by which the edge of the basic cube is multiplied	2	3	4	5
Factor, y, by which the volume of the basic cube is multiplied				

What is the relationship between x and y?

The questions in this exercise have illustrated several different ways in which varying quantities may be related. An increase in one quantity may lead to an increase or a decrease in the other.

PUZZLE

Have you ever considered 89 to be a special number? You know it's a prime number.

Now try the following and record the results:

1 Add the product of 89's digits to their sum.

2 Divide 8 by 9, and give your answer to 2 decimal places.

3 Find two consecutive positive one-digit integers, one of which is a square number and the other is a cube number.

4 Square the digits of 89 and add the answers. Continue doing this with the new numbers you get.

e.g. 89 → 145 → 42 → … Continue.

Direct variation

If two varying quantities are always in the same ratio, they are said to be directly proportional to one another.

Consider the total cost for a group of people to attend a concert. The varying costs, depending on the size of the group, are given in the table.

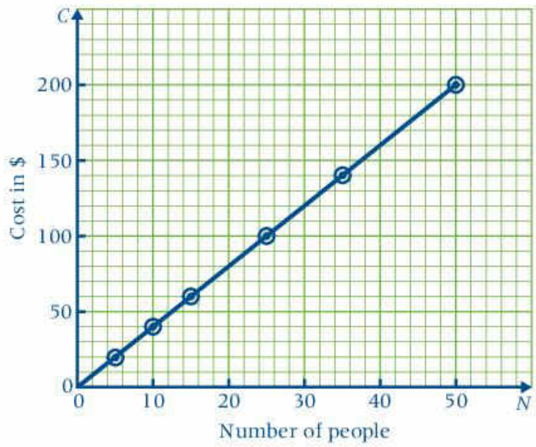
Number of people, N	5	10	15	25	35	50
Total cost in \$, C	20	40	60	100	140	200

The two quantities C and N are connected by the equation $C = 4N$ i.e. the value of C is always four times the value of N. Therefore C is directly proportional to N. This relation is called **direct proportion**.

In general, if two variables Y and X are in direct proportion then $Y = kX$ where k is the constant of proportion.

Alternatively we can say that *Y varies directly with X*. This gives exactly the same equation, i.e. $Y = kX$ where k is the constant of variation.

If we plotted the data given in the table on a graph, with N along the horizontal axis and C along the vertical axis the points would all lie on the straight line with equation $C = 4N$. Comparing the equation $C = 4N$ with the general equation $y = mx + c$, we see that the gradient of the line $C = 4N$ is 4, and this gradient represents the cost for one person to attend the concert. The fact that the straight line passes through the origin confirms that the cost if no people attend is $0.



The information in the table can also be written

Number of people attending : Cost in $ = 1 : 4

It is important to realise that we now have three different ways in which we can illustrate the relationship between the number of people attending the concert and the total cost.

In mathematics we are always looking for shorter ways of writing things. Instead of writing 'y is directly proportional to x' or 'y varies directly as x' we sometimes write $y \propto x$, where '$\propto$' means 'varies as', from which we can write the equation $y = kx$ where k is some constant.

EXERCISE **17b**

1 Copy and complete the table so that $y \propto x$.
What is the equation connecting x and y?

As $y \propto x$, $y = kx$. Use the first pair of values to find k.

x	2		8
y	20	40	

2 Copy and complete the table so that $C \propto r$.

r		3	5		8
C	6	18		36	48

What is the equation connecting C and r?

3 Copy and complete the table so that $C \propto n$.

Number of units of electricity used, n	100	120	142	260	312	460
Total cost in cents, C	600		852	1560		2760

What meaning can you give to the constant of proportion?

4 Copy and complete the table so that $Y \propto X$.

Number of oranges bought, X	2	4	7	9	11	15
Total cost in cents, Y	20	40		90		150

What meaning can you give to the constant of variation?

Example:

If y varies directly as x, and $y = 2$ when $x = 3$, find
a y when x is 9 **b** x when y is 18.

$y \propto x$ i.e. $y = kx$ where k is a constant

⇒	$2 = k \times 3$
i.e.	$3k = 2 \Rightarrow k = \frac{2}{3}$
Hence	$y = \frac{2}{3}x$
a When $x = 9$,	$y = \frac{2}{3} \times 9 = 6$
b When $y = 18$,	$18 = \frac{2}{3}x$
i.e.	$54 = 2x$
⇒	$27 = x$
i.e.	$x = 27$

We know that $y = 2$ when $x = 3$ so we can substitute 2 for y and 3 for x into $y = kx$.

5 y varies directly as x, and $y = 21$ when $x = 7$.
Find **a** y when $x = 3$ **b** x when $y = 48$.

6 $y \propto x$ and $y = 6$ when $x = 24$.
Find **a** y when $x = 6$ **b** x when $y = 5$.

7 s varies directly as t, and $s = 35$ when $t = 5$.
Find **a** s when $t = 3$ **b** t when $s = 49$.

8 P is directly proportional to Q, and $P = 15$ when $Q = 50$.
Find **a** P when Q is 70 **b** Q when P is 12.

9 W is directly proportional to S, and $W = 8$ when S is 10.
Find **a** W when S is 30 **b** S when W is 12.

10 Y is directly proportional to X, and $Y = 45$ when $X = 18$.
Find **a** Y when X is 6 **b** X when Y is 20.

11 y varies directly as $3x - 4$, and $y = 33$ when $x = 5$.
Find **a** y when $x = 2$ **b** x when $y = 15$.

Dependent and independent variables

Usually one of the quantities varies because of a change in the other. In question **3** on the previous page the total cost goes up because the number of units of electricity used goes up, that is, the variation in the first quantity, C, *depends* on the change in the other, n. C is called a **dependent variable** while n is referred to as an **independent variable**.

Similarly, when the radius of a circle increases, the area of the circle increases. Therefore, the radius is the independent variable and the area is the dependent variable.

The dependent variable is sometimes proportional to a power of the independent variable. For example, the area of a circle, A, is directly proportional to the *square* of its radius, R, and we can write

$$A \propto R^2 \text{ or } A = kR^2$$

k has a special value in this case; can you say what it stands for?

Similarly, if the safe speed, V, at which a car can round a bend varies as the square root of the radius of the bend, R, then

$$V \propto \sqrt{R} \text{ or } V = k\sqrt{R}.$$

EXERCISE 17c

Example:

Copy and complete the table for positive values of x so that $y \propto x^2$.

x	1	3	5		11
y	2		50	98	

What is the equation connecting x and y?

Since $y \propto x^2$, $y = kx^2$
When $x = 1, y = 2$ so $2 = k \times 1^2$
i.e. $k = 2$
$\therefore$ $y = 2x^2$

When $x = 3$, $y = 2 \times 9 = 18$
When $x = 11$, $y = 2 \times 121 = 242$
When $y = 98$, $98 = 2 \times x^2$
$\Rightarrow$ $x^2 = 49$
$\therefore$ $x = 7$

We are only asked for the positive value so we discard $x = -7$.

The completed table is

x	1	3	5	7	11
y	2	18	50	98	242

and the equation connecting x and y is $y = 2x^2$.

1 Copy and complete the table for positive values of x so that $y \propto x^2$.

x	0		3	4	5	
y		12	27		75	192

Use the coordinates (5, 75) to find the constant of variation.

What is the equation connecting x and y?

2 Copy and complete the table for positive values of t so that $s \propto t^2$.

t	2	4		6	10
s		80	125	180	

What is the equation connecting s and t?

3 Copy and complete the table so that $y \propto x^2$.

x	−3	−1	0	2	4	7
y				16		196

What is the equation connecting x and y?

Example:

If y is directly proportional to the square of x, and $y = 3$ when $x = 1$, find

a y when x is 4 **b** x when y is $\frac{3}{4}$.

$$y \propto x^2$$

i.e. $y = kx^2$ where k is a constant

But $y = 3$ when $x = 1$

$\therefore$ $3 = k \times 1^2$

i.e. $k = 3$

so $y = 3x^2$

a If $x = 4$, then $y = 3 \times 4^2$

$= 3 \times 16$

$= 48$

b If $y = \frac{3}{4}$, then $\frac{3}{4} = 3x^2$

$x^2 = \frac{1}{4}$

$x = \pm\frac{1}{2}$

Dividing both sides by 3.

4 y is directly proportional to the square of x, and $y = 18$ when $x = 3$. Find

a y when $x = 4$ **b** x when $y = 2$

5 y varies as the square of x, and $y = 48$ when $x = 4$. Show that $y = 3x^2$ and find

a y when $x = \frac{1}{2}$ **b** x when $y = \frac{1}{3}$

6 $P \propto Q^2$ and $P = 12$ when $Q = 4$.
Find

a P when $Q = 12$

b the positive value of Q when $P = 48$

Example:

If y is directly proportional to the cube of x, and $y = 4$ when $x = 2$, find

a y when $x = 4$ **b** x when $y = \frac{1}{2}$.

$$y \propto x^3$$

i.e. $y = kx^3$

But $y = 4$ when $x = 2$

$\therefore$ $4 = k \times 2^3$

i.e. $8k = 4$

$\Rightarrow$ $k = \frac{1}{2}$

so $y = \frac{1}{2}x^3$

a If $x = 4$, $y = \frac{1}{2} \times 4^3$

$= 32$

b If $y = \frac{1}{2}$, $\frac{1}{2} = \frac{1}{2}x^3$

i.e. $x^3 = 1$

$\Rightarrow$ $x = 1$

7 Copy and complete the table so that $V \propto H^3$.

H	2		6	8	10
V	2	16	54		

What is the equation connecting V and H?

8 Copy and complete the table so that $y \propto x^3$.

x	3	6	9	12	15
y		72		576	

What is the equation connecting x and y?

9 $y \propto x^3$, and $y = 3$ when $x = 2$.
Find **a** y when $x = 4$ **b** x when $y = 81$.

10 If y varies directly as the cube of x, and $y = 64$ when $x = 2$,
find **a** y when $x = 3$ **b** x when $y = 8$.

11 W is proportional to the cube of H, and $W = 32$ when $H = 4$.
Find **a** W when $H = 6$ **b** H when $W = 4$.

12 Copy and complete the table so that $V \propto \sqrt{R}$.

R	0	1	4		25
V			8	12	

What is the equation connecting V and R?

13 Y varies directly as the square root of X, and $Y = 1$ when $X = 100$.
Find **a** Y when $X = 400$ **b** X when $Y = 3$.

14 Plot the graph of y against x for the following data.

x	1	4	9	16	25
y	1	2	3	4	5

Is the graph a straight line? If it is not, complete the following table and plot the graph of y against $\sqrt{x}$.

$\sqrt{x}$					
y	1	2	3	4	5

Is this graph a straight line? What is the equation connecting x and y?

15 Plot the graph of y against x for the following data.

x	1	2	3	4	5
y	0.5	4	13.5	32	62.5

Is the graph a straight line? If it is not, plot the graphs of y against x^2 and y against x^3. Use your results to find the equation connecting x and y.

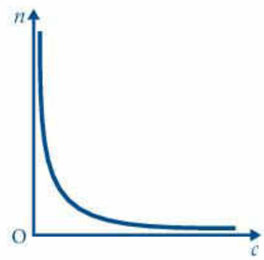
Inverse variation

When two quantities are inversely proportional their product remains constant.

Consider two varying quantities whose product is constant, for example the number of stamps of a particular value that Len can buy for 480c. At 10c each he can buy 48 while at 12c each he can buy 40. Some of the varying numbers of stamps he can buy are listed in the table.

Cost of each stamp, c cents	60	48	40	20	12	10
Number, n, of stamps Len can buy for 480c	8	10	12	24	40	48

The two quantities c and n are connected by the equation $n = \frac{480}{c}$
i.e. $nc = 480$.

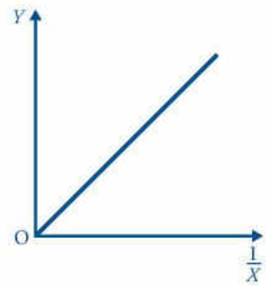


We say that n is inversely proportional to c or n varies inversely as c.

Plotting the data given in the table on a graph gives the curve shown opposite. The shape of this curve is characteristic for any two variables that are inversely proportional.

In general, if X and Y are inversely proportional,
then $Y = \frac{k}{X}$ where k is the constant of proportion.
Alternatively, we say that if Y varies inversely as X
then $Y = \frac{k}{X}$ where k is the constant of variation.

Note that if we compare the equation $Y = \frac{k}{X}$ with the equation $y = mx + c$ then plotting Y against $\frac{1}{X}$ will give a straight line passing through the origin. This means that we can test whether or not two quantities are inversely proportional by plotting one quantity against the reciprocal of the other. If the points lie on a straight line that passes through the origin then one quantity varies inversely as the other, and the gradient of the line gives the constant of variation.



EXERCISE 17d

In questions **1** to **3** complete the table given that the product of the varying quantities is constant. Write down the equation connecting these varying quantities.

1

Cost of a birthday card in cents, C	25	50	100	125
Number of cards that can be bought for \$5, N	20		5	

As $CN = k$, use the first pair of values to find k.

2

Number of similar magazines a boy could buy with his pocket money, N cents	12	9	8	6
Cost of one magazine in cents, C	60		90	

3

Pressure, P pascals	4	5	6	8	12
Volume, V m^3	30		20		10

In questions **4** and **5** write down the equations connecting x and y.

4

x	36	24	18	12	8
y	2	3	4	6	9

Start with $xy = k$.

5

x	0.8	0.9	1.2	1.8	2.7
y	2.7	2.4	1.8	1.2	0.8

Example:

x	1	2	3	4	6	12
y	12	6	4	3	2	1

a Write down the equation connecting x and y.

b Construct a table showing the values of $\frac{1}{x}$ and y.

c Plot the values of y against the corresponding values of $\frac{1}{x}$. Do these points lie on a straight line that passes through the origin? If they do, what does this confirm about the relationship between x and y? What is the gradient of this line? Interpret the gradient.

a The table shows that $xy = 12 \quad \Rightarrow \quad y = \frac{12}{x}$

b

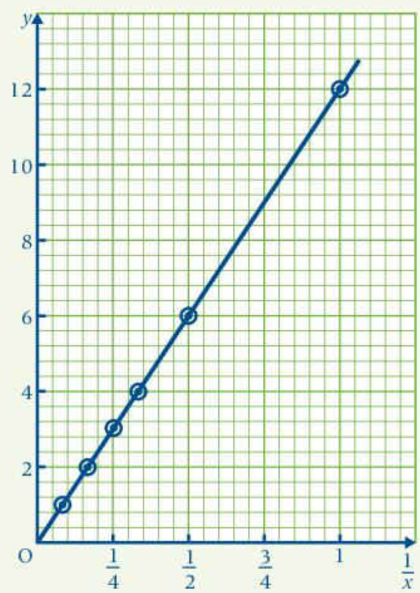
$\frac{1}{x}$	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{12}$
y	12	6	4	3	2	1

c The points lie on a straight line passing through the origin. This confirms that y varies inversely as x.

Gradient of line

$$= \frac{\text{increase in } y}{\text{increase in } \frac{1}{x}} = \frac{12}{1} = 12$$

The gradient of the line is the constant of variation.



In questions **6** and **7** repeat the worked example for the data given in the table.

Use one pair of values to find y in terms of x. Use another pair of values to check your answer.

6

x	10	5	1	0.5	0.25
y	0.1	0.2	1	2	4

7

x	2.4	1.8	1.2	0.9	0.8
y	3	4	6	8	9

INVESTIGATION

What happens to the sequence of areas of squares as we keep increasing the length of the side by 1 unit?

Copy and complete the following table of values, where L is the length of the side of the square and A is the area.

L	1	2	3	4	5	6	7	8	9	10
A	1	4			25					

a Now list the sequence of numbers formed by taking the differences between successive areas.

b What do you observe about the differences?

c Find an expression for the difference between the nth term and the $(n-1)$th term of the sequence in part **a**.

Equations for inverse variation

To summarise this work so far, if two quantities p and q are inversely proportional, we say that p varies inversely as q.

This relationship can be written as $p \propto \frac{1}{q}$, or as $p = \frac{k}{q}$ which implies that $pq = k$.

The idea can then be extended so that if two quantities x and y are such that y varies inversely as the square of x

then $\quad y \propto \frac{1}{x^2} \quad \Rightarrow \quad y = \frac{k}{x^2}$

Similarly, y varies inversely as $\sqrt{x} \quad \Rightarrow \quad y \propto \frac{1}{\sqrt{x}} \quad \Rightarrow \quad y = \frac{k}{\sqrt{x}}$.

EXERCISE **17e**

Example:

Copy and complete the table for positive values of x so that $y \propto \frac{1}{x^2}$

x	3	5		10	15
y	100	36	25		

If $y \propto \frac{1}{x^2}$ then $y = \frac{k}{x^2}$

But $y = 100$ when $x = 3$

$\therefore$ $100 = \frac{k}{9}$, i.e. $k = 900$ so $y = \frac{900}{x^2}$

Check: When $x = 5$, $y = \frac{900}{25} = 36$

If $x = 10$, $y = \frac{900}{100} = 9$ and if $x = 15$, $y = \frac{900}{225} = 4$

If $y = 25$, $25 = \frac{900}{x^2}$

i.e. $25x^2 = 900 \Rightarrow x^2 = \frac{900}{25} = 36$

i.e. $x = 6$

$\therefore$ the completed table is

x	3	5	6	10	15
y	100	36	25	9	4

1 Copy and complete the table so that $y \propto \frac{1}{x}$.

x	2	4	6	9	12	
y	18	9			3	2

What is the equation connecting x and y?

As $y \propto \frac{1}{x}$, $xy = k$.
There are two points of values for x and y. Use one pair to find k. Use the other pair to check your answer.

2 Copy and complete the table for positive values of x so that $y \propto \frac{1}{x^2}$.

x	0.5		2	3	6	10
y		36	9			

What is the equation connecting x and y?

3 Copy and complete the table so that $q \propto \frac{1}{\sqrt{p}}$.

p	0.25		4	9		25
q	120	60	30		15	12

What is the equation connecting p and q?

4 If y is inversely proportional to x, and $y = 8$ when $x = 5$, find
a y when x is 10 **b** x when y is 2 **c** y when x is -4.

5 $y \propto \frac{1}{\sqrt{x}}$ and $y = 2$ when $x = 4$.
Find
a y when $x = 9$ **b** x when $y = 1$.

6 If p is inversely proportional to v, and $p = 15$ when $v = 20$, find
a p when $v = 30$ **b** v when $p = 7.5$.

7 If P varies inversely as $Q + 2$, and $P = 5$ when $Q = 4$, find
a P when $Q = 3$ **b** Q when $P = 15$.

8 If y is inversely proportional to the square of x, and $y = 4$ when $x = 5$, find
a y when $x = 2$ **b** x when $y = 1$.

9 If y varies inversely as x, and $y = 6$ when $x = 8$, find
a y when $x = 12$ **b** x when $y = 4$.

10 If y varies inversely as the cube of x, and $y = 7$ when $x = 6$, find
a y when $x = 3$ **b** x when $y = 189$.

Example:

If y is the constant speed of a train and x is the time it takes to travel a fixed distance k, find the value of n if x and y are related by a law of the form $y \propto x^n$.

Since distance travelled = constant speed × time

$$k = y \times x$$

i.e. $$xy = k$$

and $$y = \frac{k}{x}$$

or $$y = kx^{-1}$$

$\therefore$ $y \propto x^n$ where $n = -1$

11 In each of the following cases, x and y are related by a law of the form $y \propto x^n$.
Find the value of n.
a y is the area of a square and x is the length of one side.
b y is the area of a circle and x is its radius.
c y is the length of a rectangle of constant area and x is its breadth.
d y is the radius of a circle and x is its area.
e y is the length of a line in centimetres and x is its length in millimetres.
f y is the speed and x is the time taken to travel round one lap of a racing track.

EXERCISE 17f

The questions in this exercise are a mixture of direct and inverse variation.

1 p varies directly as the square of q, and $p = 9$ when $q = 6$.
Find p when q is
a 2 **b** -2 **c** 5.

2 A is directly proportional to L, and $A = 28$ when $L = 4$.
Find
a A when $L = 3$ **b** L when $A = 42$.

3 $y \propto x^3$ and $y = 48$ when $x = 4$.
Find
a the formula for y in terms of x **b** y when $x = 2$
c x when $y = 6$.

4 y varies inversely as x, and $y = 7$ when $x = 6$.
Find
a y when $x = 3$ **b** x when $y = 14$.

5 y is inversely proportional to x^2, and $y = 4.5$ when $x = 4$.
Find
a y when $x = 3$ **b** x when $y = 8$.

6 R is directly proportional to the positive square root of S, and $R = 4$ when $S = 64$.

a Calculate R when S is **i** 16 **ii** 6.25

b Calculate S when R is **i** 2 **ii** 3.5

7 Copy and complete the table so that $y \propto x^2$.

x	0	1		4	8
y		0.25	1		16

8 Copy and complete the table so that $t \propto \sqrt{s}$.

s		4	9		
t	0	0.5		1	2

9 Given that $y \propto \sqrt{x}$

a what is the effect on y if

i x is multiplied by 4 **ii** x is divided by 4?

b what is the effect on x if

i y is multiplied by 3 **ii** y is divided by 3?

10 Given that y varies inversely as x^2

a what is the effect on y if

i x is multiplied by 2 **ii** x is divided by 2?

b what is the effect on x if

i y is multiplied by 9 **ii** y is divided by 9?

11 Given that y varies as x^n, write down the value of n in each of the following cases.

a y is the area of a square of side x

b y is the volume of a cube of edge x

c y is the volume of a cylinder with constant base area A and height x

d y and x are the sides of a rectangle with a given area.

Example:

A stone falls from rest down a mine shaft. It falls D metres in T seconds where D varies as the square of T. If it falls 20 m in the first 2 seconds and takes 5 seconds to reach the bottom, how deep is the shaft?

$$D \propto T^2$$

i.e. $$D = kT^2$$

But $D = 20$ when $T = 2$

$\therefore$ $$20 = k \times 2^2$$

i.e. $$4k = 20$$

$$k = 5$$

$\therefore$ $$D = 5T^2$$

If $T = 5$, $D = 5 \times 5^2$

$= 125$

Therefore the shaft is 125 metres deep.

12 The mass, M kg, of a circular disc of constant thickness varies as the square of its radius, R cm. If a disc of radius 5 cm has a mass of 1 kilogram find

a the mass of a disc of radius 10 cm

b the radius of a disc of mass 25 kg.

13 The safe speed, V km/h, at which a car can round a bend of radius R metres varies as $\sqrt{R}$. If the safe speed on a curve of radius 25 m is 40 km/h, find the radius of the curve for which the safe speed is 64 km/h.

14 The time of swing, T seconds, of a simple pendulum is directly proportional to the square root of its length, L cm.
If $T = 2$ when $L = 100$ find

a T when $L = 64$ **b** L when $T = 1\frac{1}{2}$.

15 The extension, x cm, of an elastic string varies as the force, F newtons, used to extend it. If a force of 4 newtons gives an extension of 10 cm find

a the extension given by a force of 10 newtons

b the force required to give an extension of 12 cm.

16 The cost of buying a rectangular carpet is directly proportional to the square of its longer side. If a carpet whose longer side is 3 m costs $180 find

a the cost of a carpet with a longer side of 4 m

b the length of the longer side of a carpet costing $405.

17 The radius of a circle, r cm, varies as the square root of its area, A cm^2. How does the radius change if the area is increased by

a a factor of 4 **b** a factor of 25 **c** 44%?

18 Mathematically similar jugs have capacities that vary as the cubes of their heights. If a jug 10 cm high holds $\frac{1}{8}$ litre find

a the capacity of a jug that is 12 cm high

b the height of a jug that will hold 1 litre.

19 For a given mass of gas at a given temperature the pressure p varies inversely as the volume, v. If $p = 100$ when $v = 2.4$ find

a v when $p = 80$ **b** p when $v = 2$.

20 For a vehicle travelling between two gas stations the time taken is inversely proportional to its speed. If it takes $2\frac{1}{2}$ hours when its speed is 48 mph find

a its average speed if it takes 3 hours

b by how much its average speed must increase if the journey time is to be reduced to 2 hours.

21 The number, n, of plastic squares of edge x cm, and with a fixed thickness, that can be made from a given volume of plastic can be found using the formula $n = \frac{k}{x^2}$ where k is a constant.

a Given that 1000 squares of side 2 cm can be made from a given volume of the material, calculate how many squares can be made from an equal volume of material if the edge of the square is

i 4 cm **ii** 10 cm **iii** 3 cm.

b Rearrange $n = \frac{k}{x^2}$ to make x the subject.

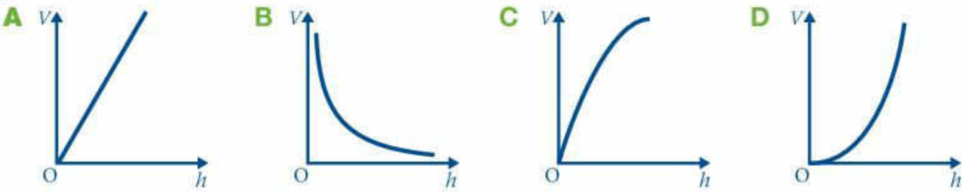
c 800 squares are to be made using the same volume of plastic. Calculate, correct to two decimal places, the length of an edge of the largest square. State, giving reasons, whether you have rounded your answer up or down.

22 The energy generated by a solar panel varies directly as the area of the panel. At midday on a Saturday in December a rectangular panel measuring 1.2 m by 0.8 m produced 0.84 units of energy.

a How many units of energy would be produced by a solar panel measuring **i** 1.8 m by 0.8 m **ii** 2 m by 1.4 m?

b The solar panels supplied by one manufacturer are all 1 m wide but can be made to any length. What length of panel is needed if the amount of energy produced is **i** 1.4 units **ii** 1.05 units?

23 The volume of perfume, V cm^3, contained in a perfume bottle varies as the cube of the height of the bottle, h mm. The graph representing the way that V varies with h for a collection of these perfume bottles could be



24 By what factor does y change when x is doubled if

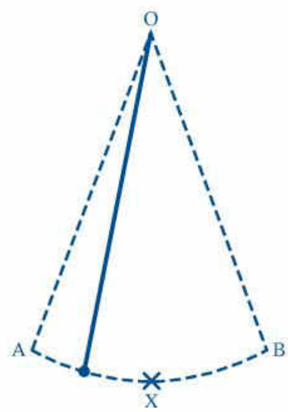
a $y \propto x$ **b** $y \propto \frac{1}{x}$ **c** $y \propto x^2$ **d** $y \propto x^3$?

25 State the percentage change in y when x is increased by 25% if

a $y \propto x$ **b** $y \propto \frac{1}{x}$ **c** $y \propto x^2$.

PRACTICAL WORK

This is an experiment to find out how the time of swing of a simple pendulum is related to the length of the pendulum.



You will need

a length of fine string about 1 metre long

a small heavy mass to fix to one end of the string

a means of suspending the other end of the string from a fixed point (A ruler and some Blu Tack should help you to do this.)

a stopwatch which measures time in seconds.

Attach the pendulum to a fixed point and measure its length, L centimetres.
Start the pendulum swinging with a small angular displacement and measure the time it takes to pass a fixed point X marked behind the pendulum, say 50 times. From this you can calculate the time it takes to swing from one extreme position A to the other extreme position B and back again. Suppose it takes T seconds.

Repeat the experiment for different lengths of string and collect your results in the following table.

Length of pendulum, L cm						
Time of swing, T seconds						

Draw four graphs by plotting

a T against L **b** T against $\frac{1}{L}$ **c** T against L^2 **d** T against $\sqrt{L}$

Use your graphs to decide, with reasons, which of the following statements is true.

The time of swing of a simple pendulum varies directly as its length.

The time of swing of a simple pendulum varies inversely as its length.

The time of swing of a simple pendulum varies directly as the square of its length.

The time of swing of a simple pendulum varies directly as the square root of its length.

Repeat using different masses. Does it make any difference to the time of swing? Use an appropriate graph to find the time of swing for a given length or the length for a given time of swing.

Can you tell why the angular displacement needs to be small?

MIXED EXERCISE 17

Several answers are given for these questions.
Write down the letter that corresponds to the correct answer.

1 Given that y varies inversely as the square of z, the relationship between y and z can be described as

A $y \propto z^2$ **B** $y \propto \sqrt{z}$ **C** $y \propto \frac{1}{z^2}$ **D** $y \propto \frac{1}{\sqrt{z}}$

2 Given that $x \propto y$, and $y = 1$ when $x = 2$, then

A $2y = x$ **B** $y = 2x$ **C** $y = x$ **D** $y = 3x$

3 If y is inversely proportional to x^2, and $y = 5$ when $x = 2$, then when $x = 5$, $y =$

A $\frac{4}{5}$ **B** $\frac{5}{4}$ **C** 2 **D** 4

4 The table shows some related values of x and y. Which of the following statements are true?

x	2	3	4	5
y	6	9	12	15

1 y varies as x 2 $xy = k$ 3 $y = 3x$

A 1 and 2 **B** 1 only **C** 1 and 3 **D** 3 only

5 Given that $y \propto \sqrt{x}$, when x is multiplied by 9, y is multiplied by

A 1 **B** 3 **C** 9 **D** 81

6 A circle with a diameter d units has an area of A square units. The relationship between A and d can be written

A $A \propto \frac{1}{d^2}$ **B** $A \propto \frac{1}{d}$ **C** $A \propto d$ **D** $A \propto d^2$

7 Given that y varies as x times the square root of z,

A $y \propto x\sqrt{z}$ **B** $y \propto \frac{x}{\sqrt{z}}$ **C** $y \propto \sqrt{xz}$ **D** $y \propto z\sqrt{x}$

MATHS IS OUT THERE

It was known by the early Egyptians that the true length of the solar year is $365\frac{1}{4}$ days. However, they retained the Civil Year of 365 days.

This meant that their Civil Year slipped steadily backwards through the solar year in the following way:

1 day every 4 years
30 days every 120 years.
12 months or 1 year every 1440 years.

Clearly this means that their seasons were slowly changing.

Fortunately, this was not noticeable in a man's lifetime. So why worry!

IN THIS CHAPTER YOU HAVE SEEN THAT...

- when two variables, x and y, are directly proportional, then $y = kx$, where k is the constant of proportion. We say that y varies as x
- when two variables, x and y, are inversely proportional, then $yx = k$, where k is the constant of proportion. We say that y varies inversely as x
- when y varies as a function of x, $y = k\mathrm{f}(x)$. For example, when y varies as the square of x, then $y = kx^2$.

18 VECTORS IN GEOMETRY

AT THE END OF THIS CHAPTER YOU SHOULD BE ABLE TO...

1 Define a vector.
2 Represent a vector by a directed line segment.
3 Denote a vector with initial point A and final point B by **AB**.
4 Differentiate between scalar and vector quantities.
5 Determine when two vectors are equal.
6 Find the sum or difference of two vectors using the triangle law.
7 Use vectors to model practical situations.
8 Calculate the length and direction of a given vector.
9 Multiply a vector by a scalar.
10 Identify opposite vectors.
11 Determine when two vectors are collinear.

BEFORE YOU START

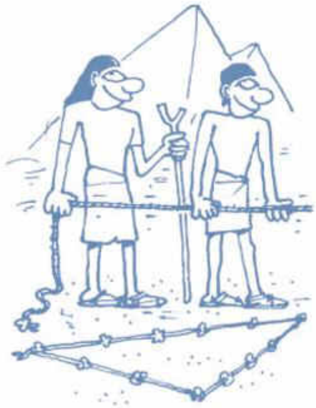
you need to know:
✓ how to draw a line segment
✓ how to find the length of a line segment by measuring
✓ how to use Pythagoras' Theorem
✓ how to represent a line segment on a grid.

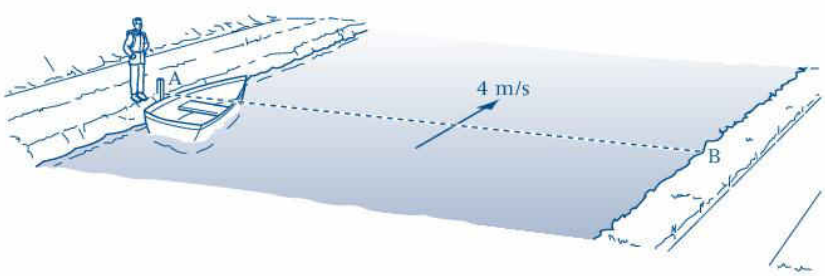
KEY WORDS

collinear, component, displacement, magnitude, position vector, resultant, scalar, triangle law, vector, vector sum

MATHS IS OUT THERE

Did you know that, in surveying their land, the ancient Egyptians measured right-angles by means of a length of knotted rope held in the form of a right-angled triangle. Read in your library or surf the net to find out how the men called 'rope stretchers' did their surveying.





A traveller hires a boat to cross a river from a point A on one bank to, hopefully, a point B on the opposite bank. The river is 100 m wide; the boat can travel in still water at 8 m/s and the current in the river is flowing at 4 m/s.

The traveller is anxious to know whether it is possible to cross the river directly from A to B, the point immediately opposite A. He would like to cross in the shortest possible time. (He is not a very good rower!)

Before he hires the boat he needs to know

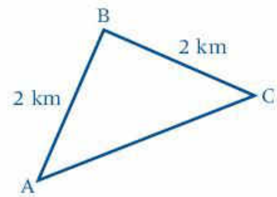
- whether or not the shortest distance necessarily takes the shortest time
- the direction in which he must point the boat if he is to cross directly to B
- the effect of the river's current combining with the velocity of the boat
- whether steering the boat directly across the river is the quickest way of crossing the river
- how long the different possible journeys will take.

To answer these questions the traveller needs to know how to combine quantities that are in different directions, that is he needs to know more about vectors.

The mathematics of vectors was first developed to simplify working with physical quantities such as force and velocity, where both magnitude and direction are of importance.

Definition of a vector

Although we usually assume that two and two make four, this is not always the case.



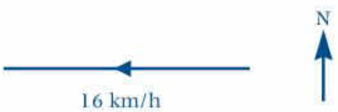
Amberley (A), Beckford (B) and Croxton (C) are three villages. It is 2 km from Amberley to Beckford and 2 km from Beckford to Croxton, but it is easy to see from the diagram that Croxton is not 4 km from Amberley.

A to B, B to C and A to C are **displacements**. Each of them has **magnitude** (that is, size), and each of them has a definite direction in space. They are all examples of a **vector**.

> A vector is a quantity that has both magnitude and a specific direction in space.

A vector can be represented by a directed line segment, i.e. a line of given length in a given direction.

We can represent 16 km/h west by a line.

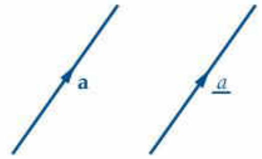


Velocity (e.g. 16 km/h west) is a vector but mass (e.g. 4 kg) is not.

The length of the line represents the magnitude of the vector. The direction of the line represents the direction of the vector; notice that an arrow is needed on the line.

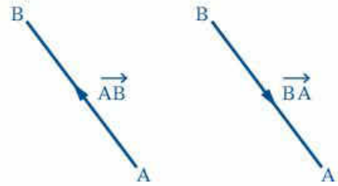
A **scalar** quantity is one that is fully defined by magnitude alone. For example, length is a scalar quantity since the length of a piece of string does not depend on its direction when it is being measured. Temperature is another quantity that does not have direction, so temperature also is a scalar.

Representation of a vector



We now know that a vector can be represented by a line segment, whose length represents the magnitude of the vector and whose direction, indicated by an arrow, represents the direction of the vector. In general, such vectors can be denoted by a letter in bold type, e.g. **a** or **b**.

It is difficult to hand-write letters in bold type so we indicate that a letter represents a vector by putting a line underneath it. Hence a vector shown in a book as **a** will be handwritten as $\underline{a}$.



Alternatively we can represent a vector by the magnitude and direction of a line joining A to B. We denote the vector from A to B by $\overrightarrow{AB}$ or **AB** (but by $\overrightarrow{AB}$ when it is handwritten). The vector in the opposite direction, i.e. from B to A, is written $\overrightarrow{BA}$ or **BA**.

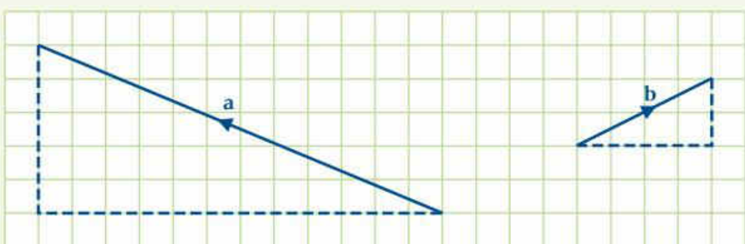
The magnitude of a vector

The magnitude of a vector **a**, which is written as |**a**| or *a*, is equal to the length of the line segment representing **a**.

EXERCISE 18a

Example:

Find the length of **a** and **b**



In the diagram, each square is of side 1 unit.

$|\mathbf{a}| = \sqrt{5^2 + 12^2}$ (Pythagoras' theorem)
$= \sqrt{25 + 144}$
$= \sqrt{169}$
$= 13$ units

$|\mathbf{b}| = \sqrt{4^2 + 2^2}$
$= \sqrt{20}$
$= 4.47$ units (3 s.f.)

1 Which of the following quantities are vectors?
- **a** The length of a piece of wire.
- **b** The force needed to move a lift up its shaft.
- **c** A move from the door to your chair.
- **d** The speed of a galloping horse.
- **e** The distance between Kingston and Montego Bay.

2 Represent each of the following vector quantities by a suitable directed line.

a A force of 6 newtons acting vertically downwards.

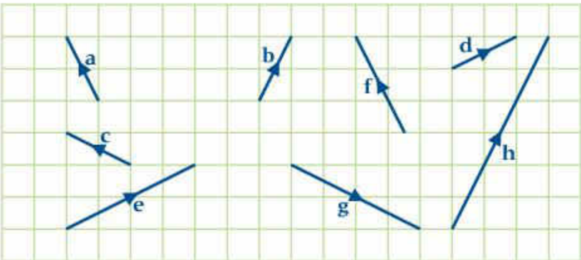
b A velocity of 3 m/s on a bearing of 035°.

c An acceleration of 2 m/s^2 north-west.

d A displacement of 7 km due south.

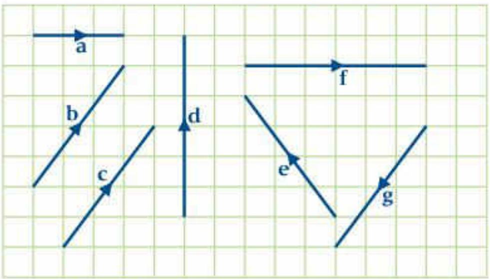
In questions **3** and **4** the vectors are drawn on a unit grid.

3



Find the magnitude of each of the vectors given in the diagram.

4



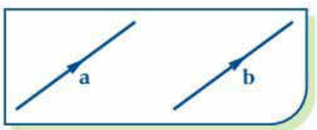
a–g Find the magnitude of each of the vectors given in the diagram.

h Which two vectors are equal in magnitude and in the same direction?

i Which two vectors are equal in magnitude but opposite in direction?

Equal vectors

Two vectors which have the *same magnitude* and are in the *same direction* are equal, i.e. **a** = **b** implies that |**a**| = |**b**| *and* that the directions of **a** and **b** are the same.

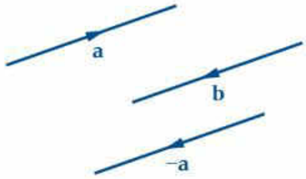


It follows that a vector can be represented by *any* line segment of the right length and direction, irrespective of its position, so each of the lines in the diagram below represents the vector **c**.

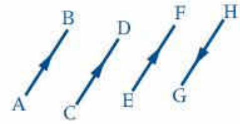
Negative vectors

If two vectors **a** and **b**, have the *same magnitude* but are in *opposite directions* we say that $\mathbf{b} = -\mathbf{a}$, i.e. $-\mathbf{a}$ is a vector of the magnitude of **a** and in the direction opposite to that of **a**.

We say that **a** and **b** are equal and opposite.

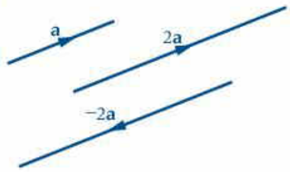


It follows that
$\overrightarrow{AB} = \overrightarrow{CD} = \overrightarrow{EF}$
and $\overrightarrow{AB} = -\overrightarrow{HG}$



Multiplication of a vector by a scalar

If k is a positive number then $k\mathbf{a}$ is a vector in the same direction as **a** and of magnitude $k|\mathbf{a}|$.

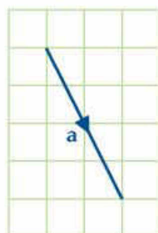


In the diagram, for example, the line representing 2**a** is twice the length of the line representing **a** and parallel to it.
It follows that $-k\mathbf{a}$ is a vector in the opposite direction with magnitude $k|\mathbf{a}|$.

EXERCISE 18b

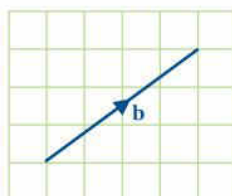
1 Copy the vector **a** onto squared paper, and on the same grid draw line segments to represent the vectors

a $2\mathbf{a}$ b $\frac{1}{2}\mathbf{a}$ c $\frac{3}{2}\mathbf{a}$
d $4\mathbf{a}$ e $-3\mathbf{a}$ f $-\mathbf{a}$



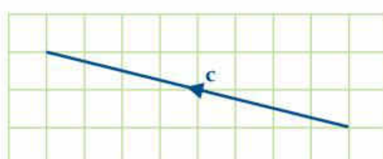
2 Copy the vector **b** onto squared paper, and on the same grid draw line segments to represent the vectors

a $-\mathbf{b}$ b $3\mathbf{b}$ c $-\frac{1}{2}\mathbf{b}$
d $\frac{5}{2}\mathbf{b}$ e $-2\mathbf{b}$ f $4\mathbf{b}$



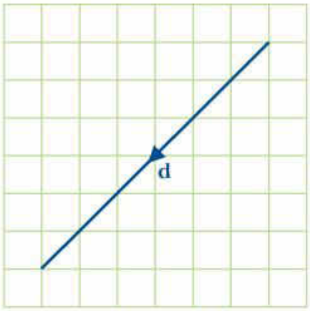
3 Copy the vector **c** onto squared paper, and on the same grid draw line segments to represent the vectors

a $3\mathbf{c}$ b $-2\mathbf{c}$ c $-\frac{3}{2}\mathbf{c}$ d $\frac{3}{4}\mathbf{c}$

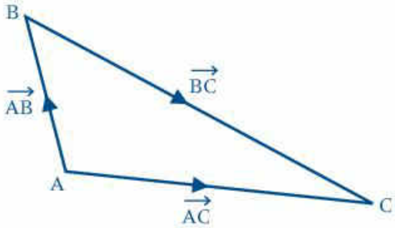


4 Copy the vector **d** onto squared paper, and on the same grid draw line segments to represent the vectors

a $-\frac{1}{2}\mathbf{d}$ **b** $-\frac{5}{2}\mathbf{d}$ **c** $-3\mathbf{d}$ **d** $\frac{2}{3}\mathbf{d}$



Equivalent displacements



The displacement from A to B, followed by the displacement from B to C, is equivalent to the displacement from A to C.

This is written as the vector equation

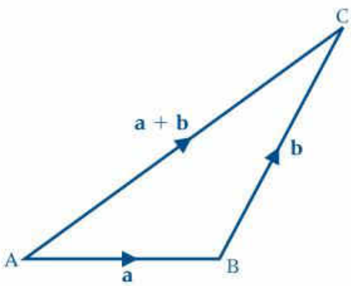
$$\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$$

All vector quantities can be combined in this way.

Note that, in vector equations like this one, + means 'together with' and = means 'is equivalent to'

Addition of vectors

We can see in the diagram above that, in a triangle ABC, the displacement from A to B followed by the displacement from B to C, is equivalent to the displacement from A to C.



If the sides $\overrightarrow{AB}$ and $\overrightarrow{BC}$ of a triangle ABC represent the vectors **a** and **b** then the third side $\overrightarrow{AC}$ is equivalent to **a** followed by **b**. We say that $\overrightarrow{AC}$ represents the **vector sum**, or **resultant**, of **a** and **b**.

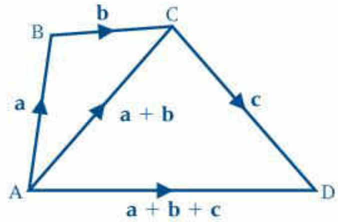
We write $\overrightarrow{AC} = \mathbf{a} + \mathbf{b}$.

This is known as the **triangle law** for the addition of vectors and can be extended to the addition of more than two vectors.

Note that **a** and **b** follow each other round this triangle (they go anticlockwise) whereas the resultant **a** + **b** goes the opposite way.

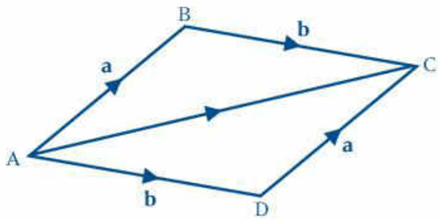
In the diagram below

$\overrightarrow{AB} + \overrightarrow{BC} = \mathbf{a} + \mathbf{b} = \overrightarrow{AC}$ and $\overrightarrow{AC} + \overrightarrow{CD} = (\mathbf{a} + \mathbf{b}) - \mathbf{c} = \overrightarrow{AD}$



Note again that **a**, **b** and **c** go one way round the quadrilateral (clockwise) while the resultant, **a** + **b** + **c**, goes in the opposite sense.

In the parallelogram ABCD, if $\overrightarrow{AB}$ represents the vector **a** then so does $\overrightarrow{DC}$ (the opposite sides of a parallelogram are parallel and equal in length).



Likewise, if $\overrightarrow{BC}$ represents the vector **b**, so does $\overrightarrow{AD}$.

Using the triangle law for addition in triangle ABC

$$\mathbf{a} + \mathbf{b} = \overrightarrow{AC}$$

and using the law again in △ADC we have

$$\mathbf{b} + \mathbf{a} = \overrightarrow{AC}$$

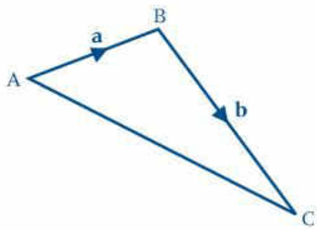
Hence $\quad \mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$

i.e. the order in which we add two vectors does not matter; so vector addition is commutative.

If the vector **c** is represented by the line $\overrightarrow{AC}$ then $\mathbf{c} = \mathbf{a} + \mathbf{b}$ and $\mathbf{c} = \mathbf{b} + \mathbf{a}$.

Subtraction of vectors

This diagram shows how to find the sum of two vectors **a** and **b**.



The vector $-\mathbf{b}$ can be shown on a diagram by changing the direction of the arrow on BC, but we cannot use this diagram to find the sum of **a** and $-\mathbf{b}$ since the arrows on the lines representing the two vectors are not in the same sense.

We can overcome this problem by moving the line representing $-\mathbf{b}$ to a new position.

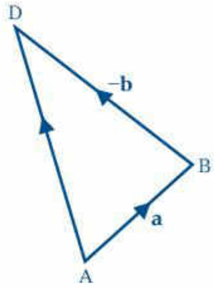
> Remember that a vector is not changed if the line representing it is moved to a different position, provided that its magnitude and direction are unchanged.

This new position is shown opposite.

Then the resultant of **a** and −**b** is represented by the line $\overrightarrow{AD}$

so $\quad \overrightarrow{AD} = \mathbf{a} + (-\mathbf{b})$

i.e. $\quad \overrightarrow{AD} = \mathbf{a} - \mathbf{b}$

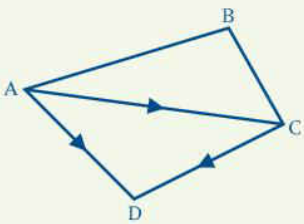


The line representing −**b** has moved to a new position so that the end of −**b** originally marked C now coincides with the end of **a** marked B, In its new position the other end of −**b** is marked D.

EXERCISE **18c**

Example:

Find the single vector equivalent to $\overrightarrow{AC} + \overrightarrow{CD}$.



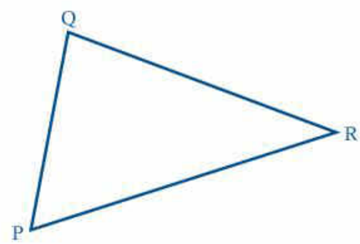
$\overrightarrow{AC} + \overrightarrow{CD} = \overrightarrow{AD}$

Marking $\overrightarrow{AC}$ and $\overrightarrow{CD}$ on the diagram we see that the displacement equivalent to $\overrightarrow{AC}$ followed by $\overrightarrow{CD}$, is $\overrightarrow{AD}$.

1 Find the single vector that is equivalent to

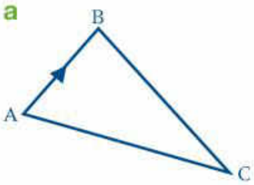
a $\overrightarrow{PQ} + \overrightarrow{QR}$ **b** $\overrightarrow{PR} + \overrightarrow{RQ}$

c $\overrightarrow{RQ} + \overrightarrow{QP}$ **d** $\overrightarrow{QP} + \overrightarrow{PR}$

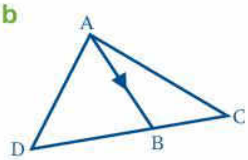


2 Give an alternative route for $\overrightarrow{AB}$.

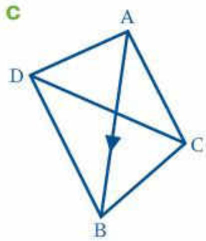
a



b



c

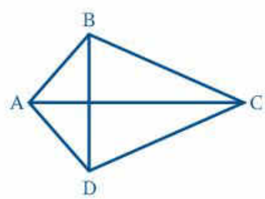


3 Find the single vector that is equivalent to

a $\overrightarrow{AB} + \overrightarrow{BC}$ **b** $\overrightarrow{BC} + \overrightarrow{CD}$

c $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$ **d** $\overrightarrow{AB} + \overrightarrow{BD}$

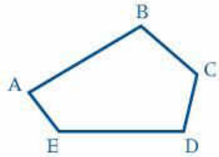
e $\overrightarrow{DA} + \overrightarrow{AC}$ **f** $\overrightarrow{AB} + \overrightarrow{BD} + \overrightarrow{DC}$



4 Find the single vector that is equivalent to

a $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CD}$ **b** $\overrightarrow{BC} + \overrightarrow{CD} + \overrightarrow{DA}$

c $\overrightarrow{AE} + \overrightarrow{EC} + \overrightarrow{CD}$ **d** $\overrightarrow{DA} + \overrightarrow{AB} + \overrightarrow{BC}$

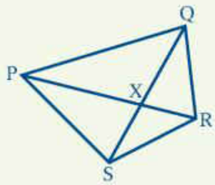


Example:

Which single vector is equivalent to

a $\overrightarrow{PS} - \overrightarrow{PQ}$

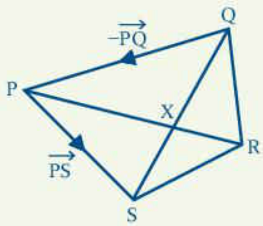
b $\overrightarrow{QR} - \overrightarrow{QX}$?



a $(-\overrightarrow{PQ}) + \overrightarrow{PS} = \overrightarrow{QS}$

or $\overrightarrow{PS} + (-\overrightarrow{PQ}) = \overrightarrow{QS}$

so $\overrightarrow{PS} - \overrightarrow{PQ} = \overrightarrow{QS}$

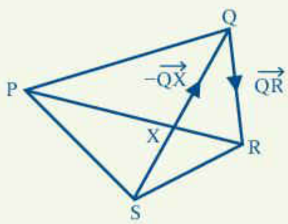


Marking $\overrightarrow{PS}$ and $-\overrightarrow{PQ}$ on the diagram we see that, as the arrows follow round, $(-\overrightarrow{PQ}) + \overrightarrow{PS}$ is equivalent to the third side of △PQS in the direction Q to S.

b *Similarly, marking* $-\overrightarrow{QX}$ *and* $\overrightarrow{QR}$ *on the diagram we see that*

$$-\overrightarrow{QX} + \overrightarrow{QR} = \overrightarrow{XR}$$

i.e. $$\overrightarrow{QR} - \overrightarrow{QX} = \overrightarrow{XR}$$

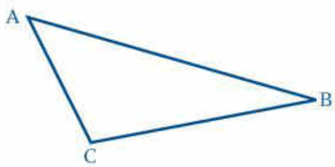


5 Which single vector is equivalent to

a $\overrightarrow{AC} - \overrightarrow{BC}$

b $\overrightarrow{BA} - \overrightarrow{CA}$

c $\overrightarrow{BC} - \overrightarrow{BA}$?



Remember that $-\overrightarrow{BC}$ is in the direction C to B.

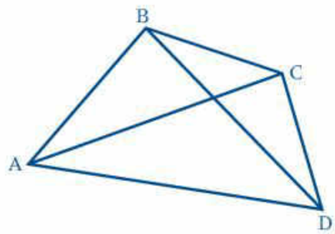
6 Which single vector is equivalent to

a $\overrightarrow{AC} - \overrightarrow{AD}$

b $\overrightarrow{DC} - \overrightarrow{BC}$

c $\overrightarrow{AD} + \overrightarrow{DC} - \overrightarrow{BC}$

d $\overrightarrow{AB} - \overrightarrow{DC} - \overrightarrow{CB}$?

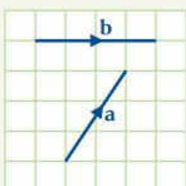


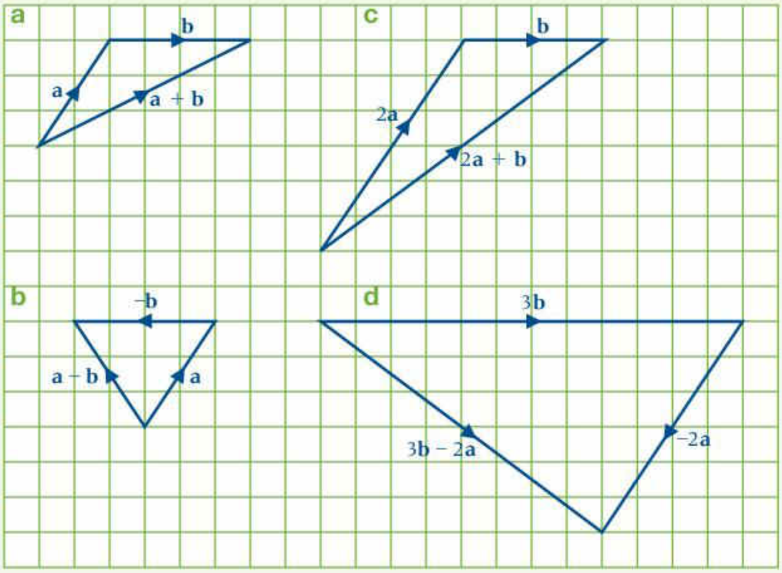
Example:

Copy the vectors **a** and **b** onto a grid. On your grid draw line segments to represent the vectors

a **a** + **b** **b** 2**a** + **b**

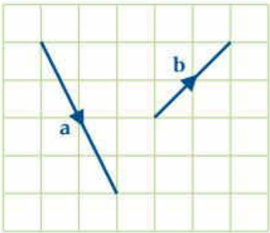
c **a** − **b** **d** 3**b** − 2**a**





7 Copy the vectors **a** and **b** onto a grid. On your grid draw line segments to represent the vectors

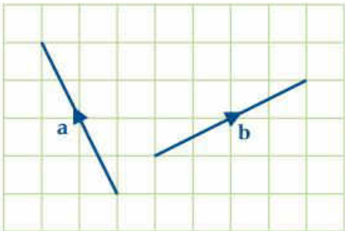
a **a** + **b** b 2**a** + **b** c **a** − **b** d **a** − $\frac{1}{2}$**b**



Make sure that the vectors you are combining follow each other in the same sense, i.e. clockwise or anticlockwise.

8 Copy the vectors **a** and **b** onto a grid. On your grid draw line segments to represent the vectors

a **a** + **b** b **a** − 2**b** c 2**a** + 3**b**
d **b** − 2**a** e $\frac{5}{2}$**b** − $\frac{3}{2}$**a**



PUZZLE

100 can be composed using each of the digits 0 to 9, once only.

Example: 0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + (8 × 9) = 100

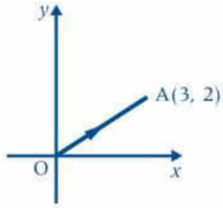
or $78\frac{3}{6} + 21\frac{45}{90} = 100$

Can you find others?

Position vectors

We can use coordinates to locate a point on a set of *xy* axes.

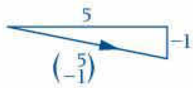
We can also use the vector from the origin to the point.



In the diagram, A is the point (3, 2).

The vector $\overrightarrow{OA}$ also gives the position of A and it is called the **position vector** of A and can be denoted by the column vector $\begin{pmatrix}3\\2\end{pmatrix}$.

The magnitude of a column vector is equal to the length of the line it represents.



In the diagram, the magnitude of $\begin{pmatrix}5\\-1\end{pmatrix}$ is $\sqrt{5^2+1^2}=\sqrt{26}$ (Pythagoras' theorem).

Two column vectors are parallel if one is a scalar multiple of the other.

For example, $\begin{pmatrix}2\\3\end{pmatrix}$ and $\begin{pmatrix}-4\\-6\end{pmatrix}$ are parallel since $\begin{pmatrix}-4\\-6\end{pmatrix}=-2\begin{pmatrix}2\\3\end{pmatrix}$.

EXERCISE 18d

Example:

The vertices of a triangle ABC are (3, 2), (1, 4) and (0, 1) respectively.

a Express in the form $\begin{pmatrix}a\\b\end{pmatrix}$ **i** $\overrightarrow{OA}$ **ii** $\overrightarrow{CA}$

b Using a vector method, show that triangle ABC is isosceles.

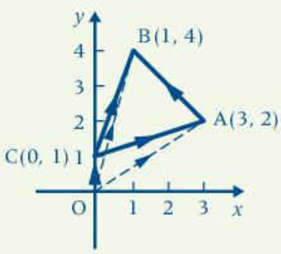
Start by drawing a diagram.

a i $\overrightarrow{OA}=\begin{pmatrix}3\\2\end{pmatrix}$

ii $\overrightarrow{CA}=\overrightarrow{OA}-\overrightarrow{OC}$

From the diagram, $\overrightarrow{OC}=\begin{pmatrix}0\\1\end{pmatrix}$

$\therefore\ \overrightarrow{CA}=\begin{pmatrix}3\\2\end{pmatrix}-\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}3\\1\end{pmatrix}$



b $|\overrightarrow{CA}|=\left|\begin{pmatrix}3\\1\end{pmatrix}\right|=\sqrt{9+1}=\sqrt{10}$

$\overrightarrow{CB}=\overrightarrow{OB}-\overrightarrow{OC}=\begin{pmatrix}1\\4\end{pmatrix}-\begin{pmatrix}0\\1\end{pmatrix}=\begin{pmatrix}1\\3\end{pmatrix}$

$|\overrightarrow{CB}|=\left|\begin{pmatrix}1\\3\end{pmatrix}\right|=\sqrt{1+9}=\sqrt{10}$

$\therefore$ CA = CB, so triangle ABC is isosceles.

Column vectors are 2 × 1 matrices so they can be added and subtracted in the same ways as matrices.

1 The vertices of a quadrilateral ABCD are (0, 1), (6, 1), (4, 4) and (2, 4) respectively.

a Express in the form $\begin{pmatrix} a \\ b \end{pmatrix}$, i $\overrightarrow{OA}$ ii $\overrightarrow{OB}$ iii $\overrightarrow{OC}$ iv $\overrightarrow{OD}$ v $\overrightarrow{AC}$ vi $\overrightarrow{BD}$

b Hence show that the diagonals of this quadrilateral are equal in length,

2 The vertices of a quadrilateral ABCD are (4, 0), (4, 2), (1, 4) and (−2, 4) respectively.

a Express in the form $\begin{pmatrix} x \\ y \end{pmatrix}$, i $\overrightarrow{OB}$ ii $\overrightarrow{OC}$

b Use vector methods to prove two geometric facts about AD and BC.

3 The vertices of a quadrilateral ABCD are (2, 1), (3, 2), (−1, 4) and (−2, 3) respectively.

a Express in the form $\begin{pmatrix} x \\ y \end{pmatrix}$, i $\overrightarrow{AB}$ ii $\overrightarrow{DC}$

b Use vector methods to prove that ABCD is a parallelogram.

4 The vertices of a triangle PQR are (0, −4), (4, 0) and (1, 3) respectively.

a Express in the form $\begin{pmatrix} a \\ b \end{pmatrix}$, i $\overrightarrow{PQ}$ ii $\overrightarrow{QR}$ iii $\overrightarrow{PR}$

b Hence show that triangle PQR has a right-angle at Q.

5 The vertices of a quadrilateral ABCD are (−2, 4), (1, 3), (−1, 1) and (−4, 2) respectively. The diagonals intersect at E.

a Use vector methods to show that DE = EB and that AE = EC.

b State which special quadrilateral ABCD is.

6 The vertices of a triangle ABC are (0, 1), (5, 2) and (2, 4) respectively. E is the point (2.5, 1.5).

a Express in the form $\begin{pmatrix} x \\ y \end{pmatrix}$, i $\overrightarrow{AE}$ ii $\overrightarrow{BE}$

b Hence show that E is the midpoint of AB.

Vectors and geometry

Two coplanar vectors (vectors in the same plane) are said to be **collinear** (lie in a straight line) if and only if one is a scalar multiple of the other and they have a point in common.

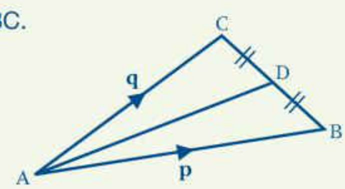
EXERCISE **18e**

Example:

$\overrightarrow{AB} = \mathbf{p}$, $\overrightarrow{AC} = \mathbf{q}$ and D is the mid-point of BC.

Give $\overrightarrow{BC}$ and $\overrightarrow{AD}$ in terms of **p** and **q**.

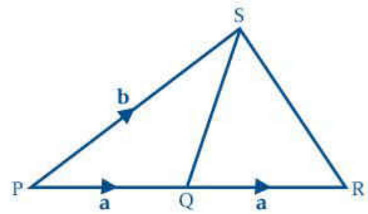
$$\overrightarrow{BC} = \overrightarrow{BA} + \overrightarrow{AC}$$
$$= -\mathbf{p} + \mathbf{q}$$
$$= \mathbf{q} - \mathbf{p}$$



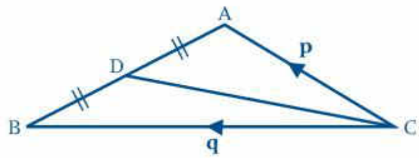
Give an alternative route from B to C first.

$$\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BD}$$
$$= \overrightarrow{AB} + \tfrac{1}{2}\overrightarrow{BC}$$
$$= \mathbf{p} + \tfrac{1}{2}(\mathbf{q} - \mathbf{p})$$
$$= \tfrac{1}{2}\mathbf{p} + \tfrac{1}{2}\mathbf{q}$$
$$= \tfrac{1}{2}(\mathbf{p} + \mathbf{q})$$

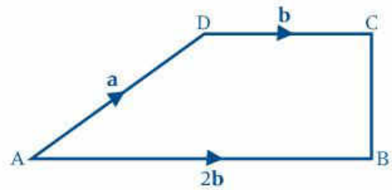
1 **a** Is PQR a straight line? Justify your answer.
b Give, in terms of **a** and **b**,
i $\overrightarrow{QS}$ ii $\overrightarrow{SR}$ iii $\overrightarrow{RS}$



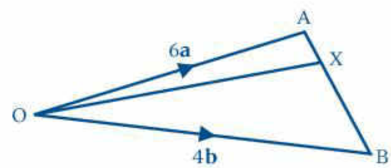
2 If D is the mid-point of AB give, in terms of **p** and **q**,
a $\overrightarrow{AB}$ **b** $\overrightarrow{AD}$
c $\overrightarrow{DB}$ **d** $\overrightarrow{CD}$



3 **a** What type of quadrilateral is ABCD?
b Give, in terms of **a** and **b**,
i $\overrightarrow{BC}$ ii $\overrightarrow{BD}$ iii $\overrightarrow{AC}$

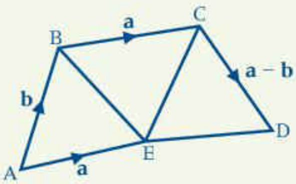


4 In the diagram X is the point on AB such that AX = $\frac{1}{3}$XB. Given that $\overrightarrow{OA}$ = 6**a** and $\overrightarrow{OB}$ = 4**b**, express, in terms of **a** and/or **b**,
a $\overrightarrow{AB}$ **b** $\overrightarrow{AX}$ **c** $\overrightarrow{OX}$



Example:

$\overrightarrow{AB}$ = **b**, $\overrightarrow{AE}$ = **a**, $\overrightarrow{BC}$ = **a**, and $\overrightarrow{CD}$ = **a** − **b**.
a Find $\overrightarrow{BE}$ and $\overrightarrow{ED}$ in terms of **a** and **b**.
b What type of quadrilateral is BEDC?
c Prove that A, E and D lie in a straight line.



a $\overrightarrow{BE} = \overrightarrow{BA} + \overrightarrow{AE}$, $= -\mathbf{b} + \mathbf{a}$ $= \mathbf{a} - \mathbf{b}$

$\overrightarrow{ED} = \overrightarrow{EB} + \overrightarrow{BC} + \overrightarrow{CD}$ $= -(\mathbf{a} - \mathbf{b}) + \mathbf{a} + (\mathbf{a} - \mathbf{b})$ $= \mathbf{a}$

b BEDC is a parallelogram because $\overrightarrow{BE} = \overrightarrow{CD}$
i.e. CD is parallel to BE and CD = BE.

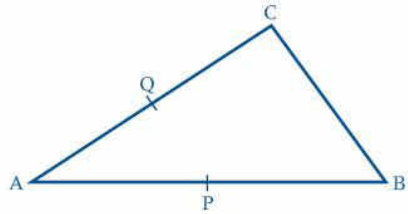
c Both $\overrightarrow{AE}$ and $\overrightarrow{ED}$ represent **a**.
∴ ED is parallel to AE.
Also the point E is on both lines
∴ A, E and D lie in a straight line.

5 $\overrightarrow{AB} = \mathbf{b}$ and $\overrightarrow{AC} = \mathbf{c}$. P and Q are the mid-points of AB and AC respectively. Give, in terms of **b** and **c**,

a $\overrightarrow{AP}$ **b** $\overrightarrow{AQ}$ **c** $\overrightarrow{BC}$ **d** $\overrightarrow{PQ}$

e Show that PQ is parallel to BC.

f What is the relationship between the lengths of PQ and BC? Justify your answer.

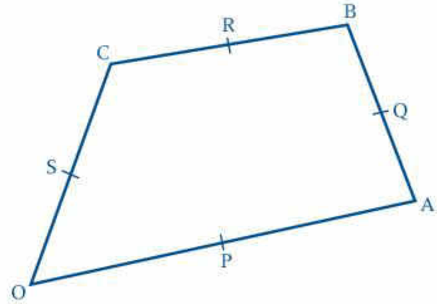


6 $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$. P, Q, R and S are the mid-points of OA, AB, BC and OC respectively. Give, in terms of **a**, **b** and **c**,

a $\overrightarrow{OP}$ **b** $\overrightarrow{AB}$ **c** $\overrightarrow{AQ}$ **d** $\overrightarrow{PQ}$ **e** $\overrightarrow{SR}$

f Show that PQ is parallel to SR.

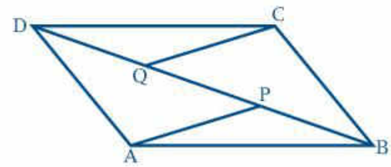
g What type of quadrilateral is PQRS? Give reasons for your answer.



7 ABCD is a parallelogram. $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AD} = \mathbf{b}$. P and Q are points on BD such that BP = PQ = QD. Give, in terms of **a** and **b**,

a $\overrightarrow{BD}$ **b** $\overrightarrow{BP}$ **c** $\overrightarrow{BQ}$ **d** $\overrightarrow{AP}$ **e** $\overrightarrow{QC}$

f Show that APCQ is a parallelogram.

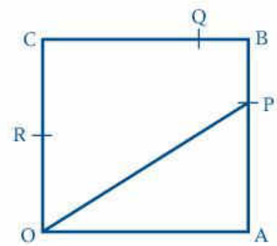


8 OABC is a square. $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OC} = \mathbf{b}$.
P is the point on AB such that AP : PB = 2 : 1.
Q is the point on BC such that BQ : QC = 1 : 3.
R is the mid-point of OC.
Find, in terms of **a** and **b**,

a $\overrightarrow{AB}$ **b** $\overrightarrow{AP}$ **c** $\overrightarrow{OP}$ **d** $\overrightarrow{OR}$ **e** $\overrightarrow{CQ}$ **f** $\overrightarrow{RQ}$

g Show that RQ is parallel to OP.

h How do the lengths of RQ and OP compare?



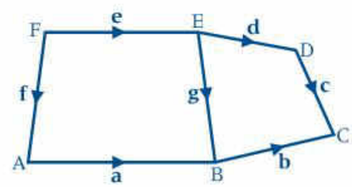
9 The diagram shows a rough sketch of two quadrilaterals.

a If $\mathbf{a} = 2\mathbf{b}$ what can you say about A, B and C?

b If $\mathbf{a} = \mathbf{b} = \mathbf{e} = \mathbf{d}$ what type of figure is ABCDEF?

c If $\mathbf{g} = 2\mathbf{c}$ what type of figure is BCDE?

d If $\mathbf{d} + \mathbf{c} = \mathbf{e} + \mathbf{g}$ name four points that are vertices of a parallelogram.

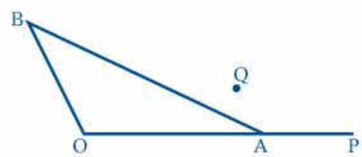


10 $\overrightarrow{OA} = 4\mathbf{a}$, $\overrightarrow{OB} = 2\mathbf{b}$, $\overrightarrow{AP} = \frac{1}{2}\overrightarrow{OA}$ and $\overrightarrow{OQ} = 3\mathbf{a} + \mathbf{b}$.
Give, in terms of **a** and **b**,

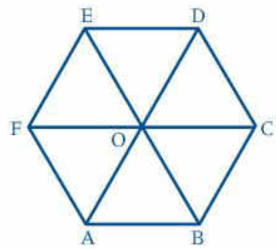
a $\overrightarrow{BP}$ **b** $\overrightarrow{BQ}$

c Show that B, Q and P lie in a straight line.

d Find BQ : BP.



11 ABCDEF is a regular hexagon whose diagonals intersect at O.



$\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

a Find, in terms of **a** and **b**,

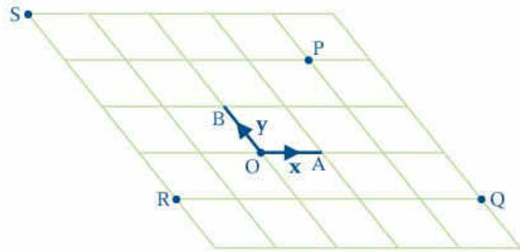
i $\overrightarrow{OC}$ **ii** $\overrightarrow{OD}$

iii $\overrightarrow{OE}$ **iv** $\overrightarrow{OF}$

b Give, in terms of **a** and **b**, the vectors

i $\overrightarrow{AB}$ **ii** $\overrightarrow{BC}$ **iii** $\overrightarrow{CD}$ **iv** $\overrightarrow{DE}$ **v** $\overrightarrow{EF}$ **vi** $\overrightarrow{FA}$

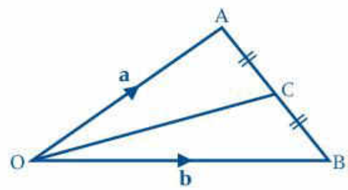
12



In the diagram $\overrightarrow{OA} = \mathbf{x}$ and $\overrightarrow{OB} = \mathbf{y}$. Express each of the following vectors in the form $h\mathbf{x} + k\mathbf{y}$. For each vector give the values of h and k.

a $\overrightarrow{OP}$ **b** $\overrightarrow{OQ}$ **c** $\overrightarrow{OR}$ **d** $\overrightarrow{OS}$

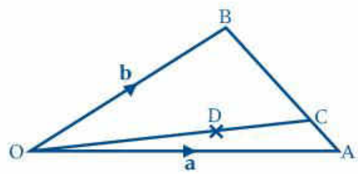
13 In the diagram $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and C is the mid-point of AB. Give, in terms of **a** and **b**,



a $\overrightarrow{BA}$ **b** $\overrightarrow{AB}$ **c** $\overrightarrow{BC}$ **d** $\overrightarrow{AC}$

e $\overrightarrow{OC}$ can be given as $\overrightarrow{OB} + \overrightarrow{BC}$ or $\overrightarrow{OA} + \overrightarrow{AC}$. Use each of these two versions to find $\overrightarrow{OC}$ in terms of **a** and **b**. Are your two answers the same?

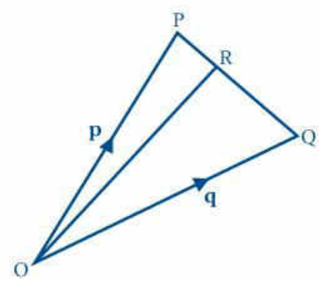
14 $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. C is on AB such that AC : CB = 1 : 3. D is on OC such that OD : DC = 2 : 1. Give the following vectors in terms of **a** and **b**.



a $\overrightarrow{AB}$ **b** $\overrightarrow{BA}$ **c** $\overrightarrow{AC}$ **d** $\overrightarrow{BC}$

e $\overrightarrow{OC}$ **f** $\overrightarrow{OD}$ **g** $\overrightarrow{DC}$

15 $\overrightarrow{OP} = \mathbf{p}$ and $\overrightarrow{OQ} = \mathbf{q}$. R is the point on PQ such that $PR = kPQ$. Give, in terms of **p** and **q**,



a $\overrightarrow{PQ}$ **b** $\overrightarrow{PR}$

c $\overrightarrow{RQ}$ **d** $\overrightarrow{OR}$

PUZZLE

Baldrick has enjoyed a relaxing day on the river and is rowing downstream back towards his starting point. As he passes under a bridge, unknown to him his hat falls into the river. Luckily his hat floats. He continues to row downstream and after 20 minutes realises what has happened. The river is flowing at 4 km/h and Baldrick can row in still water at 5 km/h. How long will it take him to row upstream to recover his hat and how far will he be downstream from the bridge at this time?

Using vectors to model practical situations

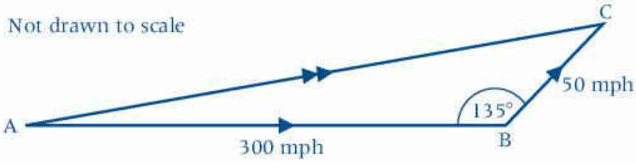
So far we have visualised vectors as displacements but there are many other quantities that have magnitude and direction, e.g. velocities, accelerations, forces, magnetic fields. Problems involving vector quantities can often be solved by representing the vector quantities by lines.

If an athlete runs 100 m with a wind immediately behind him the time will be faster than it would be without the wind. On the other hand a wind blowing into his face will give a slower time than a wind blowing on his back. Between these two extremes any wind blowing across the track has an effect on the actual speed of the runner and hence on the time taken. It is for this reason that, for a time to be acceptable as a record, the wind speed must be very low. The effect of a wind on a moving object can be found by using vectors.

For example, when an aeroplane is being steered due east but a crosswind is blowing from the south-east, the actual flight path of the aircraft will be somewhere between the two directions.

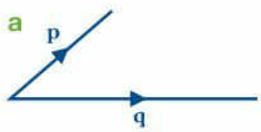
The actual flight path and speed are found using vector addition to combine the direction and speed of the wind with that of the aircraft due to its own power. The speed of the aircraft due to its own power is called its 'speed in still air'. We refer to the two velocities that combine to give the resultant velocity of the aircraft as *component velocities* or simply **components**.

If $\overrightarrow{AB}$ represents the velocity of the aircraft and $\overrightarrow{BC}$ the velocity of the wind then $\overrightarrow{AC}$ represents the resultant or actual velocity, that is, it gives the actual speed and direction of the aircraft over the ground below.



EXERCISE 18f

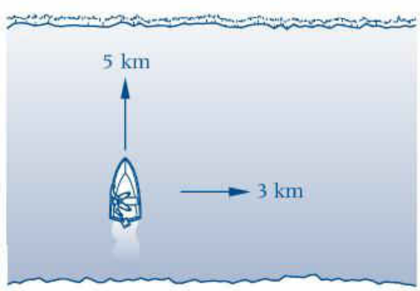
1 The motion of a body is made up of two velocities, **p** and **q**. Draw a diagram to show the resultant velocity, **p** + **q**.

a


b


2 a Sketch a vector diagram to show the two velocities and their resultant.
  b Calculate the magnitude of the resultant velocity.

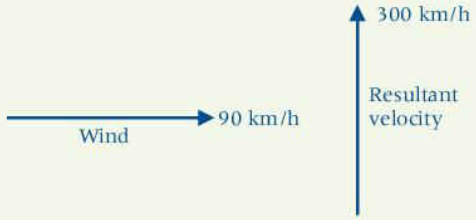
3 The diagram shows a boat crossing a river. Its motion is made up of two velocities. One is the current in the river. The other is the speed and direction set by the boatman.
  a Draw an accurate vector diagram to show the resultant velocity.
  b Find the angle between the resultant velocity and the direction of the current.
  c Check the accuracy of your drawing by calculation.



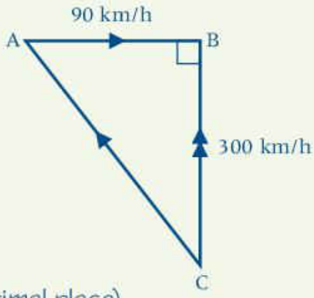
4 a Use the information given above this exercise to make an accurate drawing of ABC.
  b Hence find the direction in which the aircraft moves. Give your answer as a three-figure bearing.
  c Find the speed of the aircraft over the ground.
  d Check your answer to part **c** by calculation.

Example:

A plane needs to travel due north at 300 km/h. There is a crosswind of 90 km/h blowing from the west. Find the direction which the pilot should set.



We know the resultant velocity (300 km/h due north) of the plane and one of the two component velocities. First sketch the two known velocities separately.



Now we can draw a second combined diagram.

$\overrightarrow{CB}$ is the resultant,

$\overrightarrow{AB}$ is one component velocity

so $\overrightarrow{CA}$ is the second component velocity.

To find the direction of $\overrightarrow{CA}$ we need to calculate $A\hat{C}B$.

In $\triangle ABC$, $\tan \hat{C} = \frac{90}{300} = 0.3$

$\hat{C} = 16.69\ldots°$

$= 16.7°$ (correct to 1 decimal place)

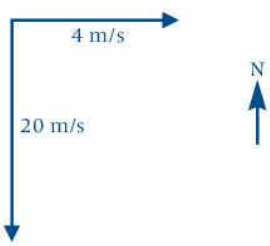
The pilot has to set course on a bearing

$360° - 16.69...° = 343.31...°$

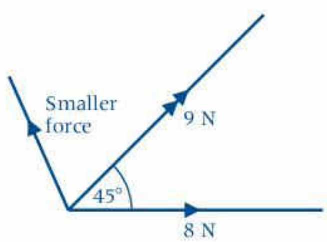
$= 343.3°$ (correct to 1 decimal place)

Notice that, because of the wind, the speed of the plane due to its engine power has to exceed 300 km/h.

5 A ball rolls, at a speed of 4 m/s, across the floor of a carriage of a train which is travelling due south at 20 m/s. Find the speed and direction of the movement of the ball in relation to the ground.



6 The resultant of two forces is 9 newtons at an angle of 45° to the larger force. If the larger force is 8 newtons find, by drawing an accurate diagram, the magnitude and direction of the smaller force.



7 A pilot sets course due south at 120 km/h but, because of the wind, the plane actually flies at 130 km/h on a bearing of 150°. Find the speed and direction of the wind by drawing an accurate diagram.

8 A pallet with bricks loaded on it is pulled by two ropes, each inclined at 30° to the direction in which the pallet moves. The force in one rope is 10 newtons. Find the force in the other rope.

9 A helicopter tries to fly at 60 km/h on a bearing of 085°, but is blown off course by a wind blowing at 30 km/h in the direction 175°. Make a scale drawing, and use it to estimate the magnitude and direction of the resultant velocity.

10 A boat, whose speed is set to 3 m/s (this is the speed it would have in still water) is driven across a river which is flowing at 1.5 m/s. Use scale drawings to answer the following questions.

a If the boat is pointed straight across the river, in what direction does it actually move?

b The boat needs to go across the river at right angles to the bank. In which direction should it be pointed?

c The speed of the current changes so that when the boat is pointed upstream at 70° to the bank, it actually moves downstream at 80° to the bank. What is the speed of the current?

11 Solve the problem stated at the beginning of the chapter.

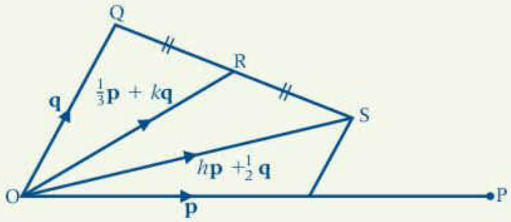
Further problems

If **a** and **b** are two non-parallel vectors, a vector **c** that is made up from **a** and **b** can be expressed in the form $h\mathbf{a} + k\mathbf{b}$, where h and k are numbers. If information is given that allows **c** to be found in another form, e.g. $3\mathbf{a} - \frac{1}{2}\mathbf{b}$, then we know that $h = 3$ and $k = -\frac{1}{2}$ because the coefficients of **a** and **b** must be the same in both forms. In the following exercise you will see how this idea can be used in solving problems.

EXERCISE 18g

Example:

$\overrightarrow{OQ} = \mathbf{q}$, $\overrightarrow{OP} = \mathbf{p}$, $\overrightarrow{OR} = \frac{1}{3}\mathbf{p} + k\mathbf{q}$, $\overrightarrow{OS} = h\mathbf{p} + \frac{1}{2}\mathbf{q}$ and R is the mid-point of QS. Find h and k.



We will find $\overrightarrow{QR}$ and $\overrightarrow{RS}$ in terms of **p**, **q**, h and k.

$$\begin{aligned}\overrightarrow{QR} &= \overrightarrow{QO} + \overrightarrow{OR}\\ &= -\mathbf{q} + \frac{1}{3}\mathbf{p} + k\mathbf{q}\\ &= \frac{1}{3}\mathbf{p} + (k - 1)\mathbf{q}\end{aligned}$$

$$\overrightarrow{RS} = \overrightarrow{RO} + \overrightarrow{OS}$$
$$= -\left(\tfrac{1}{3}\mathbf{p} + k\mathbf{q}\right) + h\mathbf{p} + \tfrac{1}{2}\mathbf{q}$$
$$= \left(h - \tfrac{1}{3}\right)\mathbf{p} + \left(\tfrac{1}{2} - k\right)\mathbf{q}$$

But $\overrightarrow{QR} = \overrightarrow{RS}$

R is the mid-point of QS.

$\therefore$ $\quad \tfrac{1}{3}\mathbf{p}\,(k - 1)\,\mathbf{q} = \left(h - \tfrac{1}{3}\right)\mathbf{p} = \left(\tfrac{1}{2} - k\right)\mathbf{q}$

Comparing coefficients of $\mathbf{p}$, $\frac{1}{3} = h - \frac{1}{3}$ $\quad$ i.e. $h = \frac{2}{3}$

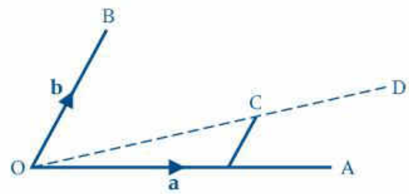
Comparing coefficients of $\mathbf{q}$, $k - 1 = \frac{1}{2} - k$

$$2k = 1\tfrac{1}{2} \quad \Rightarrow \quad k = \tfrac{3}{4}$$

So $\quad h = \frac{2}{3}$ and $k = \frac{3}{4}$

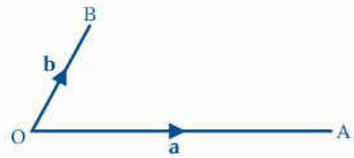
1 $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$, $\overrightarrow{OC} = \frac{2}{3}\mathbf{a} + \frac{1}{3}\mathbf{b}$.
D is the point such that $\overrightarrow{OD} = k\overrightarrow{OC}$.

- **a** Find $\overrightarrow{OD}$ and $\overrightarrow{BD}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.
- **b** If BD is parallel to OA, find the value of k.
- **c** Find the ratio OC : CD.



2 $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$,
C and D are the points such that $\overrightarrow{OC} = \mathbf{a} - \frac{1}{2}\mathbf{b}$
and $\overrightarrow{OD} = k\mathbf{a} + \frac{3}{4}\mathbf{b}$.

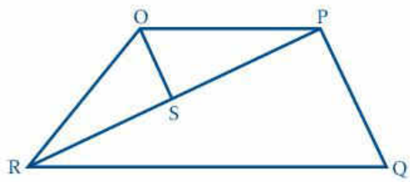
- **a** Find $\overrightarrow{BD}$ in terms of $\mathbf{a}$ and $\mathbf{b}$.
- **b** If BD is parallel to OC find the value of k.
- **c** Find $\frac{BD}{OC}$.



3 OPQR is a trapezium with OP parallel to RQ.
$\overrightarrow{OP} = \mathbf{p}$, $\overrightarrow{OR} = \mathbf{r}$, RQ = hOP and PS = kPR.
Express, in terms of $\mathbf{p}$ and $\mathbf{r}$,

a $\overrightarrow{RQ}$ **b** $\overrightarrow{PR}$ **c** $\overrightarrow{PQ}$ **d** $\overrightarrow{PS}$ **e** $\overrightarrow{OS}$

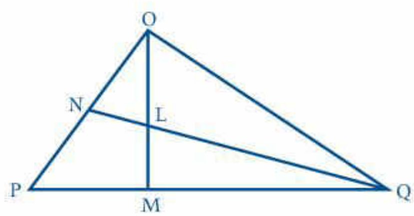
- **f** If OS is parallel to PQ, find h in terms of k.
- **g** If, in addition, $\frac{PS}{PR} = \frac{1}{2}$, find k and h.



4 $\overrightarrow{OP} = \mathbf{p}$, $\overrightarrow{OQ} = \mathbf{q}$, M is a point on PQ such that PM : MQ = 1 : 2. N is the mid-point of OP. LQ = hQN.
Give, in terms of $\mathbf{p}$, $\mathbf{q}$ and h,

a $\overrightarrow{PQ}$ **b** $\overrightarrow{PM}$ **c** $\overrightarrow{OM}$ **d** $\overrightarrow{ON}$
e $\overrightarrow{QN}$ **f** $\overrightarrow{QL}$ **g** $\overrightarrow{OL}$

- **h** If OL = kOM, express $\overrightarrow{OL}$ in terms of $\mathbf{p}$, $\mathbf{q}$ and k.
- **i** Using the two versions of $\overrightarrow{OL}$, find the values of h and k.



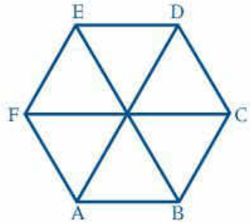
5 ABCDEF is a regular hexagon, $\overrightarrow{AB} = \mathbf{a}$ and $\overrightarrow{AC} = \mathbf{b}$. G is the point such that $\overrightarrow{CG} = \mathbf{b}$ and H is the point such that $\overrightarrow{CH} = 2\mathbf{a} - \mathbf{b}$.

a Find, in terms of **a** and **b**,

i $\overrightarrow{AD}$ ii $\overrightarrow{BE}$ iii $\overrightarrow{EG}$ iv $\overrightarrow{HG}$

b Show that HG is parallel to EF.

c What type of quadrilateral is ADGH?

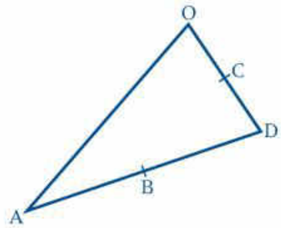


6 $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OC} = \mathbf{c}$. B and C are the mid-points of AD and OD.

a Express $\overrightarrow{OD}$ and $\overrightarrow{AD}$ in terms of **a** and **c**.

b Find **b** in terms of **a** and **c**.

c E is a point on OA produced such that $\overrightarrow{OE} = 4\overrightarrow{AE}$. If $\overrightarrow{CB} = k\overrightarrow{AE}$ find the value of k.

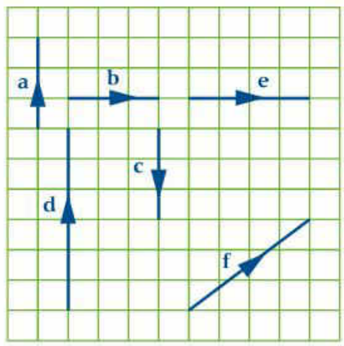


7 O, A and B are the points (0, 0), (3, 4) and (4, −6) respectively. C is the point such that $\overrightarrow{OA} = \overrightarrow{OC} + \overrightarrow{OB}$. Find the coordinates of C.

MIXED EXERCISE 18

Several answers are given for these questions.
Write down the letter that corresponds to the correct answer.

Questions **1** to **3** refer to this diagram.



1 Which of the following statements are true?

1 $\mathbf{a} = \mathbf{b}$ 2 $\mathbf{a} = \mathbf{e}$ 3 $\mathbf{d} = 2\mathbf{a}$

A 1 B 2 C 2 and 3 D 3

2 $|\mathbf{a} + \mathbf{c}| =$

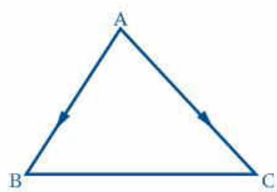
A 5 B 7 C 18 D 25

3 $\mathbf{e} - \mathbf{c} =$

A 5 B **f** C 2**f** D 10

4 In the diagram, **BC** =

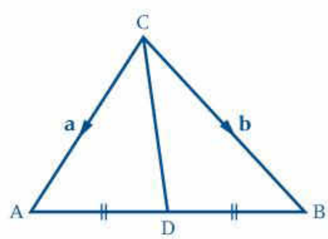
A **AB** + **AC** B **AB** + **CA** C **BA** + **AC** D **BA** + **CA**



5 In the diagram, **D** is the mid-point of **AB**.

CD =

A $\mathbf{a} + \mathbf{b}$ B $\frac{1}{2}\mathbf{a} + \mathbf{b}$ C $\mathbf{a} + \frac{1}{2}\mathbf{b}$ D $\frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$



6 A is the point (2, 5) and B is the point (7, 17).
$|\mathbf{AB}| =$

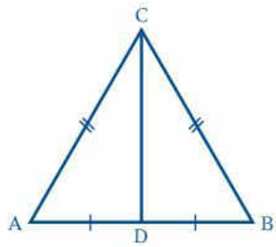
A 9 **B** 13 **C** 21 **D** 30

7 The translation $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ followed by the translation $\begin{pmatrix} -4 \\ 5 \end{pmatrix}$ is equivalent to the translation

A $\begin{pmatrix} 6 \\ 6 \end{pmatrix}$ **B** $\begin{pmatrix} -2 \\ 4 \end{pmatrix}$ **C** $\begin{pmatrix} 2 \\ -4 \end{pmatrix}$ **D** $\begin{pmatrix} -8 \\ -5 \end{pmatrix}$

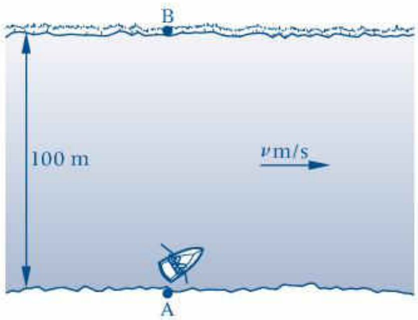
8 In the diagram, ABC is an equilateral triangle and D is the midpoint of AB.
$|\mathbf{AC} + \mathbf{BC}| =$

A AD **B** 2AD **C** CD **D** 2CD



INVESTIGATION

A river flows at v m/s and is 100 m wide. The greatest speed at which a boatman can row in still water is $2v$ m/s. The boatman wishes to cross the river, starting at A and landing at B, the point on the other bank immediately opposite A.



a Investigate whether or not he can travel directly from A to B. If he can, must he row at one particular speed or is there a range of speeds that he can choose from? Justify your answer.

b If the speed of the river increases to $1.5v$ m/s can he still travel directly across the river from A to B? What happens if the speed of the river increases to $2v$ m/s?

c Investigate the maximum speed of the river that allows him to travel directly from A to B.

d He points his boat towards B and rows at the same speed as the water is flowing in the river. How far downstream from B will he land?

e Will your answers to the previous parts change if the width of the river is 150 m? Justify your answers.

MATHS IS OUT THERE

The computer language Pascal, which was designed to facilitate the teaching of programming as a systematic language, was named after the French mathematician and philosopher Blaise Pascal (1623–1662).

IN THIS CHAPTER YOU HAVE SEEN THAT...

- a vector has magnitude and direction whereas a scalar has magnitude only
- when you multiply a vector by a scalar, you change the magnitude but not the direction of the vector
- the vector $-\mathbf{a}$ has the same magnitude as **a** but is in the opposite direction
- when vectors **a** and **b** are represented by two sides of a triangle such that they go in the same sense round the triangle, the vector represented by the third side of the triangle and going in the opposite sense round the triangle is called the resultant of **a** and **b**, and is denoted by $\mathbf{a} + \mathbf{b}$. This is known as the triangle law for the addition of vectors
- vectors can be used to solve problems involving quantities such as displacement, velocity, acceleration and force.

19 TRANSFORMATIONS AND MATRICES

AT THE END OF THIS CHAPTER YOU SHOULD BE ABLE TO...

1 Give the position vector of a point (*x*, *y*).

2 Illustrate with a sketch the position vector of a point (*x*, *y*).

3 Give the coordinates of a point whose position vector is given.

4 Find the image of a position vector under a given transformation.

5 Use a transformation matrix to find the image of a given point.

6 Classify a given transformation as a reflection, a rotation, an enlargement or some other transformation.

7 Recognise the identity transformation.

8 Find the inverse of a given transformation.

9 Find the inverse matrix of a given transformation matrix.

10 Identify the following as transformations which cannot be described by a matrix:
- a a rotation about a point other than the origin
- b a reflection in a line not passing through the origin
- c an enlargement whose centre is not the origin
- d a translation
- e a glide reflection.

MATHS IS OUT THERE

Did you know that digital pictures are made up of pixels? These are individual squares of colour. They are usually so small that you cannot see them.

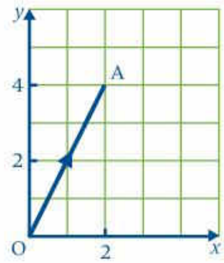
BEFORE YOU START

you need to know:
- ✓ how the perform a rotation, reflection, translation and enlargement and how to describe them
- ✓ the meaning of vectors and how to describe them
- ✓ how to draw a vector
- ✓ how to multiply a vector by a matrix
- ✓ how to multiply matrices
- ✓ what the unit matrix is
- ✓ how to find the inverse of a matrix

KEY WORDS

enlargement, glide reflection, identity transformation, image, invariant point, inverse transformation, matrix (plural matrices), object, position vector, reflection, rotation, transformation, transformation matrix, translation, vector

The position vector of a point



Matrices can be used for several different purposes; in Chapter 11 they were used for solving simultaneous equations. Now we will see how they can be used for defining some **transformations**.

If a point, A, has coordinates (2, 4), then the **vector** $\overrightarrow{OA}$ is $\begin{pmatrix}2\\4\end{pmatrix}$ and this is called the **position vector** of A relative to the origin O.

EXERCISE 19a

Give the position vectors of the points in questions **1** to **6**. Illustrate each with a sketch.

> The position vector goes from (0, 0) to the point.

1 (4, 5) **2** (3, −2) **3** (−7, 5)

4 (−3, −5) **5** (5, 3) **6** (−5, 2)

Give the coordinates of the points whose position vectors are given in questions **7** to **12**. Illustrate each with a sketch.

7 $\begin{pmatrix}5\\2\end{pmatrix}$ **8** $\begin{pmatrix}1\\-3\end{pmatrix}$ **9** $\begin{pmatrix}-2\\-4\end{pmatrix}$

10 $\begin{pmatrix}2\\-3\end{pmatrix}$ **11** $\begin{pmatrix}-6\\2\end{pmatrix}$ **12** $\begin{pmatrix}2\\-6\end{pmatrix}$

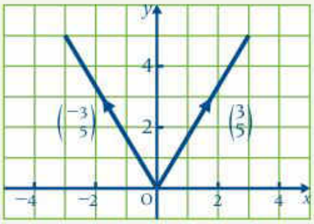
The image of a position vector

EXERCISE 19b

Example:

Find the image of the position vector $\begin{pmatrix}3\\5\end{pmatrix}$ under a reflection in the *y*-axis.

> First sketch the vector then draw its reflection in the *y*-axis.



> Now you can see that the coordinates of the image point are (−3, 5).

The image is the vector $\begin{pmatrix}-3\\5\end{pmatrix}$

For each of the following questions, draw x and y axes, marking values from -5 to 5 on each axis.

1 Find the image of the position vector $\begin{pmatrix}5\\2\end{pmatrix}$ under an anticlockwise rotation of 90° about the origin.

2 Find the image of the position vector $\begin{pmatrix}-4\\3\end{pmatrix}$ under a reflection in the x-axis.

3 Find the image of the position vector $\begin{pmatrix}4\\-2\end{pmatrix}$ under a clockwise rotation of 180° about the origin.

4 Find the image of the position vector $\begin{pmatrix}3\\-5\end{pmatrix}$ under a rotation of 90° anticlockwise about the origin.

5 Find the image of the position vector $\begin{pmatrix}5\\3\end{pmatrix}$ under a reflection in the line $y = x$.

6 Find the image of the position vector $\begin{pmatrix}2\\1\end{pmatrix}$ under an enlargement with scale factor 2, centre the origin.

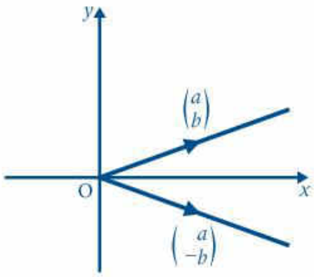
Transformation matrix

Consider the transformation which is a reflection in the x-axis. If the **object** is the position vector $\begin{pmatrix}3\\2\end{pmatrix}$, the **image** is $\begin{pmatrix}3\\-2\end{pmatrix}$.

We can say that this transformation changes $\begin{pmatrix}3\\2\end{pmatrix}$ to $\begin{pmatrix}3\\-2\end{pmatrix}$.

This transformation changes any position vector in the same way: it leaves the x-coordinate alone but it changes the sign of the y coordinate,

i.e. it changes $\begin{pmatrix}a\\b\end{pmatrix}$ to $\begin{pmatrix}a\\-b\end{pmatrix}$.



We also know that pre-multiplying by a matrix changes a vector,

e.g. $\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}4\\3\end{pmatrix} = \begin{pmatrix}4\\-3\end{pmatrix}$ and $\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}\begin{pmatrix}a\\b\end{pmatrix} = \begin{pmatrix}a\\-b\end{pmatrix}$

So we can use the matrix $\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$ to define the transformation 'reflection in the x-axis'.

We call $\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix}$ a **transformation matrix**.

MATHS IS OUT THERE

Did you know that it is believed that the ancient Chinese, over 2000 years ago, solved systems of equations using a method similar to the modern elementary transformations and matrices?

Finding the image

If we are given a transformation matrix $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ and a point A (2, 3), then we can find the image A9 under this transformation, by multiplying the position vector of A by the transformation matrix.

i.e. $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\overset{A}{\begin{pmatrix} 2 \\ 3 \end{pmatrix}} = \overset{A'}{\begin{pmatrix} 7 \\ 5 \end{pmatrix}}$

Note that the transformation matrix *must* come first. The position vector of A′ is then $\begin{pmatrix} 7 \\ 5 \end{pmatrix}$ and A′ is the point (7, 5).

EXERCISE **19c**

Example:

Find the image, A′, of the point A(1, 2) under the transformation defined by the matrix $\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}$.

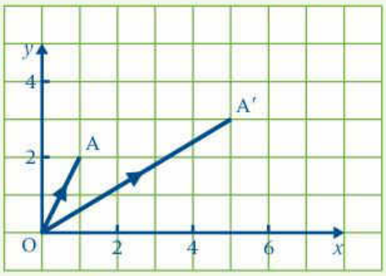
Illustrate with a sketch.

The position vector of A is $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$

Pre-multiplying $\overrightarrow{OA}$ by the matrix.

$$\begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}\overset{A}{\begin{pmatrix} 1 \\ 2 \end{pmatrix}} = \overset{A'}{\begin{pmatrix} 5 \\ 3 \end{pmatrix}}$$

The position vector of A′ is $\begin{pmatrix} 5 \\ 3 \end{pmatrix}$, i.e. A′ is the point (5, 3).



In A′, 5 comes from multiplying the top row of the matrix by $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and 3 comes from multiplying the bottom row of the matrix by $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Find the image, A′, of the given point A, under the transformation defined by the given matrix. Illustrate with a diagram.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} ae + bf \\ ce + df \end{pmatrix}$$

1 The point A is (1, 2) and the transformation matrix is $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

2 The point A is (4, 1) and the transformation matrix is $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

3 The point A is (−2, 3) and the transformation matrix is $\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$

4 The point A is (−1, −3) and the transformation matrix is $\begin{pmatrix} -1 & 1 \\ 2 & 1 \end{pmatrix}$

5 The point A is (2, 3) and the transformation matrix is $\begin{pmatrix} 4 & -1 \\ -1 & 2 \end{pmatrix}$

6 The point A is (2, −1) and the transformation matrix is $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$

Sometimes, there are several object and image points on one diagram. If so, then it is clearer to mark only the points and to leave out the lines representing the position vectors.

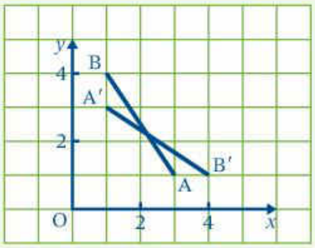
Example:

Find the images, A′ and B′, of the points A(3, 1) and B(1, 4) under the transformation defined by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Mark the points on a sketch. Join AB and A′B′.

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \overset{A}{\begin{pmatrix} 3 \\ 1 \end{pmatrix}} = \overset{A'}{\begin{pmatrix} 1 \\ 3 \end{pmatrix}}$$

and $$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \overset{B}{\begin{pmatrix} 1 \\ 4 \end{pmatrix}} = \overset{B'}{\begin{pmatrix} 4 \\ 1 \end{pmatrix}}$$

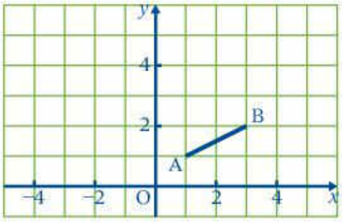
A′ is the point (1, 3) and B′ is the point (4, 1).



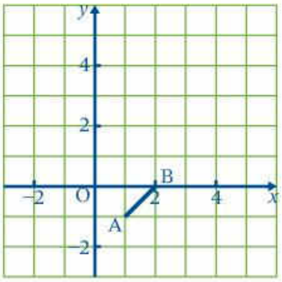
Pre-multiplying $\overrightarrow{OA}$ and $\overrightarrow{OB}$ by the matrix

In questions **7** to **12** you are given the coordinates of two points A and B and a transformation matrix. Find the coordinates of the images, A′ and B′, of A and B under the transformation defined by the matrix. Mark all the points on a diagram, but do not draw the position vectors. In each case join AB and A′B′.

7 A is the point (1, 1), B is the point (3, 2) and the transformation matrix is $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$



8 A is the point (1, −1), B is the point (2, 0) and the transformation matrix is $\begin{pmatrix} 1 & 2 \\ 2 & 3 \end{pmatrix}$



9 A is the point (1, 3), B is the point (1, −2) and the transformation matrix is $\begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$

10 A is the point (1, 1), B is the point (−2, 0) and the transformation matrix is $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$

11 A is the point (1, 4), B is the point (4, 1) and the transformation matrix is $\begin{pmatrix} 1 & -1 \\ -2 & 1 \end{pmatrix}$

12 A is the point (3, 2), B is the point (−3, 2) and the transformation matrix is $\begin{pmatrix} 1 & 2 \\ 2 & -1 \end{pmatrix}$

Identifying a transformation

To find out whether a transformation is a reflection, a rotation, an enlargement or some other transformation, we need a simple object to which the transformation can be applied.

A rectangle or a triangle is usually the most convenient.

Reflections

A **reflection** is defined by the mirror line.

Mirror line

EXERCISE 19d

Example:

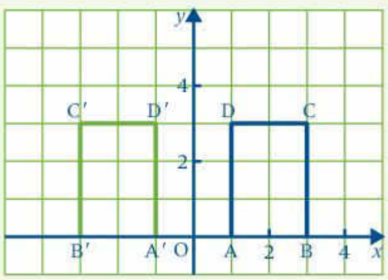
A, B, C and D are the points (1, 0), (3, 0), (3, 3) and (1, 3). Draw a diagram, mark the given points and join them up.
Find the image of each point under the transformation defined by the matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$, marking the points on the diagram. Join up the image points in order. What is the transformation?

First find the images of the points.

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\overset{A}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} = \overset{A'}{\begin{pmatrix} -1 \\ 0 \end{pmatrix}} \qquad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\overset{B}{\begin{pmatrix} 3 \\ 0 \end{pmatrix}} = \overset{B'}{\begin{pmatrix} -3 \\ 0 \end{pmatrix}}$$

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\overset{C}{\begin{pmatrix} 3 \\ 3 \end{pmatrix}} = \overset{C'}{\begin{pmatrix} -3 \\ 3 \end{pmatrix}} \qquad \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\overset{D}{\begin{pmatrix} 1 \\ 3 \end{pmatrix}} = \overset{D'}{\begin{pmatrix} -1 \\ 3 \end{pmatrix}}$$

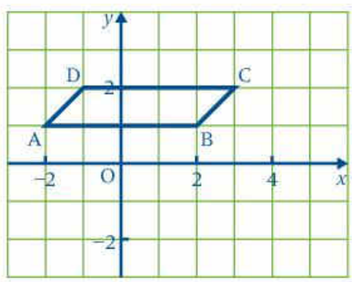
Now plot the image points.



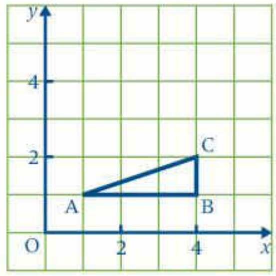
We can see that the transformation is a reflection. The mirror line is the y-axis.

Draw x and y axes, each using a scale from -4 to 4. Mark the given points and join them up in order. Find the image of each point under the transformation defined by the given matrix and join up the image points in order. You will see that the transformation is a reflection. What is the mirror line?

1 The given points are A(-2, 1), B(2, 1), C(3, 2) and D(-1, 2). The transformation matrix is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$



2 The given points are A(1, 1), B(4, 1) and C(4, 2). The transformation matrix is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$



3 The given points are A(2, −3), B(5, −3) and C(3, 2). The transformation matrix is $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

4 The given points are A(4, 1), B(3, 3) and C(2, 0). The transformation matrix is $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

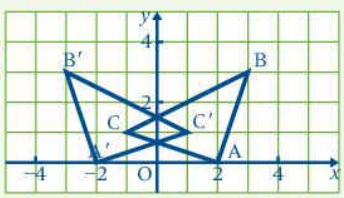
To save writing so many matrices, we can combine two or three or more position vectors into one matrix.

Example:

Find the images of A(2, 0), B(3, 3) and C(−1, 1) under the transformation defined by the matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Mark the object and the image points on a diagram. What is the transformation?

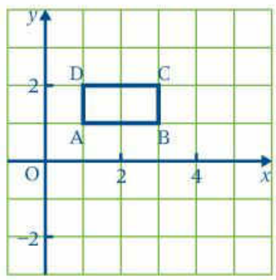
$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\overset{\quad A \quad B \quad C}{\begin{pmatrix} 2 & 3 & -1 \\ 0 & 3 & 1 \end{pmatrix}} = \overset{\quad A' \quad B' \quad C'}{\begin{pmatrix} -2 & -3 & 1 \\ 0 & 3 & 1 \end{pmatrix}}$$

The transformation is a reflection.
The mirror line is the *y*-axis.

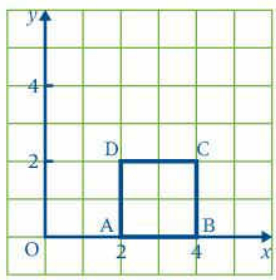


Remember, multiply rows by columns.

5 The given points are A(1, 1), B(3, 1), C(3, 2) and D(1, 2). The transformation matrix is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$



6 The given points are A(2, 0), B(4, 0), C(4, 2) and D(2, 2). The transformation matrix is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$



7 The given points are A(1, 1), B(2, 1), C(2, 2) and D(1, 2). The transformation matrix is $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

8 The given points are A(1, 0), B(4, 0) and C(4, 2). The transformation matrix is $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

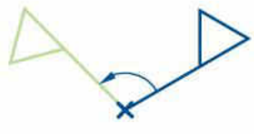
9 The given points are A(2, 1), B(3, 1), C(3, 4) and D(2, 4). The transformation matrix is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

10 The given points are A(1, 1), B(3, 1), C(4, 3) and D(3, 3). The transformation matrix is $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

11 The given points are A(2, 4), B(4, 5) and C(3, 2). The transformation matrix is $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

Rotations

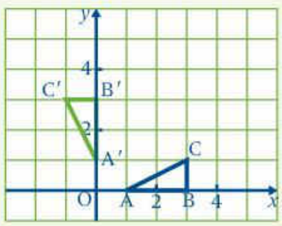
A **rotation** is defined by the centre and the direction and angle of rotation.


Centre of rotation

EXERCISE 19e

Example:

A, B and C are the points (1, 0), (3, 0) and (3, 1). Draw a diagram, mark the points and join them up. Find the image of each point under the transformation defined by the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. Mark the image points on the diagram and join them up. What is the transformation?

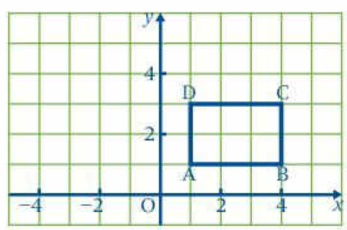


$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\overset{\quad A \quad B \quad C}{\begin{pmatrix} 1 & 3 & 3 \\ 0 & 0 & 1 \end{pmatrix}} = \overset{\quad A' \quad B' \quad C'}{\begin{pmatrix} 0 & 0 & -1 \\ 1 & 3 & 3 \end{pmatrix}}$$

The transformation is a rotation.
Its centre is O. The angle of rotation is 90° anticlockwise.

Draw *x* and *y* axes, marking values from −4 to 4 on each axis. Mark the given points and join them up in order. Find the image of each point under the transformation defined by the given matrix and join up the image points. You will see that the transformation is a rotation. Describe the rotation.

> You may find tracing paper helps you identify the rotation.



1 The given points are A(1, 1), B(4, 1), C(4, 3) and D(1, 3). The transformation matrix is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

2 The given points are A(1, 1), B(4, 1), C(4, 2) and D(1, 2). The transformation matrix is $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

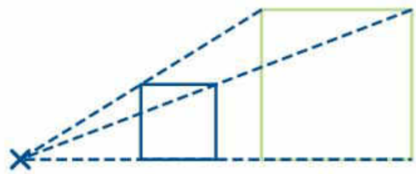
3 The given points are A(1, 0), B(3, 0) and C(4, 4). The transformation matrix is $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

4 The given points are A(1, 1), B(4, 1) and C(4, 4). The transformation matrix is $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

5 The given points are A(3, 2), B(4, 3) and C(1, 4). The transformation matrix is $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

Enlargements

An **enlargement** is defined by the centre of enlargement and the scale factor.

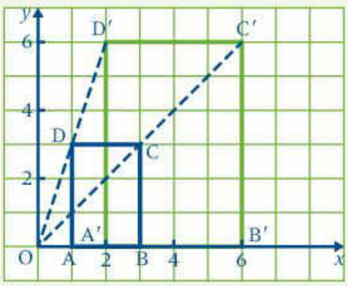

Centre of enlargement

EXERCISE 19f

Example:

A, B, C and D are the points (1, 0), (3, 0), (3, 3) and (1, 3). Draw a diagram, mark the points and join them up in order.

Find the image of each point under the transformation defined by the matrix $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and mark each point of the diagram. Join up the image points. What is the transformation?

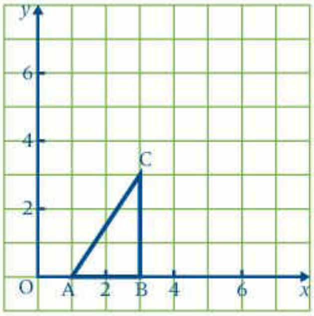


$$\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \overset{\quad A \quad B \quad C \quad D}{\begin{pmatrix} 1 & 3 & 3 & 1 \\ 0 & 0 & 3 & 3 \end{pmatrix}} = \overset{\quad A' \quad B' \quad C' \quad D'}{\begin{pmatrix} 2 & 6 & 6 & 2 \\ 0 & 0 & 6 & 6 \end{pmatrix}}$$

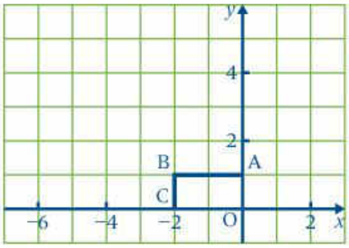
The transformation is an enlargement, centre O, with scale factor 2.

Draw x and y axes, marking values from 0 to 10 on each. Mark the given points and join them up in order. Find the image of each of the given points under the transformation defined by the given matrix. Mark the image points and join them up in order. Describe the enlargement.

1 The given points are A(1, 0), B(3, 0) and C(3, 3). The transformation matrix is $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$



2 The given points are A(0, 1), B(−2, 1), C(−2, 0) and O(0, 0). The transformation matrix is $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$



3 The given points are A(2, 2), B(2, 4), C(4, 4) and D(4, 2). The transformation matrix is $\begin{pmatrix} 1\frac{1}{2} & 0 \\ 0 & 1\frac{1}{2} \end{pmatrix}$

4 The given points are A(4, 2), B(4, 4) and C(−4, 4). The transformation matrix is $\begin{pmatrix} 2\frac{1}{2} & 0 \\ 0 & 2\frac{1}{2} \end{pmatrix}$

For questions **5** and **6**, mark the axes with values from −5 to 5.

5 The given points are O(0, 0), A(0, 1), B(−1, 1) and C(−1, 0). The transformation matrix is $\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$

6 The given points are A(0, 2), B(3, 2), C(3, 5) and D(0, 5). The transformation matrix is $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

Is there another transformation which produces the same image?

PUZZLE

Mrs Weekes had no money on her so she cashed a cheque. The bank teller accidentally transposed the dollars and cents, giving her dollars instead of cents and cents instead of dollars. Mrs Jones then bought a candy bar for 50 cents. When Mrs Jones was given her change, she realised that the amount she had left was exactly three times the amount on the cheque she had written. What was the amount written on the cheque?

MATHS IS OUT THERE

Did you know that the tetrahedron, the dodecahedron, and the cube were discovered by the Pythagoreans?

Transformations that cannot be described simply

Some transformation matrices define transformations which are different from any of the previous transformations and which sometimes cannot be described adequately. The next exercise gives some examples of these.

EXERCISE **19g**

Draw x and y axes, marking values from -8 to 8 on each. Mark the given points and join them up. Find the image of each point under the transformation defined by the given matrix. Mark the image points and join them up. Do *not* try to describe the transformation.

1 The given points are A(−1, 0), B(1, 0), C(1, 2) and D(−1, 2).

The transformation matrix is $\begin{pmatrix} 1 & 2 \\ 3 & 1 \end{pmatrix}$

2 The given points are A(−1, 0), B(1, 0), C(1, 2) and D(−1, 2).

The transformation matrix is $\begin{pmatrix} 1 & -2 \\ -3 & 1 \end{pmatrix}$

3 The given points are A(−2, −1), B(1, −1), C(1, 2) and D(−2, 2).

The transformation matrix is $\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$

4 The given points are A(−2, −1), B(1, −1), C(1, 1) and D(−2, 1).

The transformation matrix is $\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$

5 The given points are A(−2, −1), B(2, −1), C(2, 1) and D(−2, 1).

The transformation matrix is $\begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}$

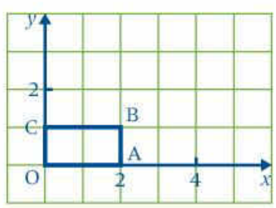
6 The given points are A(−2, −2), B(1, −1), C(1, 2) and D(−2, 2).

The transformation matrix is $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Mixed transformations

EXERCISE **19h**

In each of the questions **1** to **12**, O is point (0, 0), A is (2, 0), B is (2, 1) and C is (0, 1). The rectangle OABC is the object. Draw x and y axes, marking values from -8 to 8 on each axis. Find the image of OABC under the transformation defined by the given matrix and (where possible) describe the transformation.



1 $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ 2 $\begin{pmatrix} 1 & 1\frac{1}{2} \\ 0 & 1 \end{pmatrix}$ 3 $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

4 $\begin{pmatrix} -3 & 0 \\ 0 & 3 \end{pmatrix}$ 5 $\begin{pmatrix} 2 & -3 \\ 3 & 2 \end{pmatrix}$ 6 $\begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$

7 $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$ 8 $\begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix}$ 9 $\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$

10 $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$ 11 $\begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$ 12 $\begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$

13 What are the simplest objects you could use to identify a transformation?

Find and describe the transformation given by $\begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}$, using the simplest object you can.

14 Draw *x* and *y* axes, marking values from −5 to 5 on each axis. Use 1 cm to 1 unit. The object is the quadrilateral OABC where O is (0, 0), A is (3, 0), B is (5, 2) and C is (2, 2). Find and draw the eight images of OABC under the transformations defined by the following eight matrices.

a $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ b $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ c $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ d $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

e $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ f $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ g $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ h $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

15 Find the images of the following objects under the transformations defined by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

a OABC: O(0, 0), A(2, 0), B(2, 2) and C(0, 2)
b △PQR: P(−1, 1), Q(−3, 1) and R(−3, 4)
c Parallelogram WXYZ: W(1, −1), X(0, −3), Y(−3, −3) and Z(−2, −1).

What do you notice about the results?

16 A is the point with position vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and B is the point with position vector $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

A′ and B′ are the images of A and B under the transformations defined by the following matrices.

a $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ b $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ c $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ d $\begin{pmatrix} 2 & 5 \\ 4 & -1 \end{pmatrix}$

In each case write down the position vectors of A′ and B′ and illustrate with a diagram. What do you notice about the position vectors of A′ and B′ and the columns of the matrix which produced them?

The identity transformation

We saw in questions **14** and **15** in the last exercise that, under the transformation defined by the unit matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ the image is the same as the object.

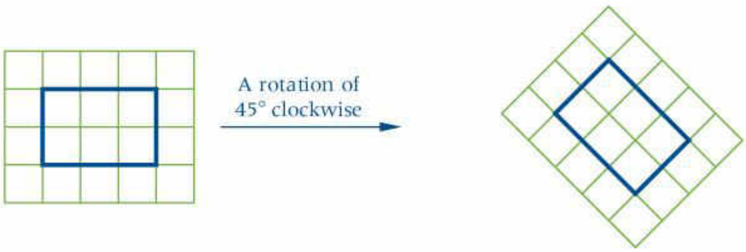
If the image is identical to the object the transformation is called an **identity transformation**.

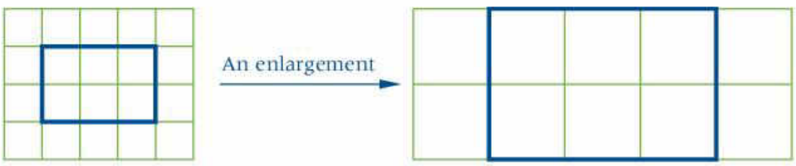
A rotation of 360° about the origin is an example of an identity transformation.

Transformations and images

Remember that a transformation is the *operation* or *process* that changes an object into its image; it is *not* the resulting image. If we use the same matrix to transform several different objects, we will obtain different images but the *transformation* is the same in each case.

A transformation transforms the whole space we are using and carries the object with it to become the image.





Inverse transformations

An **inverse transformation** is one that will map an image back to its object.

Suppose, for example, that we start with an enlargement of scale factor 2. An enlargement of scale factor $\frac{1}{2}$ will then shrink the image down to the size of the object.

If we produce an image by rotating an object through 60° anticlockwise about a point P, then a rotation of 60° clockwise will return it to its original position.

EXERCISE 19i

1 What is the inverse of a rotation of 90° clockwise about the origin?

2 What is the inverse of an enlargement of scale factor 3 and centre the origin?

3 What is the inverse of a reflection in the x-axis?

4 What is the inverse of a rotation of 45° anticlockwise about the origin?

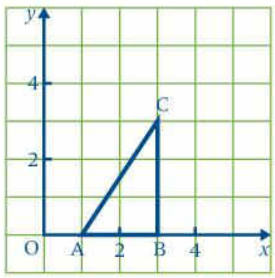
Inverse matrix transformations

EXERCISE 19j

1 Draw x and y axes, marking values from -4 to 4 on each axis.

A is the point (1, 0), B is (3, 0) and C is (3, 3).

M is the matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$



a Transform $\triangle$ABC using the matrix **M** and label the image A′B′C′.
b Describe the transformation.
c Find the inverse of the matrix **M**.
d Use the inverse matrix to transform $\triangle$A′B′C′. What happens?
e Describe the transformation defined by the inverse matrix.

> To find the inverse of a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$, interchange the entries in the leading diagonal: $\begin{pmatrix} d & \\ & a \end{pmatrix}$. Then change the sign of the entries in the other diagonal: $\begin{pmatrix} & -b \\ -c & \end{pmatrix}$. This gives $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. Finally, divide each entry by the determinant of $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$; this is $ad - bc$.

2 Draw x and y axes, marking values from 0 to 6 on each axis.

A is the point (1, 1), B is (2, 1), C is (2, 2) and D is (1, 2).

M is the matrix $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$

a Transform the square ABCD using the matrix **M** and label the image A′B′C′D′.
b Describe the transformation.
c Find the inverse of the matrix **M**.
d Use the inverse matrix to transform A′B′C′D′. What happens?
e Describe the transformation defined by the inverse matrix. Is this the inverse of the transformation described in **b**?

3 Draw x and y axes, marking values from -4 to 4 on each axis. Mark the points A(1, 0), B(3, 1), C(3, 2) and D(1, 1).

a Transform the parallelogram using the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and label the image A′B′C′D′.
b Describe the transformation.
c Find the inverse of the matrix $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.
d Use the inverse matrix to transform A′B′C′D′. What happens?
e Describe the transformation defined by the inverse matrix. Is this the inverse of the transformation defined by $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$?

4 Draw x and y axes, marking values from 0 to 9 on each axis. Mark the points A(1, 1), B(3, 3) and C(1, 3).

a Transform $\triangle$ABC using the matrix $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. Label the image A′B′C′.
b Find the inverse of the matrix $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.
c Use the inverse matrix to transform $\triangle$A′B′C′. What happens?

5 Draw x and y axes, marking values from 0 to 13 on each. Mark the points A(1, 0), B(3, 0), C(3, 2) and D(1, 2).
   a Transform the square ABCD using the matrix $\begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}$.
   b Find the inverse of the matrix $\begin{pmatrix} 1 & 1 \\ 3 & 2 \end{pmatrix}$.
   c Use the inverse matrix to transform A′B′C′D′. What happens?

6 Draw x and y axes, marking values from 0 to 10 on each. Mark the points O(0, 0), A(1, 0), B(1, 2) and C(0, 2).
   a Find the image of rectangle OABC under the transformation defined by the matrix $\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$. What happens?
   b Find, if possible, the inverse of the matrix $\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$.
   c Comment on your answers to **a** and **b**. Does the transformation in **a** have an inverse?

Inverse matrices and inverse transformations

We conclude that if a transformation is defined by a matrix $\mathbf{M}$, the inverse transformation is defined by the inverse matrix $\mathbf{M}^{-1}$.

If $\mathbf{M}^{-1}$ does not exist then the transformation does not have an inverse.

An invariant point

There is one point which is **invariant** under every transformation defined by a matrix. This is found in the next exercise.

EXERCISE **19k**

For each question draw x and y axes, marking values from -5 to 5 on each axis. Mark the points O(0, 0), A(2, 0), B(2, 2) and C(0, 2) and use the square OABC as the object.

1 Find the image of OABC under the transformation defined by the matrix $\begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$. Which of the four points are invariant?

2 Find the image of OABC under the transformation defined by the matrix $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$. Which of the four points are invariant?

3 Find the image of OABC under the transformation defined by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Which of the four points are invariant?

4 Find the image of OABC under the transformation defined by the matrix $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Which point is invariant?

5 Which one point is always invariant whichever transformation is used? Find the image of this point under a transformation defined by a matrix of your choice. Is it still invariant?

We see from the last exercise that, for any transformation defined by a 2×2 matrix, *the origin is invariant.*

Transformations not using matrices

Some of the transformations that can be defined by matrices are:

rotations with centres at the origin
reflections whose mirror lines pass through the origin
enlargements whose centres are at the origin.

There is no transformation matrix that will produce a rotation about (1, 1) or a reflection in the line $x = 2$ or an enlargement with centre (0, 6), because in each of these cases, the origin changes.

In a **translation** in particular, the origin is not an invariant point and therefore a translation cannot be produced by a matrix. We can describe a translation only by stating what the movement or displacement is. The easiest way to do this is to give the vector that describes the displacement.

A **glide reflection** combines a reflection with a translation so it cannot be produced by a matrix.

MATHS IS OUT THERE

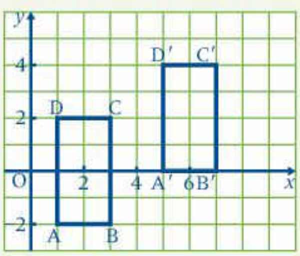
Did you know that the common name for a regular hexahedron is 'a cube'?

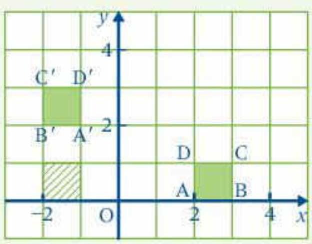
Find the derivation of the word hexahedron.

EXERCISE 19I

Example:

Describe the transformation that maps ABCD to A′B′C′D′.

a


b


a The transformation is a translation given by the vector $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$

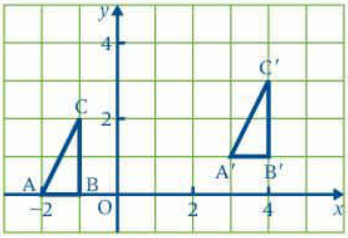
Consider the displacement from D to D′.

b The transformation is a glide reflection in the line $x = \frac{1}{2}$ by $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$

A′B′C′D′ is not a single translation because B′C′ is to the left of A′D′ whereas BC is to the right of AD.

Describe the transformations in questions **1** to **8**.

1



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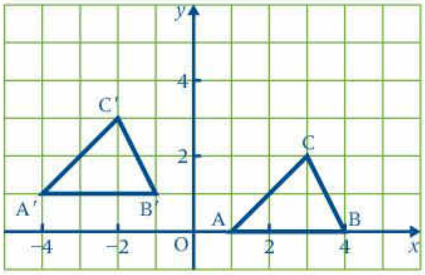

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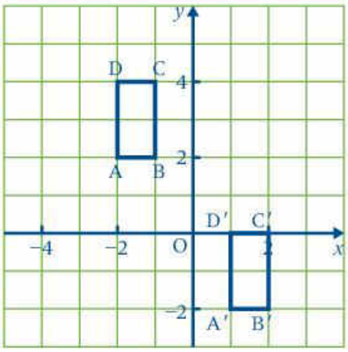

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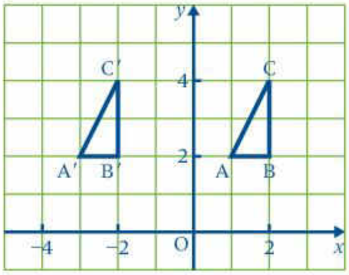

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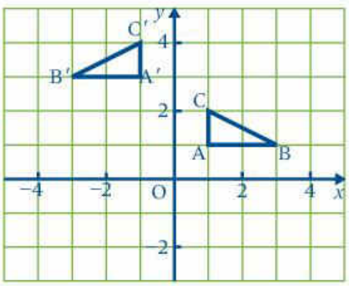

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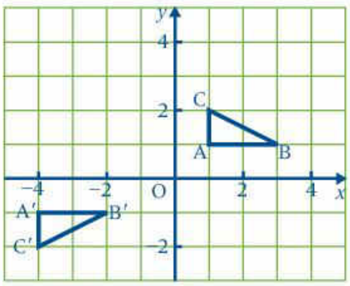

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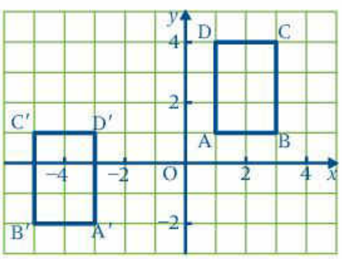

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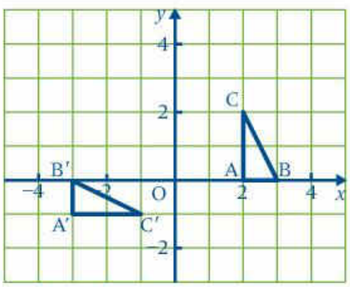
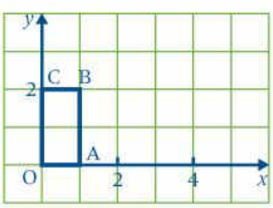


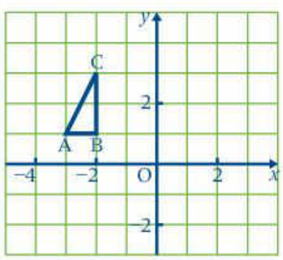
In questions **9** to **12**, find the image of the given object under a translation defined by the given vector.

9



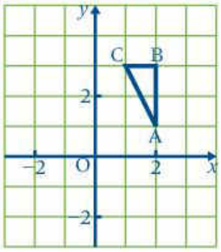
The vector is $\begin{pmatrix} 3 \\ 1 \end{pmatrix}$

10



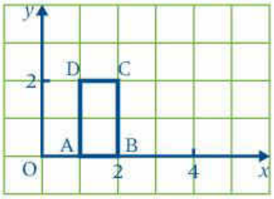
The vector is $\begin{pmatrix} 4 \\ -2 \end{pmatrix}$

11



The vector is $\begin{pmatrix} -3 \\ -3 \end{pmatrix}$

12



The vector is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

13 Describe the inverse of the transformations given in questions **1** to **4**.

MIXED EXERCISE 19

Several answers are given for these question.
Write down the letter that corresponds to the correct answer.

1 The image of the position vector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ under a reflection in the line $y = x$ is the position vector

A $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ **B** $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ **C** $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ **D** $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$

2 The image of the position vector $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ under the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is

A $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ **B** $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ **C** $\begin{pmatrix} 2 \\ -1 \end{pmatrix}$ **D** $\begin{pmatrix} 2 \\ -2 \end{pmatrix}$

3 The transformation defined by the matrix $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ is

A an enlargement
B a reflection
C a rotation
D a glide reflection

4 An enlargement with scale factor 2 and centre the origin can be defined by the matrix

A $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ **B** $\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ **C** $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ **D** $\begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$

5 The inverse of a reflection in the y-axis is

A a reflection in the x-axis
B a reflection in the y-axis
C a rotation of 90° about the origin
D a rotation of 180° about the origin

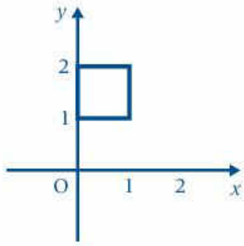
6 The image of the point (a, b) under the matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ is the point

A (0, 0) **B** $(-a, -b)$ **C** (1, 1) **D** (a, b)

7 The square in the diagram is transformed by the matrix $\begin{pmatrix}1 & 1\\2 & 2\end{pmatrix}$.
The coordinates of the image can be found from

A $\begin{pmatrix}1 & 1\\2 & 2\end{pmatrix}\begin{pmatrix}0 & 1 & 2 & 1\\1 & 1 & 1 & 0\end{pmatrix}$ **B** $\begin{pmatrix}1 & 1\\2 & 2\end{pmatrix}\begin{pmatrix}1 & 0 & 1 & 0\\1 & 1 & 2 & 2\end{pmatrix}$

C $\begin{pmatrix}1 & 1\\2 & 2\end{pmatrix}\begin{pmatrix}1 & 1\\1 & 2\end{pmatrix}$ **D** $\begin{pmatrix}1 & 0 & 1 & 1\\1 & 1 & 2 & 0\end{pmatrix}\begin{pmatrix}1 & 1\\2 & 2\end{pmatrix}$



MATHS IS OUT THERE

THEY CAME IN THREES

9 February D. Bernoulli (1700–1783); W.F. Bolyai (1775–1856); L. Fejer (1880–1959)

15 February Galileo (1564–1642); C. Navier (1785–1826); A. Whitehead (1861–1949)

Two sets of three mathematicians shared the same birthday in February.

IN THIS CHAPTER YOU HAVE SEEN THAT...

- the position vector of a point A, is the vector from the origin to A
- when a position vector is pre-multiplied by a matrix, the resulting vector is the image of the position vector
- matrices can be used to perform transformations
- the identity transformation leaves the object unchanged
- an inverse transformation maps the image back to the object
- when **M** is the transformation matrix, $\mathbf{M}^{-1}$ is the inverse transformation matrix. If $\mathbf{M}^{-1}$ does not exist, there is no inverse transformation.

MATHS IS OUT THERE

Teacher: Where do mathematicians who want to reflect or to enlarge their figure sleep?

Student: On matrices.

20 COMPOUND TRANSFORMATIONS

AT THE END OF THIS CHAPTER YOU SHOULD BE ABLE TO...

1 Find the image of an object under the product of two given transformations.

2 Name a single transformation equivalent to the product of two transformations.

BEFORE YOU START

you need to know:
✓ how to add, subtract and multiply matrices
✓ how to perform the transformations of reflection, rotation, translation and enlargement
✓ how to find the determinant of a matrix.

KEY WORDS

area scale factor, base vector, compound transformation, enlargement, identity transformation, reflection, rotation

MATHS IS OUT THERE

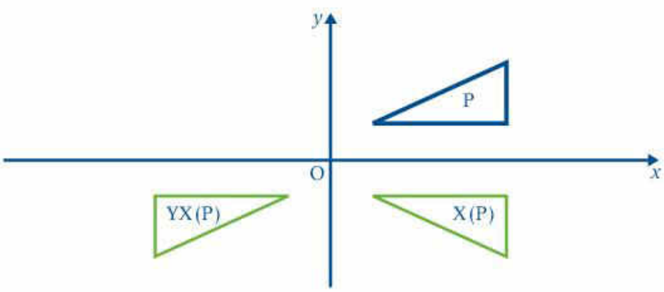
Did you know that points in the coordinate plane that are arranged like the pegs on a geoboard are called *lattice points*?

Compound transformations

If we reflect the object P in the *x*-axis and then reflect its image in the *y*-axis, we are carrying out a **compound transformation**.

If we denote the reflection in the *y*-axis by Y, and reflection in the *x*-axis by X, then the image of P in the *y*-axis is named Y(P); similarly the image of P in the *x*-axis will be denoted by X(P).

Using this notation, the compound transformation described above will be written as YX(P).



Notice that the letter X denoting the first transformation is nearer to the object P.

The 'product' YX of two transformations X and Y is equivalent to that obtained by performing the transformation X first and following this with transformation Y.

EXERCISE 20a

Example:

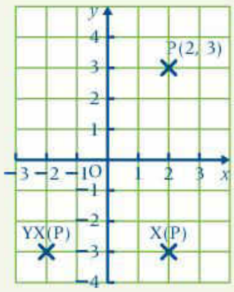
Find the image of the point P(2, 3), under the compound transformation YX, where X and Y are defined as above.

First, we find X(P), i.e. reflection of (2, 3) in the x-axis.

Therefore X(P) = (2, −3).

Then, find YX(P), i.e. reflection of (2, −3) in the y-axis

i.e. (−2, −3).



In this exercise,

R_1 is a rotation of 90° anticlockwise about O

R_2 is a rotation of 180° about O

R_3 is a rotation of 90° clockwise about O

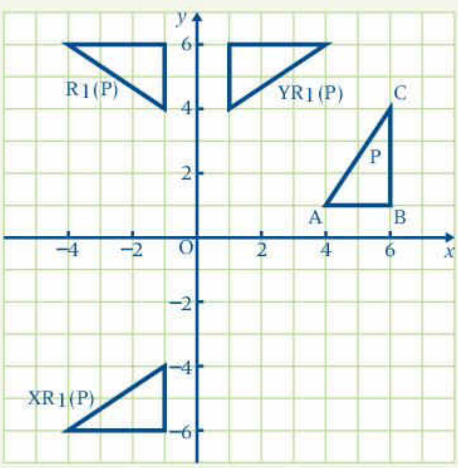
X is a reflection in the x-axis

Y is a reflection in the y-axis

T is a translation defined by the vector $\begin{pmatrix} 4 \\ 3 \end{pmatrix}$

Example:

A, B and C are the points (4, 1), (6, 1) and (6, 4).
Draw △ABC and label it P. Draw and label R_1(P), XR_1(P) and YR_1(P)



In each of the following questions draw x and y axes, each for values from −6 to 6.

1 P, Q and R are the points (1, 3), (3, 3) and (1, 6). Draw △PQR and label it A. Draw and label
   a $R_3(A)$ b $XR_3(A)$ c X(A)   d $R_3X(A)$
   e Describe the single transformation that will map A to $R_3X(A)$.

2 L, M and N are the points (−2, 1), (−4, 1) and (−2, 5). Draw △LMN and label it P. Draw and label
   a $R_1(P)$ b $R_2R_1(P)$ c $R_3R_1(P)$
   d What is the single transformation that will map P to $R_2R_1(P)$?
   e What is the single transformation that will map $R_2R_1(P)$ to $R_1(P)$?

3 A, B and C are the points (−5, 2), (−2, 2) and (−5, 4). Draw △ABC and label it P. Draw and label
   a X(P)   b YX(P)   c $R_2(P)$ d Y(P)   e XY(P)
   f Is YX(P) the same triangle as XY(P)?
   g Is $R_2(P)$ the same triangle as XY(P)?
   h What single transformation is equivalent to a reflection in the *x*-axis followed by a reflection in the *y*-axis?

4 L, M and N are the points (−3, 0), (−1, 0) and (−1, 3). Draw △LMN and label it Q. Find
   a T(Q)   b XT(Q)   c X(Q)   d TX(Q)
   e Describe the single transformation that will map X(Q) to XT(Q).
   f Describe the single transformation that will map XT(Q) to TX(Q).

MATHS IS OUT THERE

Did you know that the *pentangle* and *pentagram* are names for the same shape?

This figure was used as a symbol of mystery by the Greeks and various societies.

In the Middle Ages many people believed that the *pentacle* had the power to keep away evil spirits.

Equivalent single transformations

We have seen that if we reflect an object P in the *x*-axis and then reflect the image X(P) in the *y*-axis we get the same final image as if we had rotated P through 180° about O. YX(P) is the same as $R_2(P)$ and the effect of YX is the same as the effect of R_2.

We can write $YX(P) = R_2(P)$ referring to the images and $YX = R_2$ referring to the transformations.

In the following exercise notice that $X^2 = XX$, i.e. the transformation X is used twice in succession.

EXERCISE 20b

In each question draw *x* and *y* axes, each for values from −6 to 6. Use 1 cm for 1 unit.

1 A is a reflection in the line $x = -1$ and B a reflection in the line $y = 2$. Label as Z the triangle PQR where P is the point (1, 4), Q(4, 6) and R(1, 6).
   a Find A(Z), B(Z), AB(Z) and BA(Z).
   b Describe the single transformations given by AB and BA. Is AB equal to BA?
   c Find $A^2(Z)$ and $B^2(Z)$.

2 T is a reflection in the line $x = 1$.
   U is a reflection in the line $y = 2$.
   V is a rotation of 180° about the point (1, 2).

Label with A the triangle PQR where P is the point (1, 1), Q is (3, 1) and R is (3, −2).

a Draw T(A), U(A), TU(A) and UT(A).
b Are TU and UT the same transformation?
c Is it true that V = TU?

3 R_1 is a rotation of 90° anticlockwise about O.
R_2 is a rotation of 180° about O.
R_3 is a rotation of 90° clockwise about O.
Label with P the triangle ABC where A is the point (1, 2), B is (4, 2) and C is (1, 4).

a Draw $R_1(P)$, $R_1^2(P)$, $R_2(P)$, $R_2R_1(P)$ and $R_3(P)$.
Complete the following statements

$R_1^2 =$ and $R_2R_1 =$

b Draw whatever images are needed and complete the following statements:

$R_3^2 =$ $R_2R_3 =$ and $R_3R_2 =$

The identity transformation

If an object is rotated through 360° or translated using the vector $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$, the final image turns out to be the same as the original object. We are back where we started and might as well not have performed a transformation at all. This operation is called the **identity transformation** and is usually denoted by I.

EXERCISE 20c

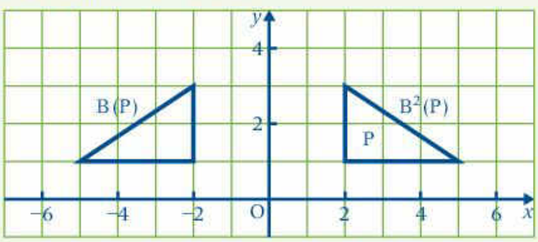
In this exercise,
R_1 is a rotation of 90° anticlockwise about O
R_2 is a rotation of 180° about O
R_3 is a rotation of 90° clockwise about O
A is a reflection in the x-axis
B is a reflection in the y-axis
C is a reflection in the line $y = x$
D is a reflection in the line $y = -x$
I is the identity transformation

Example:

P is the triangle with vertices (2, 1), (5, 1) and (2, 3).
Find B(P) and $B^2(P)$. Name the single transformation which is equal to B^2.

$B^2(P) = P$

$\therefore \quad B^2 = I$



For each question draw x and y axes, each for values from -6 to 6.
Use 1 cm to 1 unit.

1 P is the triangle with vertices (2, 1), (5, 1) and (5, 5).

a Find $R_1(P)$, $R_2(P)$, $R_3(P)$, $R_1R_3(P)$ and $R_3R_1(P)$.
Name the single transformation which is equal to both R_1R_3 and R_3R_1.

b Complete the following statements with a single letter.
i $R_2^2 =$ **ii** $R_2R_3 =$ **iii** $R_1R_2 =$

2 Q is the triangle with vertices $(-2, 1)$, $(-5, 1)$ and $(-4, 5)$.

a Find A(Q), B(Q), AB(Q), $R_2(Q)$ and $B^2(Q)$.

b Complete the following statements with a single letter.
i $B^2 =$ **ii** $AB =$

3 N is the triangle with vertices (1, 3), (1, 6) and (5, 6).

a Find C(N), DC(N), $C^2(N)$, AC(N), BC(N) and IC(N).

b Complete the following statements with a single letter.
i $DC =$ **ii** $C^2 =$ **iii** $AC =$ **iv** $BC =$ **v** $IC =$

4 M is the triangle with vertices (3, 2), (5, 2) and (5, 6).

a Find $R_1(M)$, A(M), $A^2(M)$, $AR_1(M)$ and $R_1A(M)$.

b Complete the following statements with a single letter.
i $A^2 =$ **ii** $AR_1 =$ **iii** $R_1A =$

c Is the statement $AR_1 = R_1A$ true or false?

5 L is the triangle with vertices (3, 1), (4, 4) and (1, 4).

a Find I(L), AI(L), BI(L), IA(L) and IB(L).

b Simplify AI, BI, IA and IB.

6 P is the rectangle with vertices $(-2, -2)$, $(-5, -2)$, $(-5, -4)$ and $(-2, -4)$.

a Find I(P), CI(P), DI(P) and $D^2(P)$.

b Simplify CI, DI and D^2.

7 Q is the rhombus with vertices $(-3, -3)$, $(-4, -1)$, $(-3, 1)$ and $(-2, -1)$.

a Find $R_1(Q)$, $R_2R_1(Q)$ and $R_1R_2R_1(Q)$.

b Simplify R_2R_1 and $R_1R_2R_1$.

c Is it true that $R_1R_3 = R_3R_1 = R_2^2$?

Common transformation matrices

It is useful to know the following facts.

- An **enlargement** matrix is of the form $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$.
- **Reflection** in the x- or y-axis or lines $y = \pm x$ and rotations of multiples of 90° about the origin are produced by matrices with zeros in one diagonal and 1 or -1 elsewhere, e.g. $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ gives a **rotation** of 180° about O.
- There are no matrices that produce reflections in lines other than lines through the origin. Nor are there any matrices that produce rotation or enlargement about points other than the origin.
- The unit matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ maps an object to itself. This matrix gives the identity transformation (for instance, a rotation of 360° about O).
- A translation is *not* produced by a 2×2 matrix but is defined by a vector.

Area scale factor

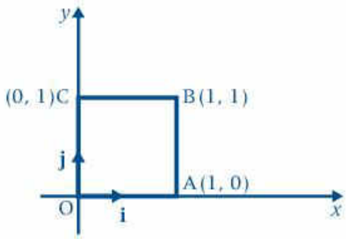
The fraction $\frac{\text{area of the image}}{\text{area of the object}}$ is the **area scale factor**, and it is given by the determinant of the transformation matrix.

If we know the area of the object, the determinant can be used to calculate the area of the image.

Unit vectors and the unit square

The square OABC where A is the point (1, 0) and C is (0, 1) is called the unit square.

This is a particularly easy object to use when we want to identify a transformation, as the position vectors of A and C are $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ and, lined up together, they form the unit matrix $\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$.



The vectors $\begin{pmatrix}1\\0\end{pmatrix}$ and $\begin{pmatrix}0\\1\end{pmatrix}$ are called the **base vectors** for the coordinates and are denoted by **i** and **j**.

Often O, A′ and C′ are enough to identify the image of the unit square but, if the shape of the image is still not clear, we can also find the image of B(1, 1).

EXERCISE **20d**

Using the unit square as the object, in each question from **1** to **6** find the image under the transformation defined by the given matrix. If it is possible to do so, identify the transformation, describing it fully.

1 $\begin{pmatrix}0 & -1\\1 & 0\end{pmatrix}$ **2** $\begin{pmatrix}1 & -2\\0 & 1\end{pmatrix}$ **3** $\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}$

4 $\begin{pmatrix}2\frac{1}{2} & 0\\0 & 2\frac{1}{2}\end{pmatrix}$ **5** $\begin{pmatrix}1 & 2\\0 & 1\end{pmatrix}$ **6** $\begin{pmatrix}3 & 12\\1 & 4\end{pmatrix}$

7 Find the area scale factor of each transformation in questions **1** to **6**.

8 Given the matrix $\begin{pmatrix}4 & -8\\-1 & 2\end{pmatrix}$ find

a the image of the unit square under the transformation given by the matrix

b the determinant of the matrix and the area scale factor of the transformation

c the area of the image.

9 A rectangle ABCD has area 6 square units. It is transformed using the matrix $\begin{pmatrix}3 & 1\\1 & 2\end{pmatrix}$

a Find the determinant of the matrix and the area scale factor of the transformation.

b Find the area of the image of ABCD.

10 A, B, C and D are the points (0, 2), (0, 4), (3, 4) and (3, 2) respectively.

a Find the area of rectangle ABCD.

b ABCD is transformed using the matrix $\begin{pmatrix}4 & 5\\1 & 2\end{pmatrix}$

Calculate the area of the image of ABCD.
(Do *not* find and draw the image.)

Finding a transformation matrix

EXERCISE 20e

Example:

A is the point (2, 1), B is (3, 4), P is (5, 3) and Q is (10, 12).

Find the matrix of the transformation under which AB is mapped to PQ.

Let the transformation matrix be $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

then $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \overset{\text{A} \quad \text{B}}{\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}} = \overset{\text{P} \quad \text{Q}}{\begin{pmatrix} 5 & 10 \\ 3 & 12 \end{pmatrix}}$

$\begin{pmatrix} 2a + b & 3a + 4b \\ 2c + d & 3c + 4d \end{pmatrix} = \begin{pmatrix} 5 & 10 \\ 3 & 12 \end{pmatrix}$

Comparing entries in the first row gives

$2a + b = 5 \qquad (1)$

$3a + 4b = 10 \qquad (2)$

(1) × 4 $\qquad 8a + 4b = 20 \qquad (3)$

$3a + 4b = 10 \qquad (2)$

(3) − (2) $\qquad 5a = 10$

$a = 2$

In (1) $\qquad b = 1$

Comparing entries in the second row gives

$2c + d = 3 \qquad (4)$

$3c + 4d = 12 \qquad (5)$

(4) × 4 $\qquad 8c + 4d = 12 \qquad (6)$

$3c + 4d = 12 \qquad (5)$

(6) − (5) $\qquad 5c = 0$

$c = 0$

In (5) $\qquad d = 3$

∴ the matrix is $\begin{pmatrix} 2 & 1 \\ 0 & 3 \end{pmatrix}$

In each question from **1** to **6** find the matrix of the transformation which maps AB to PQ.

1 A is (1, 1), B is (2, 1), P is (1, 2) and Q is (2, 2)

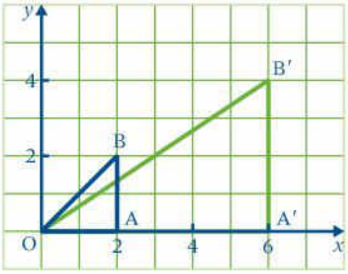
2 A is (1, 1), B is (3, 1), P is (3, 2) and Q is (5, 6)

3 A is (4, 2), B is (1, 1), P is (12, 2) and Q is (3, 2)

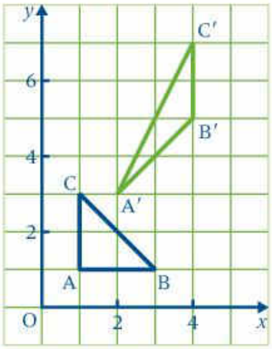
4 A is (2, 1), B is (1, 1), P is (1, 5) and Q is (0, 5)

5 A is (2, 2), B is (1, 3), P is (−6, 6) and Q is (−11, 9)

6 A is (1, 2), B is (1, −2), P is (1, 0) and Q is (17, −4)

7


A transformation maps △OAB to △OA′B′. Find the matrix that defines this transformation. (Use AB and its image A′B′.)

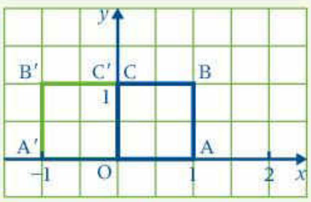
8


A transformation maps △ABC to △A′B′C′. Find the matrix that defines this transformation.

Use two points to find the matrix. The third point may be used for a check.

Example:

Find the matrix that defines reflection in the *y*-axis.



Let the matrix be $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\overset{\text{A}\quad\text{C}}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} = \overset{\text{A}'\quad\text{C}'}{\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}}$$

$$\therefore \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

i.e. the transformation matrix is $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

We may choose our own object so we use the unit square. This is useful because we can use the unit matrix.

In each question from **9** to **12** , find the matrix that defines the transformation.

9 Reflection in the *x*-axis.

10 Rotation of 90° clockwise about O.

11 Rotation of 90° anticlockwise about O.

12 Reflection in the line $y = x$.

Compound transformations

EXERCISE **20f**

1 A square ABCD has vertices at A(2, 0), B(4, 0), C(4, 2) and D(2, 2).

$$\mathbf{P} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \mathbf{Q} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

- **a** Find the images of ABCD under the two transformations given by **P** and **Q** and identify the transformations. Label the two images $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ for **P** and **Q** respectively.
- **b** Find the image of $A_1B_1C_1D_1$ under the transformation given by Q. Label the image $A_3B_3C_3D_3$.
- **c** Describe the transformation that maps ABCD to $A_3B_3C_3D_3$.
- **d** Find the matrices **R** and **S** where **R** = **QP** and **S** = **PQ**.
- **e** Find the image of ABCD under the transformation given by **R**.
- **f** Find the image of ABCD under the transformation given by **S**.
- **g** Comment on the results of **b**, **e** and **f**. Explain the significance of the order in which **P** and **Q** occur.

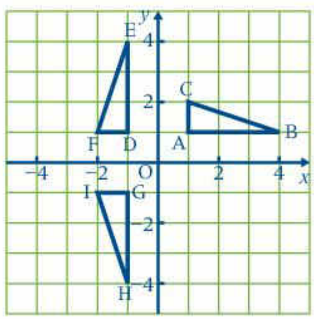
2 The vertices of a triangle P are (2, 1), (4, 1) and (4, 4).

$\mathbf{M} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ and $\mathbf{N} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ define transformations M and N respectively.

- **a** Find the matrix **L**, given that **L** = **NM**.
- **b** Find **i** M(P) and **ii** NM(P).
- **c** Describe the transformations M and NM.

3 Using the figure find the matrices **P**, **Q** and **R** if

- **a** **P** is the matrix of the transformation which maps △ABC to △DEF.
- **b** **Q** is the matrix of the transformation which maps △DEF to △GHI.
- **c** **R** is the matrix of the transformation which maps △ABC to △GHI.
- **d** Give an equation linking **P**, **Q** and **R** .



Defining transformations as matrix equations

The matrix $\begin{pmatrix} 2 & 3 \\ 0 & -1 \end{pmatrix}$ maps the point A whose position vector is $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ to the point A′ whose position vector is given by the product $\begin{pmatrix} 2 & 3 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 2 \\ 4 \end{pmatrix}$ i.e. $\begin{pmatrix} 16 \\ -4 \end{pmatrix}$.

In general the matrix will map any point P whose position vector is $\begin{pmatrix} x \\ y \end{pmatrix}$ to the point P′ whose position vector is $\begin{pmatrix} x' \\ y' \end{pmatrix}$ where $\begin{pmatrix} 2 & 3 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$

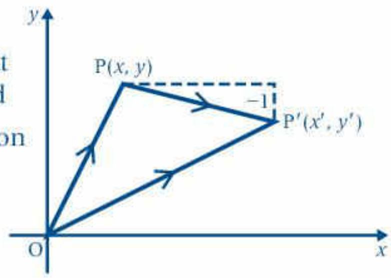
We can use this matrix equation to define the transformation.

A translation cannot be defined in terms of a 2×2 matrix, but we can still use a matrix equation if we consider a position vector in the form $\begin{pmatrix} x \\ y \end{pmatrix}$ as a 2×1 matrix.

Consider a translation defined by the vector $\begin{pmatrix} 3 \\ -1 \end{pmatrix}$.

This vector will move any point 3 units to the right and 1 unit down, i.e. it will move P to P′. Using vector addition, the relationship between OP and OP′ is $\overrightarrow{OP} + \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \overrightarrow{OP'}$, i.e. the translation can be defined by the equation

$$\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 3 \\ -1 \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$



Compound transformations and matrices

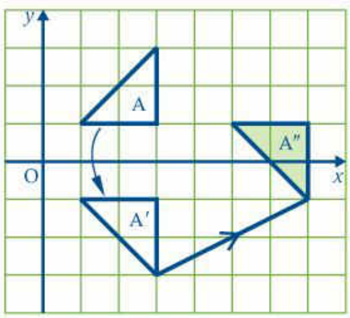
From the last exercise we can see that the matrix for a compound transformation ST is given by the matrix product **ST**.

For example, if S is defined by the matrix $\begin{pmatrix} 2 & 3 \\ -1 & 0 \end{pmatrix}$ and T is defined by $\begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}$ then the matrix for the compound transformation ST is given by the product $\begin{pmatrix} 2 & 3 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ -1 & 0 \end{pmatrix}$, which is equal to the matrix $\begin{pmatrix} -3 & 4 \\ 0 & -2 \end{pmatrix}$.

Hence the transformation ST is defined by $\begin{pmatrix} -3 & 4 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$.

This is true *only* if **S** and **T** are both 2×2 matrices.

Now consider the compound transformation of a reflection in the x-axis followed by a translation given by the vector $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$. The diagram shows the effect on a triangle A.



The matrix which maps A to A′ is $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

This matrix maps any point $\begin{pmatrix} x \\ y \end{pmatrix}$ to the point given by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ which is then translated to the point given by $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \end{pmatrix}$.

Hence if $\begin{pmatrix} x \\ y \end{pmatrix}$ is mapped to $\begin{pmatrix} x' \\ y' \end{pmatrix}$ then the combined transformation can be expressed as the matrix equation

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$$

EXERCISE **20g**

Questions **1** to **5** refer to △ABC with A(1, 1), B(4, 1) and C(2, 3).

1 Draw the image of △ABC under the transformation defined by

a $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$ **b** $\begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$

c $\begin{pmatrix} 3 & 0 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$ **d** $\begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$

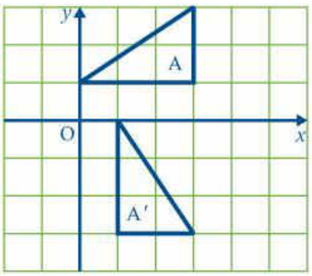
2 A transformation is defined by the matrix equation
$\begin{pmatrix}-1 & 4\\ 1 & 2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}x'\\ y'\end{pmatrix}$. Draw a diagram to show the image of △ABC under this transformation.

3 Draw a diagram to show the image of △ABC under the transformation defined by $\begin{pmatrix}2 & 0\\ 0 & 2\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}+\begin{pmatrix}3\\ -1\end{pmatrix}=\begin{pmatrix}x'\\ y'\end{pmatrix}$
Describe this transformation in words.

4 Repeat question **4** with the transformation defined by

$$\begin{pmatrix}0.5 & 0\\ 0 & -0.5\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}+\begin{pmatrix}-3\\ 2\end{pmatrix}=\begin{pmatrix}x'\\ y'\end{pmatrix}$$

Using the diagrams given in questions **5–8**, answer the following
a describe the transformation that maps A to A′
b give a matrix equation that defines the transformation.
The side of one square is one unit.

5


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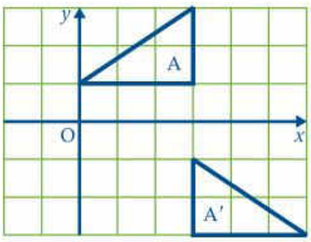

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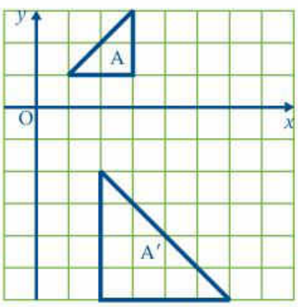

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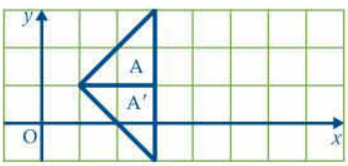


9 A point whose coordinates are (2, 5) is translated by the vector $\begin{pmatrix}2\\ 4\end{pmatrix}$ and then reflected in the y-axis. What is the image of the point? Express the transformation as a matrix equation.

MIXED EXERCISE 20

Sereral answers are given tor these questions.
Write down the letter that corresponds to the correct answer.

1 The determinant of the matrix $\begin{pmatrix}4 & 6\\ 2 & 4\end{pmatrix}$ is

A 16 **B** 4 **C** $\frac{1}{4}$ **D** −16

2 The inverse of the matrix $\begin{pmatrix} 3 & 1 \\ 4 & 2 \end{pmatrix}$ is

A $\begin{pmatrix} 1 & -\frac{1}{2} \\ -2 & 1\frac{1}{2} \end{pmatrix}$ **B** $\begin{pmatrix} 2 & -1 \\ -4 & 3 \end{pmatrix}$ **C** $\begin{pmatrix} 1\frac{1}{2} & \frac{1}{2} \\ 2 & 1 \end{pmatrix}$ **D** $\begin{pmatrix} \frac{1}{5} & \frac{1}{10} \\ -\frac{2}{5} & \frac{3}{10} \end{pmatrix}$

3 The inverse of a rotation of 270° clockwise about O is
A a rotation of 90° clockwise about O
B a rotation of 270° anticlockwise about (1, 0)
C a rotation of 90° anticlockwise about O
D a rotation of 270° clockwise about O.

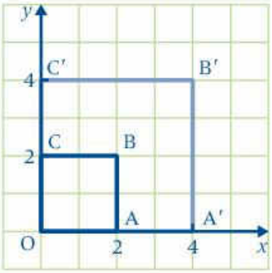
4 The area scale factor of the transformation defined by

$\begin{pmatrix} 3 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$ is

A $\frac{1}{5}$ **B** 1 **C** 2 **D** 5

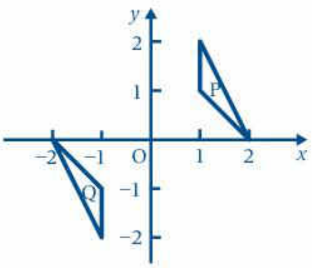
5 The matrix which maps OABC to OA′B′C′ is

A $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ **B** $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ **C** $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ **D** $\begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$



6 $\mathbf{A} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $\mathbf{A}^n$ is

A $\begin{pmatrix} 2n & 0 \\ 0 & 2n \end{pmatrix}$ **B** $\begin{pmatrix} n^2 & 0 \\ 0 & n^2 \end{pmatrix}$ **C** $\begin{pmatrix} 2^n & 0 \\ 0 & 2^n \end{pmatrix}$ **D** $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$



7 The compound transformation 'reflecton in the y-axis' followed by 'reflection in the x-axis' is equivalent to
A a rotation of 90° about O
B a translation parallel to $y = x$
C a reflection in the line $y = -x$
D a glide reflection parallel to the y-axis

8 The matrix transformation that maps P to Q is

A $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ **B** $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ **C** $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ **D** $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

IN THIS CHAPTER YOU HAVE SEEN THAT...

- the image of an object P under the transformation X followed by the transformation Y is denoted by YX(P). The transformation nearest P is the one that is performed first
- the area scale factor is the area of the image as a fraction of the area of the object and it is given by the determinant of the transformation matrix
- the matrix for the compound transformation X followed by Y, is the matrix product **YX**.

MATHS IS OUT THERE

Did you know that a *median* is a line joining a vertex of a triangle to the mid-point of the opposite side?

Did you know that the *centroid* of a triangle is the point in the triangle through which the three medians pass? It is the centre of gravity of the triangular figure.

Multiple choice questions

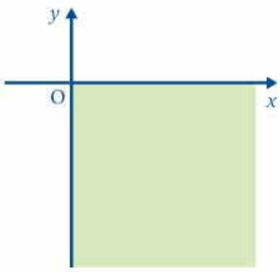
Several possible answers are given.
Write down the letter that corresponds to the correct answer.

1 If F varies as the product of m and M over the square of r then F may be expressed as

A $F \propto \frac{r^2}{mM}$ **B** $F \propto \frac{mM}{r^2}$ **C** $F \propto \frac{mM}{\sqrt{r}}$ **D** $F \propto \frac{\sqrt{r}}{mM}$

2 The shaded region shown in the above diagram is bounded by

A $\{x \leqslant 0\} \cap \{y \leqslant 0\}$ **B** $\{x \geqslant 0\} \cap \{y \leqslant 0\}$
C $\{x \leqslant 0\} \cap \{y \geqslant 0\}$ **D** $\{x \geqslant 0\} \cap \{y \geqslant 0\}$



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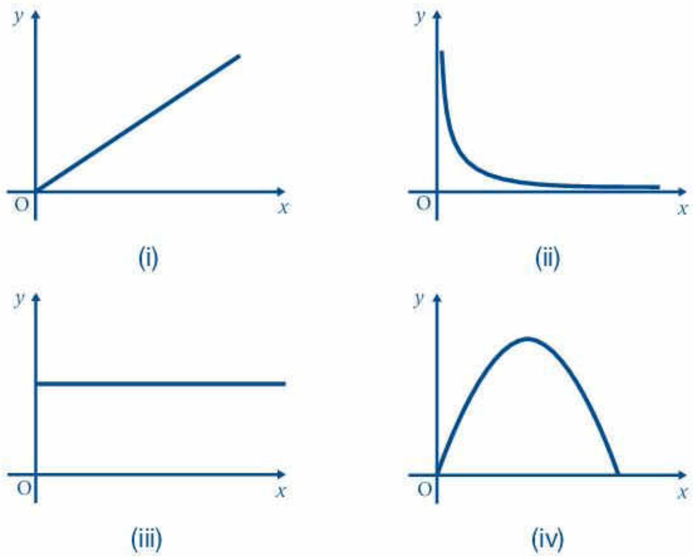


y varies inversely as x is best illustrated in diagram

A (i) **B** (ii) **C** (iii) **D** (iv)

4 The matrix $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ represents

A an anticlockwise rotation of 90° about O
B a clockwise rotation of 90° about O
C a rotation of 180° about O
D a reflection in the y-axis

5 Given $\mathbf{a} = 2\mathbf{i} + 3\mathbf{j}$ and $\mathbf{b} = \mathbf{i} - \mathbf{j}$. If $\mathbf{a} - 2\mathbf{b} = p\mathbf{i} + q\mathbf{j}$, then

A $p = 1$ and $q = 2$ **B** $p = 0$ and $q = 5$
C $p = 5$ and $q = 0$ **D** $p = 1$ and $q = -2$

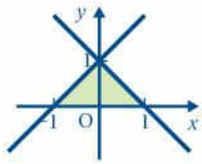
6 If $\mathbf{M} = \begin{pmatrix} 3 & x \\ 2 & 4 \end{pmatrix}$ and **M** is singular, then the value of x is
A 12 B 6 C −2 D −6

7 Point P is mapped onto P′ (−2, 3) under the translation **T** where **T** is $\begin{pmatrix} -1 \\ 2 \end{pmatrix}$.
P has coordinates
A (−3, 5) B (−1, 1) C (1, −1) D (2, 0)

8 y is inversely proportional to the sum of a and b. $y =$
A $y \propto \frac{a}{b}$ B $y \propto \frac{1}{a} + \frac{1}{b}$ C $y \propto \frac{1}{a + b}$ D $y \propto a + b$

9 The region enclosed by the x-axis and the lines $x + y = 1$ and $y - x = 1$ is the region in the diagram denoted by
A P B Q C R D S

10 Which of the following points is NOT in the shaded region?
A $\left(\frac{1}{2}, \frac{1}{2}\right)$ B (0, 1) C (1, 1) D $\left(0, \frac{1}{2}\right)$



Questions **11** and **12** refer to this information.

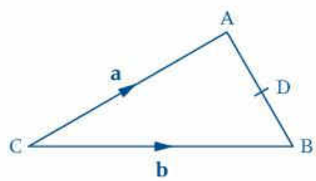
M is a transformation defined by $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and **T** is a transformation defined by $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$

11 The transformation **MT** is defined by
A $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ B $\begin{pmatrix} 2 \\ 3 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ C $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \end{pmatrix}$ D none of these

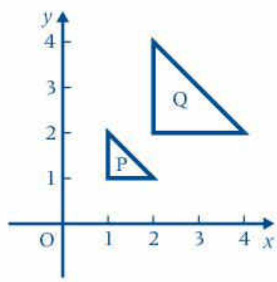
12 The single transformation equivalent to **MT** is
A an enlargement B a rotation
C a glide reflection D a translation

13 A is the point (3, −1) and B is the point (−2, 2). $\overrightarrow{AB} =$
A $\begin{pmatrix} 5 \\ -3 \end{pmatrix}$ B $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ C $\begin{pmatrix} -5 \\ 3 \end{pmatrix}$ D $\begin{pmatrix} -1 \\ -1 \end{pmatrix}$

14 In the diagram, D is the mid-point of AB. $\overrightarrow{AD} =$
A **b** − **a** B $\frac{1}{2}$**a** + **b** C $\frac{1}{2}$**a** − $\frac{1}{2}$**b** D $\frac{1}{2}$**b** − $\frac{1}{2}$**a**



15 The matrix transformation that maps P to Q is
A $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ B $\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ C $\begin{pmatrix} 4 & 0 \\ 0 & 4 \end{pmatrix}$ D $\begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$



16 The transformation matrix $P = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ represents
A a reflection in the line $x = 3$
B a reflection in the line $y = 2$
C an enlargement by scale factor 3
D a 180° rotation about the point (0, 2)

17 The transformation matrix $Q = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ represents
A a reflection in the line $x = 1$
B a reflection in the line $y = 1$
C an enlargement by scale factor 3
D a 90° rotation anticlockwise about O

18 The transformation matrix $P = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ represents
A a rotation of 90° anticlockwise about O
B a reflection in the line $y = 1$
C an enlargement by scale factor -1
D a 180° rotation about O

19 Given that $2y \geqslant 3x - 4$, which of the following values of x and y do not satisfy the inequality?
A (2, 1) **B** (2, −2) **C** (−2, 4) **D** (−2, −4)

20 Given that $y \propto \sqrt{x}$, when x is multiplied by 4, y is multiplied by
A 1 **B** 2 **C** 4 **D** 16

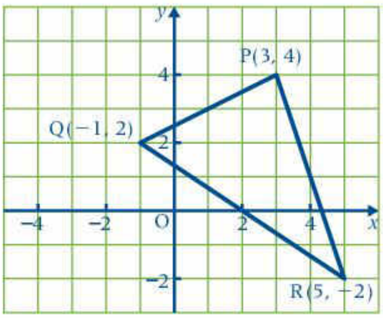
21 Given that $y \propto \frac{1}{\sqrt{z}}$, when z is multiplied by 4, y is multiplied by
A $\frac{1}{4}$ **B** $\frac{1}{2}$ **C** 2 **D** 4

22 y varies inversely as x^3 and $y = 8$ when $x = 2$. When $y = 1$ the value of x is
A 1 **B** 2 **C** 4 **D** 8

23 The translation $\begin{pmatrix} 3 \\ -2 \end{pmatrix}$ followed by the translation $\begin{pmatrix} 1 \\ 4 \end{pmatrix}$ is equivalent to the translation
A $\begin{pmatrix} 2 \\ 2 \end{pmatrix}$ **B** $\begin{pmatrix} 2 \\ -6 \end{pmatrix}$ **C** $\begin{pmatrix} 4 \\ -2 \end{pmatrix}$ **D** $\begin{pmatrix} 4 \\ 2 \end{pmatrix}$

24 If $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$ is the position vector of the point P and $\begin{pmatrix} 9 \\ -4 \end{pmatrix}$ is the position vector of the point Q, then $|\mathbf{PQ}| =$
A 6 **B** 8 **C** 10 **D** 12

Use this diagram for questions **25** and **26**.



25 The vector $\overrightarrow{PR}$ is
A $\begin{pmatrix} -2 \\ -6 \end{pmatrix}$ **B** $\begin{pmatrix} -2 \\ 6 \end{pmatrix}$ **C** $\begin{pmatrix} 2 \\ 6 \end{pmatrix}$ **D** $\begin{pmatrix} 2 \\ -6 \end{pmatrix}$

26 If M is the midpoint of PR the position vector of M is
A $\begin{pmatrix} -4 \\ -1 \end{pmatrix}$ **B** $\begin{pmatrix} -4 \\ 1 \end{pmatrix}$ **C** $\begin{pmatrix} 4 \\ -1 \end{pmatrix}$ **D** $\begin{pmatrix} 4 \\ 1 \end{pmatrix}$

27 The transformation $P = \begin{pmatrix} -3 & 0 \\ 0 & -3 \end{pmatrix}$ represents
A a reflection in the line $x = 3$
B a reflection in the line $y = 3$
C an enlargement by a scale factor of -3
D a 90° rotation about (3, 0)

General proficiency questions

1 A variable y varies directly as the square of x and inversely as the square root of z. If $y = 2$ when $x = -1$ and $z = 9$, find
  a y when $z = 4$ and $x = 2$
  b x when $z = 36$ and $y = 4$

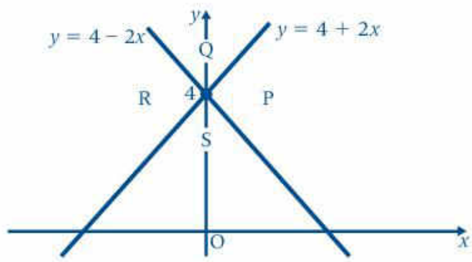
2 A farmer wishes to buy 'c' chickens at \$2.00 each and 'd' ducklings at \$5.00 each. He can accommodate a maximum of 60 chickens and ducklings, but the number of ducklings is not to be more than twice the number of chickens.
  a If the farmer can spend up to \$210, write down three inequalities, other than $c > 0$ and $d > 0$, that satisfy these conditions.
  b By drawing the graphs of the inequalities, identify the region, R, that satisfies all three.
  c If the ducklings and chickens are later sold at a profit of \$20 and \$12 respectively, find the total profit, P, in terms of c and d.
  d By using the graph, find the number of chickens and ducklings that would give a maximum profit and the maximum profit expected.

3 Express $x + 3 < 9 < 4x + 1$ in the form $a < x < b$

4 The two lines $y = 4 - 2x$ and $y = 4 + 2x$, shown in the diagram, divide the x-y plane into four regions P, Q, R and S.

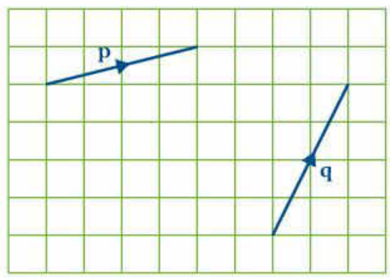
State the inequalities that define each of these regions.

Determine the region in which the point $(-1\frac{1}{2}, 3)$ lies, clearly explaining your reasoning.

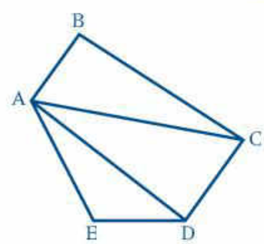


5 Copy the vectors **p** and **q** onto a grid.

On your grid draw line segments to represent the vectors
  a $\mathbf{p} + \mathbf{q}$
  b $\mathbf{p} - \mathbf{q}$
  c $3\mathbf{p} - \frac{1}{2}\mathbf{q}$
  d $2\mathbf{p} + 3\mathbf{q}$

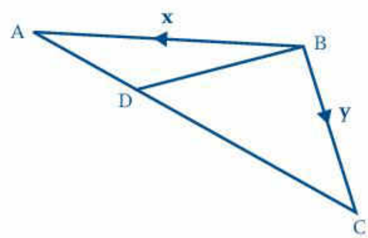


6 Find the single vector that is equivalent to
  a $\overrightarrow{AE} + \overrightarrow{ED}$
  b $\overrightarrow{AC} + \overrightarrow{CD} + \overrightarrow{DE}$
  c $\overrightarrow{AC} - \overrightarrow{BC}$
  d $\overrightarrow{AE} + \overrightarrow{ED} - \overrightarrow{CD}$



7 $\overrightarrow{BA} = \mathbf{x}$ and $\overrightarrow{BC} = \mathbf{y}$. D is the point on AC such that AD : DC = 1 : 2. Give, in terms of **x** and **y**,
  a $\overrightarrow{AC}$
  b $\overrightarrow{AD}$
  c $\overrightarrow{CA}$
  d $\overrightarrow{CD}$
  e $\overrightarrow{BD}$ can be given as $\overrightarrow{BA} + \overrightarrow{AD}$ or $\overrightarrow{BC} + \overrightarrow{CD}$.

  Find $\overrightarrow{BD}$ in terms of **x** and **y** by the two different ways.



8 A ferry boat, which needs to sail due west, is affected by a current of 6 km/h flowing from a direction of 20° east of north. The speed of the ferry in still water is 24 km/h.

a On what course should the ferry be steered to travel in the desired direction?

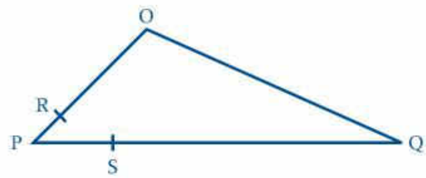
b What is the actual speed of the ferry through the water?

9 Find the value of a for which the matrix $\begin{pmatrix} a-2 & 0 \\ 0 & 3 \end{pmatrix}$

a is singular

b represents an enlargement. State the scale factor of this enlargement.

10 In the diagram $OR = \frac{4}{5}OP$, $\overrightarrow{OP} = \mathbf{p}$, $\overrightarrow{OQ} = \mathbf{q}$ and $PS : SQ = 1 : 4$.



a Express $\overrightarrow{OR}$, $\overrightarrow{RP}$ and $\overrightarrow{PQ}$ in terms of $\mathbf{p}$ and $\mathbf{q}$.

b Express $\overrightarrow{PS}$ and $\overrightarrow{RS}$ in terms of $\mathbf{p}$ and $\mathbf{q}$.

c What conclusion do you draw about RS and OQ?

d What type of quadrilateral is ORSQ?

e The area of $\triangle PRS$ is $5\,cm^2$. What is the area of ORSQ?

11 Under a certain transformation, (x', y') is the image of (x, y) and

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 3 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

a Find the coordinates of the image of the point (2, 3).

b Find the coordinates of the image of the point (−1, 2).

c Find the coordinates of the point of which (7, 3) is the image.

12 A transformation T is defined by the matrix $\begin{pmatrix} 4 & 2 \\ 7 & 4 \end{pmatrix}$

a Find the inverse of the matrix.

b Given that T maps the point A to the point (8, 15), find the coordinates of A.

13 Find the matrix product $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$ and describe the transformation defined by $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$

14 A transformation is defined by the matrix $\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}$

a Express as a single matrix $\begin{pmatrix} 1 & 0 \\ 3 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$

b Find the coordinates of the image of the point (p, p) under the transformation.

c The transformation maps the line $y = x$ to the line $y = mx$. Find the value of m.

15 $\mathbf{R}_y$ is the 2×2 matrix that describes a reflection in the y-axis.
$\mathbf{R}_o$ is the 2×2 matrix that describes a 90° clockwise rotation about O.

a **i** State $\mathbf{R}_y$

ii State $\mathbf{R}_o$

b A, B and C have coordinates (−5, 1), (3, 6) and (1, 1), respectively. Triangle ABC is mapped onto triangle A′B′C′ under the transformation $\mathbf{R}_y$. Find A′, B′ and C′.

c Triangle A′B′C′ is mapped onto triangle A″B″C″ under the transformation $\mathbf{R}_o$. Find A″, B″ and C″.

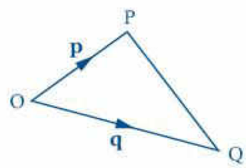
d Find the single transformation $\mathbf{R}_c$ that maps triangle ABC onto triangle A″B″C″.
Prove that $\mathbf{R}_c$ maps triangle ABC onto triangle A″B″C″.

16 Points A and B have position vectors **a** and **b** respectively, relative to a fixed point O. AB is produced to C so that $\overrightarrow{BC} = k\,\overrightarrow{AB}$, where k is a scalar.

Express $\overrightarrow{BC}$ in terms of k, **a** and **b**.

If $\overrightarrow{OC} = -2\mathbf{a} + 2\mathbf{b}$, find the value of k.

17 In the diagram $\overrightarrow{OP} = \mathbf{p}$ and $\overrightarrow{OQ} = \mathbf{q}$.

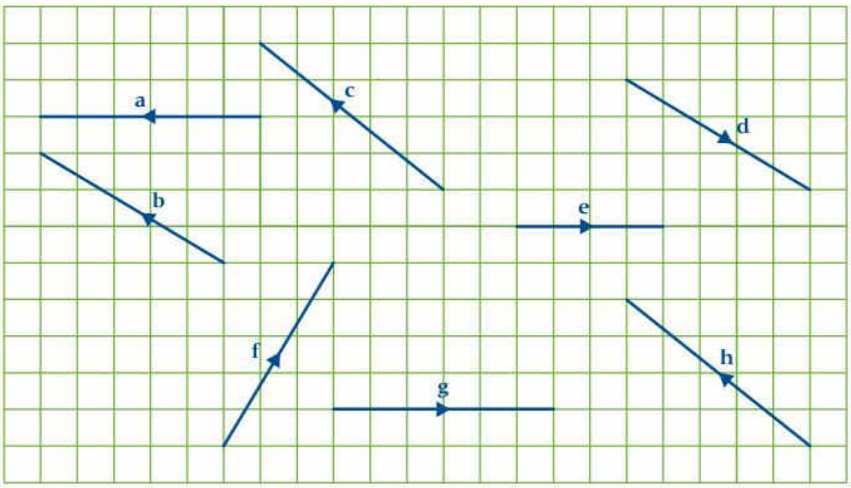


M and N are the mid-points of OP and OQ, respectively. By finding MN, show that the straight line joining the mid-points of two sides of a triangle is

a parallel to the third side

b equal to one-half of the third side.

18



Look at the vectors in the diagram.

a Find the magnitude of the vector i **a** ii **h**

b Which two vectors are equal?

c Which two vectors are equal in magnitude but opposite in direction?

d Which vector is 50% larger than **e**?

19 a Copy and complete the table so that $y \propto \frac{1}{x+1}$

x	1	3	
y	4		1

b Find the equation connecting x and y.

20 Find the percentage change in y if x is increased by 50% when

a $y \propto x$ b $y \propto \frac{1}{x}$

21 Given that p varies directly as the sum of q and r and that $p = 4$ when $q = 3$ and $r = 5$, find the equation connecting p, q and r.

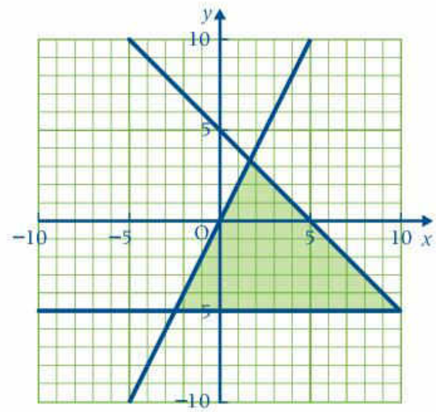
22 Given that T varies inversely as the square of s, use the table to find the values of a and b.

s	1	a	0.25
T	8	2	b

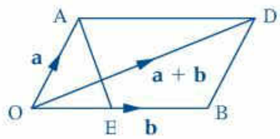
23 R is the transformation 'reflection in the line $y = x$'. P is the transformation 'rotation by 90° anticlockwise about O'.
Find the matrix for
a transformation R
b transformation P
c transformation R followed by transformation P.

24 A shopkeeper buys x kg of sugar and y kg of salt from a wholesale company.
To get a discount, the shopkeeper must buy at least 50 kg in total.
To avoid overloading his van, the shopkeeper must buy not more than 100 kg in total.
a Write down four inequalities that must be satisfied by x and y.
b Illustrate with a sketch the region of the x-y plane that satisfies all four inequalities.

25 The diagram shows a triangular region of the plane.
Find the set of inequalities that define the shaded region.



26 In the diagram, $\overrightarrow{OA} = \mathbf{a}$, $\overrightarrow{OB} = \mathbf{b}$ and $\overrightarrow{OD} = \mathbf{a} + \mathbf{b}$. E is the mid-point of OB.
a Prove that OADC is a parallelogram.
b Given that $\mathbf{a} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$
show that triangle OAE is isosceles.



27 A(1, 1), B(2, 1) and C(1, 2) are the vertices of a triangle ABC.
The triangle is mapped to triangle A′B′C′ by the transformation matrix $\begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}$
a Draw a diagram on graph paper to show triangle ABC and its image.
b Explain why this transformation does not have an inverse.

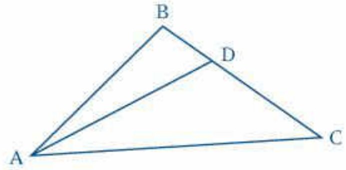
28 The position vectors of the points A, B and C relative to the origin O are given by
$\overrightarrow{OA} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\overrightarrow{OB} = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$, $\overrightarrow{OC} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$
Find $\overrightarrow{AC}$ and $\overrightarrow{CB}$.

29 In the diagram, D is a point on BC such that BD : DC = 1 : 2

$\overrightarrow{AB} = 6\mathbf{p} + 2\mathbf{q}$ and $\overrightarrow{AC} = 3\mathbf{p} - \mathbf{q}$.

a Find $\overrightarrow{AD}$ in terms of **p** and **q**.

b E is a point outside triangle ABC such that $\overrightarrow{AE} = 10\mathbf{p} + 2\mathbf{q}$.
Show that A, D and E are collinear.



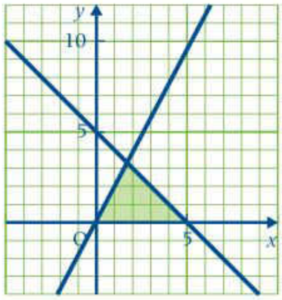
30 Given that v varies inversely as the product of w and u, find the values of a and b in the table.

This means that v is inversely proportional to wu.

w	2	a	3
u	2	3	b
v	3	6	0.5

31 y varies inversely as the cube of x.
Find the percentage change in y when x decreases by 80%.

32 Find the greatest value of $2x + y$ in the shaded region of the diagram given that $(x, y) \in \mathbb{Z}$.



33 The matrix $\begin{pmatrix} 2 & 3 \\ 3 & 4 \end{pmatrix}$ maps the point A to the point A' (4, 6).

Find the coordinates of A.

34 The variable w varies inversely as $v - 1$.
Given that $w = 6$ when $v = 4$, find

a w when $v = 7$

b v when $w = 9$.

35 Given that $\overrightarrow{OA} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$ and $\overrightarrow{OB} = \begin{pmatrix} 7 \\ 6 \end{pmatrix}$ where O is the origin, show that $\overrightarrow{AB}$ makes an angle of 45° with the x-axis.

36 x and y satisfy the inequalities $0 < x < y < 1 - x$.
Draw a diagram on graph paper to show the region of the plane satisfied by these inequalities.

21 SINE AND COSINE FORMULAE

AT THE END OF THIS CHAPTER YOU SHOULD BE ABLE TO...

1 Find the sine and cosine of angles greater than 90 degrees.

2 Use the sine and cosine formulae.

3 Solve problems using sine and cosine rules.

4 Find the area of a triangle, given the lengths of two sides of a triangle and the included angle.

5 Find the area of a segment of a circle.

BEFORE YOU START

you need to know:

✓ how to find trigonometric ratios for angles less than 90°
✓ how to use a calculator to find values of trigonometric ratios of given angles
✓ how to draw graphs of trigonometric functions
✓ how to use Pythagoras' theorem
✓ how to use three-figure bearings
✓ how to use angles of elevation and depression.

KEY WORDS

cosine rule, obtuse angle, sine rule, supplementary angle

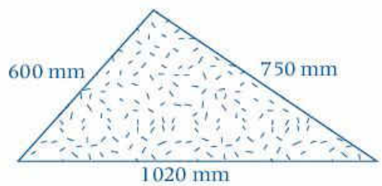
MATHS IS OUT THERE

Did you know that the Greeks were very interested in astronomy? They collected data which they used in estimating distances and predicting the positions of heavenly bodies. This was done indirectly and usually by means of similar triangles.

Trigonometry means 'triangle measurement'.

The diagram, which is not drawn to scale, shows a piece needed to fill in an awkward corner in a kitchen worktop.

To cut this exactly, it would be useful to know the angles of this triangle.



- The triangle is not right-angled so the angles can be found by making a scale drawing but we know from experience that this method is slow and not very accurate. There are, however, formulae that can be used to find sides and angles in non-right-angled triangles and these are introduced in this chapter.

Many triangles, such as the one above, contain obtuse angles, so we start by investigating the sines and cosines of **obtuse angles**.

EXERCISE 21a

1 **a** Copy and complete the following table.
Use a calculator to find each value of sin $x°$ correct to 2 decimal places.

x	0	15	30	45	60	75	90	105	120	135	150	165	180
sin $x°$													

b Using scales of 1 cm for 15 units on the horizontal axis and 1 cm for 0.2 on the vertical axis, plot these points on a graph and draw a smooth curve through the points.

c This curve has a line of symmetry. About which value of x is the curve symmetrical?

d From your graph, find the two angles for which
i $\sin x° = 0.8$ **ii** $\sin x° = 0.6$ **iii** $\sin x° = 0.4$
Find, in each case, a relationship between the two angles.

2 Use a calculator to complete the following statements:
a $\sin 30° = \square$, $\sin 150° = \square$, $150° = \square - 30°$
b $\sin 40° = \square$, $\sin 140° = \square$, $140° = \square - 40°$
c $\sin 72° = \square$, $\sin 108° = \square$, $108° = \square - 72°$

Sines of obtuse angles

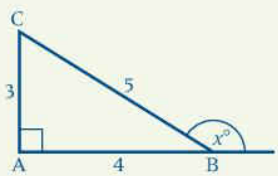
The results from Exercise 21a demonstrate that, when two angles are supplementary (i.e. they add up to 180°) their sines are the same.

$$\sin x° = \sin(180° - x°)$$

EXERCISE 21b

Example:

Find sin $x°$.

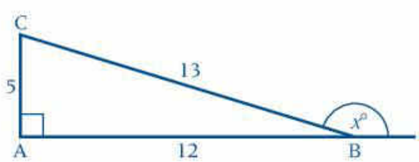


$\sin x° = \sin(180° - x°)$
$= \sin A\hat{B}C = \frac{3}{5}$

$\sin x° = \sin$ (angle supplementary to $x°$).

In questions **1** to **6**, find sin $x°$.

1



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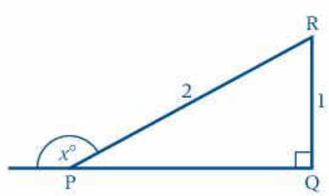

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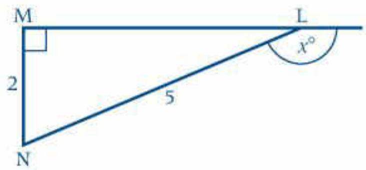

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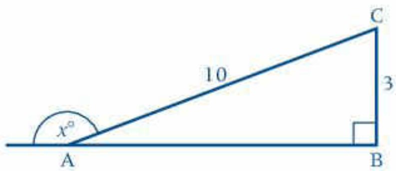

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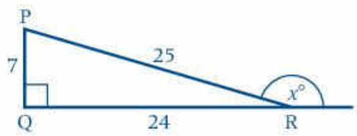

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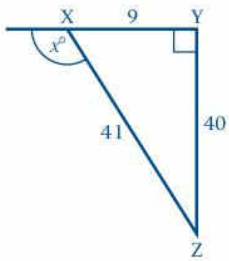


In questions **7** to **12**, $x°$ is an acute angle. Find x.

7 $\sin x° = \sin 165°$

8 $\sin x° = \sin 140°$

9 $\sin x° = \sin 152°$

10 $\sin x° = \sin 100°$

11 $\sin x° = \sin 175°$

12 $\sin x° = \sin 91°$

In the next exercise we investigate the cosines of obtuse angles.

EXERCISE 21c

1 **a** Copy and complete the following table. Use a calculator to find each value of $\cos x°$ correct to 2 decimal places.

x	0	15	30	45	60	75	90	105	120	135	150	165	180
$\cos x°$													

b Using scales of 1 cm for 15 units on the horizontal axis and 1 cm for 0.2 on the vertical axis, plot these points on a graph and draw a smooth curve through them.

c This curve has a point of rotational symmetry. What is the value of x at this point?

d What do you notice about the sign of the cosine of an obtuse angle?

e From your graph find the angles for which

i $\cos x° = 0.8$ **ii** $\cos x° = -0.8$

What is the relationship between these two angles?

2 Use a calculator to complete the following statements:

a $\cos 30° = \square$, $\cos 150° = \square$, $150° = \square - 30°$

b $\cos 50° = \square$, $\cos 130° = \square$, $130° = \square - 50°$

c $\cos 84° = \square$, $\cos 96° = \square$, $96° = \square - 84°$

Cosines of obtuse angles

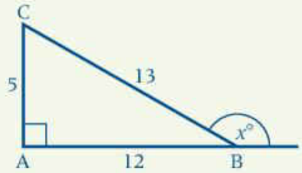
The results from the last exercise demonstrate that the cosine of an obtuse angle is negative, the numerical value (i.e. ignoring the sign) of the cosines of supplementary angles is the same.

$$\cos x° = -\cos (180° - x°)$$

EXERCISE 21d

Example:

Find $\cos x°$.

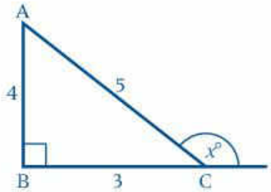


$$\cos x° = -\cos(180° - x°)$$
$$= -\cos A\hat{B}C = -\frac{12}{13}$$

In questions **1** to **4**, find $\cos x°$.

Remember to change the sign.

1



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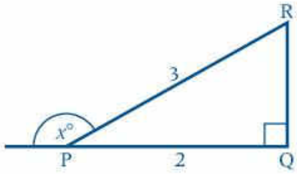

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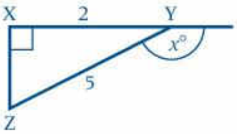

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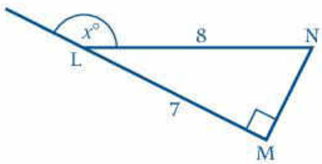
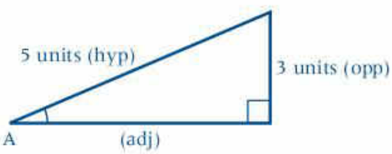


5 Find $\hat{A}$ if **a** $\cos\hat{A} = -\cos 20°$ **b** $\cos\hat{A} = -\cos 50°$

Trigonometric ratios as fractions

For an angle A in a right-angled triangle, if one of $\sin\hat{A}$, $\cos\hat{A}$ or $\tan\hat{A}$ is given as a fraction, we can draw a right-angled triangle and mark in the lengths of two sides.

For example, if $\sin\hat{A} = \frac{3}{5}$, this triangle can be drawn.

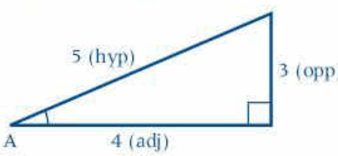


Then, using Pythagoras' theorem, the length of the third side can be calculated. In this case it is of length 4 units.

Now the cosine of angle A can be written down as a fraction,

i.e. $\cos\hat{A} = \frac{4}{5}$

Similarly $\tan\hat{A} = \frac{3}{4}$



MATHS IS OUT THERE

Did you know that Halley's Comet derived its name from that of Edmund Halley (1656–1742) who described it?

EXERCISE 21e

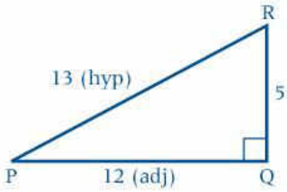
Example:

If $\cos \hat{P} = \frac{12}{13}$, draw a suitable right-angled triangle and hence find $\sin \hat{P}$ and $\tan \hat{P}$.

$QR = 5$ (Using Pythagoras' theorem or recognising a 5, 12, 13 triangle.)

Therefore $\sin \hat{P} = \frac{5}{13}$

and $\tan \hat{P} = \frac{5}{12}$



All angles are acute.

Draw a triangle and label the vertices and the side. Mark the lengths given in the sine ratio.

1 If $\sin \hat{A} = \frac{7}{25}$ find $\cos \hat{A}$ and $\tan \hat{A}$.

2 If $\cos \hat{A} = \frac{5}{13}$ find $\sin \hat{A}$ and $\tan \hat{A}$.

3 If $\tan \hat{P} = \frac{3}{4}$ find $\sin \hat{P}$ and $\cos \hat{P}$.

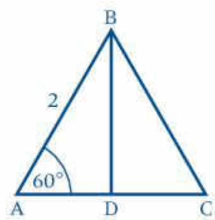
4 If $\cos \hat{D} = \frac{3}{5}$ find $\tan \hat{D}$ and $\sin \hat{D}$.

5 If $\sin \hat{X} = \frac{9}{41}$ find $\cos \hat{X}$ and $\tan \hat{X}$.

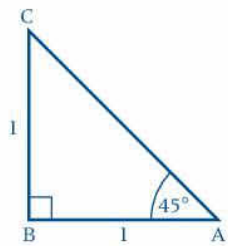
6 If $\tan \hat{A} = 1$ find $\sin \hat{A}$ and $\cos \hat{A}$.

Remember that $1 = \frac{1}{1}$ and leave the square root in your answer.

7 ABC is an equilateral triangle of side 2 units. D is the mid-point of AC.
- a Show that AD = 1 unit and BD = $\sqrt{3}$ units.
- b Find the value, in square root form where necessary, of
  - i $\sin 60°$ ii $\cos 60°$ iii $\tan 60°$
- c Find in square root form the value of
  - i $\sin 30°$ ii $\cos 30°$ iii $\tan 30°$



8 In $\triangle ABC$, AB = BC = 1 unit and $\angle ABC = 90°$.
- a Show that CA = $\sqrt{2}$ units.
- b Find, in square root form where necessary, the value of
  - i $\sin 45°$ ii $\cos 45°$ iii $\tan 45°$



Trigonometric ratios of 30°, 45° and 60°

As we have seen from questions **7** and **8** in the last exercise, it is possible to give the trig ratios of 30°, 45° and 60° in an exact form as square roots. It is useful to gather the results together in a table.

	sin	cos	tan
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
45°	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$

EXERCISE 21f

1 If $\sin \widehat{A} = \frac{1}{2}$ and $\widehat{A} < 90°$ find $\cos \widehat{A}$, giving your answer in square root form. Hence find, in square root form, the value of

a $2 \sin \widehat{A} \cos \widehat{A}$ **b** $2(\cos \widehat{A})^2 - 1$

2 Use the table given in the text to write down the values of sin 30°, cos 30°, sin 60° and cos 60°.

a Hence find, in square root form, the value of

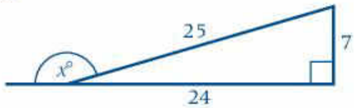
i $\sin 30° \cos 60° + \cos 30° \sin 60°$

ii $\sin 60° \cos 30° - \cos 60° \sin 30°$

b What angle has a sine equal in value to your answer to part **a i**?

c What angle has a sine equal in value to your answer to part **a ii**?

The remaining questions use all the work considered so far in this chapter.



3 Write down, as a fraction

a $\cos x°$ **b** $\sin x°$

4 Find two angles each with a sine of 0.5

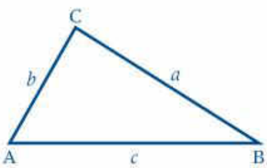
5 If cos 59° = 0.515 what is cos 121°?

6 Find an obtuse angle whose cosine is equal to −cos 72°.

7 If $\sin x° = \frac{4}{5}$, find as a fraction

a $\sin (180° - x°)$ **b** $\cos x°$ **c** $\cos (180° - x°)$

Triangle notation



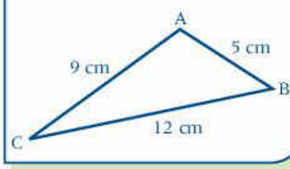
In a triangle ABC, the sides can be referred to as AB, BC and CA. It is convenient, however, to use a single letter to denote the number of units in the length of a side and the standard notation uses

a for the side opposite angle A

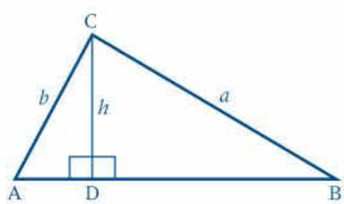
b for the side opposite angle B

c for the side opposite angle C.

In this triangle, for example,
$a = 12$
$b = 9$
$c = 5$



The sine rule



Consider a triangle ABC in which there is no right angle.

If a line is drawn from C, perpendicular to AB, the original triangle is divided into two right-angled triangles ADC and BDC as shown.

In △ADC $\quad \sin \widehat{A} = \frac{h}{b} \Rightarrow h = b \sin \widehat{A}$

In △BDC $\quad \sin \widehat{B} = \frac{h}{a} \Rightarrow h = a \sin \widehat{B}$

Equating the two expressions for h gives

$$a \sin \hat{B} = b \sin \hat{A}$$

Hence $\quad \dfrac{a}{\sin \hat{A}} = \dfrac{b}{\sin \hat{B}}$ (dividing both sides by $\sin \hat{A} \sin \hat{B}$).

Now if we were to divide △ABC into two right-angled triangles by drawing the perpendicular from A to BC the similar result would be $\dfrac{b}{\sin \hat{B}} = \dfrac{c}{\sin \hat{C}}$

Combining the two results gives

$$\frac{a}{\sin \hat{A}} = \frac{b}{\sin \hat{B}} = \frac{c}{\sin \hat{C}}$$

This result is called the **sine rule** and it enables us to find angles and sides of triangles which are *not* right-angled.

The sine rule can be used in any triangle, but if the triangle is right-angled it is easier to use the simple trig ratios.

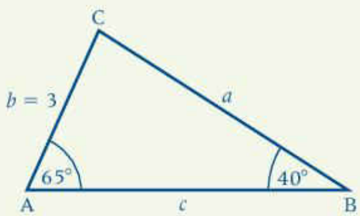
The sine rule is made up of three equal fractions, but only two of them can be used at a time. When using the sine rule we choose the two fractions in which three quantities are known and only one is unknown.

In all the following exercises, unless instructed otherwise, give angles correct to 1 decimal place and lengths to 3 significant figures.

EXERCISE 21g

Example:

In △ABC, AC = 3 cm, ∠B = 40° and ∠A = 65°.
Find the length of BC.



$$\frac{a}{\sin \hat{A}} = \frac{b}{\sin \hat{B}} = \frac{c}{\sin \hat{C}}$$

$$\frac{a}{\sin 65°} = \frac{3}{\sin 40°}$$

$$a = \frac{3 \sin 65°}{\sin 40°}$$

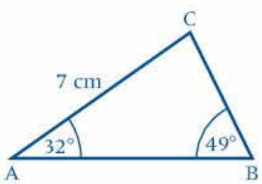
$$= 4.2298\ldots$$

Therefore BC is 4.23 cm correct to 3 s.f.

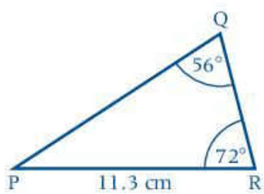
Label the sides a, b, c.

We want to find a and we know b, ∠A and ∠B so use the first two fractions in the sine rule.

1 Find BC.

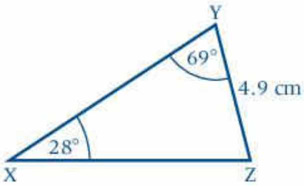


2 Find PQ.

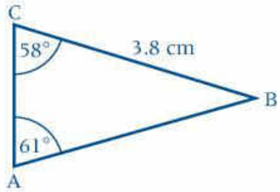


Mark the sides of the triangles with the appropriate letters.

3 Find XZ.



4 Find AB.

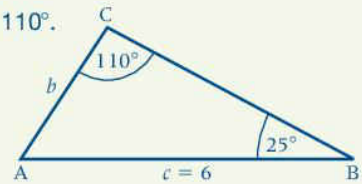


5 In △LMN, LM = 17.7 cm, ∠N = 73° and ∠L = 52°. Find MN.

The sine rule can be used when one angle is obtuse.

Example:

In △ABC, AB = 6 cm, ∠B = 25° and ∠C = 110°.
Find AC.



From the sine rule

$$\frac{a}{\sin \hat{A}} = \frac{b}{\sin \hat{B}} = \frac{c}{\sin \hat{C}}$$

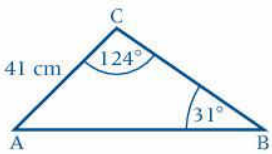
$$\frac{b}{\sin 25°} = \frac{6}{\sin 110°}$$

$$b = \frac{6 \sin 25°}{\sin 110°}$$

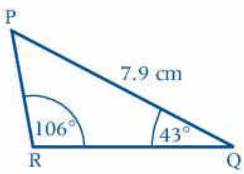
$$= 2.698\ldots$$

Therefore AC = 2.70 cm correct to 3 significant figures.

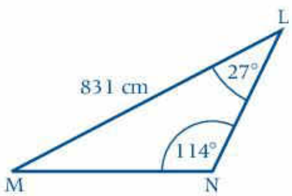
6 Find AB.



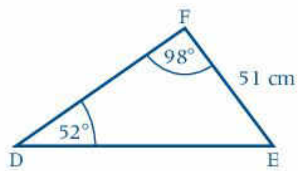
7 Find PR.



8 Find MN.



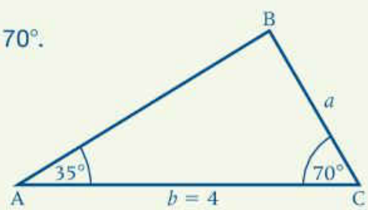
9 Find DE.



Sometimes the third angle must be found before a suitable pair of fractions can be selected from the sine rule.

Example:

In △ABC, AC = 4 cm, ∠A = 35° and ∠C = 70°.
Find BC.



$$\angle B = 180° - (35° + 70°) = 75°$$

$$\frac{a}{\sin \hat{A}} = \frac{b}{\sin \hat{B}} = \frac{c}{\sin \hat{C}}$$

$$\frac{a}{\sin 35°} = \frac{4}{\sin 75°}$$

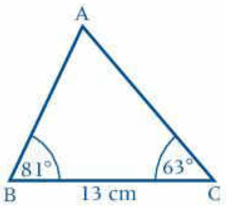
$$a = \frac{4 \sin 35°}{\sin 75°}$$

$$= 2.375\ldots$$

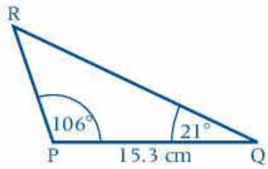
Therefore BC is 2.38 cm correct to 3 s.f.

The two sides involved are a and b so we must use $\angle A$ and $\angle B$ in the sine rule; first we must find $\angle B$.

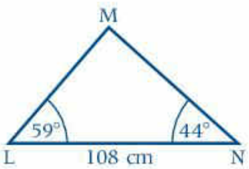
10 Find AB.



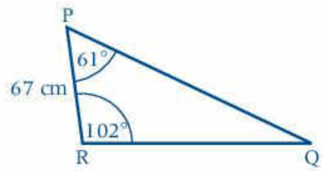
11 Find QR.



12 Find LM.

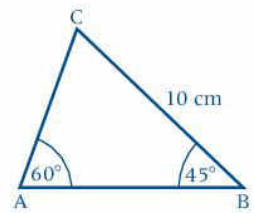


13 Find PQ.

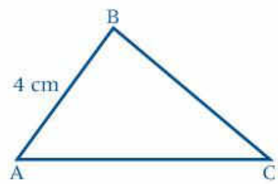


14 In triangle ABC, given in question **10**, the length of BC is given correct to the nearest centimetre and angles B and C are given correct to the nearest degree. Find the smallest possible value for the length of AB.

15 In triangle ABC, $\angle A = 60°$, $\angle B = 45°$ and BC = 10 cm. Find the length of AC, giving your answer as an irrational number in surd, i.e. square root, form.



16 In triangle ABC, $\sin \hat{A} = \frac{2}{3}$ and $\sin \hat{C} = \frac{2}{5}$. Find, without using a calculator, the length of BC.



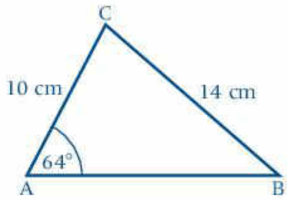
Using the sine rule to find an angle

The sine rule can be used to find an angle in a triangle. In this case it is more convenient to use the sine formula in the form

$$\frac{\sin \hat{A}}{a} = \frac{\sin \hat{B}}{b} = \frac{\sin \hat{C}}{c}$$

However care must be taken when two sides and the non-included angle is the only information given about a triangle.

In this triangle, for example, using the sine rule to find ∠B gives



$$\frac{\sin \widehat{B}}{10} = \frac{\sin 64°}{14}$$

$$\Rightarrow \quad \sin \widehat{B} = \frac{10 \times \sin 64°}{14} = 0.641\ldots$$

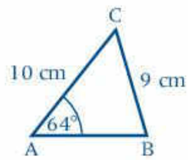
Using a calculator gives $\angle B = 39.9°$

But, from the work at the beginning of the chapter, we know that the sines of supplementary angles are equal, i.e. $\sin(180° - 39.9°) = 0.641\ldots$

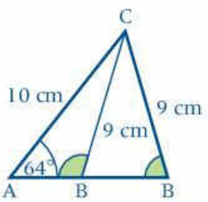
Therefore another possible value of ∠B is $(180° - 39.9°) = 140.1°$.

In this case we can reject this possibility because the side opposite ∠B is shorter than the side opposite ∠A, therefore $\angle B < \angle A$, so ∠B is acute.

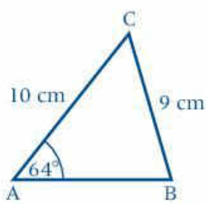
Now consider this triangle. This time the side opposite ∠B is longer than the side opposite ∠A, so it is possible that ∠B may be acute or obtuse.



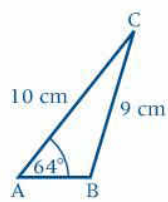
This diagram shows the two possible triangles that can be drawn from the information given.



i.e.



and



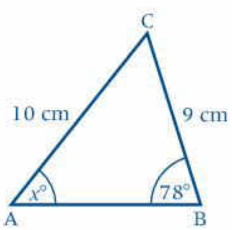
MATHS IS OUT THERE

Did you know that the Mayans, who lived on the island of Cozumel in the Republic of Mexico since AD300, are most noted for their complex system of mathematics and astrology?

EXERCISE 21h

In questions **1** to **6** find the size of the marked angle. If there are two possible values for this angle, give both and illustrate the two possible triangles with a sketch.

1



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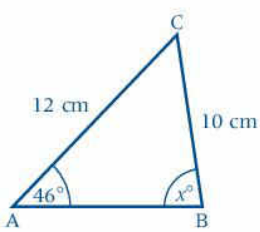

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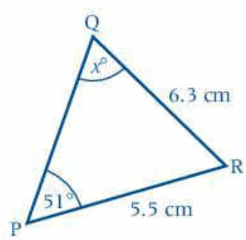

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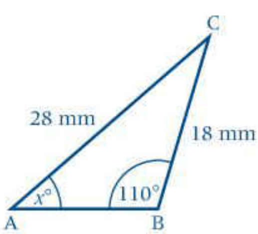

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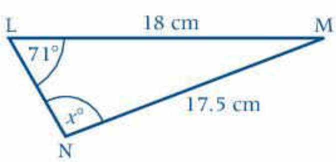
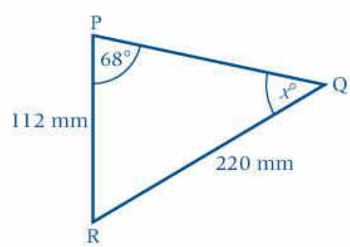
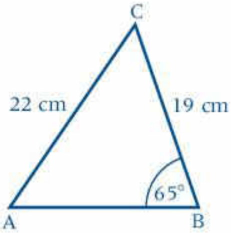


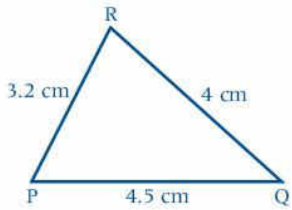
6



7 In triangle ABC, each measurement is given correct to 2 significant figures. Find the smallest possible size of ∠A.



8 **a** Which is the largest angle in triangle PQR?
b Explain why angle P cannot be obtuse.
c Can you use the sine rule to find angle P? Explain your answer.



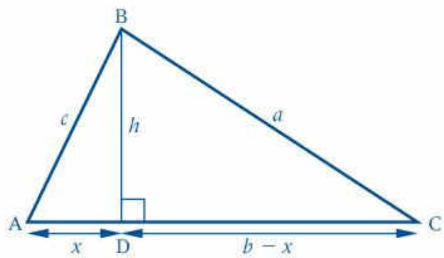
The cosine rule

When we know the lengths of the three sides of a triangle but none of the angles, we cannot use the sine formula to find an angle because there will be two unknown angles whichever pair of fractions we choose.

For cases like this where the sine rule fails we need a different formula.

Consider a triangle ABC divided into two right-angled triangles by a line BD, perpendicular to AC. Let the length of AD be x so that DC = $b - x$.

Using Pythagoras' theorem in triangles ABD and CBD gives



$$c^2 = h^2 + x^2$$

and

$$\begin{aligned} a^2 &= h^2 + (b - x)^2 \\ &= h^2 + b^2 - 2bx + x^2 \\ &= b^2 + (h^2 + x^2) - 2bx \\ &= b^2 + c^2 - 2bx \end{aligned}$$

In △ABD, $x = c \cos \hat{A}$. Therefore

$$a^2 = b^2 + c^2 - 2bc \cos \hat{A}$$

This result is called the **cosine rule**.

If we were to draw a line from A perpendicular to BC, or from C perpendicular to AB, similar equations could be obtained,

i.e.

$$b^2 = a^2 + c^2 - 2ac \cos \hat{B}$$

$$c^2 = a^2 + b^2 - 2ab \cos \hat{C}$$

Note that, in each version of the cosine rule, the side on the left and the angle on the right have the same letter.

When using the cosine rule it is a good idea to put brackets round the term containing the cosine

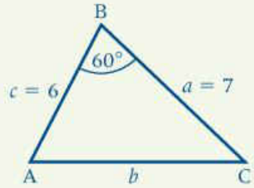
i.e.

$$a^2 = b^2 + c^2 - (2bc \cos \hat{A})$$

EXERCISE 21i

Example:

In a triangle ABC, AB = 6 cm, BC = 7 cm and $\angle B = 60°$. Find AC.



$$b^2 = a^2 + c^2 - (2ac\cos\hat{B})$$
$$= 7^2 + 6^2 - (2 \times 7 \times 6\cos 60°)$$
$$= 49 + 36 - (84 \times \cos 60°)$$
$$= 85 - 42 = 43$$
$$\Rightarrow \quad b = \sqrt{43} = 6.557\ldots$$

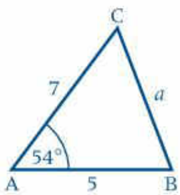
Therefore AC is 6.56 cm correct to 3 s.f.

The sine rule fails here because we do not know either of the angles opposite given sides.

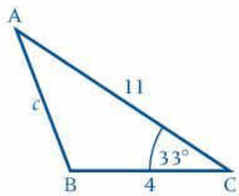
The unknown side is b, and $\angle B$ is known, so we use the version of the cosine rule that starts with b^2.

In this exercise the lengths of the sides of each triangle are measured in centimetres.

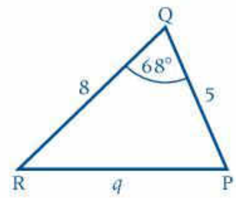
1 Find *a*.



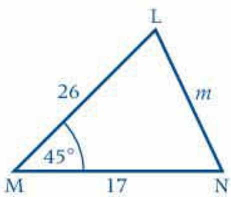
2 Find *c*.



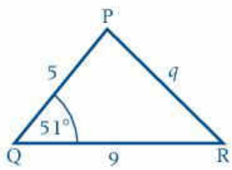
3 Find *q*.



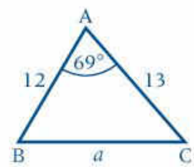
4 Find *m*.



5 Find *q*.



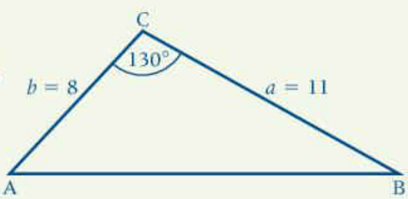
6 Find *a*.



If the given angle is obtuse its cosine is negative and extra care is needed in using the cosine rule; brackets are even more helpful in this case.

Example:

Triangle ABC is such that BC = 11 cm, AC = 8 cm and $\angle C = 130°$. Find AB.

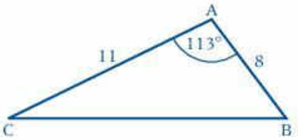


Using the cosine rule

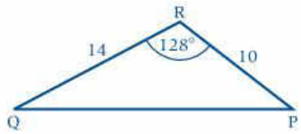
$$c^2 = a^2 + b^2 - (2ab\cos\hat{C})$$
$$= 11^2 + 8^2 - (2 \times 11 \times 8\cos 130°)$$
$$= 298.13\ldots$$
$$\Rightarrow \quad c = \sqrt{298.13\ldots} = 17.26\ldots$$

Therefore AB is 17.3 cm correct to 3 s.f.

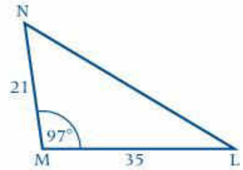
7 Find BC.



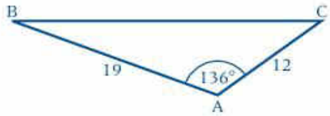
8 Find PQ.



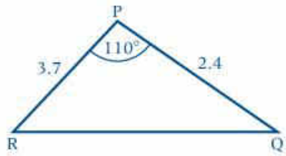
9 Find LN.



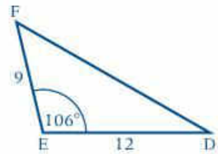
10 Find BC.



11 Find *p*.



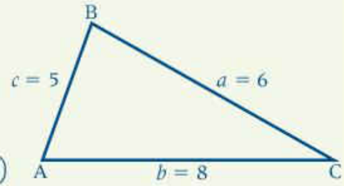
12 Find *e*.



The cosine rule can be used to find an angle when three sides of a triangle are known.

Example:

In triangle ABC, AB = 5 cm, BC = 6 cm and AC = 8 cm. Find the smallest and the largest angle in this triangle.



$$c^2 = a^2 + b^2 - (2ab\cos\hat{C})$$

$$5^2 = 6^2 + 8^2 - (2 \times 6 \times 8\cos\hat{C})$$

$$25 = 36 + 64 - (96\cos\hat{C})$$

$$96\cos\hat{C} + 25 = 100 \text{ (Adding } 96\cos\hat{C} \text{ to each side.)}$$

$$96\cos\hat{C} = 75 \text{ (Subtracting 25 from each side.)}$$

$$\cos\hat{C} = \frac{75}{96} = 0.7812\ldots$$

Therefore ∠C = 38.6° (correct to 1 d.p.).

i.e. the smallest angle is 38.6° correct to 1 d.p.

The smallest angle is opposite to the shortest side, so we are looking for ∠C and will use the cosine rule starting with c^2.

$$b^2 = a^2 + c^2 - (2ac\cos\hat{B})$$

$$8^2 = 6^2 + 5^2 - (2 \times 5 \times 6\cos\hat{B})$$

$$64 = 36 + 25 - (60\cos\hat{B})$$

$$60\cos\hat{B} = 61 - 64 = -3$$

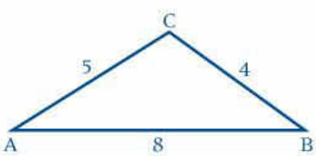
$$\cos\hat{B} = \frac{-3}{60} = -0.05$$

Therefore the largest angle, ∠B, is 92.9° correct to 1 d.p.

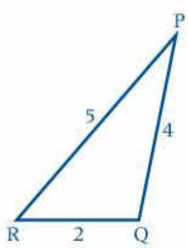
The largest angle is opposite to the longest side, so we are looking for ∠B and will use the cosine rule starting with b^2.

$\cos\hat{B}$ is negative so ∠B is obtuse.

13 Find ∠A.



14 Find ∠Q.



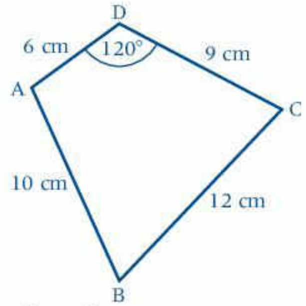
15 In $\triangle$LMN, LM = 8 cm, MN = 5 cm and LN = 6 cm. Find $\angle$N.

16 In $\triangle$ABC, AB = 3 cm, BC = 2 cm and AC = 4 cm. Find the smallest angle in $\triangle$ABC.

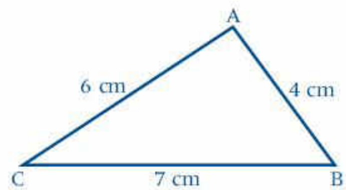
17 In $\triangle$XYZ, XY = 7 cm, XZ = 9 cm and YZ = 5 cm. Find the largest angle in $\triangle$XYZ.

18 In $\triangle$DEF, DE = 2.1 cm, EF = 3.6 cm and DF = 2.7 cm. Find the middle-sized angle in $\triangle$DEF.

19 Use the information given in the diagram to find
  a the length of AC
  b $\angle$ABC



20 a Use the information in the diagram to find, as a fraction, the cosine of $\angle$A.
  b What does your answer in part **a** tell you about the size of $\angle$A?



21 In $\triangle$ABC, AB = 41 mm, BC = 28 mm and AC = 36 mm. Each measurement is correct to the nearest millimetre. Find the smallest possible value of $\angle$A.

Mixed questions

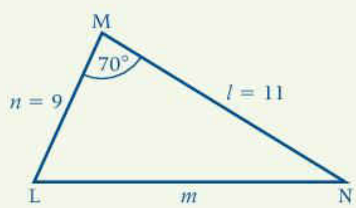
If three independent facts are given about the sides and/or angles of a triangle and we are asked to find one or more of the unknown quantities, first look for a right-angled triangle or an isosceles triangle (which can be divided into two right-angled triangles). When no such triangle exists, we must then decide whether to use the sine rule or the cosine rule.

The sine rule is the easier to work out so it is chosen whenever possible and that is when the given information includes a side and the angle opposite to it (remember that, if the two angles are given, the third angle is also known). The cosine rule is chosen only when the sine rule cannot be used.

EXERCISE 21j

Example:

In the triangle LMN, LM = 9 cm, MN = 11 cm and $\angle$M = 70°. Find LN.



Using the cosine rule, $m^2 = l^2 + n^2 - (2ln\cos\widehat{M})$

$$= 11^2 + 9^2 - (2 \times 11 \times 9\cos 70°)$$

$$= 134.28\ldots$$

$\Rightarrow \quad m = \sqrt{134.28\ldots} = 11.58\ldots$

Therefore LN is 11.6 cm correct to 3 s.f.

We are not given an angle and the side opposite to it so we cannot use the sine rule.

Each question from **1** to **8** refers to a △ABC. Fill in the empty spaces.

	a	b	c	$\widehat{A}$	$\widehat{B}$	$\widehat{C}$
1	11.7		╳	39°	66°	╳
2		128	86	63°		╳
3	╳		65	╳	79°	55°
4	16.3	12.7		╳	╳	106°
5		263		╳	47°	74°
6	14			53°	82°	╳
7		╳	17.8	107°		35°
8		16	16	81°	╳	╳

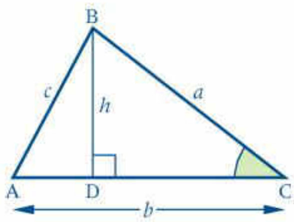
The area of a triangle

We already know that the area, *A*, of a triangle can be found by multiplying half the base, *b*, by the perpendicular height, *h*.

In some cases, however, the perpendicular height is not given, so an alternative formula is needed. There are two other formulae that can be used.

Sine formula for area of a triangle

Consider a triangle ABC in which the lengths of BC and CA, and the angle C, are known.



The line BD, drawn from B perpendicular to AC, is a perpendicular height of the triangle.

Therefore the area of the triangle is $\frac{1}{2}bh$.

But, in triangle BDC, $\sin\widehat{C} = \frac{h}{a}$ i.e. $h = a\sin\widehat{C}$.

Therefore the area of the triangle is $\frac{1}{2}ab\sin\widehat{C}$, i.e. $A = \frac{1}{2}ab\sin\widehat{C}$.

Alternatively we could draw perpendicular heights from A or from C, giving similar expressions for the area,

i.e. $\quad A = \frac{1}{2}bc\sin\widehat{A}$ and $A = \frac{1}{2}ac\sin\widehat{B}$

Each of these expressions involves two sides and the included angle

i.e. $\quad$ Area $\triangle = \frac{1}{2} \times$ product of two sides $\times$ sine of the included angle

Heron's formula

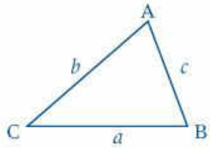
If we know the lengths of the three sides of a triangle, we can find its area by first using the cosine formula to find an angle, then we can use the sine formula for the area.

Heron's formula gives a direct method:

The area of triangle ABC $= \sqrt{s(s-a)(s-b)(s-c)}$

where s is half the perimeter of the triangle,

i.e. $s = \frac{1}{2}(a+b+c)$.



MATHS IS OUT THERE

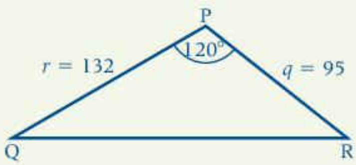
Heron came up with this formula some time in the first century BC, although it may have been known earlier. He also extended it to the area of quadrilaterals and higher-order polygons.

The formula is also known as Hero's formula.

EXERCISE 21k

Example:

Find the area of △PQR if ∠P = 120°, PQ = 132 cm and PR = 95 cm.



Area $= \frac{1}{2}qr\sin\hat{P}$

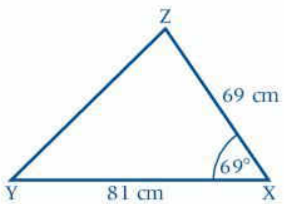
$= \frac{1}{2} \times 95 \times 132 \times \sin 120° = 5429.9\ldots$

Therefore area of △PQR = 5430 cm² correct to 3 s.f.

r and *q* are given and ∠P is the included angle.

Find the area of each triangle.

1



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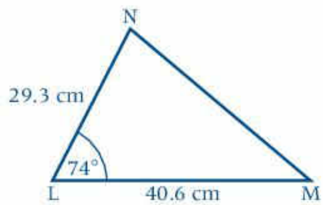

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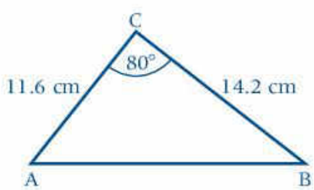

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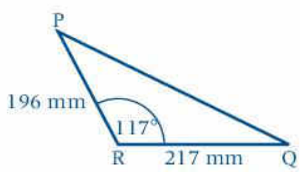


5 △ABC; a = 18.1 cm, c = 14.2 cm, ∠B = 101°

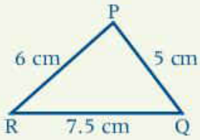
6 △PQR; PQ = 234 cm, PR = 196 cm, ∠P = 84°

7 △XYZ; x = 9 cm, z = 10 cm, ∠Y = 52°

8 △ABC; AC = 3.7 m, AB = 4.1 m, ∠A = 116°

Example:

In triangle PQR, PQ = 5 cm, QR = 7.5 cm and PR = 6 cm.
Find the area of triangle PQR.



$$s = \frac{1}{2}(6 + 5 + 7.5) = 9.25$$

$$\text{Area} = \sqrt{9.25(9.25 - 5)(9.25 - 6)(9.25 - 7.5)}$$

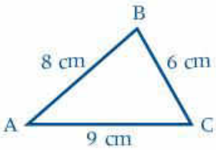
$$= \sqrt{9.25 \times 4.25 \times 3.25 \times 1.75}$$

$$= 14.95\ldots = 15.0\,\text{cm}^2 \text{ to 3 s.f.}$$

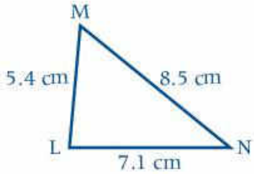
We know the lengths of the three sides so we can use Heron's formula.

In questions **9** to **13**, find the area of each triangle.

9



10



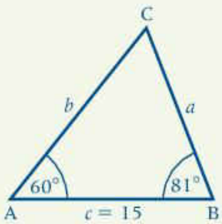
11 Triangle ABC where a = 12 cm, b = 8 cm and c = 6 cm.

12 Triangle XYZ where XY = 4.8 cm, YZ = 3.9 cm and XZ = 3.7 cm.

13 Triangle PQR where PQ = 25 cm, QR = 18 cm and PR = 21 cm.

Example:

In triangle ABC, AB = 15 cm, ∠A = 60° and ∠B = 81°.
Find the area of △ABC.



$$\angle C = 180° - 60° - 81° = 39°$$

$$\boxed{\frac{a}{\sin \hat{A}}} = \frac{b}{\sin \hat{B}} = \boxed{\frac{c}{\sin \hat{C}}}$$

$$\frac{a}{\sin 60°} = \frac{15}{\sin 39°}$$

$$a = \frac{15 \sin 60°}{\sin 39°} = 20.64\ldots$$

Using area $= \frac{1}{2} ac \sin \hat{B}$ gives

$$\text{area} = \frac{1}{2} \times 20.64\ldots \times 15 \times \sin 81° = 152.9\ldots$$

Therefore area △ABC is 153 cm² correct to 3 s.f.

To use the formula to find the area of △ABC, we need two sides and the included angle. We know only one side, so we will use the sine formula to find a. First we must calculate ∠C.

14 In △ABC, BC = 7 cm, AC = 8 cm, AB = 10 cm. Find ∠C and the area of the triangle.

15 △PQR is such that PQ = 11.7 cm, ∠Q = 49° and ∠R = 63°. Find PR and the area of △PQR.

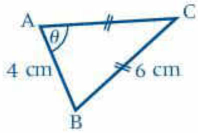
16 In △LMN, LM = 16 cm, MN = 19 cm and the area is 114.5 cm². Find ∠M and LN.

17 The area of △ABC is 27.3 cm². If BC = 12.8 cm and ∠C = 107° find AC.

18 **a** Find the area of a triangle whose sides are 6 cm, 8 cm and 10 cm long.

b Find the length of a side of an equilateral triangle which has the same area as the triangle in part **a**.

19 In the diagram, triangle ABC is isosceles, AB = 4 cm and BC = 6 cm.

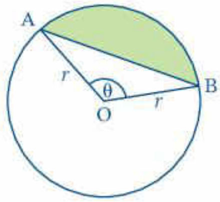


a Show that the area of triangle ABC is $8\sqrt{2}$ cm^2.

b Hence show that $\sin\theta = \frac{2\sqrt{2}}{3}$

20 In triangle PQR, PQ = 4 cm, QR = 5 cm and PR = 6 cm.
Find, in square root form, the altitude of the triangle from P to QR.

The area of a segment of a circle



A segment of a circle is the area enclosed by a chord and an arc of the circle.

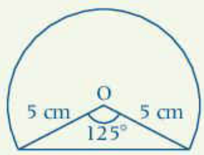
The area of a **segment** of a circle is found by subtracting the area of the triangle AOB from the area of the sector.

Therefore the area of the shaded segment is given by $A = \frac{\pi r^2\theta}{360} - \frac{1}{2}r^2\sin\theta$

EXERCISE 21l

Example:

The diagram shows a washer which is a circular disc from which a segment has been removed.
Find the area of the washer.



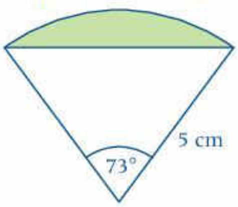
The area of the complete disc is $\pi r^2 = \pi \times 25 = 78.539\ldots$

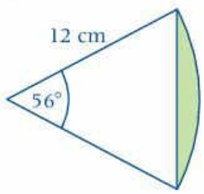
The area of the segment removed is given by

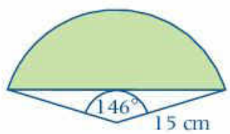
$$A = \frac{\pi r^2\theta}{360} - \frac{1}{2}r^2\sin\theta = \frac{25 \times 125 \times \pi}{360} - \frac{25}{2}\sin 125^\circ$$
$$= 17.031\ldots$$

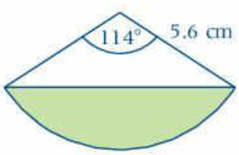
Therefore the area of the washer is $78.539 - 17.031 = 61.5$ cm^2.

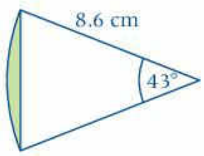
1 Find the area of each shaded segment.

a


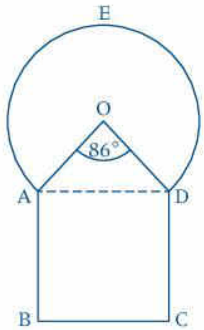
b


c


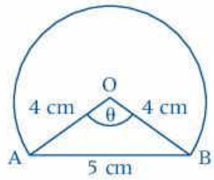
d


e


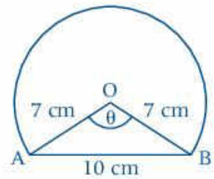
2 ABCD is a rectangle and AED a sector of a circle, centre O, radius 4 cm with $\angle AOD = 86°$.
Find
- **a** the length of AD
- **b** the area of triangle AOD
- **c** the area of the major segment AED
- **d** the area of the complete shape, given that AB = 7 cm.



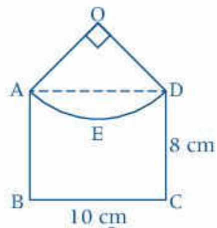
3 A machine part is in the form of a circle with a segment cut off.
Using the measurements given on the diagram determine
- **a** the measure of angle θ
- **b** the area of triangle AOB
- **c** the area of the segment that was cut off.



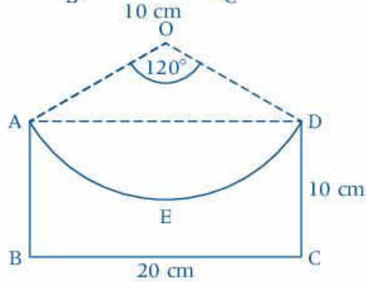
4 A washer for a mixer tap is in the form of a circle with a segment removed.
Using the measurements given on the diagram determine
- **a** the measure of angle θ
- **b** the area of triangle AOB
- **c** the area of the segment that was cut off.



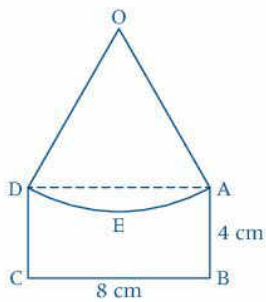
5 AODE is a quadrant of a circle centre O.
ABCD is a rectangle with BC = 10 cm and CD = 8 cm.
Find
- **a** the length of OA
- **b** the area of triangle AOD
- **c** the area of the segment AED
- **d** the area of the shape AEDCB.



6 AODE is a sector of a circle centre O containing an angle of 120°.
ABCD is a rectangle with BC = 20 cm and CD = 10 cm.
The segment AED is removed from the rectangle.
Find
- **a** the length of OA (Hint: drop a perpendicular from O to AD.)
- **b** the area of triangle AOD
- **c** the area of the segment AED
- **d** the area of the complete shape AEDCB.



7 AEDO is a sector of a circle centre O. Angle AOD = 60°.
ABCD is a rectangle with AB = 4 cm and BC = 8 cm.
Find
- **a** the length of OA
- **b** the area of triangle ADO
- **c** the area of the segment AED
- **d** the area of ABCDE.



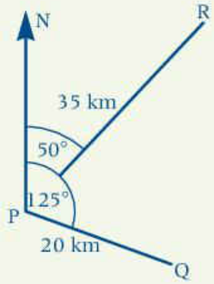
Applications

In many problems a description of a situation is given which can be illustrated by a diagram. Our aim is to find, in this diagram, a triangle in which three facts about sides and/or angles are known. A second diagram, showing only this triangle, can then be drawn and the appropriate rules of trigonometry applied to it.

EXERCISE 21m

Example:

From a port P a ship Q is 20 km away on a bearing of 125° and a ship R is 35 km away on a bearing of 050°. Find the distance between the two ships.

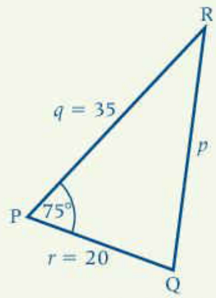


$\triangle PQR$ makes a suitable triangle since PR and PQ are known and $\angle QPR$ can be found.

$\angle QPR = 125° - 50° = 75°$

Using the cosine rule in $\triangle PQR$ gives

$$p^2 = q^2 + r^2 - (2qr\cos\widehat{P})$$
$$= 35^2 + 20^2 - (2 \times 35 \times 20 \times \cos 75°)$$
$$= 1625 - 362.3\ldots = 1262.6\ldots$$
$$\Rightarrow \quad p = 35.53\ldots$$



The distance between the ships is QR so we must calculate p in $\triangle PQR$.

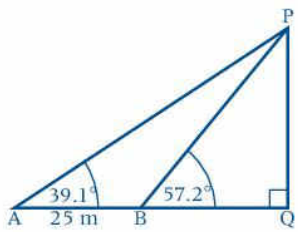
Therefore the distance between the ships is 35.5 km correct to 3 s.f.

1 Starting from a point A, an aeroplane flies for 40 km on a bearing of 169° to B, and then for 65 km on a bearing of 057° to C. Find the distance between A and C.

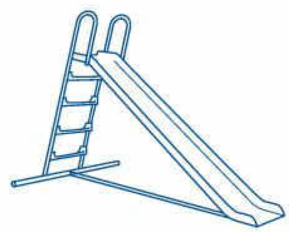
2 From two points A and B, on level ground, the angles of elevation of the top of a radio mast PQ are found to be 39.1° and 57.2°. If the distance between A and B is 25 m, find

a $\angle APB$

b the height of the radio mast.



3 A children's slide has a flight of steps of length 2.7 m and the length of the straight slide is 4.2 m. If the distance from the bottom of the steps to the bottom of the slide is 4.9 m find, to the nearest degree, the angle between the steps and the slide.

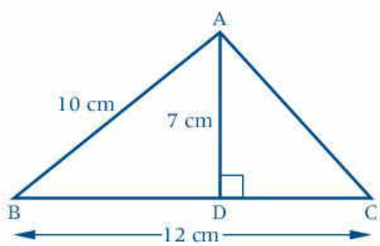


4 Using the information given in the diagram

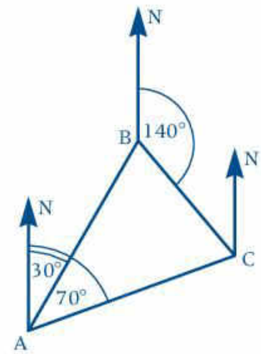
a find the area of $\triangle ABC$

b use the formula area $\triangle ABC = \frac{1}{2}ac\sin\widehat{B}$ to find $\angle B$

c find AC.

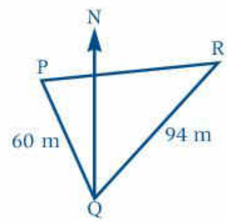


5 The diagram shows the position of three villages A, B and C. From A the bearing of B is 030° and the bearing of C is 070°. The bearing of C from B is 140°.
Determine
- a angle BAC
- b the bearing of B from C
- c angle ACB.



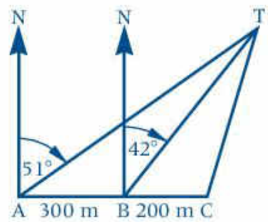
6 A helicopter leaves a heliport A and flies 2.4 km on a bearing of 154° to a checkpoint B. It then flies due east to its base C. If the bearing of C from A is 112° find the distances AC and BC. The helicopter flies at a constant speed throughout and takes 5 minutes to fly from A to C. Find its speed.

7 P, Q and R are three points on level ground. From Q, P is 60 m away on a bearing of 325° and R is 94 m away on a bearing of 040°.
- a Find cos Q.
- b Find the distance between P and R.

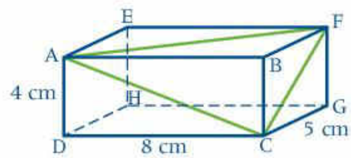


8 ABCD is a quadrilateral in which AB = 4.1 cm, BC = 3.7 cm, CD = 5.3 cm, ∠ABC = 66° and ∠ADC = 51°. Find
- a the length of the diagonal AC
- b ∠CAD
- c the area of quadrilateral ABCD, considering it as split into two triangles by the diagonal AC.

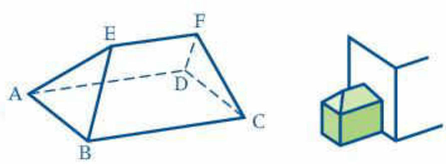
9 The diagram shows three survey points, A, B and C, which are on an east–west line on level ground. From point A the bearing of the foot of a tower T is 051°, while from B the bearing of the tower is 042°. Find
- a ∠TAB and ∠ATB b AT c CT



10 ABCDEFGH is a cuboid measuring 4 cm by 5 cm by 8 cm.
- a Find the length of
  - i AC ii CF iii AF
- b Use the cosine rule and your answers to part **a** to find ∠ACF.



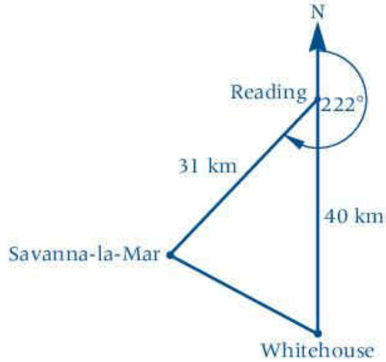
11 The diagram shows the roof of a house extension. BC = 4 m, FC = 3 m, BE = 3.35 m and EF = 2.5 m.
- a Find the length of BF.
- b Use the cosine rule to find the angle between the hip of the roof (EB) and the ridge of the roof (EF).



12 The points A, B and C are on the circumference of a circle with centre O and radius 10 cm. The lengths of the chords AB and BC are 8 cm and 3 cm respectively.
Calculate **a** ∠AOB **b** ∠BOC
c the length of the chord AC
d the area of quadrilateral ABCO.

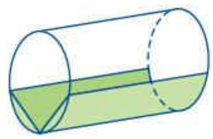
Remember that OA = OB = OC = 10 cm.

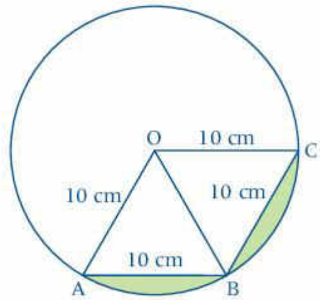
13 The diagram represents the positions of Reading, Whitehouse and Savanna-la-Mar. Whitehouse is 40 km due south of Reading. Savanna-la-Mar is 31 km from Reading on a bearing of 222°.
a Calculate the distance and bearing of Savanna-la-Mar from Whitehouse.
b How far is Savanna-la-Mar
i north of Whitehouse
ii west of Reading?



14 In triangle ABC, AB = 21.4 cm, BC = 14.6 cm and AC = 18.2 cm.
Find **a** ∠ABC **b** the area of the triangle.

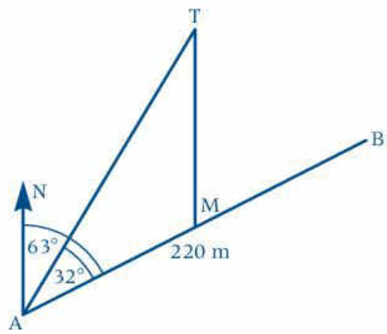
15 The cross-section of a length of packing to keep the contents of a cylinder in place is formed from two identical segments cut from a circle and jointed at B.
The radius of the circle is 10 cm and the tube is 20 cm in length.
Determine
a the angle AOB
b the area of triangle AOB
c the area of the two segments
d the capacity of the tube with the packing in place.





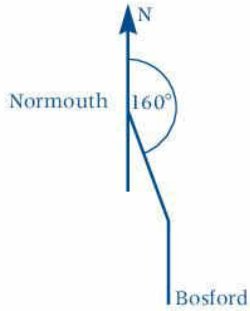
16 ABCD is a rectangle with AB = 12 cm and BC = 5 cm. The bisector of ∠ABC intersects the diagonal AC at E and the bisector of ∠ABE cuts AC in F.
Calculate **a** ∠BAC **b** the lengths of AE and EC
c the length of BE **d** the length of AF.

17 Two brothers, George and James, stand at A and B, two points 220 metres apart on a straight path which runs on a bearing of 063°. George sights a television mast (T) on a bearing of 032° which is due north of M, the mid-point of AB.
Calculate
a George's distance from the mast
b the distance of the nearest point (N) of the path from the mast
c the distance James must walk to arrive at N.



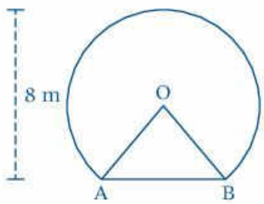
18 In a triangular field OAB, the bearing of B from O is 040°, OA = 160 metres, OB = 170 metres, AB = 70 metres and A is on the eastern side of B.
Calculate **a** ∠OBA
b the bearing of A from B
c the area of the field in square metres.

19 A car ferry leaves Normouth at midnight to sail to Bosford. It sets a course of 160° from Normouth and sails at a steady 12 knots for three hours by which time it has reached a point due north of Bosford. At this point it changes course and sails due south to Bosford, at the same speed, arriving there at 9.15 a.m.
Calculate
a the distance sailed before changing course
b the distance travelled by the ferry between the two ports
c the direct distance between the ports.



20 A farm (F) lies on a road AB. TA and TB are two roads which enable the farmer to travel from his farm F to the nearest town T via either junction A or junction B. If ∠TAB = 84°, ∠TBA = 42°, and the farm is 1.7 km from A and 0.5 km from B, which route is the shorter and by how much?

21 The cross-section through a tunnel is the major segment of a circle of radius 5 m.
The tunnel is 200 m long and 8 m high.
Determine
a the length of the line segment AB
b the angle AOB
c the area of triangle AOB
d the area of the cross-section of the tunnel
e the volume of material removed when the tunnel was excavated.



22 Three heavy guns are situated at points A, B and C and all are directed towards a target T. The distance of the target from A is 1200 metres on a bearing of 222.7°. From B the target is 1500 metres on a bearing of 289.6°, and from C the target is 800 metres on a bearing of 056.3°.
Draw a diagram to illustrate the respective positions of the target and the three guns, and calculate the distance and direction of
a B from A **b** C from A.

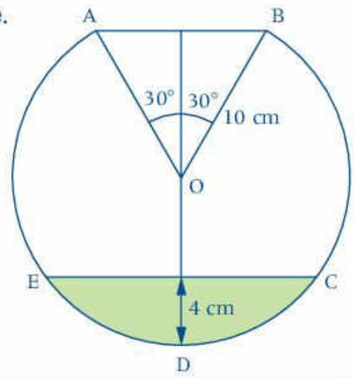
23 Three hill-top cell phone masts A, B and C are such that B is 6.84 km from A on a bearing of 197.5° and C is 11.52 km from A on a bearing of 147.3°. Assuming that all three masts stand the same height above sea-level, find the distance and bearing of B from C.

24 At noon the positions of three aircraft X, Y and Z from an airport control tower, towards which they are flying, are given as 30 km on a bearing of 293°, 45 km on a bearing of 042° and 60 km on a bearing of 153° respectively. If all three aircraft are flying at the same height at a speed of 300 km/h find the position and bearing of Z from Y when X is over the control tower.

25 The diagram shows the cross-section, ABCDE, of a storm drainage pipe. The top of the drain AB is level with the ground. Silt has accumulated in the bottom of the pipe to a depth of 4 cm. OA = OB = 10 cm and $\angle AOB = 60°$.

Determine

- **a** the length of AB, which is open to the air
- **b** the length of CE, which shows the surface level of the silt
- **c** the area of the segment CDE
- **d** the area of cross-section of water when the pipe is full
- **e** the percentage of the area of the cross-section that is water.



MIXED EXERCISE 21

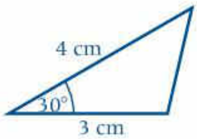
Several answers are given for these questions.
Write down the letter that corresponds to the correct answer.

1 $\sin\theta = \frac{3}{5}$, $\cos\theta =$

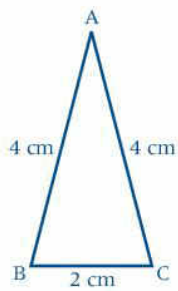
A $\frac{1}{5}$ **B** $\frac{2}{5}$ **C** $\frac{3}{5}$ **D** $\frac{4}{5}$

2 The area of this triangle is

A $3\,\text{cm}^2$ **B** $3\sqrt{3}\,\text{cm}^2$ **C** $6\,\text{cm}^2$ **D** $6\sqrt{3}\,\text{cm}^2$



Questions **3** and **4** refer to this diagram.



3 The area of this triangle is

A $\sqrt{5}\,\text{cm}^2$ **B** $\sqrt{15}\,\text{cm}^2$ **C** $2\sqrt{5}\,\text{cm}^2$ **D** $20\,\text{cm}^2$

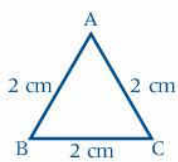
4 $\cos A =$

A $\frac{1}{4}$ **B** $\frac{1}{2}$ **C** $\frac{7}{8}$ **D** $\frac{7}{4}$

5 $\tan\theta = \frac{2}{3}$, $\cos\theta =$

A $\frac{2}{\sqrt{13}}$ **B** $\frac{3}{\sqrt{13}}$ **C** $\frac{1}{3}$ **D** $\frac{3}{\sqrt{5}}$

Questions **6** and **7** refer to this diagram.



6 $\sin B =$

A $\frac{1}{\sqrt{3}}$ **B** $\frac{1}{2}$ **C** $\frac{\sqrt{3}}{2}$ **D** $\sqrt{3}$

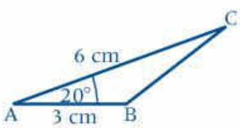
7 The area of the triangle, in square units, is

A $\sqrt{3}$ **B** 2 **C** $2\sqrt{3}$ **D** 4

8 Which of the following statements about this triangle are true?

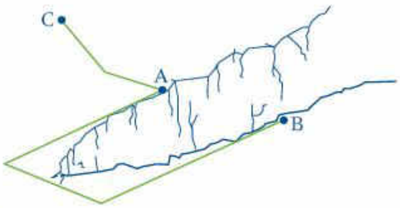
I $BC^2 = 45 - 36\cos 20°$ **II** $\cos 20° = \frac{1}{2}$ **III** area $= 18\sin 20°$

A **I** only **B** **I** and **III** **C** **II** only **D** **I** and **II**



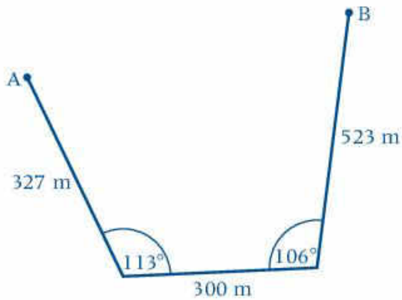
INVESTIGATION

Brendan and May are asked if they can find the distance across a canyon at a particular point where their civil engineering company would like to construct a bridge.



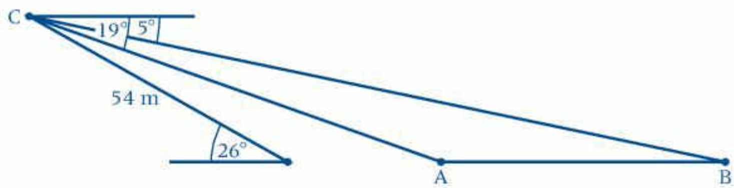
Brendan discovers that the ground from A and B to the end of the canyon is level. He makes several measurements all on the same level and these are given in the following diagram.

Investigate whether or not Brendan has sufficient information to find the distance AB. If he has, find it. If he has not, can you suggest another measurement which he needs? Investigate the minimum information needed to 'fix' a quadrilateral.



May's approach is quite different. She finds that a short distance back from A in the line BA the ground slopes up at a constant angle. She walks 54 metres up the slope to a point C and from this point she measures the angles of depression of A and B. Her measurements are given below.

Does May have enough information to find AB? If she has, find it. If she has not, can you suggest another measurement she could make which will then make it possible?



Are there other ways of finding the distance AB using your present knowledge?

MATHS IS OUT THERE

Did you know that the ancient Egyptians built large pyramids to serve as tombs for their kings?

The Great Pyramid of Giza in Egypt took 100 000 workers over twenty years to build.

IN THIS CHAPTER YOU HAVE SEEN THAT...

- the sine of an obtuse angle is equal to the sine of the supplementary angle, but the cosine of an obtuse angle is equal to *minus* the cosine of the supplementary angle
- when you know one of the trig ratios of an angle as a fraction, you can find the other trig ratios of the same angle
- it is worth remembering that $\sin 30° = \cos 60° = \frac{1}{2}$
- given just the lengths of two sides and one angle of a triangle where the angle is *not* between the two sides, you may find that two different triangles can be drawn
- the sine rule is easier to work with than the cosine rule; the sine rule can be used when you know the length of a side of a triangle and the size of the angle opposite to that side.
- the area of a triangle can be found from the formula $A = \frac{1}{2}ab \sin C$
- the area of a segment of a circle can be found from the formula $A = \frac{\pi r^2 \theta}{360} - \frac{1}{2}r^2 \sin \theta$

22 CONSTRUCTIONS AND TRANSFORMATIONS

AT THE END OF THIS CHAPTER YOU SHOULD BE ABLE TO...

1 Construct a triangle, given the necessary dimensions.
2 Bisect angles and line segments using ruler and compasses.
3 Construct angles of 90, 30, 45 and 60 degrees using ruler and compasses only.
4 Draw a perpendicular from a point to a given line.
5 Make a scale drawing of an object.
6 Construct the centre of rotation, given an object and its image.
7 Construct parallel lines.
8 Construct tangents from a given point to a circle.

MATHS IS OUT THERE

'If the Greek civilization explored the universe with geometry, the Maya did so with arithmetic and time'. Ronald Wright, *Time Among the Maya*.

Ask your history teacher to tell you more about the Maya people.

BEFORE YOU START

you need to know:
- ✓ how to use a pair of compasses, a ruler, and a protractor
- ✓ the properties of the special quadrilaterals
- ✓ how to recognise reflections and rotations
- ✓ the meaning of angles of elevation and depression
- ✓ how to find the gradient of a straight line
- ✓ how to find the equation of a straight line.

KEY WORDS

centre of rotation, circumcircle, construction, inscribed circle, mirror line, parallel lines, perpendicular, tangent

Constructions using ruler and compasses

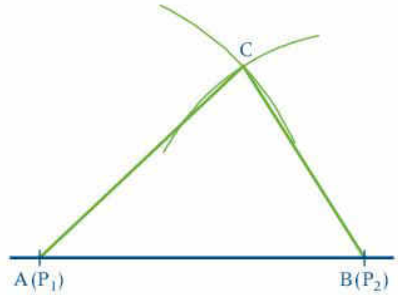
Sometimes a **construction** has to be done without the aid of a protractor or set square. Here is a reminder of the most useful 'ruler and compasses only' constructions. In these diagrams, the positions where the point of the compasses has to be placed are marked P_1, P_2, ... for the first, second, ... positions.

To construct a triangle given the lengths of the three sides

Start by drawing one side, AB.

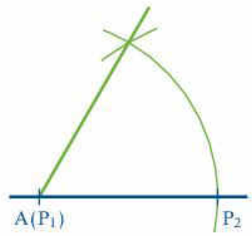
With the point of the compasses at A, and the radius equal to AC, draw an arc.

Then with the point of the compasses at B and the radius equal to BC, draw an arc to cut the first arc.



To construct an angle of 60° at A

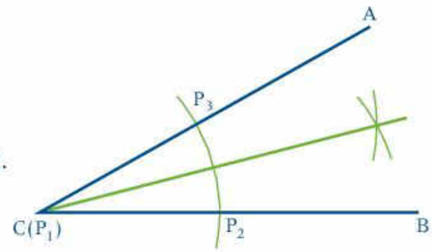
This involves constructing an equilateral triangle but drawing only two sides. Keep the radius the same throughout.



To bisect angle ACB

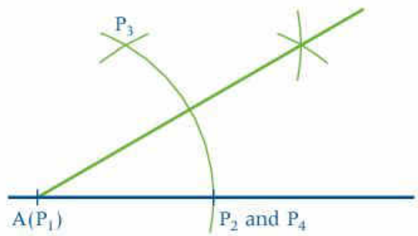
Keep the radius the same throughout.

With the point of the compasses at C draw an arc to cut both arms of the angle. Move the point of the compasses to P_2 and draw an arc. Then, move the point to P_3 and draw an arc to cut the first arc.



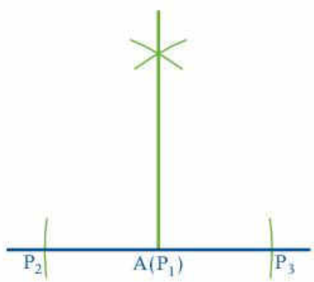
To construct an angle of 30° at A

Construct an angle of 60° at P_1 and then bisect it. Keep the radius the same throughout.



To construct an angle of 90° at A

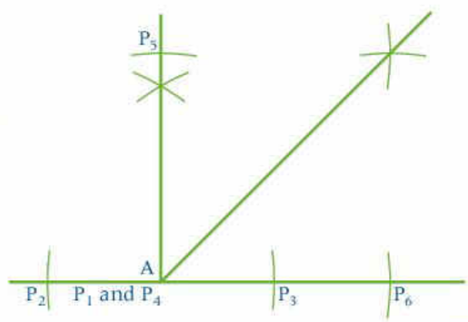
Bisect an angle of 180° at P_1. Enlarge the radius for the arcs drawn from P_2 and P_3.



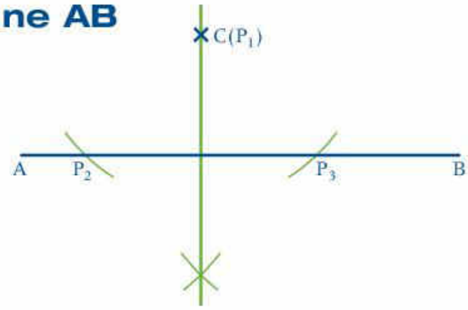
To construct an angle of 45° at A

Construct an angle of 90° at A and then bisect it.

By combining these we can construct a triangle given the lengths of two sides and the included angle when that angle is 90°, 60°, 45°, 30° or 15°.



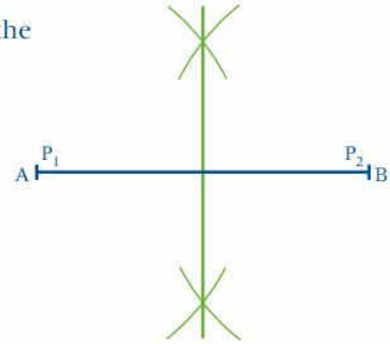
To draw a perpendicular from C to the line AB



Keep the radius the same throughout.

With the point of the compasses on C, draw an arc to cut AB in two places (P_{-2} and P_3). Move the point to P_2 and draw an arc below AB. Then move the point to P_3 and draw an arc to cut the previous arc.

To bisect a line AB



Choose a radius that is more than half the length of the line, and keep the radius the same throughout.

With the point fo the compasses on A, draw arcs above and below AB. Move the compass point to B and draw arcs to cut the first two arcs.

Accuracy

When they are done properly, ruler and compasses constructions give more accurate results than those using the protractors and set squares commonly available. That accuracy depends, however, on

- using compasses that are reasonably stiff (a loose joint means that the radius will change in use)
- using a *sharp* pencil, preferably an H, and keeping it sharp
- making the construction as large as is practical, in particular *not* using too small a radius on the compasses.

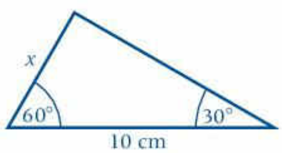
> Aim for a minimum radius of about 5 cm and do not rub out any arcs.

EXERCISE 22a

Construct the following figures using only a ruler and a pair of compasses.

Check your construction by measuring the length marked x. Your result should be within 1 mm of the value given in the answers.

1



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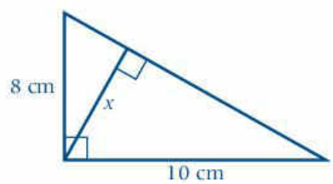

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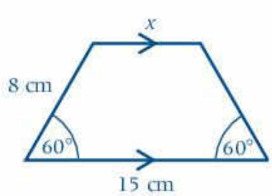

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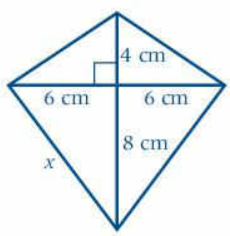

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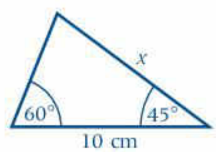


In questions **1** to **5**, a sketch was given of the shape to be drawn accurately. Sometimes, however, only a description of the figure is given. In this case it is essential to draw a sketch first because it gives some understanding of the shape and properties of the accurate drawing that has to be produced.

A sketch should be drawn freehand, but neatly. It should be big enough for essential information to be marked. Sometimes a first attempt at a sketch does not fit the given information. When this happens make another sketch.

6 **a** Sketch a quadrilateral ABCD in which AB = 10 cm, BC = 8 cm, BD = 10 cm and ∠ABC = ∠BCD = 90°.
b What other properties of ABCD can you deduce from your sketch? Justify your answer.

7 ABC is an equilateral triangle of side 6 cm. The line which bisects ∠B cuts AC at D and BD is produced to E so that DE = BD.
a Sketch the quadrilateral ABCE.
b What type of quadrilateral is ABCE? Give reasons.

8 Draw a base line AC about 16 cm long. Mark B on AC so that AB = 8 cm. Show on a sketch the position of D if ∠DAC = 30° and ∠DBC = 60°.
a Write down the sizes of the angles in △ABD.
b What type of triangle is △ABD?
c Construct the diagram and measure BD. Does its length confirm your answer to part **b**?

9 Construct a triangle ABC with AB = 9.5 cm, BC = 7 cm and ∠ABC = 30°.
By construction, find a point D such that ABCD is a parallelogram. Measure and record the lengths of the two diagonals of this parallelogram.

10 Construct a parallelogram ABCD whose diagonals intersect at X given that AC = 10.6 cm, BD = 8.2 cm and ∠AXD = 60°.
Measure the lengths of the sides of the parallelogram.

11 Construct a rectangle with diagonals of length 11.2 cm containing an angle of 45°.
Measure and record the lengths of the sides of the rectangle.

12 Construct a quadrilateral ABCD with AB = 10 cm, BC = 6.5 cm, AC = 10 cm, AD = 8 cm and CD = 4.5 cm.
Measure and record the length of DB.

13 Construct a quadrilateral ABCD with AB = 9.2 cm, BC = 7.8 cm, AD = 5.3 cm, ∠ABC = 60° and ∠CAD = 60°.
Measure and record the length of CD.

14 Construct a quadrilateral PQRS given PQ = 7.8 cm, PR = 9.4 cm, PS = 7.2 cm, ∠QPR = 30° and ∠SPR = 45°.
Measure the length of the diagonal QS.

15 **a** Draw a line AC that is 10 cm long.
b Draw the perpendicular bisector of AC.
c Hence construct the square for which AC is one diagonal.

16 **a** Draw a line DF that is 10 cm long.

b Construct the perpendicular bisector, EG, of DF so that DF and EG intersect at H. Mark the points E and G so that DEFG is a rhombus whose shorter diagonal is 8 cm long.

17 **a** Draw a line AC that is 15 cm long and mark a point E on AC such that AE = 10 cm.

b Construct a line at E that is perpendicular to AC and mark points B and D on this line such that BE = ED = 4 cm.

c Join A, B, C and D to form the quadrilateral ABCD.
State what type of quadrilateral this is and give a reason for your answer.

Scales

The construction of a scale drawing usually requires the choice of a scale and the calculation of lengths to be used. The next exercise gives practice in producing useful sketches and getting information from them. Some of the questions also involve using scales and interpreting information from scale drawings.

EXERCISE 22b

1 A first-floor window A is 4 m above ground level and overlooks a level garden with a tree PQ standing in front of the window. From A the angle of depression of Q, the foot of the tree, is 30° and the angle of elevation of P, the top of the tree, is 60°.

a Draw a sketch to represent this data. Mark B the point on the ground vertically beneath A, and C the foot of the perpendicular from A to PQ.

b Construct this figure with AB = 4 cm. What scale has been used to represent the real-life situation described above?

c Use your diagram to find

i the distance from B to the foot of the tree

ii the height of the tree.

2 A quadrilateral PQRS is to be drawn to scale. PQ = 92 m and QR = 76 m.

a Find what the lengths of PQ and QR should be if the scale is

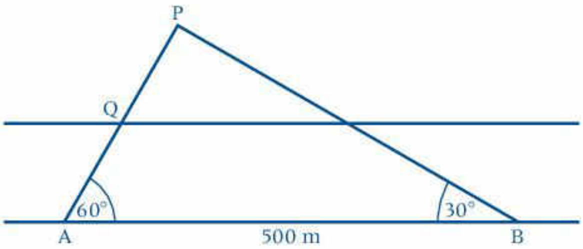
i 1 cm to 10 m **ii** 1 cm to 8 m.

b On a drawing using a scale of 1 cm to 10 m, the measured length of PR is 6.8 cm. What is the real length of PR?

c On a drawing using a scale of 1 cm to 8 m, the measured length of QS is 7.2 cm. What is the real length of QS?

d Which scale was easier to use? Would a scale of 1 cm to 5 m be easier to use than 1 cm to 8 m? Give your reasons.

3 A car travels for 5 kilometres along a straight road towards a crossroads. Here it turns right and travels for 4 kilometres in a perpendicular direction before stopping. Draw a scale diagram for the journey using 2 cm ≡ 1 km. How far is the car in a straight line from its starting point?

4 

Tom stands on a river bank and sees a crow P in a field on the opposite bank. A and B are two points on his river bank which are 500 m apart. From A he measures ∠PAB as 60° and from B he measures ∠PBA as 30°.

a Using ruler and compasses only and using a scale of 1 cm to represent 50 m, make a scale drawing to show these data. Leave all construction lines clearly visible.

b Measure the shortest distance from the crow to the river bank on which Tom is standing.

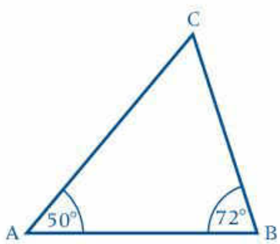
c Calculate, using trigonometry, the shortest distance from the crow to the river bank on which Tom is standing.

d Discuss how reliable your answers to parts **b** and **c** are.

e If the point Q at which AP crosses the opposite river bank is the mid-point of AP, find the width of the river.

5 From a ship (S) the distance and bearing of a lighthouse (L) is 500 m on a bearing of 033°, and the distance and bearing of a trawler (T) is 750 m on a bearing of 123°. Taking 2 cm ≡ 100 m make a scale drawing and use it to find the distance and bearing of the lighthouse from the trawler.

6 Using 1 cm ≡ 1 m, draw the triangle ABC to scale given that AB = 14 m, ∠ABC = 72° and ∠BAC = 50°.
Use this diagram to find the lengths of AC and BC.



7 From a point 160 m from the base of a church tower the angles of elevation of the base and top of the spire are 32° and 46° respectively. Make a scale drawing using 1 cm ≡ 10 m and use it to find the height of the spire.

8 Standing at one corner A of a triangular field the other two corners B and C are 132 m on a bearing of 320° and 94 m on a bearing of 055° respectively. Draw a plan of the field taking 1 cm ≡ 10 m and use it to find

a the distance BC　　**b** the bearing of C from B.

9 Viewed from the top of a cliff 45 metres high the angles of depression of two boats directly out to sea are 42° and 29°. Using 1 cm ≡ 5 m make a scale drawing and use it to find

a the distance of each boat from the base of the cliff

b the distance between the boats.

10 An aeroplane takes off from an airfield and follows the following route: 50 km on a bearing of 047°, 40 km on a bearing of 155°, 150 km on a bearing of 143°.

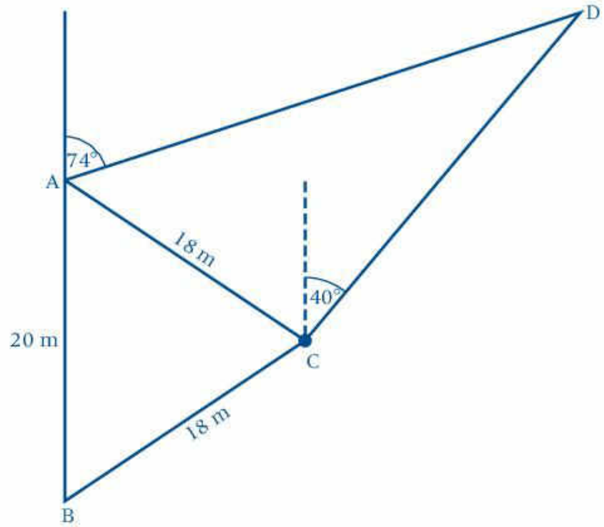
a Draw a scale diagram to illustrate this flight. Take 1 cm ≡ 10 km.

b Use your drawing to find the bearing the aeroplane must set to return to its starting point.

c How long will it take if the aeroplane flies at 500 km/h?

11 A cross-country course starts at the clubhouse, takes a south-easterly path for 4.5 km, changes direction to a bearing of 038° for 7.5 km before running due north for 3 km. The direction then changes yet again to 290° for 10 km.
Taking 1 cm ≡ 1 km make a plan of the route.
Hence find the distance and bearing of the clubhouse from the point where a runner finally turns for home.

12 A and B show the positions of the wickets, which are 20 m apart, on a cricket pitch. The batsman at A strikes the ball towards D where AD makes an angle of 74° with the line of the wickets BA. A fielder standing at C, which is 18 m from each wicket, runs along the path CD in order to cut off the shot, CD making an angle of 40° with the direction BA. The fielder intercepts the ball at D.

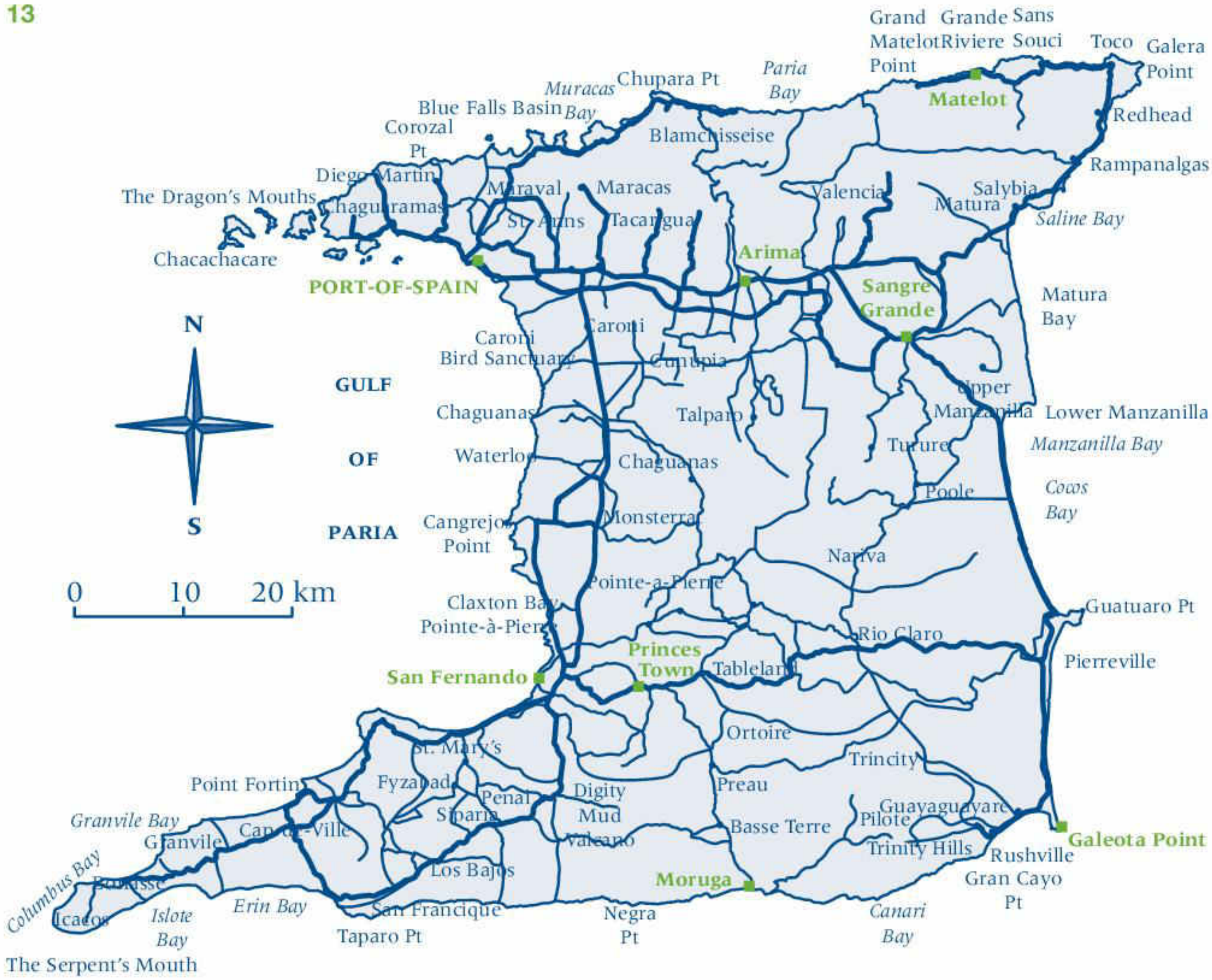


Make a scale drawing and from it find

a the distance the fielder runs before he retrieves the ball

b the distance travelled by the ball along the ground from bat to hand. (Use 1 cm ≡ 2 m)

13



Use this map of Trinidad and the given scale to find

a the direct distance from San Fernando to Port of Spain

b the distance and direction of Arima from Port of Spain

c the distance and direction from Matelot in the north to Moruga in the south

d the distance and direction of Sangre Grande from Princes Town

e the distance and bearing of Galeota Point in the south from Port of Spain.

14 On a map the area of a piece of land is 5.6 cm^2. The scale of the map is 1 : 10 000.
What is the actual area of the land?

15 The scale of a map is 1 : 50 000.

a Find the actual distance between two places that are 12 cm apart on the map.

b The area of a farm is 100 hectares. What is its area on the map?

16 The scale of a map is 1 : 50 000.

a Find the actual distance between two places that are 4.2 cm apart on the map.

b On the map the area of an airfield is 11.2 cm^2. Find, in square kilometres, the actual area of the airfield.

17 The scale of the model of a building is 5 cm to 150 metres.
  a Find this scale in the form 1 : n.
  b How high is the building if the height of the model is 7.2 cm?
  c The area of one wall of the building is 900 m^2. What is the area of this wall on the model?

Transformations

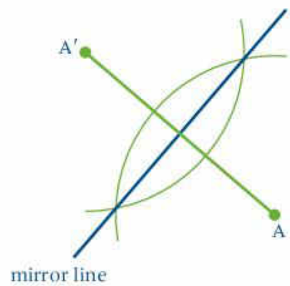
In previous chapters we have found the mirror line for a reflection by inspection.

We also found the image of a shape when it was rotated about a given centre, but unless it was obvious, we could not find the centre of rotation from the original position of a shape and its image.

We are now in a position to determine a **mirror line** and a **centre of rotation**.

Finding the mirror line

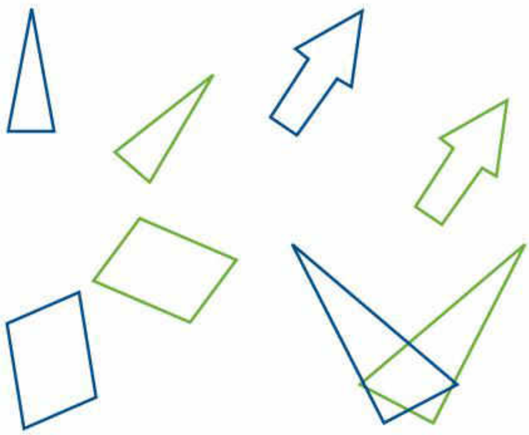
When an object is reflected, we can use the fact that the mirror line is the **perpendicular** bisector of the line joining an object to its image. This means that we can construct the mirror line.



22c

1 In these diagrams, the green shape is the reflection of the blue shape. Trace these diagrams and construct the mirror line.

Identify one point and its image and construct the perpendicular bisector of the line joining them. Check your construction by identifying a second point and its image; the line you have constructed should also be the perpendicular bisector of the line joining this pair of points.



Use 5 mm squared paper and 1 cm for 1 unit for questions **2** and **3**.

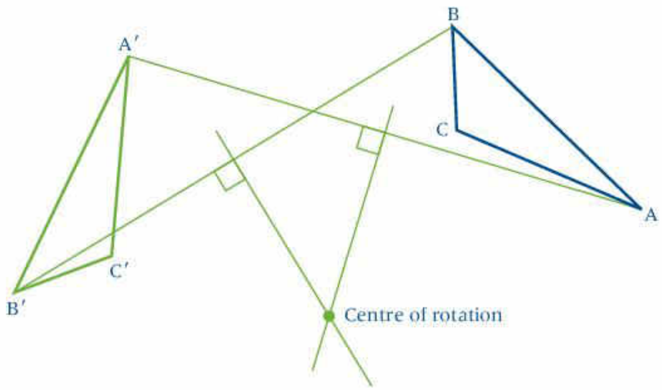
2 Plot the points A(2, 3) and A′(−4, 6).
  Construct the mirror line so that A′ is the reflection of A.
  Use your construction to find the equation of the mirror line.

Find the gradient of the line and its y-intercept. You can then use $y = mx + c$.

3 Repeat question **2** for the points A(6, −1) and A′(−4, 5).

Finding the centre of rotation

When an object is rotated we can quite often locate the centre of rotation simply by looking at the object and its image. This is not always possible, however, and then we use the fact that the centre of rotation is the same distance from a point on the object and from the image of that point.



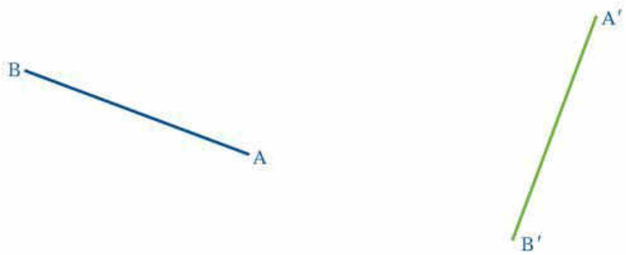
To locate the centre of rotation in this diagram, first we find the point midway between A and its image A′. Then, through this point, a line is drawn at right angles to AA′. Any point on this perpendicular bisector is equidistant from A and A′. (This fact can be checked by measurement.)

This process is repeated with either B and B′ or C and C′.
The perpendicular bisectors meet at the centre of rotation.

EXERCISE 22d

In each diagram the green shape is the image by rotation of the blue shape. Trace the diagram and find the centre of rotation and the angle of rotation.

1



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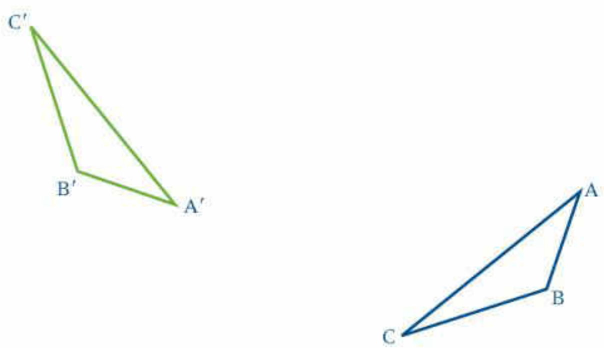

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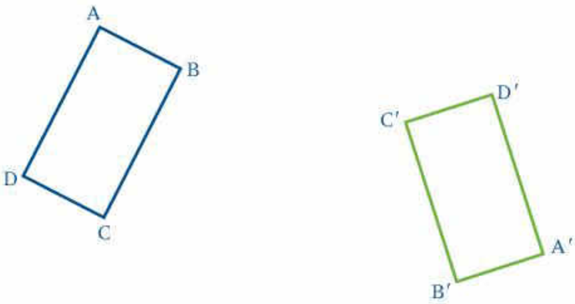
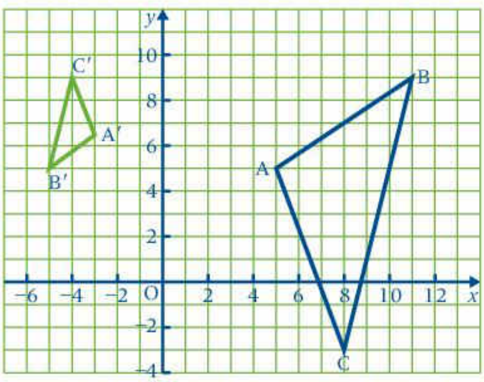


4

Reflections, translations, enlargements and rotations all appear in the remaining questions in this exercise.

In questions **5** to **8** name the transformation that maps the blue shape to the green one (A′ is the image of A, etc.). Describe each transformation as fully as possible; for example, if the transformation is a rotation give the centre and angle of rotation.

5



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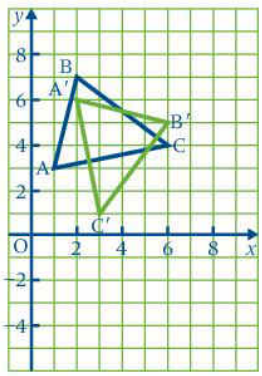

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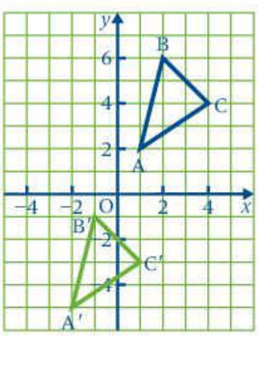

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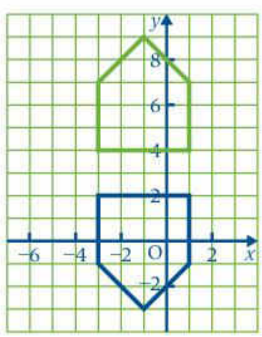


For questions **9** and **10** draw x- and y-axes, each for values from -6 to 6. Use 1 cm for 1 unit.

9 Draw $\triangle PQR$ with P(−1, 2), Q(−1, 5) and R(−3, 2).
Draw the image of $\triangle PQR$

a under a reflection in the line $y = x$. Label it $\triangle P_1Q_1R_1$

b under a reflection in the y-axis. Label it $\triangle P_2Q_2R_2$

c under a reflection in the x-axis. Label it $\triangle P_3Q_3R_3$.

Describe the transformation

d that maps $\triangle P_2Q_2R_2$ onto $\triangle P_3Q_3R_3$

e that maps $\triangle P_3Q_3R_3$ onto $\triangle P_1Q_1R_1$.

10 Draw $\triangle LMN$ with L(3, 2), M(5, 2) and N(5, 5).
Draw the image of $\triangle LMN$

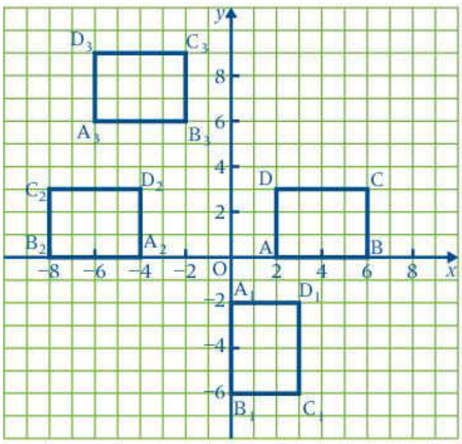
a under a reflection in the line $y = -x$. Label it $\triangle L_1M_1N_1$

b under a rotation of 180° about (0, 2). Label it $\triangle L_2M_2N_2$

c under a translation described by the vector $\begin{pmatrix} -4 \\ 1 \end{pmatrix}$. Label it $\triangle L_3M_3N_3$.

What is the image of $\triangle LMN$ under a rotation of 360° about O?

11



a Give the transformation that maps rectangle ABCD to

i $A_1B_1C_1D_1$

ii $A_2B_2C_2D_2$

iii $A_3B_3C_3D_3$

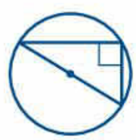
b $A_1B_1C_1D_1$ is mapped to $A_3B_3C_3D_3$ by a rotation.
Copy the diagram and use it to find the centre of rotation.

Circles and tangents

Ruler and compasses only constructions involving circles use these properties.

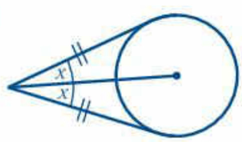
A **tangent** to a circle is perpendicular to the radius at the point of contact.

The centre of a circle lies on the perpendicular bisector of any chord.

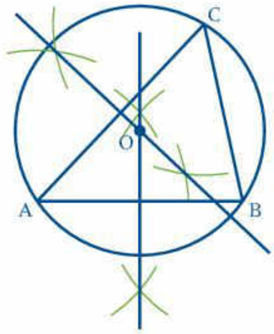


The angle in a semicircle is equal to 90°.

Tangents from an external point to a circle are equal in length and the line from the point to the centre of the circle bisects the angle between the tangents.



To construct the circumcircle of a triangle



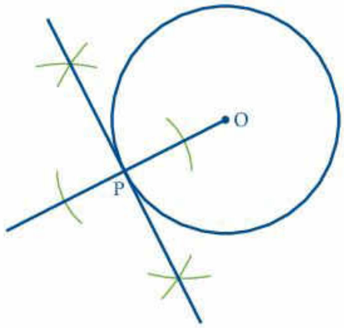
The **circumcircle** of a triangle goes through each vertex of the triangle, so the sides of the triangles are chords of the circle.

Construct the perpendicular bisectors of AB and AC. Where they cross, at O, is the centre of the circle.

With centre O and radius equal to OA, draw a circle.

If the construction is accurate, the circle will go through the three vertices of the triangle.

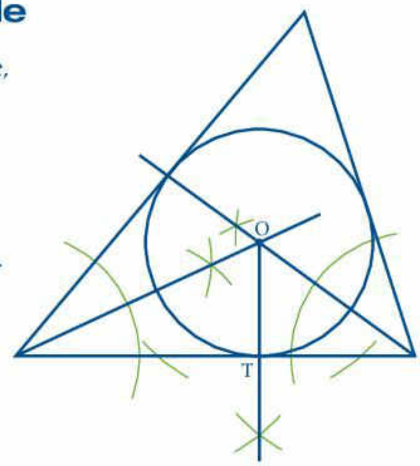
To construct a tangent to a circle



To construct a tangent to the circle at P, draw the radius from the centre of the circle O to P and extend the line.

Then construct a line perpendicular to OP at P.

To construct the inscribed circle of a triangle



The **inscribed** circle of a triangle touches the three sides of the triangle, so the sides of the triangle are tangents to the circle.

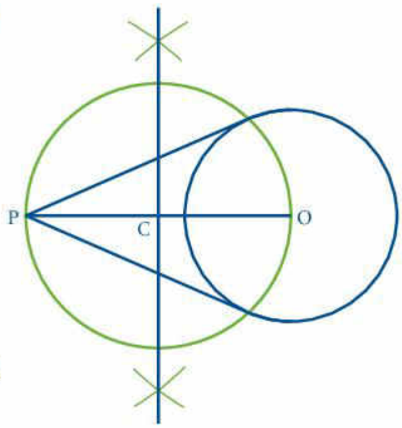
To find the centre of the inscribed circle, bisect two of the angles of the triangle; the centre, O, of the circle is where these bisectors intersect.

To find the radius of the circle, construct a line from O that is perpendicular to one of the sides of the triangle to meet that side at T.

Then OT is the radius of the circle.

If the construction is accurate, the circle will touch each side of the triangle.

To construct the tangents from an external point to a circle



To construct the tangents from P to the circle with centre O, join OP.

Then construct the perpendicular bisector of OP.

Then with the point at C (where the perpendicular bisector cuts OP) and radius equal to CP, draw a circle.

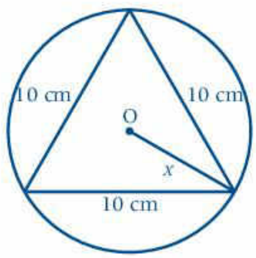
The points where this circle cuts the circle centre O are where the tangents from P meet the circle.

This construction uses the fact that the angle in a semicircle is 90°, and the fact that a tangent to a circle is perpendicular to the radius at the point of contact.

EXERCISE 22e

Construct the following figures using only a ruler and a pair of compasses. Check your construction by measuring the length marked x; it should be within 1 mm of the given answer.

1



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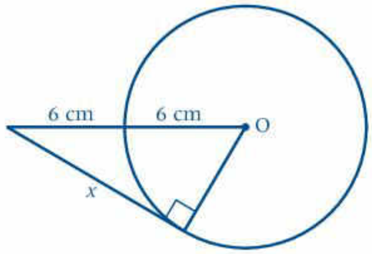

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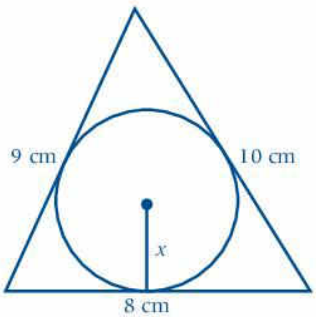

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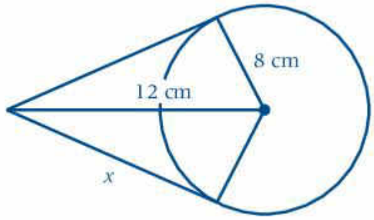


5 Construct a right-angled triangle with sides 6 cm, 8 cm and 10 cm. Construct the circumcircle and the inscribed circle of this triangle. Find the difference between the radii of these two circles.

6 Construct a triangle PQR in which PR = 8.4 cm, ∠QPR = 30° and ∠QRP = 45°.
Construct the circumcircle to this triangle. Measure and record its radius.

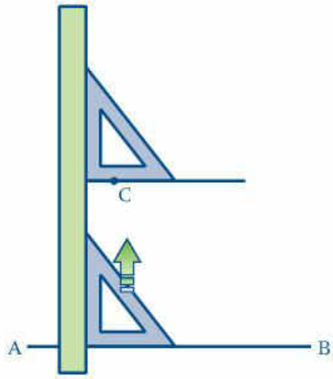
Parallel lines

When we need to construct a figure with **parallel lines**, we can use a set square to draw the parallel lines.

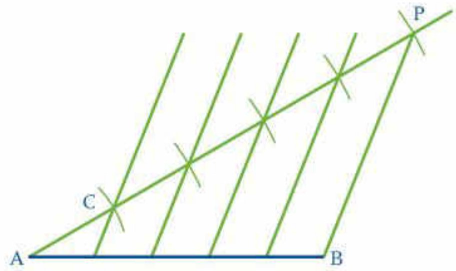
To draw a line through a given point parallel to a given line

To draw a line through C parallel to AB, place one edge of a set square along AB.

Then place a ruler along another edge of the set square and slide the set square along the ruler until the edge of the set square that was on AB is on the point C. Draw a line along this edge of the set square.



To divide a line into a given number of equal parts



To divide the line AB into, say, five equal parts, draw a line at an angle to AB from A. (Make this line fairly long.)

With the point of the compasses on A, and any radius (but not too large) draw an arc to cut your line at C. Keep the same radius, move the point to C, then draw another arc to cut your line.

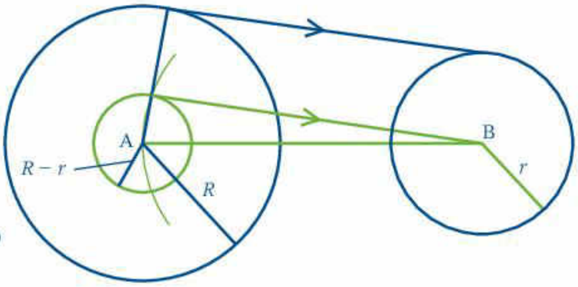
Repeat this, until there are 5 arcs cutting your line.

(This creates five equal divisions along your line.)

Join the point where the fifth arc cuts the line (P) to B.

Then, using a ruler and a set square, draw lines parallel to PB through the other four divisions.

To construct a common tangent to a pair of circles



To construct a common tangent to two circles whose centres are A and B, join AB. Then with centre A and radius equal to the *difference* in the radii of the two circles, draw another circle, centre A.

Now construct the tangent from B to this circle.

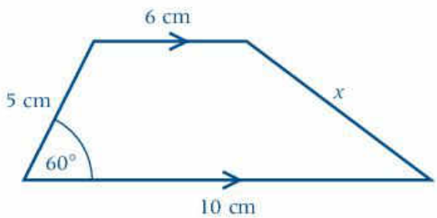
Produce the radius through the point of contact of the tangent to cut the larger circle and draw a line parallel to the tangent through this point.

If the construction is accurate, this line will touch the other circle.

EXERCISE **22f**

Construct the following figures using a ruler, a pair of compasses and a set square. Check your construction by measuring the length marked x; it should be within 1 mm of the given answer.

1



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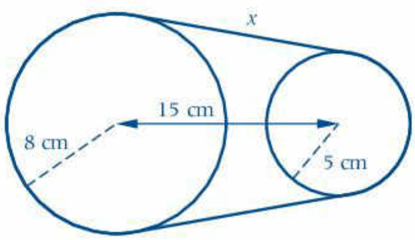

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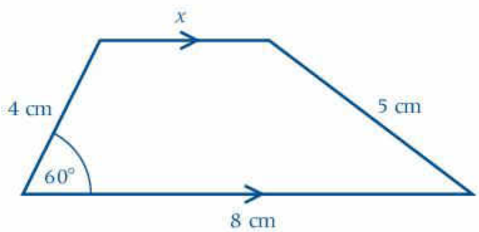

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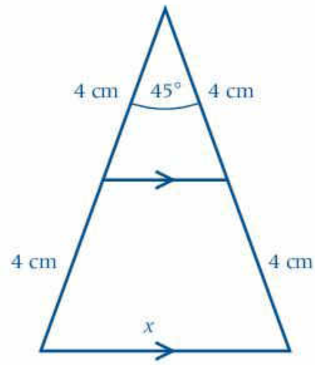


5 Draw a line AB that is 12 cm long. Using a ruler and set square, divide this line into 7 equal parts.
Measure one of these parts and write down its length.

6 Construct an equilateral triangle with sides 8 cm long. Use a ruler, a pair of compasses and a set square to construct a line parallel to one side of the triangle that divides the other two sides in the ratio 1 : 4.
Measure this line and write down its length.

7 Draw AB = 11.2 cm. Using a constructional method, find the point C which divides AB in the ratio 3 : 5. Measure and record the length of AC.

8 Construct a quadrilateral ABCD with AB = 5 cm, BD = 7 cm, AD = 6.3 cm, ∠DBC = 60° and ∠BDC = 45°.
Construct a line through C parallel to BD cutting AB produced at E.
Measure and record the length of AE.

9 Construct a trapezium ABCD with AB parallel to DC in which AB = 12.2 cm, BC = 7.3 cm, DC = 8.5 cm and ∠ABC = 60°.
Measure and record the lengths of AD, AC and BD.

INVESTIGATION

The diagram shows two squares.

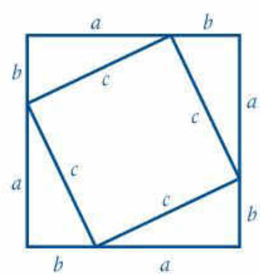
Find the area of the larger square

a in terms of *a* and *b*

b in terms of *a*, *b* and *c*.

Hence find a relationship between *a*, *b* and *c*.

What have you proved?



MATHS IS OUT THERE

Did you know that the Greeks treated geometry as an intellectual game? One of their rules was that *only such constructions were allowed as could be made with a straight edge and a pair of compasses.*

IN THIS CHAPTER YOU HAVE SEEN THAT...

- to make an accurate construction, you need to use a sharp pencil and a pair of compasses with a fairly stiff joint
- many constructions can be done using a ruler and a pair of compasses, but a set square is useful when parallel lines need to be drawn
- given a point and its image under a reflection, the mirror line is the perpendicular bisector of the line joining the points. This can be constructed using only a ruler and a pair of compasses
- given an object and its image under a rotation, the centre of rotation is found by constructing the perpendicular bisectors of the lines joining a pair of points on the object with their corresponding points on the image. The centre of rotation is the point where these bisectors meet
- drawing a sketch of the shape you need to construct will help give an idea of its shape and also enable you to identify properties of the shape that may help with the construction
- when you need to choose your own scale for a drawing, use one that is easy to work with. For example, choose 1 cm to 10 m rather than 1 cm to 9 m.

23 GROUPED DATA

AT THE END OF THIS CHAPTER YOU SHOULD BE ABLE TO...

1 Organise data into a grouped frequency table.
2 Understand class intervals and class boundaries.
3 Find estimates for the mean of a grouped frequency distribution.
4 Draw histograms to represent data in a frequency table.
5 Get information from a histogram and use it to find probabilities.
6 Decide which statistical measure to use.

BEFORE YOU START

you need to know:
✓ how to find the mean of a grouped frequency distribution
✓ how to find a probability
✓ what a cumulative frequency polygon represents.

KEY WORDS

class interval, class boundary, class limit, class mid-point, class width, continuous, discrete, frequency polygon, frequency table, histogram, lower class boundary, mid-class value, modal group, raw data, upper class boundary

MATHS IS OUT THERE

Did you know that the subject of statistics has its beginnings in the census counts around the first century AD? It did not become a scientific discipline until the 18th century. The word 'statistics' comes from the Latin word *status* which means 'condition'.

Collecting information

'Information Technology' is the name used for modern methods of dealing with information. Before information can be distributed, it has to be collected and organised.

The local health authority wanted some information about the heights of five-year-old children in its area. At their first school medical examination, the height of each child was recorded. The following figures are the recorded heights (in cm to the nearest cm) of 90 five-year-olds from one infant school:

99	107	102	98	115	95	106	110	108	105
118	102	114	108	94	104	113	102	105	95
105	110	109	101	106	108	107	107	101	109
108	105	116	109	114	110	97	110	113	116
112	101	92	105	104	115	111	103	110	99
93	104	103	113	107	94	102	117	116	104
99	114	106	114	98	109	107	114	106	107
109	113	112	100	109	113	118	104	94	114
107	96	108	103	112	106	115	111	115	101

This set of figures was written down in the same order as the children came into the medical examination, so the heights are listed in a random order. Disorganised figures like these are called **raw data**. They need organising before we can make sense of them.

Frequency tables

The heights given above range from 92 cm to 118 cm and there are 18 different values. It is tedious to count the number of times each individual height occurs. It is better to organise them into groups. About five to ten groups is sensible. We will group them as follows:

90–94 cm, 95–99 cm, 100–104 cm, 105–109 cm, 110–114 cm, 115–119 cm.

Counting the number of heights in each group gives the following table which is called a **frequency table**:

Group	Tally	Frequency
90–94 cm	𝍸	5
95–99 cm	𝍸 IIII	9
100–104 cm	𝍸 𝍸 𝍸 II	17
105–109 cm	𝍸 𝍸 𝍸 𝍸 𝍸 III	28
110–114 cm	𝍸 𝍸 𝍸 𝍸 I	21
115–119 cm	𝍸 𝍸	10
		Total: 90

When you make a frequency table from a set of raw data, work down the columns, making a tally mark for each value in the tally column next to the appropriate group. Do *not* go through the data looking for values that fit into the first group and the second group and so on.

Class intervals

The heights in the list are continuous data and are given correct to the nearest centimetre.

The smallest height in the first group is 90 cm to the nearest centimetre, but the smallest height that rounds to 90 cm is 89.5 cm.

The largest height in this group is 94 cm to the nearest centimetre, but heights above 94 cm up to, but not including, 94.5 cm round to 94 cm.

So the group 90 cm to 94 cm includes all heights from 89.5 cm up to, but not including, 94.5 cm. 90 cm is called the lower **class limit** and 94 cm is called the upper class limit.

The interval from 89.5 cm to 94.5 cm is called the **class interval**.

89.5 cm is called the **lower class boundary** and 94.5 cm is called the **upper class boundary**.

The **class width** is the difference between the upper and lower class boundaries. In this case, the class width is 94.5 cm − 89.5 cm = 5 cm.

Numerical data may be **discrete** or **continuous**. Discrete data has distinct and exact values, whereas continuous data can have any value within a range. At a professional athletics meeting a time may be given as 8.21 seconds. This is not exact, it is correct to the nearest one-hundredth of a second, so it lies in the range 8.205 to 8.215 seconds.

EXERCISE 23a

1 Fifty junior school children joined the school's computer club. Their ages were recorded:

10	8	9	10	7	8	8	11	10	9
7	8	9	9	10	11	11	10	9	8
8	7	9	7	10	7	10	8	9	11
10	11	8	10	9	8	9	7	11	10
9	10	10	11	10	11	7	11	10	9

a Make a frequency table showing the number of children of each age.
b State the class width.

2 The local ice-cream parlour had 56 customers one Saturday evening. They bought the following amounts, in ml:

270	110	45	96	250	490	325	45
382	136	125	450	420	380	150	250
85	250	320	525	218	210	216	120
155	430	250	40	510	150	510	245
320	120	316	150	260	45	180	310
273	280	85	280	318	45	210	282
462	316	218	316	325	45	560	315

a Use groups 0–99 ml, 100–199 ml, 200–299 ml, 300–399 ml, 400–499 ml, 500–599 ml, to make a frequency table.
b Write down
 i the class width of the group 400–499 ml.
 ii the class boundaries of the group 400–499 ml.

3 The weights, to the nearest gram, of 30 bags of popcorn sold at a fête are given below:

69	83	75	65	68	68	73	70	80	79
70	76	63	86	69	65	66	74	66	68
70	60	67	74	65	65	67	88	81	63

a Choose your own groups and make a frequency table.
b **i** Write down the class interval that includes the weight 75 grams.
 ii State the class widths and the class boundaries.

Measures of central tendency of a grouped frequency distribution

When data is grouped, individual values are not known. This means we cannot give exact values for the mode, the median or the mean.

Consider the following frequency table, which was compiled from the masses (in kilograms to the nearest kilogram) of 100 people.

Mass (kg)	50–59	60–69	70–79	80–89	90–99	100–109	
Frequency	15	30	35	15	3	2	Total = 100

The modal group

The group with the highest frequency is called the **modal group**.
The modal group is 70 kg to 79 kg as this is the group with the highest frequency.

The group containing the median

The median is the value of the middle item. For the 100 masses in the table, this is the 50.5th mass. Counting tells us this is also in the group 70 kg to 79 kg.

> The next chapter shows a method for finding an estimated value for the median.

The mean

> The middle value of a group is called the **class mid-point** or the **mid-class value.**

We know that 15 people have masses between 50 kg and 59 kg but we do not know the individual masses of these people. If we assume that the mean mass of the 15 people is the middle value of the group, i.e. 55.5 kg, we can estimate the total mass in that group as 15×55.5 kg. Doing this for each group gives an estimate for the total in each group which we can then add to find an estimate for the total of the 100 masses.

This calculation is best done in an organised way by adding two rows to the table:

Mass (kg)	50–59	60–69	70–79	80–89	90–99	100–109	
Frequency, f	15	30	35	15	3	2	total: 100
Mid-class value, x	54.5	64.5	74.5	84.5	94.5	104.5	
$f \times x$	817.5	1935	2607.5	1267.5	283.5	209	total: 7120

> The mid-class value of a group is found by adding the upper and lower class boundaries and dividing by 2.
> The class boundaries of the first group are 49.5 and 59.5 so the mid-class value is
> $\frac{1}{2}(49.5 + 59.5) = 54.5$

Therefore the estimated value of the mean mass is $\frac{7120}{100} = 71.2$ kg.

EXERCISE 23b

Example:

The heights, in centimetres, of a sample of tomato seedlings were recorded. The results are shown in the table.

Height, cm	1–3	4–6	7–9	10–12	13–15
Number of plants	2	10	25	20	3

a Calculate **i** the total number of seedlings in the sample
ii the group that contains the median
iii an estimate of the mean height of these seedlings.

b One seedling in the sample is chosen at random.
Calculate the probability that its height is at least 10 cm.

a **i** 60

ii

Height, cm	1–3	4–6	7–9	10–12	13–15	
Number of plants, f	2	10	25	20	3	total: 60
Mid-class value, x	2	5	8	11	14	
$f \times x$	4	50	200	220	42	total: 516

Add two rows to the table: one for the mid-class value and one for (mid-class value) × (number of plants)

An estimate for the mean height is $\frac{516}{60}$ cm = 8.6 cm.

Mean = $\frac{\text{sum of the values}}{\text{number of values}}$

iii The group 7–9 cm contains the median.

The median is the 30.5th height.

b Probability that a seedling is at least 10 cm high = $\frac{23}{60}$ = 0.38 to 2 d.p.

There are 20 + 3 seedlings that are at least 10 cm high out of a total of 60 seedlings.

1 The heights (in cm) of a sample of plants were recorded and grouped as shown in the table.

Height (cm)	3–7	8–12	13–17	18–22	23–27
Number of seedlings	4	13	20	10	3

Calculate
a the total number of plants
b **i** the group containing the median height
ii the modal group
c an estimate of the mean height.

2 The table shows the grouped frequency distribution of the marks, out of 50, scored by the students in a class.

Mark	1–10	11–20	21–30	31–40	41–50
Frequency	2	6	12	9	3

a Find
i the number of students in the class
ii the modal group
iii the group containing the median.
b Estimate the mean mark.
c A student is chosen at random.
What is the probability that this student scored more than 30?

3 The heights, in centimetres, of the boys in a class are given in the table.

Height (cm)	150–154	155–159	160–164	165–169	170–174
Frequency	2	6	12	7	6

Find **a** the modal group
b an estimate for the mean.

4 The masses of 30 men are given in the table.

Mass (kg)	45–49	50–54	55–59	60–64	65–69
Frequency	1	5	9	12	3

a Write down the modal group.
b Estimate the mean mass of the group.

5 The number of deaths in road accidents of people under 30 in a city for last year are shown in the table.

Age (completed years)	5–9	10–14	15–19	20–24	25–29
Frequency	2	6	17	23	14

Estimate the mean age at death.

A five-year-old child is at least 5 years old but less than 6 years old. The class containing 5- to 9-year-olds is $5 \leqslant \text{age} < 10$ so the class boundaries of this group are 5 and 10 years.

6 Bags of sweets are weighed to the nearest gram. Their masses are recorded in the table.

Mass (g)	90–99	100–109	110–119	120–129	130–139
Frequency	4	7	19	12	8

a How many bags are weighed?
b Estimate the mean mass.
c A bag is chosen at random.
What is the probability that its mass is less than 120 g?

7 In a survey the length of the index finger on the right hand of a sample of adults was measured (to the nearest mm). The results are shown in the table.

Length (mm)	45–54	55–64	65–74	75–84	85–94
Frequency	3	7	30	18	2

a How many people took part in the survey?
b Estimate the mean length of the index fingers in the group.
c What is the probability that the length of the index finger of one of these adults taken at random is less than 75 mm?

8 The journey times to college, t minutes, for a group of college students are given in the table.

Time, t minutes	**Frequency**
$0 \leqslant t < 10$	10
$10 \leqslant t < 20$	22
$20 \leqslant t < 30$	16
$30 \leqslant t < 40$	24
$40 \leqslant t \leqslant 50$	8

The class boundaries are given in the table. The class boundaries of the first group are 0 and 10.

a Estimate the mean time.
b A student is selected at random.
What is the probability that this student took at least 30 minutes to get to college?

9 The table shows the number of employees on the pay rolls of the businesses in a town.

Number on roll	0–29	30–59	60–89	90–119	120–149
Frequency	39	34	20	5	2

a How many businesses are there in the town?
b Estimate the mean number of employees per business.

These are discrete data so there is no value (number of employees) between 29 and 30. The upper and lower boundaries of the first group are 0 and 29.

10 Alex counted the number of words in the sentences of the first few paragraphs in a book he was reading. This information is shown in the table.

Number of words	0–2	3–5	6–8	9–11	12–14	15–17	18–20
Frequency	5	9	11	18	8	5	4

a For how many sentences did Alex count the number of words?
b Estimate the mean number of words per sentence.
c A sentence was chosen at random.
What is the probability that there were more than 11 words in this sentence?

11 A survey was taken to find the time that the patients in a surgery had to wait before they saw the doctor. The following table shows the result of the survey during one week.

Waiting time (minutes)	1–5	6–10	11–15	16–20	21–25
Number of patients	8	43	57	33	27

a Estimate the mean waiting time.
b What is the probability that a patient had to wait more than 15 minutes?

12 The hand-spans of a group of adults were measured. The results are shown in the table.

Hand-span (l mm)	$180 \leqslant l < 190$	$190 \leqslant l < 200$	$200 \leqslant l < 210$	$210 \leqslant l < 220$	$220 \leqslant l < 230$
Frequency	4	8	12	10	6

a How many adults were there in the group?
b Estimate the mean width of a hand-span.

13 Thirty-five students sat a test. The test was marked out of 100 and the results are shown in the table.

Mark	1–20	21–40	41–60	61–80	81–100
Frequency	2	7	10	11	5

Estimate the mean mark scored on the test.

14 Denesh analysed the first few pages of a novel to find out how many times the letter ‘e’ occurred in each sentence. The result of his analysis is given in the table.

Number of e’s	0–2	3–5	6–8	9–11	12–14	15–17
Frequency	17	24	8	9	1	1

a How many sentences did Denesh look at?
b Estimate the mean number of e’s per sentence.

c What is the probability that a sentence selected at random contains either 3, 4 or 5 e's?
d Is it possible to give the number of sentences containing less than 4 e's? Give a reason for your answer.

15 The distances jumped by the competitors in a long jump competition are given in the table.

Length (l m)	$5 \leqslant l < 5.8$	$5.8 \leqslant l < 6.6$	$6.6 \leqslant l < 7.4$	$7.4 \leqslant l < 8.2$	$8.2 \leqslant l < 9.0$
Frequency	2	9	5	3	1

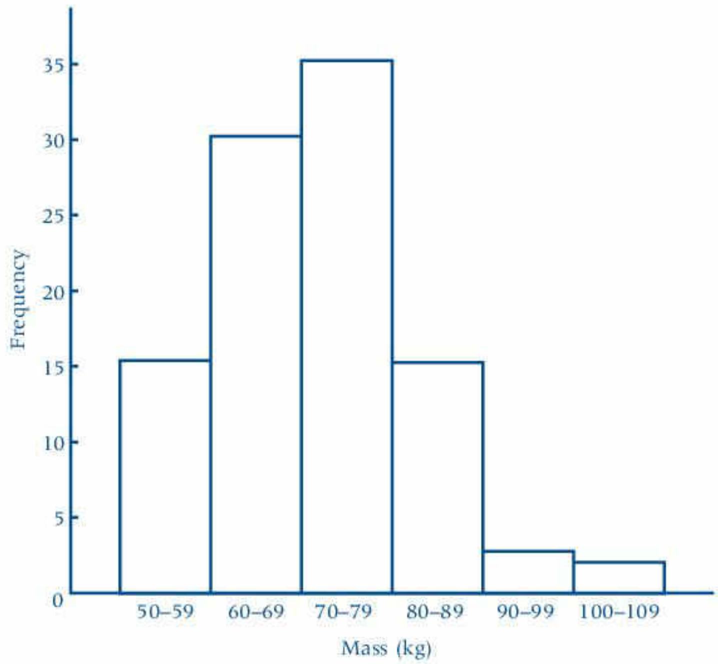
a How many competitors were there?
b What is the shortest possible distance jumped?
c Estimate the mean distance jumped.
d Sena said that 9 competitors jumped further than 6.6 metres. Can Sena be certain that this statement is true? Give a reason for your answer.

Histograms

Consider the following frequency table, which was compiled from the masses (in kilograms to the nearest kilogram) of 100 people:

Mass, kg	50–59	60–69	70–79	80–89	90–99	100–109	
Frequency	15	30	35	15	3	2	Total = 100

A frequency table tells us the number of items in each group. In this example, there are 15 people with weights between 50 kg and 59 kg, 30 people with weights between 60 kg and 69 kg, and so on. The chart illustrates this frequency table.



Notice that the bars touch. Gaps between the bars would not be suitable for illustrating the data because if there were a gap between, say, the first two bars, then a mass might be in that gap. This is because the data are continuous, so a mass could be anywhere between 49.5 kg and 109.5 kg.

It is the *area* of a bar that gives the impression of the number of items in a group.

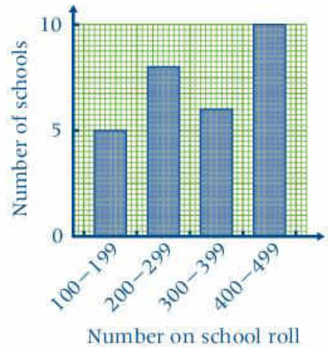
When a bar chart is constructed so that the bars touch and the *area* of each bar is proportional to the number of items in each group it is called a **histogram**.

The diagram above is a histogram: the class intervals are 49.5–59.5 kg, 59.5–69.5 kg …, i.e. all the class widths are 10 kg so each bar has the same width. Hence the area of each bar is proportional to its height. In this case, therefore, the height of each bar is proportional to the number of items in the group.

The data in the following frequency table are discrete.

Numbers of students on school rolls on one island

Number on school roll	100–199	200–299	300–399	400–499
Frequency	5	8	6	10



The data are discrete, i.e. it is not possible to have 199.5 pupils, so a histogram is not a suitable illustration for this distribution but we can use a bar chart, with gaps between the bars.

EXERCISE 23c

Draw a histogram to represent the frequency tables in questions **1** to **6**. Write down the class width for each table.

1 The table shows the distribution of ages of 100 people attending a school concert. The ages are given correct to the last birthday.

Age (years)	0–19	20–39	40–59	60–79	80–99
Frequency	43	24	17	10	6

2 The table shows the results of a survey on the weekly earnings, to the nearest dollar, of 100 sixteen-year-olds.

Weekly earnings ($)	20–29	30–39	40–49	50–59	60–69	70–79
Frequency	45	10	11	21	10	3

3 The table shows the distribution of the average marks of 40 children in the end-of-year examinations.

Average mark	1–20	21–40	41–60	61–80	81–100
Frequency	2	4	19	12	3

4 In a survey the length of the middle finger on the right hand of a sample of adults was measured (to the nearest mm). The results are shown in the table.

Length (mm)	45–54	55–64	65–74	75–84	85–94
Frequency	4	10	47	32	7

5 Bags of nails are weighed to the nearest gram. Their masses are recorded in the table.

Mass (g)	70–79	80–89	90–99	100–109	110–119
Frequency	3	6	21	13	7

6 A survey was taken to find the time that the patients in an Outpatients department of a hospital had to wait before they were attended to. The table shows the result of the survey for one Monday in May.

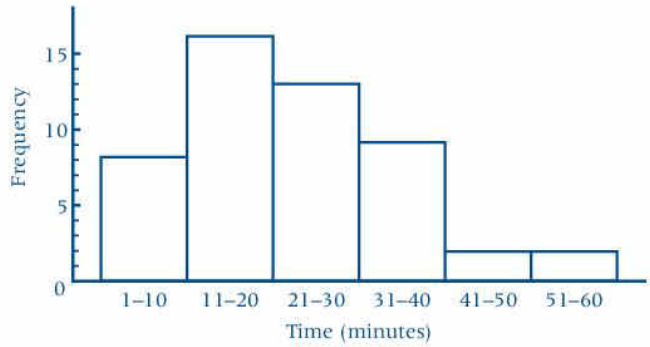
Waiting time (minutes)	1–5	6–10	11–15	16–20	21–25
Number of patients	4	38	46	42	70

7 The histogram shows the distribution of the times, to the nearest minute, taken by 50 children to get to school.

a Construct a frequency table from this histogram.

b What is the lower class boundary of the group 11–20 minutes?

c One of these children is chosen at random. What is the probability that the child took between 1 and 10 minutes to get to school.



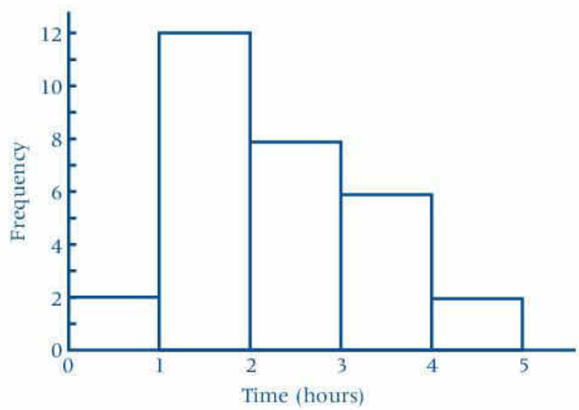
8 The histogram is based on the time to the nearest minute that 30 children spent watching television on a particular Saturday.

a Construct a frequency table from this histogram.

b What is the upper class boundary for the group 3–4 hours?

c Find the mean of this distribution.

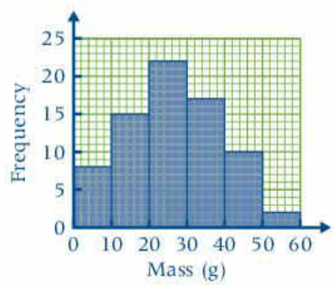
d One of these children is chosen at random. What is the probability that the child spent between 2 and 3 hours watching television that Saturday?



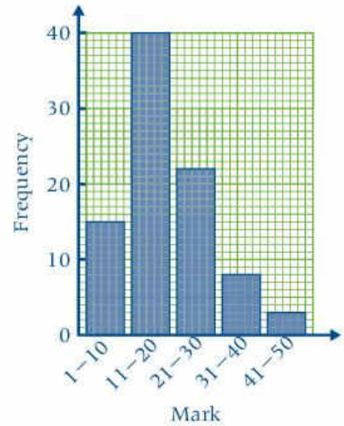
9 The histogram shows the masses of individual carrots dug up in Mr Kahai's garden.

a Use the histogram to construct a grouped frequency table.

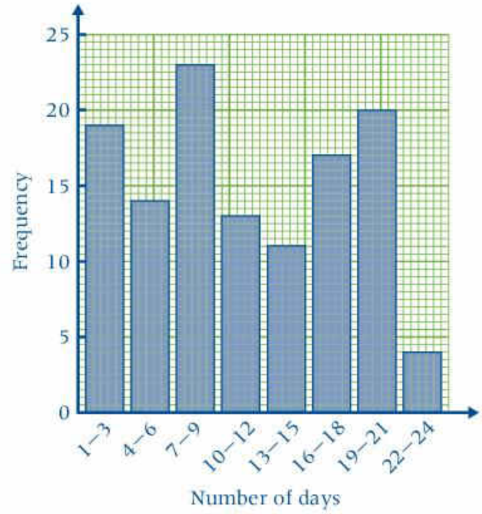
b Estimate the mean mass of a carrot.



10 The diagram shows the marks out of 50 scored by a group of students in a test.
  a How many students were there in the group?
  b Estimate the mean mark.
  c Is this diagram a histogram?
  Give a reason for your answer.



11 This diagram shows the number of days guests stayed in a hotel.
  a Explain why a histogram is not a suitable method for illustrating the data.
  b How many guests were there altogether?
  c What is the modal class?
  d Estimate the mean number of days stayed.
  e Is it more likely than not that a new guest will stay more than 12 days? Give a reason for your answer.



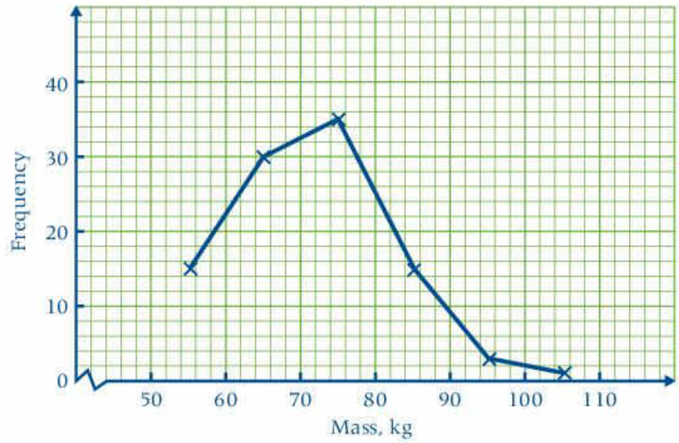
Frequency polygons

A **frequency polygon** is formed by plotting the frequencies of the groups against the mid-class values and joining the points.

For this distribution

Mass, kg	50–59	60–69	70–79	80–89	90–99	100–109
Frequency, *f*	15	30	35	15	3	2
Mid-class value, *x*	54.5	64.5	74.5	84.5	94.5	104.5

we draw a horizontal axis for the masses, scaled from 50 to 110 kg and a vertical axis for the frequencies, scaled from 0 to 40. Then we plot the points (54.5, 15), ... (104.5, 2) and join them with straight lines.



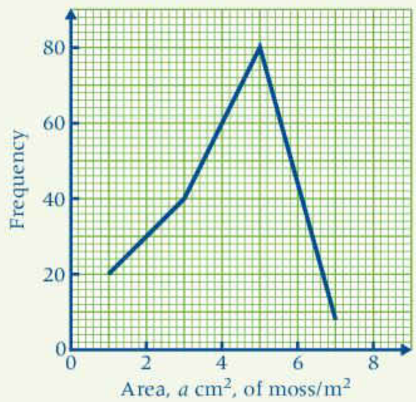
The zigzag on the x-axis shows that the scale does not start at zero.

EXERCISE 23d

Use questions **1** to **3** of **Exercise 23c** to draw a frequency polygon for each set of data.

Example:

The polygon illustrates the distribution of the area covered by moss per square metre in a meadow. Find the mean area of moss per square metre.



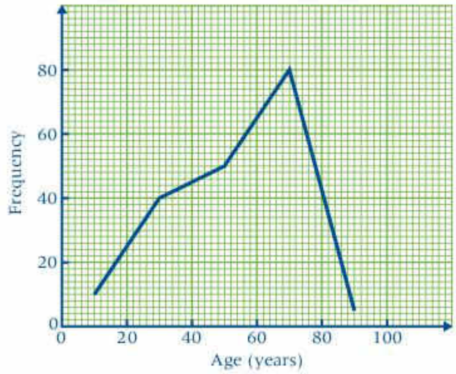
A frequency polygon is constructed by joining the mid-points of the bars representing the groups. The groups are therefore 0–2 cm², 2–4 cm², 4–6 cm² and 6–8 cm². This information can be used to construct a frequency table, from which we can calculate an estimate of the mean.

Area, a cm², of moss/m²	$0 \leqslant a < 2$	$2 \leqslant a < 4$	$4 \leqslant a < 6$	$6 \leqslant n < 8$
Frequency, f	20	40	80	8
Mid-class value, x	1	3	5	7

$$\text{Mean area of moss/m}^2 = \frac{\Sigma fx}{\Sigma f} = \frac{20 + 120 + 400 + 56}{20 + 40 + 80 + 8}\text{ cm}^2$$
$$= 4.03\text{ cm}^2 \text{ (correct to 3 s.f.)}$$

The symbol Σ means 'the sum of'.

4 The polygon opposite shows the distribution of the ages of passengers alighting from buses at Camberley Crescent during the course of one week.



a Use this frequency polygon to complete the following table.

Age, *n* years	$0 \leqslant n < 20$	$20 \leqslant n < 40$	$40 \leqslant n < 60$	$60 \leqslant n < 80$	$80 \leqslant n < 100$
Frequency	10				

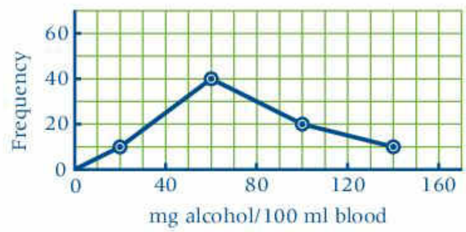
b How many passengers were there altogether?
c Estimate the mean number of passengers.

5 Similar information about the ages of passengers collected in the same week, at Bramberdown bus stop, gave the following table.

Age, *n* years	$0 \leqslant n < 20$	$20 \leqslant n < 40$	$40 \leqslant n < 60$	$60 \leqslant n < 80$	$80 \leqslant n < 100$
Frequency	60	85	40	30	0

a Superimpose the polygon for these data on a copy of the polygon for question **4**.
b Compare the two polygons. Give a possible explanation for their differences.
c Estimate the mean age of the passengers.

6 Information about the quantity of alcohol in the bloodstream of a group of 18-year-old men is given by the graph.
a The legal limit for driving is 80 mg/100 ml of blood. What percentage of this group are over the limit?
b Illustrate this information with a histogram.
c Estimate the mean.

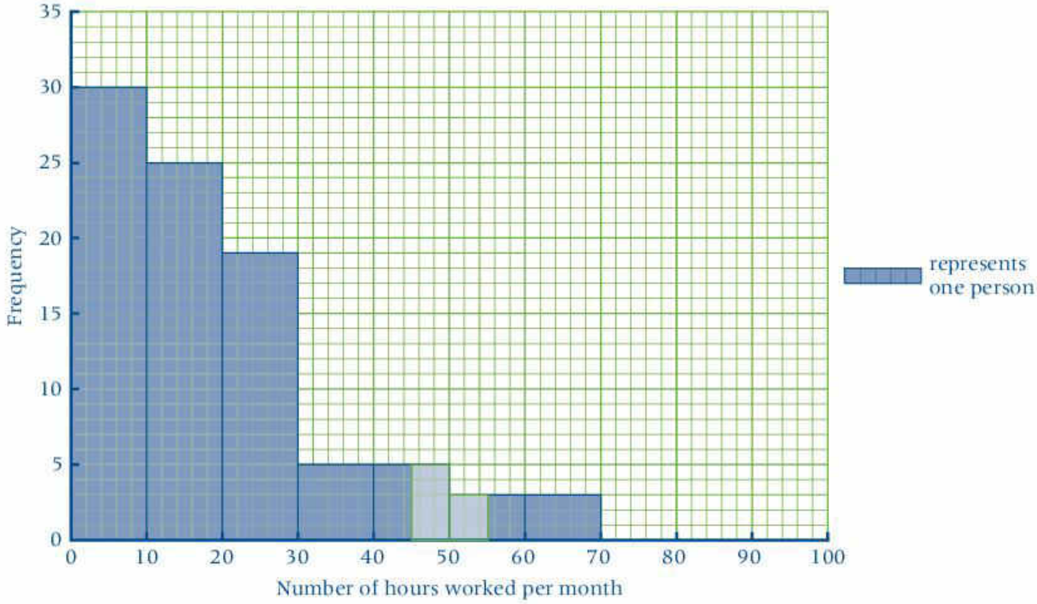


Using the area under a histogram

As the area of a bar in a histogram represents the number of items in the group.

> the total area of the histogram represents the total number of items in the distribution.

This histogram illustrates the distribution of the number of hours worked in March by the part-time members of staff in one company.



The area of the histogram is 450 small squares.
5 small squares represents 1 person.
Therefore the area represents 90 people.

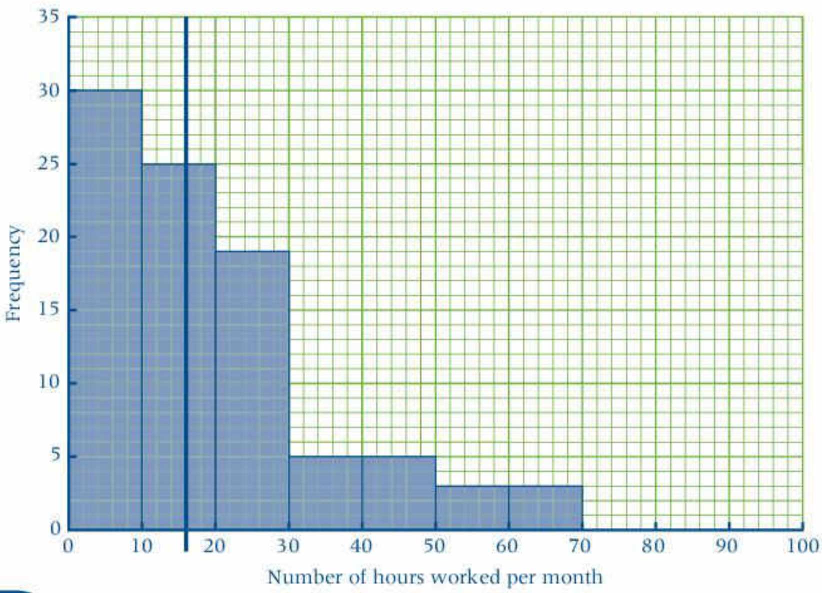
We can also use the given unit of area to estimate, for example, the number of people who worked between 45 and 55 hours in March.
To do this we need to find the area shaded green.

The required area = (2.5 × 5) + (2.5 × 3) small squares
= 20 small squares

Therefore 4 people worked between 45 and 55 hours in March.

As the median is the value of the middle ranked item, it follows that

the vertical line through the median divides the area in half.

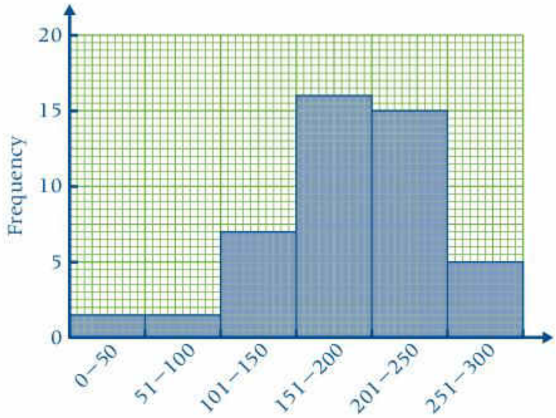


The vertical line that divides this area into two equal parts, each representing 45 people, gives an estimate of the median number of hours worked. Counting units of area, this line goes through 16 on the horizontal axis, i.e. the median is 16 hours.

EXERCISE 23e

1 Explain why the line that divides the area covered by a histogram into two equal parts gives an *estimate* of the median.

2 The histogram shows the distribution of test scores from the schools in one island.



a How many schools have their results included in this histogram?
b Estimate the median score.
c Estimate the number of schools that have a score of
i more than 270 **ii** less than 120.

3 The distribution of the masses of individual new potatoes dug from Mr Shah's allotment are summarised in the frequency table.

Mass (grams, correct to the nearest gram)	0–20	21–40	41–60	61–80	more than 80
Frequency	10	40	45	20	5

a Explain why it would be difficult to draw a histogram to illustrate this distribution.
b Explain why an estimate for the median mass can be found and find it.
c Say, with reasons, whether it is possible to estimate the mean mass.

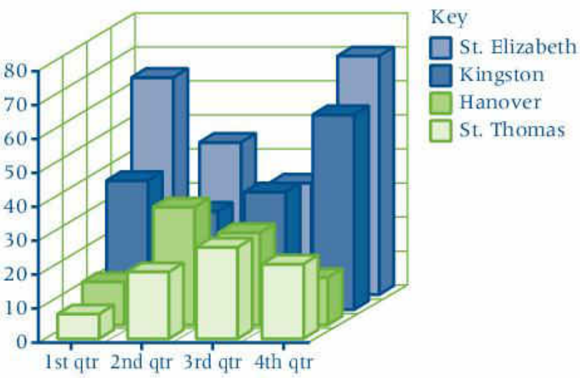
PRACTICAL WORK

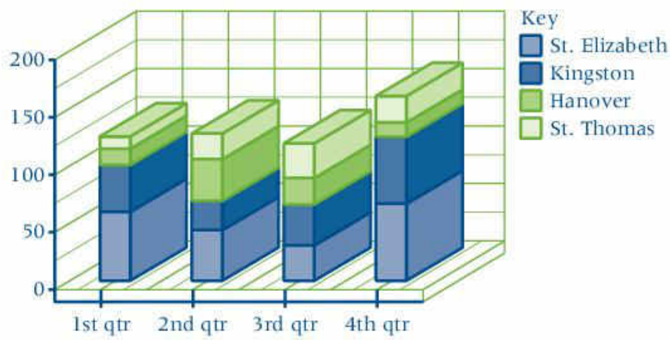
A chart-drawing package is included with most spreadsheet programs. Such a package can produce a variety of graphs from data entered in the spreadsheet.

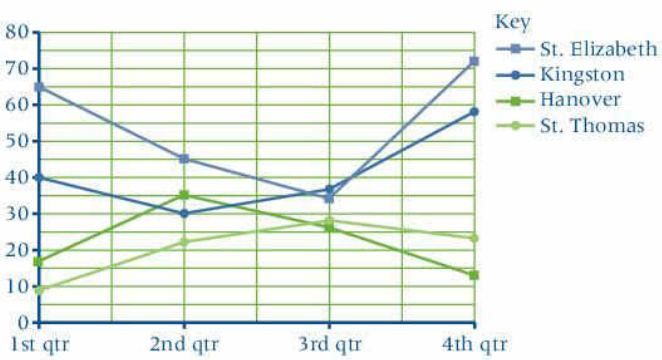
This information shows the number of air conditioning systems installed in each quarter of one year in the four regions in which a company operates.

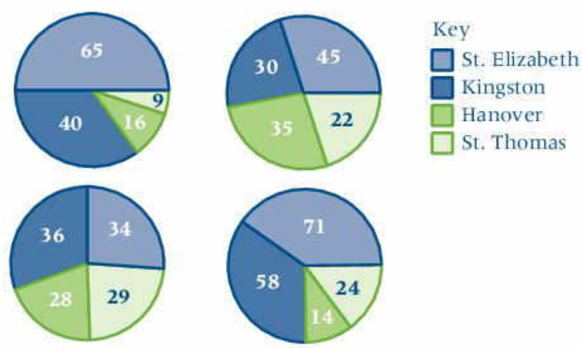
	1st quarter	2nd quarter	3rd quarter	4th quarter
St. Elizabeth	65	45	34	71
Kingston	405	30	36	58
Hanover	16	35	28	14
St. Thomas	9	22	29	24

These charts were produced using the data in a spreadsheet.







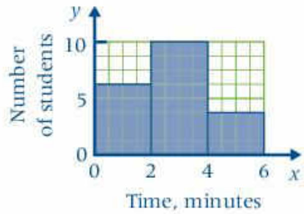


Discuss the advantages and disadvantages of each of these charts. Use these data, or some of your own, to experiment with a chart-drawing package.

ABCD MIXED EXERCISE 23

Several answers are given for these questions.
Write down the letter that corresponds to the correct answer.

This histogram shows the times taken by a sample of students to solve a problem. Questions **1** to **5** are based on this diagram.



1 The size of the sample is
A 6 **B** 10 **C** 20 **D** 120

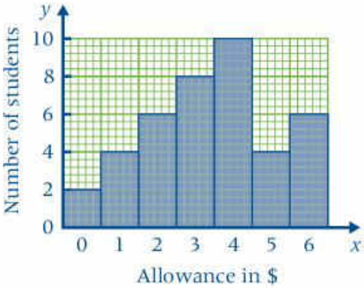
2 The mean is estimated as
A 2.8 **B** 3 **C** 3.5 **D** 5

3 The median is estimated as
A 2.5 **B** 2.8 **C** 3 **D** 3.5

4 The probability that one student, chosen at random from this sample, takes up to 2 minutes to solve this problem is
A 0.3 **B** 0.5 **C** 0.6 **D** 1

5 Another group of 100 similar students are to take this test. The number expected to take from 2 to 4 minutes to solve the problem is
A 30 **B** 40 **C** 50 **D** 60

Questions **6** to **9** are based on this diagram which shows the daily allowance, in dollars, of students in a class.



6 The modal group is
A \$2.50–\$3.50 **B** \$3.50–\$4.50 **C** \$5.50–\$6.50 **D** \$10.00

7 The class width is
A \$0.5 **B** \$1.0 **C** \$6 **D** \$10

8 The number of students in the class is
A 40 **B** 36 **C** 30 **D** 21

9 The median is
A \$3.00 **B** \$3.50 **C** \$4.00 **D** \$6.00

Questions **10** to **12** are based on this frequency table which shows the weights of 20 children in a class.

Weight in kg to the nearest kg	20	21	22	23	24	25
Number of children	4	3	2	5	4	2

10 The data in the table are
A categorical **B** continuous **C** discrete **D** none of these

11 The modal weight, to the nearest kg, is
A 20 kg **B** 22.5 kg **C** 23 kg **D** 25 kg

12 A child from the class is chosen at random. The probability that the child weighs at most 22.5 kg is
A $\frac{2}{20}$ **B** $\frac{7}{20}$ **C** $\frac{9}{20}$ **D** $\frac{11}{20}$

MATHS IS OUT THERE

Did you know that, in the mid-17th century, a simple question to Blaise Pascal sparked the birth of probability theory? Chevalier de Méré was a gambler. He bet on a roll of a die that at least one 6 would appear during a total of four rolls. He knew from experience that he was more likely than not to win money with this game. He decided to change the game to betting that he would get a double 6 on twenty-four rolls of two dice. He soon realised that he made more money with the first game. He asked Blaise Pascal why his new game was not as profitable. Pascal found that the probability of winning on the new game was only 49.1 per cent compared to 51.8 per cent on the old game.

IN THIS CHAPTER YOU HAVE SEEN THAT...

- a histogram is a bar chart where the bars touch and the area of the bar represents the frequency
- the class interval goes from the lowest possible value (the lower class boundary) to the highest possible value (upper class boundary) in a group
- the class width is equal to (upper class boundary) − (the lower class boundary)

24 MEASURES OF SPREAD

AT THE END OF THIS CHAPTER YOU SHOULD BE ABLE TO...

1 Find the range, median, upper and lower quartiles, interquartile range and semi-interquartile range of a set of data

2 Construct a cumulative frequency table

3 Draw and interpret a cumulative frequency curve

4 Decide which statsitical measure to use

MATHS IS OUT THERE

Did you know that a statistician with one hot foot and one cold foot will say that on average he feels fine?

BEFORE YOU START you need to know:

✓ the meaning of the symbols ⩽ and <
✓ how to plot points on a set of axes
✓ the meaning of percentage
✓ how to find the median of data.

KEY WORDS cumulative frequency, cumulative frequency curve, cumulative frequency polygon, extreme value, interquartile range, lower quartile, median, outlier, range, semi-interquartile range, upper quartile

Range

The **range** of a set of data is the *difference between the largest and smallest values.*

The answer is always a *single number.*

For example, the range of the weights 3 kg, 5 kg, 6 kg, 8 kg is 8 kg − 3 kg = 5 kg.

For grouped data the range is estimated as (the highest end of the last group) minus (the lowest end of the first group).

EXERCISE 24a

1 Six pupils got these marks in a test: 5, 7, 8, 4, 8, 4.
What is the range?

2 In three different shops, the price of a can of cola is 27 c, 25 c and 23 c.
What is the range of the prices?

It helps if you put the numbers in order first. Start with the smallest. Remember the answer is a single number.

3 The ages of the children in a swimming club are
9, 10, 8, 10, 11, 8, 12, 9, 10, 11, 10, 12.
Write down the range of ages.

4 This frequency table shows the number of goals scored by the teams in a football league one weekend.

Number of goals	0	1	2	3	4	5
Frequency	8	11	4	5	2	1

Find the range of the number of goals scored.

The lowest number of goals is 0 and the highest number is 5.

5 The table shows the number of children per family for some students.

Number of children	1	2	3	4	5	6	7
Frequency	2	5	8	6	5	3	1

Find the range of the number of children per family.

6 This table shows the number of letters per word in a paragraph from a newspaper.

Number of letters	1	2	3	4	5	6	7	8	9	10	11
Frequency	3	12	20	15	14	11	9	6	4	1	3

Find the range of the number of letters per word.

7 This table shows the first round scores of the golfers taking part in a competition.

Score	65	66	67	68	69	70	71	72	73	74
Frequency	3	4	3	6	9	5	3	0	2	3

a Write down the number of golfers who took part in the first round.
b Work out the range of scores.

8 This table shows the number of students in the secondary schools in a country.

Size of year group	**No. of schools**
50–100	7
101–150	32
151–200	19
201–250	5
251–300	3

a Work out the number of secondary schools in the country.
b Estimate the range of the sizes of the year groups.

The highest end of the last group is 300 and the lowest end of the first group is 50.
The estimated range is the difference between these two numbers.
Remember that the answer is a single number.

9 This table shows the prices of second-hand cars on a garage forecourt.

Price, p (\$000)	Frequency
$2 < p \leqslant 5$	4
$5 < p \leqslant 8$	5
$8 < p \leqslant 12$	8
$12 < p \leqslant 16$	5
$16 < p \leqslant 20$	2

Estimate the range of prices.

10 This table shows the distances swimmers travelled to a leisure centre.

Distance, m kilometres	Frequency
$0 < m \leqslant 4$	12
$4 < m \leqslant 8$	16
$8 < m \leqslant 12$	8
$12 < m \leqslant 16$	5
$16 < m \leqslant 20$	2

Estimate the range of distances travelled.

Cumulative frequency tables

A **cumulative frequency** is the sum of all the frequencies that have gone before. It is a running total.

A cumulative frequency table is made by adding each frequency to the sum of all those that have gone before.

This frequency table shows Ruth's marks in some tests.

Subject	Mark	Cumulative frequency (running total)
English (1st result)	54	54
Maths (2nd result)	72	126
Art (3rd result)	66	192
Science (4th result)	68	260
History (5th result)	55	315
Geography (6th result)	82	

This is the total marks for the first three tests

You find this number by adding the mark for the sixth test to the total for the first 5 tests, i.e. $315 + 82 = 397$.

The cumulative frequencies give the total marks for the first test, first two tests, and so on.

This cumulative frequency table shows the prices of some second-hand books:

Price, p \$	Frequency	Price, in \$	Cumulative frequency
$2 < p \leqslant 5$	4	$\leqslant 5$	4
$5 < p \leqslant 8$	5	$\leqslant 8$	$4 + 5 = 9$
$8 < p \leqslant 11$	8	$\leqslant 11$	$9 + 8 = 17$
$11 < p \leqslant 14$	5	$\leqslant 14$	$17 + 5 = 22$
$14 < p \leqslant 17$	2	$\leqslant 17$	$22 + 2 = 24$

This column gives the upper class boundary of each group.

You can use it to find how many of the books were priced \$11 or less: find the row with \$11 in the price column; the cumulative frequency is 17 so there were 17 books.

You can use it to find how many of the books cost more than \$8: the cumulative frequency gives totals that are *less than* or equal to a given price. There were 9 books priced at \$8 or less and 24 books altogether.

Therefore there were $24 - 9 = 15$ books priced at more than \$8.

EXERCISE **24b**

1 The table shows the number of students in Grade 10 English classes at a school.

Class	Number of students	Running total of number of students
10G	28	28
10P	32	
10R	29	
10S	19	
10Y	28	
10C	23	

a Copy and complete the table.
b Find the number of students in Grade 10.

2 The table shows the goals scored by the home sides in a football league one Saturday.

Score	Frequency	Score	Cumulative frequency
0	2	0	2
1	8	$\leqslant 1$	$2 + 8 =$
2	5	$\leqslant 2$	
3	3	$\leqslant 3$	
4	2	$\leqslant 4$	

a Copy and complete the table.
b Find how many matches were played.
c Work out the number of matches where three or more goals were scored by the home side.

Three or more means more than two.

3 Three coins were tossed several times.
The number of heads that showed are recorded in this table.

Number of heads	Frequency	Number of heads	Cumulative frequency
0	5	0	
1	11	⩽ 1	
2	18	⩽ 2	
3	6	⩽ 3	

a Copy and complete the table.
b Work out the number of coins tossed.
c Find how many times at least one head showed.

4 The table shows the sales of men's shoes in a shop one day.

Shoe size	Frequency (No. of sales)	Size	Cumulative frequency
4	6	4	
5	19	⩽ 5	
6	24	⩽ 6	
7	17	⩽ 7	
8	18	⩽ 8	
9	4	⩽ 9	
10	3	⩽ 10	

a Copy and complete the table.
b Work out the number of pairs of shoes sold that were size 8 or smaller.
c Find how many pairs of shoes sold were larger than size 8.

5 This table shows the distribution of marks scored by a group of students in a test.

Mark	Frequency	Mark	Cumulative frequency
1–10	2	⩽ 10	
11–20	6	⩽ 20	
21–30	12	⩽ 30	
31–40	7	⩽ 40	
41–50	3	⩽ 50	

a Copy and complete the table.
b Find the number of students in the group.
c Work out how many scored 40 or less.
d Calculate the number that scored more than 30.

The table gives the number that scored 30 or less: you need to take this from the total number.

6 This table shows the milk yield from a herd of cows.

Litres, c, per cow each day	Frequency	Litres, c, per cow each day	Cumulative frequency
$0 \leqslant c < 5$	2	< 5	
$5 \leqslant c < 10$	7	< 10	
$10 \leqslant c < 15$	19	< 15	
$15 \leqslant c < 20$	11	< 20	

- **a** Copy and complete the table.
- **b** Find the number of cows in the herd.
- **c** Calculate the number that produced 10 or more litres a day.
- **d** The owner claimed that his herd yielded more than 600 litres a day. Can this be true? Give a reason for your answer.

7 A sample of swimmers were asked how far they travelled to the leisure centre. The table shows the results.

Distance, m kilometres	Frequency	Distance, m kilometres	Cumulative frequency
$0 \leqslant m < 4$	12	< 4	
$4 \leqslant m < 8$	16	< 8	
$8 \leqslant m < 12$	8	< 12	
$12 \leqslant m < 16$	5	< 16	
$16 \leqslant m < 20$	2	< 20	

- **a** Copy and complete this table.
- **b** Find the number of swimmers in the sample.
- **c** Work out the number that travelled more than 8 kilometres.

Cumulative frequency curves

A **cumulative frequency curve** is shown alongside.

It shows the distribution of heights of 60 pupils in an infant school.

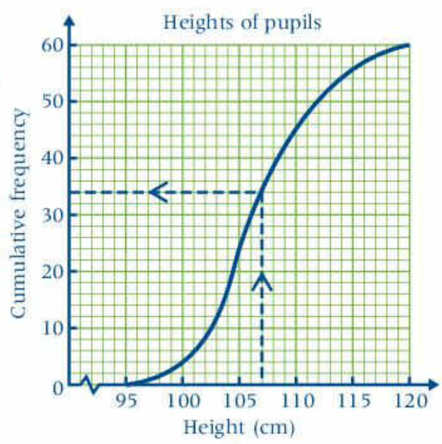
It is drawn by plotting cumulative frequencies against heights.

It can be used to estimate the number of pupils below a certain height.

For example, to estimate the number shorter than 107 cm, find 107 on the horizontal axis.

Go up to the graph. Now go across to the vertical axis and read off the value – it is 34.

So 34 pupils are shorter than 107 centimetres.



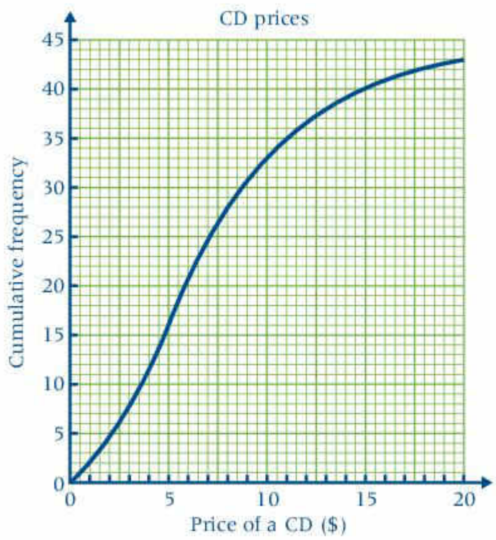
EXERCISE 24c

1 Use the cumulative frequency curve in the example above to estimate how many pupils were:
a shorter than 115 cm
b shorter than 112 cm
c shorter than 104 cm.

Read the questions carefully. Make sure that you understand what you are asked to find. Be particularly careful about the words 'more', 'less' and 'at least'.

2 Use the cumulative frequency curve in the example above to estimate how many pupils were taller than 110 cm.

3 This cumulative frequency curve shows the price, in $, of some CDs.
a Find the number of CDs.
Use the diagram to estimate how many CDs
b cost less than $6
c cost less than $10
d cost $15 or more.



Drawing a cumulative frequency curve

The heights of some plants are shown in the table.

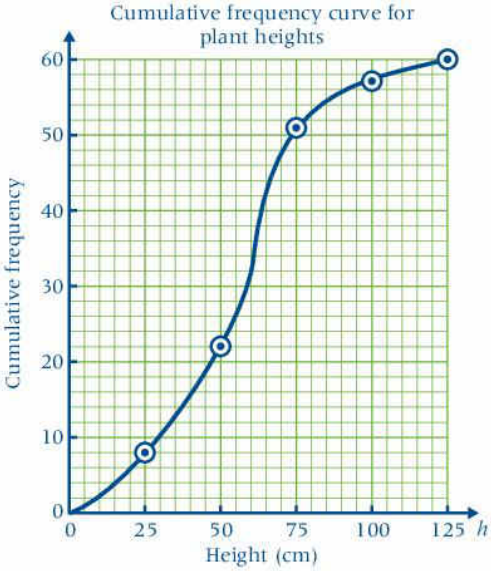
Height, *h* cm	Frequency	Height, *h* cm	Cumulative frequency
$0 \leqslant h < 25$	8	< 25	8
$25 \leqslant h < 50$	14	< 50	22
$50 \leqslant h < 75$	29	< 75	51
$75 \leqslant h < 100$	6	< 100	57
$100 \leqslant h < 125$	3	< 125	60

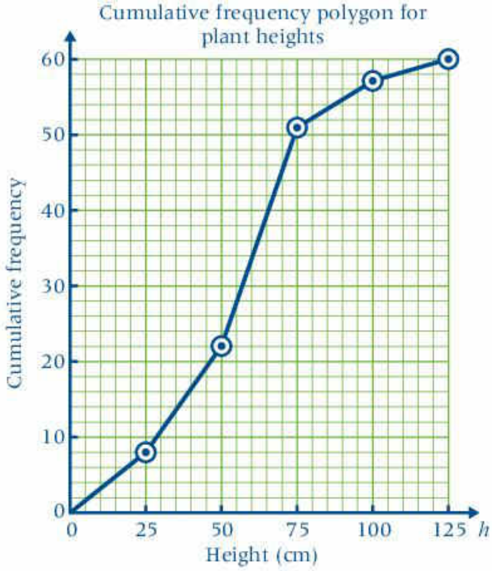
To draw a cumulative frequency curve, plot the first point at 0 for the cumulative frequency and at the *lowest boundary* of the first class. (In this case the first point is (0, 0).)

Then plot the other points by treating the *upper class boundary* of each group and the cumulative frequency as coordinates. (In this case the second point is (25, 8).)

Then draw a smooth curve through your points.

If you join the points with straight lines it is called a **cumulative frequency polygon**.





EXERCISE 24d

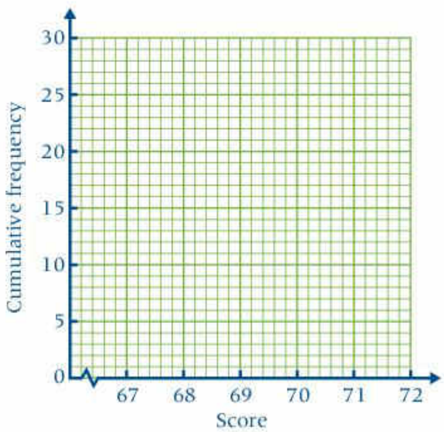
1 This cumulative frequency table shows the scores taken to complete a round of golf.

Score	< 68	< 69	< 70	< 71	< 72
Cumulative frequency	4	15	25	28	30

These are the upper class boundaries of each group.

a Write down how many rounds were completed in less than 70.

b Draw a cumulative frequency curve for this data.
Use a grid like this.
Assume that the lower end of the first group is 67.



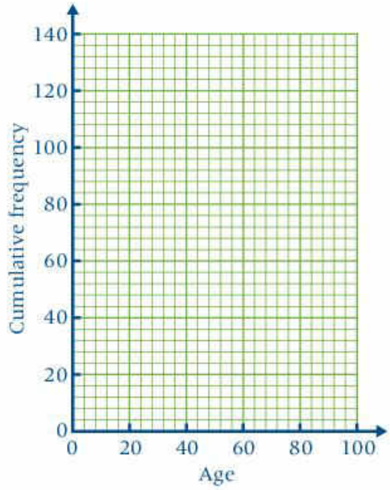
2 The table shows the distribution of the ages of people attending a concert.

Age range	< 20	< 40	< 60	< 80	< 100
Number of people attending	10	40	104	132	136

The bottom end of the first group is 0.

Draw a cumulative frequency curve for the data given in the table.

Use a grid like this.

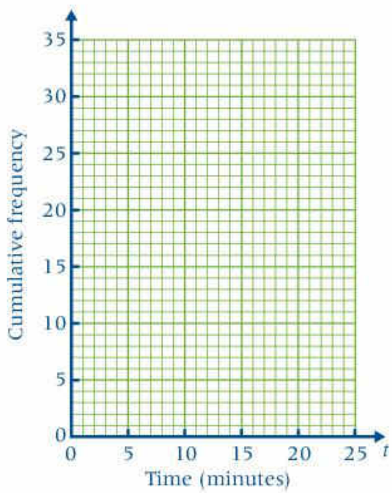


3 The times taken by 33 students to come to school are given in this table.

Time, *t* minutes	**Cumulative frequency**
$t \leqslant 5$	4
$t \leqslant 10$	10
$t \leqslant 15$	24
$t \leqslant 20$	30
$t \leqslant 22$	33

a Find how many students took between 10 and 15 minutes.
b Draw a cumulative frequency curve for this data.

Use a grid like this.



4 The masses of the loads carried by some lorries were recorded.
This frequency table shows the results:

Mass, *w* tonnes	Frequency	Mass, *w* tonnes	Cumulative frequency
$0 \leqslant w < 5$	25	< 5	
$5 \leqslant w < 10$	20	< 10	
$10 \leqslant w < 15$	15	< 15	
$15 \leqslant w < 20$	12		
$20 \leqslant w < 25$	8		

a Complete the third and fourth columns.
b Draw a cumulative frequency curve for these data.

5 This table shows the masses, in grams, of some tomatoes.

Mass, *w* grams	Frequency	Mass, *w* grams	Cumulative frequency
0–9	6	< 9.5	
10–19	12	< 19.5	
20–29	19	< 29.5	
30–39	26		
40–49	63		
50–59	18		

The upper class boundary of the group 0–9 is 9.5. Use the upper class boundaries for the cumulative frequencies.

a Copy and complete the table.
b Write down how many tomatoes were weighed.
c Write down how many tomatoes had a mass of less than 40 g.
d Draw the cumulative frequency curve.

Quartiles

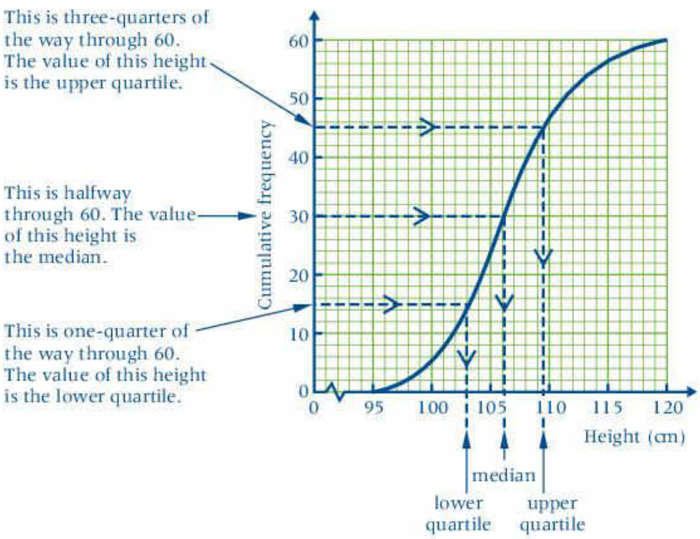
For a list of values, arranged in order of size:

The **lower quartile** is the value that is $\frac{1}{4}$ of the way through the list.

The **upper quartile** is the value that is $\frac{3}{4}$ of the way through the list.

The lower and upper quartiles can be estimated from a cumulative frequency curve. (Remember that the **median** is the middle value.)

This cumulative frequency curve shows the distribution of heights of 60 children.



To find the lower quartile go across to the graph from 15 $\left(\text{this is } \frac{1}{4} \text{ of } 60\right)$ on the vertical axis. Now go down to the bottom axis and read off the value. It is 103. So the lower quartile is 103 cm.

Similarly, the upper quartile is about 109.5 cm.

The **interquartile range** is the difference between the upper and lower quartiles. It gives a measure of the spread of the middle half of the heights.

The interquartile range of these heights is 109.5 cm − 106 cm = 3.5 cm.

Like the range, the interquartile range is always given as a *single* number.

You can use the interquartile range instead of the range when there are a few values that are very much larger (or smaller) than most.

For example, the heights, in centimetres, of a group of people in a room are: 100, 110, 112, 112, 114, 115, 116, 117, 120, 180.

The range is 180 cm − 100 cm = 80 cm.

This, on its own, is misleading because all but one of the heights is 120 cm or less. The 180 cm is called an **extreme value** or **outlier**.

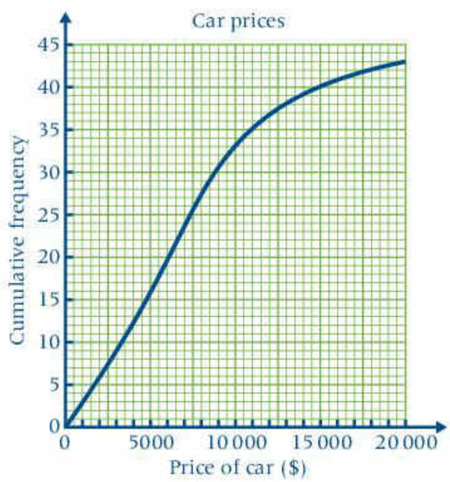
The interquartile range is 117 cm − 112 cm = 5 cm. This gives a better idea of the spread.

The **semi-interquartile range** is equal to half the interquartile range. For example, when the interquartile range is 5 cm, the semi-interquartile range is 2.5 cm.

The semi-interquartile range is often used as a measure of spread when the median is not near the middle of the range.

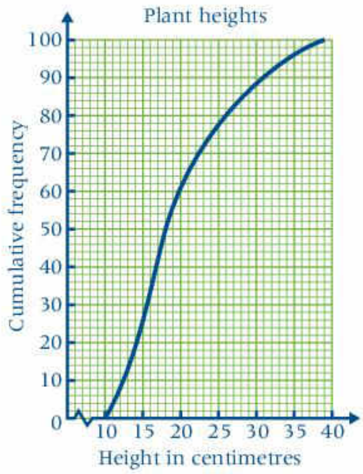
EXERCISE 24e

1 This cumulative frequency curve shows the price, in $, of some cars.
Use it to estimate:
   a the upper quartile
   b the lower quartile
   c the interquartile range
   d the median.



2 This cumulative frequency curve shows the heights, in centimetres, of 100 plants.
Use it to estimate:

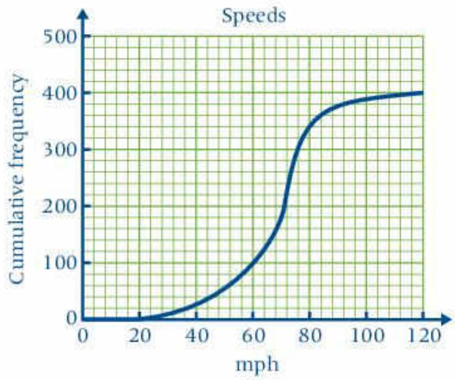
a the number of plants with a height less than 24 cm
b the upper quartile
c the lower quartile
d the interquartile range
e the semi-interquartile range.



From 24 on the bottom axis go up to the graph, then across to the vertical axis.
Read off the value.

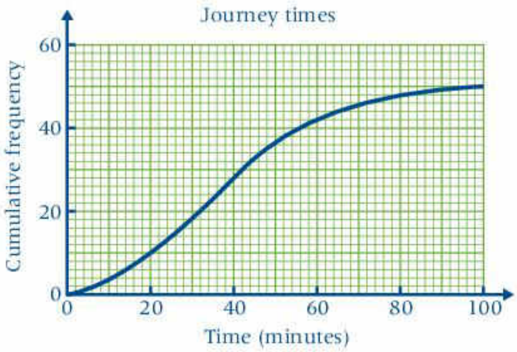
3 The speeds of vehicles travelling along a freeway were recorded as they passed a service station. This cumulative frequency curve shows the results.
Use the graph to estimate:

a the number travelling at 40 mph or less
b the number travelling at more than 90 mph
c the interquartile range.
d Would you use the range or the interquartile range to describe the spread of speeds?
Give a reason for your answer.



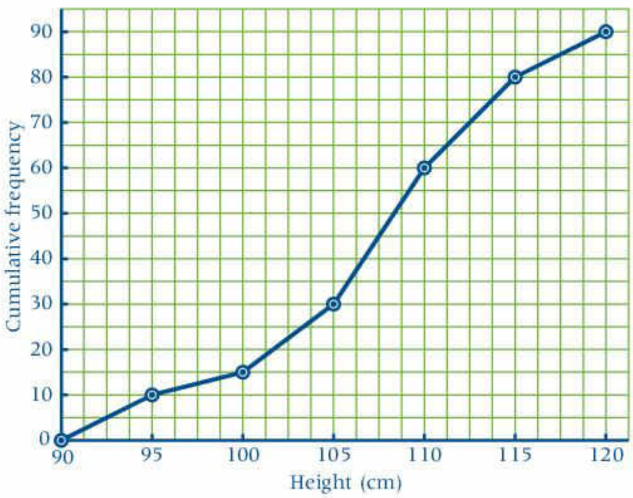
4 This cumulative frequency curve shows the journey times to college of some students. Use the curve to estimate:

a the number of students who took part in the survey
b the number who took less than 10 minutes to get to college
c the number who took between 40 minutes and 1 hour to get to college
d the semi-interquartile range.



5 This cumulative frequency polygon illustrates the distribution of the heights of 90 five-year-old children

a Use the graph to estimate the median and the interquartile range.
b Construct a frequency table from the graph.
c Illustrate the information with a histogram.
d On your histogram, draw a vertical line through the median height. Compare the areas of the parts of the histogram on each side of the line. Explain why your answer will be the same for any histogram.



e Deduce the values of the upper and lower quartiles from the histogram. Check these values from the cumulative frequency polygon. Hence give the semi-interquartile range.

6 A cricketer's scores for 200 innings were analysed. The table shows a summary of the results.

	Score
Lowest score	0
Lower quartile	32
Median	46
Upper quartile	83
Highest score	132

a i Find the interquartile range.
ii How many innings does the interquartile range cover?
b In how many innings was the score less than 32?
c Draw a cumulative frequency curve to record these scores.

7 The 100-strong workforce at a factory were asked how far they travelled to work. The table shows a summary of the results.

	Distance (km)
Shortest distance	3
Lower quartile	8.5
Median	10.5
Upper quartile	13
Furthest distance	16

Draw a cumulative frequency curve to show these results.

8 The examination results for 500 candidates are summarised in the table.

	% mark
Lowest mark	0
Lower quartile	20
Median	30
Upper quartile	50
Highest mark	80

a Draw a cumulative frequency curve to show these results.
b Estimate the number of students who scored less than 40%.

Statistical investigations

A statistical investigation usually starts with a question that requires a lot of information to answer. For example, 'Is it true that older people tend to use their vote more than younger people?'

When you collect information, you need to decide if you can use all the information available (this is called the **population**) or part of it (this is a **sample**).

It is not always sensible or possible to collect all the information. For example, it would be difficult if not impossible to ask all voters how often they voted (even if you restricted it to your own area). You can use a sample of people eligible to vote. It is important to choose a sample that represents all the variety in the population that you are investigating. For example, if you choose to interview women working in an office, your sample will be unrepresentative and any inferences from it will be **biased**.

Deciding which statistical measure to use

When data is collected and sorted, you then need to decide how to present it and which of the many statistical measures you need to try to answer the questions posed.

The statistics you can use are the mean, mode, median, range and interquartile range. Which of these you choose depends on the questions you want to answer.

For example

- If you want to know whether your score is better than most in an examination, you will need the median (i.e. middle) mark.
- If you want to compare your school's performance with another school which sat the same examination, the mean mark for each school would be useful. The ranges (or the interquartile ranges if there are a few marks that are much lower or higher than the others) give an idea of the spread.

These are called outliers.

- If you want to compare which group of marks is most common, then the modes will tell you this.

There is another measure that you can use, if it is given, called **standard deviation**.

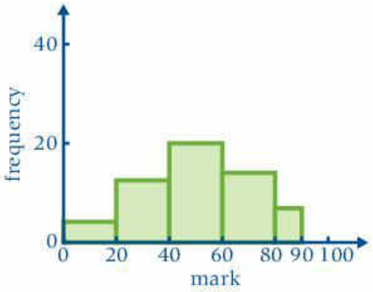
You will not be asked to calculate this.

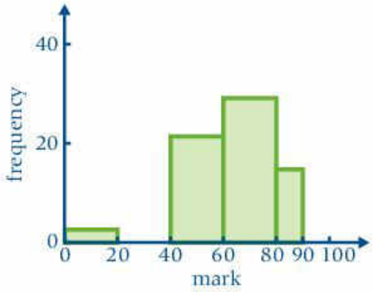
This gives a measure of how close the data is about the mean.

A small standard deviation means that the data is mainly clustered close to the mean whereas a larger standard deviation means that the data is more spread out.

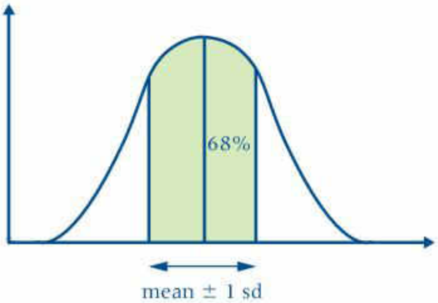
These two histograms show the results of two groups of students who sat the same examination.

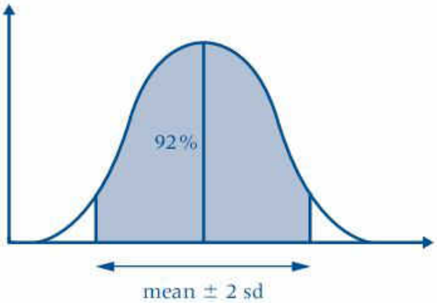
The left-hand marks are more spread out than the right-hand ones so the standard deviation of the left-hand marks is larger than that of the right-hand marks.





For most distributions, about 68% of the values lie between one standard deviation each side of the mean and about 92% of the values lie between two standard deviations each side of the mean.





EXERCISE 24f

1 Rajiv is conducting a survey among the employees in AbCo Ltd. The table shows a breakdown of the employees.

	Full time	Part time
Men	45	8
Women	21	56

Rajiv wants a sample of the employees.
Explain why choosing equal numbers of men and women would not give a representative sample.

2 Dwain wants to know whether left-handed tennis players are more successful than right-handed ones. Dwain decided to investigate the 250 top-ranked tennis players.

a What information does Dwain need to try and answer his question?
b How do you think Dwain could get the information?
c Would a sample be sensible?

3 This table shows the money that was donated by sponsors at ten different cricket clubs in a local league.

	A	B	C	D	E	F	G	H	I	J
Amount ($)	202	462	189	701	210	108	155	505	370	58

All the money collected is to be distributed equally to each club. What statistic needs to be calculated to determine this amount?

Example:

Leta scored 55 marks in a test. For the whole group the mean mark was 49 with a standard deviation of 4. In the next test, when Leta was absent, the mean mark was 62 with a standard deviation of 5. Estimate a fair mark to award Leta in the test she missed.

First put the data in a table.

	Leta's mark	Mean	Standard deviation
Test 1	57	49	4
Test 2	absent	62	5

In Test 1 the difference of Leta's mark from the mean was
$57 - 49 = 8$
i.e. 2 standard deviations above the mean.

Assuming the same difference in Leta's estimated mark from the mean in Test 2 gives an actual difference of $2 \times 5 = 10$
i.e. Leta's estimated mark in Test 2 is $62 + 10 = 72$

4 The standard deviation for a set of numbers with a mean of 50 is 10.
  a One number in the set is 70. How many standard deviations is this from the mean?
  b Another number is 40. How many standard deviations is this from the mean?

5 Garon scored 82 marks in a test. For the whole group the mean mark was 70 with a standard deviation of 6. In the next test, when Garon was absent, the mean mark was 64 with a standard deviation of 4. Estimate a fair mark to award Garon in the test he missed.

6 Two taxi companies operate using different routes between Town A and Town B. A survey of the times taken by them over a period of several months gave the following data.

	Mean time	Standard deviation
Radio Taxis	35 min	8 min
Queen's Taxis	40 min	4 min

Malik needs a taxi in Town A for 10 a.m. and wants to be as certain as he can that he arrives in Town B before 10.50 a.m. Which company would you suggest he chooses? Justify your answer.

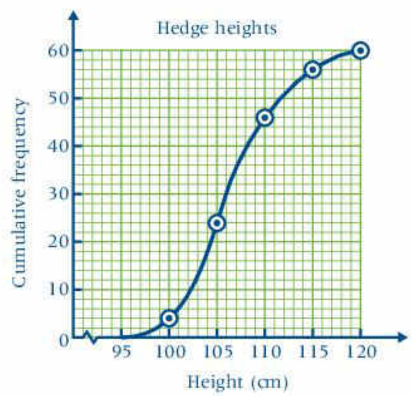
7 A shop owner keeps a record of the numbers of each drink brand sold in one week.
  a The shop owner wants to know which brand to order most of. Which of the mean, mode and median gives this information?
  b The shop owner decides not to stock so many brands. Which two statistical measures are most useful to help decide which brands to no longer stock?
  c The shop owner records this information over several weeks. Which information does he need to give useful information to track total sales each week?

8 Shona wants to buy a smart phone. She has the following information.
The mean price is about \$300.
The median price is about \$250.
The modal price is about \$200.
The prices range from \$100 to £1000.
The standard deviation of the prices is \$50.
  a Explain why the mean price is not very useful.
  b Shona does not want to spend more than \$250. What does the median price tell her?
  c How useful is the modal price group?
  d What does the standard deviation tell you about the range of prices?

MIXED EXERCISE **24**

Several answers are given for these questions.
Write down the letter that corresponds to the correct answer.

Questions **1** to **6** refer to this cumulative frequency curve which shows the distribution of the heights of some hedges.



1 The median height, in centimetres, is
A 30 B 35 C 106 D 107.5

2 The range of heights, in centimetres, is
A 25 B 30 C 35 D 120

3 The interquartile range, in centimetres, is
A 6 B 10 C 15 D 30

4 The number of hedges taller than the upper quartile is
A 10 B 15 C 30 D 45

5 The number of hedges taller than 110 cm is
A 14 **B** 25 **C** 30 **D** 46

6 The class widths, in centimetres, are
A 5 **B** 6 **C** 10 **D** 35

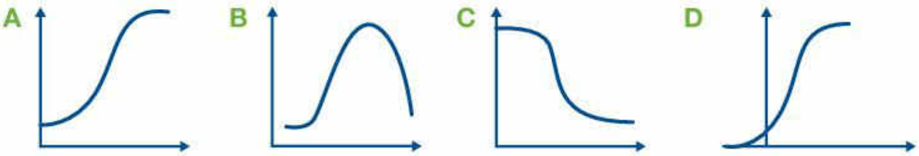
Questions **7** to **9** refer to this table which shows the number of times 0, 1, 2 or 3 heads showed when three coins were tossed.

Number of heads	0	1	2	3
Frequency	2	8	24	6

7 The number of times the three coins were tossed was
A 3 **B** 6 **C** 18 **D** 40

8 The number of times at least one head showed was
A 8 **B** 9 **C** 38 **D** 40

9 The cumulative frequency curve showing this distribution could be



IN THIS CHAPTER YOU HAVE SEEN THAT...

- the range of a set of data is the difference between the largest and smallest values
- for grouped data the range is estimated as the difference between the highest end of the last group and the lowest end of the first group
- a cumulative frequency is the running total of frequencies, i.e. the sum of all frequencies that have gone before
- you can draw a cumulative frequency curve by plotting cumulative frequencies against the values, and drawing a smooth curve through the points; for grouped frequencies, the values are plotted against the highest end of the group
- a cumulative frequency curve can be used to find the range, median, upper and lower quartiles, and interquartile range of a set of data
- the standard deviation of a distribution is a measure of how the values are spread above the mean value. For most large distributions, approximately 68% of the values lie between 1 standard deviation each side of the mean and about 92% lie between 2 standard deviations each side of the mean.

MATHS IS OUT THERE

Genetics is a branch of biology that deals with heredity. Nature's laws of heredity are mathematical and based on probability.

An Austrian monk and scientist, Gregor Mendel (1822–1884), used mathematics in his study of genetics.

AREAS AND VOLUMES

AT THE END OF THIS CHAPTER YOU SHOULD BE ABLE TO...

1 Find the volume and surface area of a pyramid.
2 Find the volume and surface area of a cylinder.
3 Find the volume and surface area of a cone.
4 Find the volume and surface area of a sphere.

BEFORE YOU START you need to know:

- ✓ how to find the areas of rectangles, squares, parallelograms and trapeziums
- ✓ how to draw a net for a three dimensional shape
- ✓ the formulae for the circumference and area of a circle
- ✓ how to find the length of an arc and the area of a sector of a circle
- ✓ the common units of length, area, volume and capacity
- ✓ how to substitute into a formula and how to change the subject of a formula

KEY WORDS cone, cylinder, density, frustum, hemisphere, pyramid, slant height, sphere, tetrahedron, vertex

MATHS IS OUT THERE

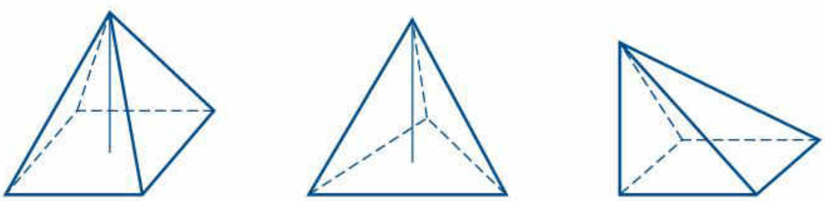
Did you know that the measurements obtained from measuring dimensions are approximate?

Hence it is impossible to find the exact dimensions of any object by measuring.

A ruler with a unit of 0.1 cm has a GREATEST POSSIBLE ERROR of 0.05 cm.

The GREATEST POSSIBLE ERROR is always half the unit of measurement.

Pyramids



Each of these solids is a **pyramid**; its shape is given by drawing lines from a single point to each corner of the base.

The first solid has a square base and is called a square-based pyramid.

The second one has a triangular base and is called a triangular pyramid. It is also called a **tetrahedron**: a special name that applies only to this shape.

In a *right pyramid* the vertex is directly above the middle point of the base.

Volume of a pyramid

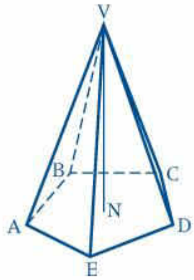
The volume of a pyramid is given by

Volume = $\frac{1}{3}$ area of base × perpendicular height

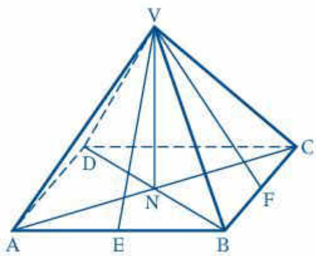
Surface area of a pyramid

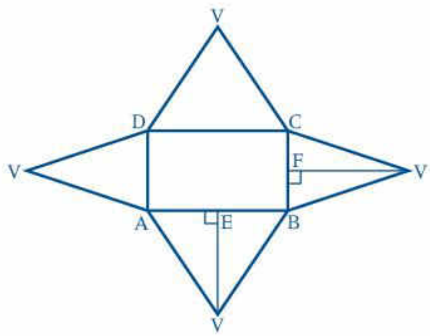
The surface area of any pyramid is found by adding the area of the base to the sum of the areas of the sloping sides.

Drawing a net often helps in finding the total surface area of a pyramid. This makes it clear what distances and areas are needed and where the right angles are.



For example, the net for a right pyramid with a rectangular base is shown below. It is easy to see that the equal sloping sides occur in pairs.





E is the middle point of AB and F is the middle point of BC.

Total surface area of this pyramid

= area base ABCD + 2 × area △VAB + 2 × area △VBC

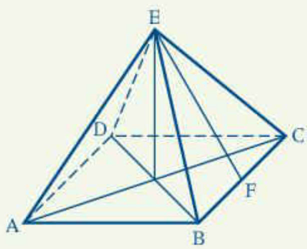
EXERCISE 25a

Example:

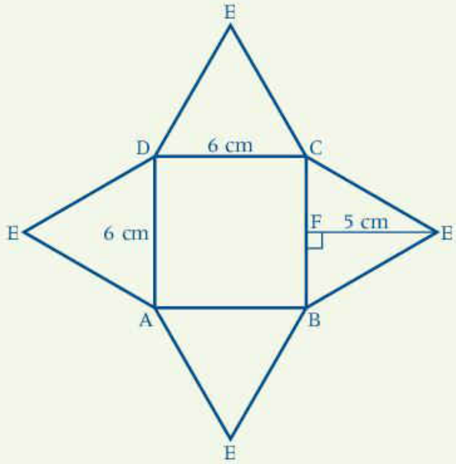
A right pyramid stands on a square base ABCD of side 6 cm and its height is 4 cm. The distance from its vertex E to the mid-point F of BC is 5 cm. Find

a the volume of the pyramid

b its total surface area.



a



First draw a net to show the actual shapes of the base and sloping sides. On it mark all the distances you need.

Area of base $= 6 \times 6\,\text{cm}^2$

$= 36\,\text{cm}^2$

Volume of pyramid

$= \frac{1}{3}$ area of base $\times$ perpendicular height

$= \frac{1}{3} \times 36 \times 4\,\text{cm}^3$

$= 48\,\text{cm}^3$

b Length of FE $= 5\,\text{cm}$

Area of one sloping side $= \frac{1}{2} \times 6 \times 5\,\text{cm}^2$

$= 15\,\text{cm}^2$

$\therefore$ area of the four sloping sides $= 4 \times 15\,\text{cm}^2$

$= 60\,\text{cm}^2$

Total surface area of pyramid

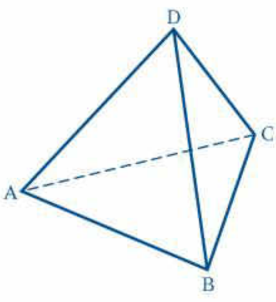
$=$ area of the base $+$ area of the four sloping sides

$= (36 + 60)\,\text{cm}^2$

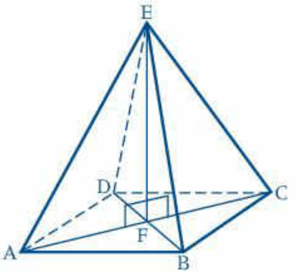
$= 96\,\text{cm}^2$

Find the volumes of the pyramids in questions **1** to **4**.

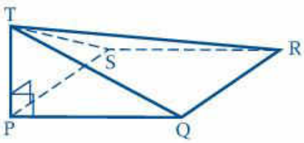
1



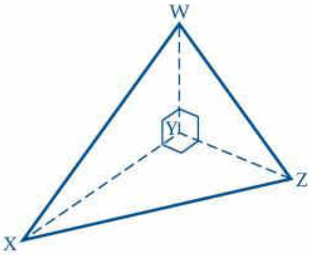
The base of the pyramid is $\triangle ABC$ whose area is $52\,\text{cm}^2$. The height of the pyramid is 6.8 cm.

2 

The base ABCD is a rectangle.
AB = 8 cm, BC = 4.5 cm and EF = 6 cm.

3 

The base PQRS is a rectangle.
PQ = 20 cm, QR = 12 cm and TP = 8 cm.

4 

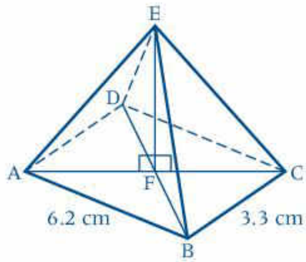
The base is triangle XYZ.
XY = 10 cm,
YZ = 8 cm and WY = 8 cm.

5 The volume of a pyramid is $76.8\,cm^3$ and the area of its base is $32.0\,cm^2$.
   a Find the height of the pyramid.
   b If the measurements given are correct to 3 significant figures, what is the least possible value for the height?

> The height is $76.8 \div (\frac{1}{3}$ of 32.0) cm. This is least when 76.8 is as small as possible, i.e. 76.75, and when 32.0 is as large as possible, i.e. 32.05

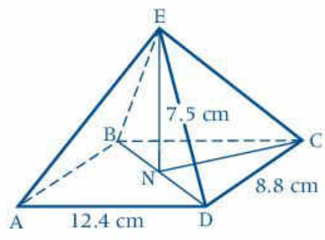
6 A solid copper pyramid with a square base of side 9 cm has a mass of 1780 g. If the mass of $1\,cm^3$ of copper is 8.9 g find
   a the volume of the pyramid
   b its height.

7 The base of a right pyramid is a rectangle ABCD where AB = 6.2 cm and BC = 3.3 cm. The vertex of the pyramid is E and EF = 5.8 cm. Each measurement is given correct to 1 decimal place.
   a Write down, correct to 2 decimal places, the greatest and least values for
      i the length of AB
      ii the area of the rectangle ABCD
      iii the volume of the pyramid.
   b By what percentage, correct to the nearest whole number, is the largest possible volume greater than the smallest possible volume?



> A value corrected to 1 d.p. lies within ± 0.05 units. For example 1.3 correct to 1 d.p. lies between 1.25 and 1.35.

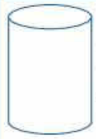
8 The dimensions of a right pyramid with a rectangular base are shown on the diagram. Each dimension is given correct to 1 decimal place. Find
   a the largest possible value for the area of the base
   b the smallest possible value for the volume of the pyramid
   c the largest possible number of pyramids that have a total volume that is less than $10\,000\,cm^3$.



Volume of a cylinder

The formula for the volume of a **cylinder** is

$$V = \pi r^2 h$$



You can think of a cylinder as a circular prism. Area of cross-section $= \pi r^2$ so volume $= \pi r^2 h$

where V cubic units is the volume, r units is the radius of the circular cross-section and h units is its height or length.

EXERCISE **25b**

Example:

Find the radius of a cylinder of volume $72\,\text{cm}^3$ and height $9\,\text{cm}$.

$$V = 72 \qquad h = 9$$

$$V = \pi r^2 h$$

$$72 = \pi \times r^2 \times 9$$

$$\frac{72}{\pi \times 9} = r^2$$

$$r^2 = 2.546\ldots \Rightarrow r = 1.595\ldots$$

The radius is 1.60 cm correct to 3 s.f.

Estimate: $r^2 \approx \frac{72}{3 \times 9} \approx 3$

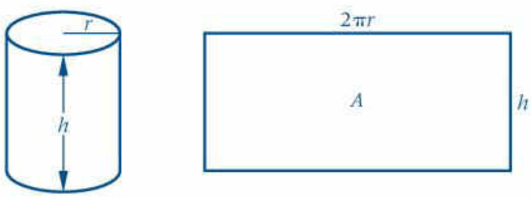
In each question from **1** to **10**, find the missing measurement of the cylinder.

Make sure the units are compatible.

	Radius	Height	Volume		Radius	Height	Volume
1	4 cm	8 cm		**2**	5.1 cm	3 cm	
3	0.5 m	1 m		**4**	25 cm	1 m	
5	3.1 cm		$72\,\text{cm}^3$	**6**		0.7 m	$9.83\,\text{m}^3$
7	11 cm		$1024\,\text{cm}^3$	**8**	3.8 cm		$760\,\text{cm}^3$
9		1.6 m	$15\,\text{m}^3$	**10**		0.12 cm	$0.56\,\text{cm}^3$

11 The diameter of a cylinder is equal to its height. If the volume of the cylinder is $30\,\text{cm}^3$ find its diameter.

Curved surface area of a cylinder



If we have a cylindrical tin with a paper label covering its curved surface, we can take off the label and flatten it out to give a rectangle whose length is equal to the circumference of the tin.

Therefore the area A of the curved surface is given by $2\pi r \times h$

i.e. $$A = 2\pi rh$$

The total surface area of a closed cylinder is given by

$$A = 2\pi rh + 2\pi r^2$$

EXERCISE 25c

In each question from **1** to **8**, find the curved surface area of the cylinder whose measurements are given. Remember to make sure that units are compatible.

1 Radius 4 cm, height 6 cm

2 Radius 30 cm, height 2 cm

3 Radius 6.2 cm, height 5.8 cm

4 Radius 2 m, height 82 cm

5 Radius 0.06 m, height 32 cm

6 Radius 5.2 cm, height 7.8 cm

7 Radius 72.6 cm, height 30 cm

8 Radius 4.2 m, height 98 cm

9 A closed cylinder has radius 6 cm and height 10 cm.
Find **a** the area of its curved surface
b the area of its base
c the total surface area.

10 A closed cylinder has radius 3.2 cm and height 4.8 cm.
Find **a** the area of its curved surface **b** the total surface area.

11 Find the area of the paper label covering the side of a cylindrical soup tin of height 9.6 cm and radius 3.3 cm. The label has an overlap of 1 cm.

12 What area of card is needed to make a cylindrical tube of length 42 cm and radius 3.2 cm? The card overlaps by 2 cm.

13 A garden roller is in the form of a cylinder of radius 0.25 m and width 0.7 m. What area of lawn does the roller cover in four revolutions?

14 Find the total area of sheet metal used to make a cylindrical oil drum with radius 24 cm and height 85 cm.

15 The diameter of a cylindrical buoy is 60 cm and its height is 150 cm. Find its total surface area.

Example:

50 litres of water are poured into a cylindrical tank of radius 0.3 m. Find the depth of water in the tank in centimetres.

1 litre = 1000 cm^3

$$\text{Volume} = 50 \text{ litres}$$
$$= 50000 \text{ cm}^3$$
$$V = 50000,\ r = 0.3 \times 100 = 30 \text{ (giving } r \text{ in cm)}$$
$$V = \pi r^2 h$$
$$50000 = \pi \times 30^2 \times h$$
$$\frac{50000}{\pi \times 30^2} = h$$
$$h = 17.68\ldots$$

The depth of water is 17.7 cm correct to 3 s.f.

16 1 m^3 of water fills a cylindrical drum of radius 50 cm. Find the height of the drum.

17 Water from a full rectangular tank measuring 1 m by 2 m by 0.5 m is emptied into a cylindrical tank and fills it to a depth of 1.2 m.
Find **a** the volume of water involved
b the diameter of the cylindrical tank.

18 A solid bronze cylinder of height 4.3 cm has a mass of 200 g. If the density of bronze is 8.96 g/cm^3 find the diameter of the cylinder.

The **density** of a substance is the mass in grams of 1 cubic centimetre of the substance.

19 A cylindrical water butt has a diameter of 80 cm and a height of 1 m. It is half full of water. If a further 20 100 cm^3 of water are poured in, find the new depth of water.

20 Water pours out of a cylindrical pipe at the rate of 1 m/s. The diameter of the pipe is 3 cm.
How much water comes out in 1 minute?

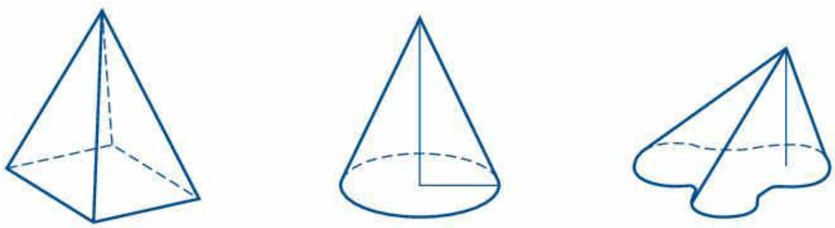
Volume of a cone

We have already stated that the volume of a pyramid is given by

$$\tfrac{1}{3} \times \text{area of base} \times \text{perpendicular height}$$

where a pyramid is a solid with a flat base and which comes up to a point called the **vertex**.

The formulae we quote for pyramids, cones and spheres cannot be proved at this stage.



This definition applies to a **cone** so the volume of a cone is given by

$$\text{Volume} = \tfrac{1}{3} \times \text{area of circular base} \times \text{perpendicular height}$$

i.e. $$V = \tfrac{1}{3}\pi r^2 h$$

A cone whose vertex is directly above the centre of the base is called a *right* circular cone; this is the only type that we deal with in this book.

EXERCISE 25d

Example:

Find the volume of a cone of base radius 3.2 cm and of height 7.2 cm.

$r = 3.2, \quad h = 7.2, \quad V = \frac{1}{3}\pi r^2 h$

$= \dfrac{\pi \times (3.2)^2 \times 7.2}{3} = 77.20\ldots$

The volume is 77.2 cm^3 correct to 3 s.f.

In each question from **1** to **6** find the volume of the cone whose dimensions are given.

1 Base radius 9 cm, height 20 cm.

2 Base radius 2.2 cm, height 5.8 cm.

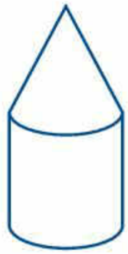
3 Base diameter 26.8 cm, height 104 cm.

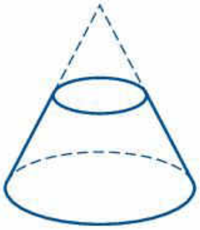
4 Base radius 0.6 cm, height 1.4 cm.

5 Base diameter 4.2 cm, height 5.9 cm.

6 Base diameter 0.62 m and height 106 cm. Give the volume in cubic metres.

7 A tower of a toy fort is formed by placing a cone on top of a cylinder. The total height of the tower is 20 cm, the common radius is 5 cm and the height of the cone is 8 cm. Find the volume of the tower.



8 

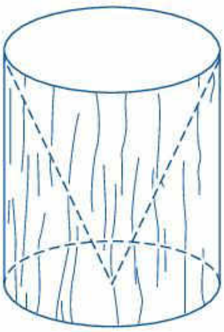
A **frustum** of a cone is formed by cutting the top off a cone. The section that remains, shown by solid lines, is the frustum. The original cone has base radius 6 cm and height 10 cm. The part cut off has base radius 3 cm and height 5 cm. Find the volume of the frustum.

9 A cylindrical piece of wood of radius 3.6 cm and height 8.4 cm has a conical hole cut in it. The cone has the same radius and the same height as the cylinder.

a Find the volume of the remaining solid.

b The mass of this solid is 1254 g. Find the density of the wood.

Density is mass per unit volume.

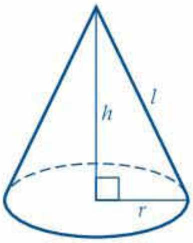


Surface area of a cone

The curved surface area of a cone is given by

$$A = \pi r l$$

where l units is the **slant height**.



If two of h, r and l are known, the third can be found using Pythagoras' theorem.

EXERCISE 25e

In each question from **1** to **4** find the area of the curved surface of the cone whose measurements are given.

1 Radius 4 cm, slant height 10 cm.

2 Radius 0.6 m, slant height 2.2 m.

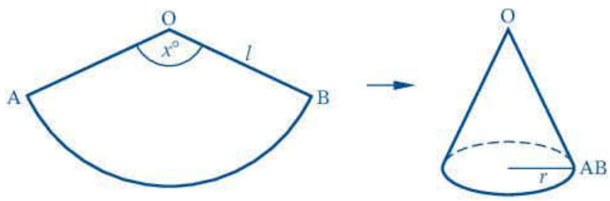
3 Radius 9.2 cm, slant height 15 cm.

4 Radius 67 mm, slant height 72 mm.

5 Find the total surface area of a cone of base radius 4 cm and with slant height 9 cm.

6 The radius of a cone is 6 cm and its perpendicular height is 8 cm. Find
- **a** the volume of the cone
- **b** its slant height
- **c** its curved surface area.

7



The sector OAB is formed into the cone as shown.
- **a** Find the length of the arc AB in terms of π, l and x.
- **b** Find the circumference of the base of the cone in terms of π and r.
- **c** Hence show that $\frac{r}{l} = \frac{x}{360}$
- **d** Find the area of the sector in terms of π, l and x. Hence show that the area, A, of the curved surface of the cone is given by $A = \pi r l$.

Volume of a sphere

The volume of a **sphere** is given by the formula

$$V = \frac{4}{3}\pi r^3$$

EXERCISE 25f

In each question from **1** to **6** find the volume of the sphere whose radius is given.

1 3 cm **2** 7.2 cm **3** 1.8 m

4 0.62 cm **5** 38 cm **6** 13 mm

7 Find the volume of a hemisphere of radius 5 cm.

A **hemisphere** is half a sphere.

8 Twenty lead spheres of radius 1.2 cm are melted down and recast into a cuboid of length 8 cm and width 4 cm.
  a Find the volume of lead involved.
  b How high is the cuboid?

9 Find, in terms of π, the volume of a sphere of radius $1\frac{1}{2}$ cm. (Do not use a calculator.)

10 Kingsley has a supply of lead pellets which are small spheres. Each pellet has a nominal diameter of 2.0 mm correct to 1 decimal place. If the density of lead is 11.4 g/cm^3 what is the minimum number of pellets, to the nearest 1000, that Kingsley must have to be certain that their total mass will exceed 1 kg?

Surface area of a sphere

The surface area, A, of a sphere of radius r is given by the formula

$$A = 4\pi r^2$$

EXERCISE 25g

In each question from **1** to **4** find the surface area of the sphere whose radius is given.

1 9 cm **2** 4.5 mm **3** 41 cm **4** 0.9 m

5 Find the curved surface area of a hemisphere of radius 23 cm.

6 240 spheres of radius 0.22 m are to be painted. Each pot of paint contains enough to cover 26 m^2. How many pots of paint are needed?

7 The surface area of a sphere is 100 cm^2.
Find
  a the radius of the sphere **b** its volume.

The next exercise involves a variety of problems concerning volumes and surface areas of cylinders, cones and spheres.

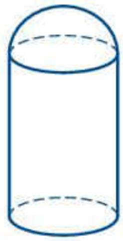
EXERCISE 25h

1 The radius of a spherical ball-bearing is 0.2 cm. How many ball-bearings can be made from 20 cm^3 of metal?

2 A toy is formed from a cone and a hemisphere. The radius of the hemisphere is 5.2 cm and the total height of the toy is 15 cm. Find the total volume.

3 Which has the greater volume, a cone of radius 3.5 cm and a perpendicular height of 12 cm or a sphere of radius 3.5 cm? What is the difference in volume?

4 A concrete bollard is in the shape of a cylinder surmounted by a hemisphere. The radius of the hemisphere and of the cylinder is 25 cm and the total height of the bollard is 130 cm. Find its volume.



5 A hollow metal sphere has an outer radius of 16 cm and its walls are 1 cm thick. Find
a the inner radius b the volume of metal.

6

Three glasses are in the shape of a cone, a cylinder and a hemisphere respectively. The radius of each is 4 cm and the depth of the cone and of the cylinder is also 4 cm.
Find, in terms of π, the capacity of
a the cone shaped glass
b the cylindrical glass
c the hemispherical glass.

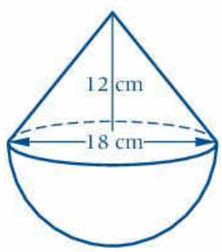
7 Find in terms of π, the volume of
a a sphere of radius 2 cm
b a sphere of radius 8 cm.

The larger sphere is made of metal. It is melted down and made into spheres of radius 2 cm.
c How many of the smaller spheres can be made from the larger sphere?

8 Which has the greater surface area, a cone of radius 3.5 cm and slant height 9 cm or a sphere of radius 3.5 cm? What is the difference between the areas?

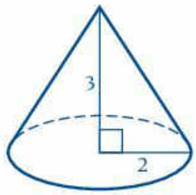
9 Find the total surface area of a hemisphere of radius 7 cm.

10 A solid is formed from a cone joined to a hemisphere as shown in the diagram. Find
a the slant height of the cone
b the total surface area of the solid.

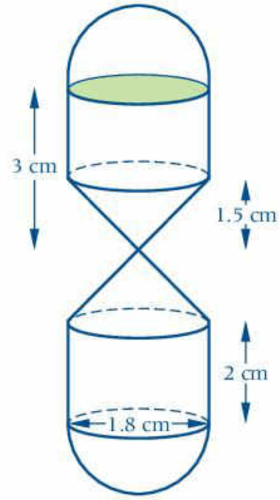


11 A sphere of radius 1.2 m and a solid cone of radius 1.2 m and slant height 2.6 m are being painted for a funfair. The tin of paint available contains enough paint to cover 30 m^2. Is there enough paint for the purpose?
Give details of the extra amount needed or the amount of paint left over.

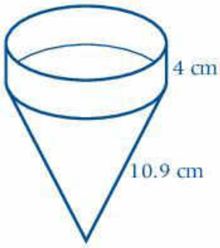
12 The radius of the base of a cone is 2 units and the height of the cone is 3 units. Find, in terms of π, exact forms for
a the curved surface area **b** the volume.



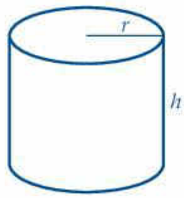
13 The diagram shows an egg-timer. It is symmetrical and is made from hollow cones, cylinders and hemispheres which are joined together as shown. When the timer starts, the depth of the sand in the upper container is 3 cm. Find
a the depth of sand in the lower container after it has all flowed through.
b the rate, in cm^3/min, at which the sand flows from one container to the other if the total time taken is 3 minutes.



14 A container is made of sheet metal in the form of an open cone joined to an open-ended cylinder. The radius of the cylinder and of the base of the cone is 8.6 cm, the depth of the cylinder is 4 cm and the slant height of the cone is 10.9 cm. Find the area of sheet metal used.



15 Write down, in terms of π, r and h, the formula for V, the volume of the cylinder and the formula for A, the curved surface area of the cylinder. Hence show that $2V = Ar$.



ABCD MIXED EXERCISE 25

Several alternative answers are given for these questions. Write down the letter that corresponds to the correct answer. Do not use a calculator. Remember that using $\pi \approx 3$ gives a quick estimate.

1 A cylinder has radius 2 cm and height 5 cm. The area of its curved surface is
A $31.4\,cm^2$ **B** $62.8\,cm^2$ **C** $98.7\,cm^2$ **D** $126\,cm^2$

2 A cylinder has radius 2 cm and height 9 cm. Its volume is
A $56.5\,cm^3$ **B** $509\,cm^3$ **C** $226\,cm^3$ **D** $113\,cm^3$

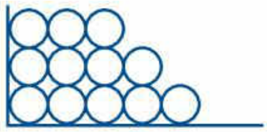
3 A cone has a base diameter of 3 cm and a height of 4 cm. The best estimate for its volume is
A 9 cm^3 **B** 12 cm^3 **C** 24 cm^3 **D** 36 cm^3

4 The best estimate for the curved surface area of the cone in question **3** is
A 18 cm^2 **B** 22.5 cm^2 **C** 36 cm^2 **D** 45 cm^2

5 The surface area of a sphere of radius 3 cm is
A 27 cm^2 **B** 28.2 cm^2 **C** 108 cm^2 **D** 113.1 cm^2

INVESTIGATION

A factory produces three-metre lengths of copper tubing which has an external diameter of 2 cm.

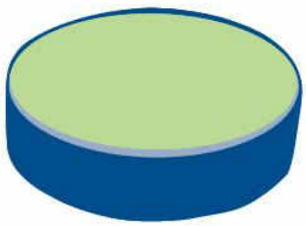


The lengths are stored in boxes whose internal dimensions are 3 m × 40 cm × 16 cm and they are arranged in the boxes as shown in the diagram.

An employee suggests that they can get more lengths into the same box if they arrange things differently. Do you agree? What is the greatest number of 3 m lengths that can be packed into a box of this size?

PUZZLE

It is possible to cut this 'cake' into 8 identical pieces in several different ways. What is the least number of cuts with a knife needed to do this?



MATHS IS OUT THERE

Proof that a page of a book is an idle dog:

1 A page of a book is an ink-lined plane.

2 An inclined plane is a slope up.

3 A slow pup is an idle dog.

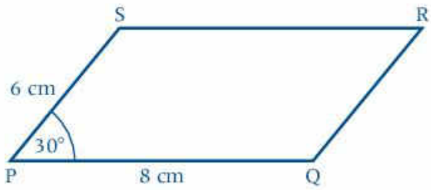
IN THIS CHAPTER YOU HAVE SEEN THAT...

- drawing a net of a pyramid makes it easier to find its surface area
- the volume of a cylinder is given by $V = \pi r^2 h$ and its curved surface area is $2\pi rh$
- the curved surface of a cylinder is equivalent to a rectangle
- the formula for the volume of a cone is $V = \frac{1}{3}\pi r^2 h$ (consistent with the formula for the volume of a pyramid)
- the curved surface area of a cone is given by πrl where l is the slant height of the cone
- the volume of a sphere is $\frac{4}{3}\pi r^3$ and the formula for the surface of a sphere is $4\pi r^2$.

Multiple choice questions

Several possible answers are given.
Write down the letter that corresponds to the correct answer.

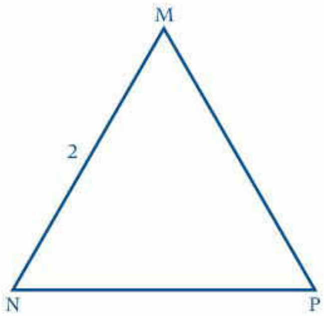
1



In the above parallelogram, PQRS, the area of the triangle PQR is

A $48\,cm^2$ **B** $36\,cm^2$ **C** $24\,cm^2$ **D** $12\,cm^2$

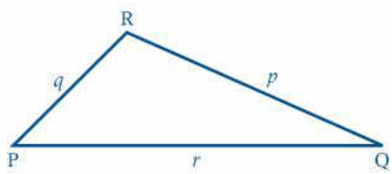
2



If MNP is an equilateral triangle of side 2 units, then the exact value of $\sin \widehat{N}$ is

A $\frac{1}{\sqrt{3}}$ **B** $\frac{1}{\sqrt{2}}$ **C** $\frac{1}{2}$ **D** $\frac{\sqrt{3}}{2}$

3



In the above triangle, $\cos \widehat{Q} =$

A $\frac{q^2 - p^2 - r^2}{2pr}$ **B** $\frac{p^2 + r^2 - q^2}{2pr}$ **C** $\frac{p^2 + q^2 + r^2}{2pq}$ **D** $\frac{q^2 - p^2 - r^2}{2pqr}$

4 If $\sin \theta = 0.6$ and θ is obtuse, then the exact value of $\cos\theta$ is

A 0.5 **B** 0.25 **C** −0.25 **D** −0.8

5 The scale on a map is 1 : 10 000. A path on the map is 4 cm long. The length of the actual path is

A 4 m **B** 400 m **C** 4 km **D** 40 km

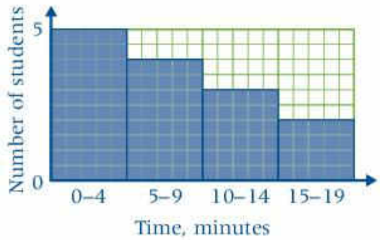
6 A circle of radius 4 cm is inscribed in a square. The area of the square is

A $4\,cm^2$ **B** $16\,cm^2$ **C** $16\pi\,cm^2$ **D** $64\,cm^2$

7 The floor plan of a room is a rectangle 5 m long. A scale drawing of the floor plan uses a scale of 1 cm to represent 50 cm. The length of the floor on the scale drawing is

A 1 cm **B** 5 cm **C** 10 cm **D** 50 cm

Questions **8** to **10** refer to this diagram which shows the times taken by a sample of students to solve a problem.

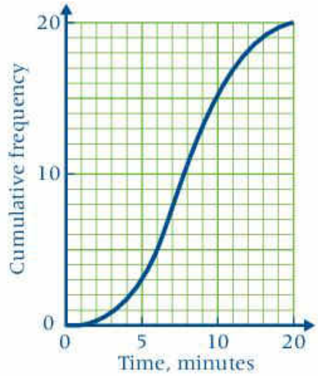


8 The lower class boundary of the first group is
A −0.5 **B** 0 **C** 0.5 **D** 1

9 The number of students in the sample is
A 4 **B** 14 **C** 20 **D** 56

10 The class width of the groups, in minutes, is
A 4 **B** 5 **C** 5.5 **D** 19

Questions **11** to **13** refer to this cumulative frequency diagram which shows the times taken by a sample of adults to fill in a form.



11 The number of adults taking more than 10 minutes is
A 1 **B** 5 **C** 10 **D** 15

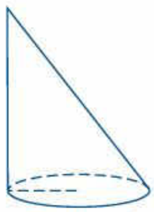
12 The median time, in minutes, is
A 2.5 **B** 5 **C** 8 **D** 10

13 The interquartile range is
A 4 minutes **B** 5 minutes **C** 10 minutes **D** 20 minutes

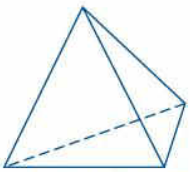
14 The diameter of a cylinder is 6 cm and the height is 10 cm. The best estimate for the area of its curved surface is
A 27 cm^2 **B** 90 cm^2 **C** 180 cm^2 **D** 270 cm^2

15 Which of the following shapes is a triangular pyramid?

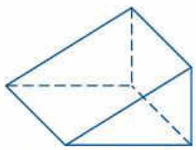
A



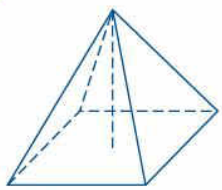
B



C



D



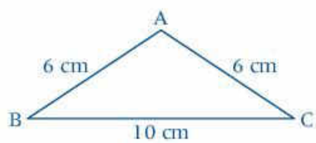
16 The radius of a cylinder is twice its height. If the height is x units, the volume is
A $2\pi x^2$ **B** πx^3 **C** $2\pi x^3$ **D** $4\pi x^3$

17 If $\cos\theta = -0.6$ and θ is obtuse the exact value of $\sin\theta$ is
A 0.5 **B** 0.6 **C** 0.8 **D** 0.9

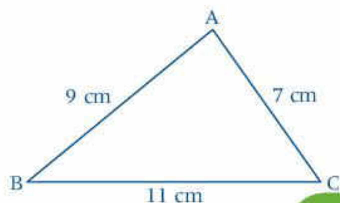
18 If $\tan\theta = -\sqrt{3}$ and θ is obtuse the exact value of $\cos\theta$ is
A 0.5 **B** −0.5 **C** $\frac{\sqrt{3}}{2}$ **D** $-\frac{\sqrt{3}}{2}$

19 Given that $\tan\theta = \frac{4}{3}$ and θ is acute, the exact value of $\sin\theta$ is
A $\frac{3}{5}$ **B** $\frac{4}{5}$ **C** $\frac{3}{4}$ **D** $\frac{4}{3}$

20 The area of this triangle is
A $\sqrt{55}$ cm^2 **B** 30 cm^2 **C** $5\sqrt{11}$ cm^2 **D** 20 cm^2

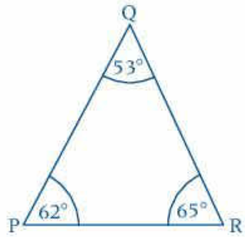


21 Which of the following statements about this triangle is false?
A Angle B is smaller than angle A.
B Angle B is smaller than angle C.
C Angle B is greater than angle C.
D Angle A is greater than angle C.

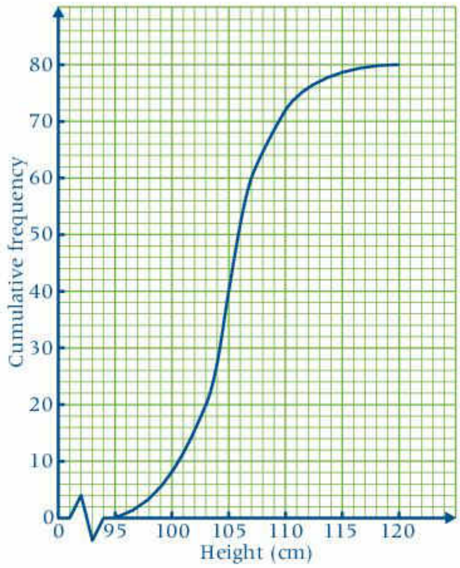


22 Which of the following statements about this triangle is false?

A The length of QR is greater than the length of PQ.
B The length of PR is smaller than the length of QR.
C The length of PQ is greater than the length of PR.
D The length of QR is smaller than the length of PQ.



Use this cumulative frequency curve, which shows the heights of 80 children, to answer questions **23** to **26**.



23 The number of children taller than 110 cm is

A 4 **B** 6
C 8 **D** 10

24 The median height of these children is

A 104 cm **B** 105 cm
C 106 cm **D** 107 cm

25 The upper quartile is

A 106 cm **B** 107 cm
C 108 cm **D** 110 cm

26 The interquartile range is

A 2.5 cm **B** 3 cm
C 3.5 cm **D** 4 cm

27 A hemisphere, a cylinder and a cone each have a circular base of radius 5 cm and a height of 5 cm.

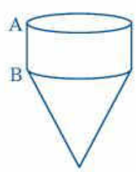
Which of the following statements is false?

A The volume of the cylinder is greater than the volume of the hemisphere.
B The surface area of the cylinder is smaller than the surface area of the cone.
C The volume of the cone is smaller than the volume of the hemisphere.
D The volume of the cylinder is equal to the combined volumes of the other two.

28 The radius of a spherical ball-bearing is 3 mm. The volume of 1000 similar ball-bearings is

A $3600\pi\,mm^3$ **B** $360\pi\,cm^3$ **C** $360\,cm^3$ **D** $36\pi\,cm^3$

29 A plastic container of volume V cm^3 is in the form of a cylinder joined to an open cone. The radius of the cone and the cylinder are both 5 cm. The depth of the cylinder is 4 cm and the overall depth of the container is 16 cm. Oil is poured into the container until it holds $\frac{1}{2}V\,cm^3$.



The level of the surface of the oil is

A at A **B** between A and B
C at B **D** below B

30 Alicia received the following marks for her end of term tests:
5, 50, 55, 65, 65, 70, 70, 75, 80, 85.

Alicia said that her average mark was 70. The mark was

A the mean **B** the median
C the mode **D** none of these

General proficiency questions

1 The heights of 5 boys are 1.2 m, 1.3 m, 1.3 m, 1.5 m and 1.9 m.
   a Find i the range ii the interquartile range.
   b Which of the range or the interquartile range would you use to describe the spread of the heights? Give a reason for your answer.

2 The area of a field on a map is $0.5\,cm^2$. The area of the actual field is 2 hectares. Find n if the scale of the map is $1 : n$.

3 The height of a cylinder is 2 cm less than the diameter of its base. If the total surface area of the cylinder is $100\,cm^2$ find
   a the diameter of its base
   b its height.

4 Given that $\sin\theta = \frac{5}{13}$ and that θ is acute, find the exact value of
   a $\cos\theta$
   b $\tan\theta$.

5 If $\cos\hat{A} = \frac{5}{13}$ and $\hat{A}$ is acute, find $\sin\hat{A}$ and $\tan\hat{A}$.

6 Find two angles between 0° and 180° such that the sine of each is 0.3667.

7 If $\cos\hat{P} = 0.43$ find, without using a calculator, $\cos(180° - \hat{P})$.

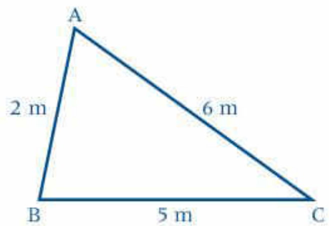
8 In $\triangle$ABC, BC = 161 cm, $\angle$B = 109° and $\angle$A = 51°. Find AC.

9 A triangle PQR is such that QR = 7.6 cm, PQ = 5.9 cm and $\angle$Q = 107°. Find PR.

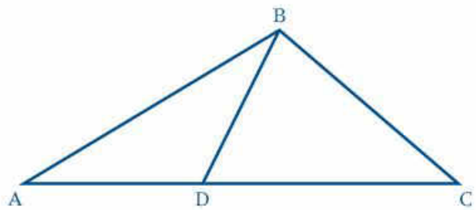
10 Find the area of $\triangle$ABC if AB = 8.2 cm, BC = 11.3 cm and $\hat{B} = 125°$.

11 A boat sails 11 km from a harbour on a bearing of 220°. It then sails 15 km on a bearing of 340°. How far is the boat from the harbour?

12 a Use the information in the diagram to find $\cos\hat{A}$ giving your answer as a fraction.
   b Hence find
      i $\sin\hat{A}$ in square root form.
      ii the area of $\triangle$ABC in square root form.



13



In the above triangle ABC, $\angle$A = 36° and $\angle$B = 95°. The point D lies on AC such that DA = DB = 7 cm. Find
   a the length of AB
   b the length of BC
   c the area of triangle ABC.

14 For an acute angle, x, prove that

$$\frac{\sin x}{\cos x} \equiv \tan x$$

Hence, or otherwise, find the acute angle x that satisfies the equation

$$2 \sin x - 3 \cos x = 0$$

giving your answer to the nearest 0.1°.

15 A vertical pole, OP, with base O, is 5 m tall. The points O, A and B lie on level ground such that OA is 8m and A lies on a bearing of 045° from O.

a Find the angle of elevation of the top, P, of the pole from A.

b B lies on a bearing of 135° from O. If the angle of depression of B from P is 20°, find the distance OB. Hence, find the distance of A from B.

c Calculate the bearing of A from B.

16 The table shows the distribution of the lengths of carpet off-cuts.

Length (m)	1–2	3–4	5–6	7–8	9–10	11–12
Number of off-cuts	2	9	15	17	10	4

a **i** Draw a cumulative frequency curve to represent the data.

ii Estimate the interquartile range.

iii One piece of carpet is selected at random. Find an estimate for the probability that is more than 10 metres long.

b Calculate an estimate for the mean length of the carpets.

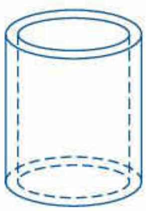
17 The figure shows a jar which is in the shape of a cylinder with a truncated cone on the top. The height of the cylinder is 10 cm and the radius is 3 cm.
The height of the truncated cone is 4 cm and the radius of the top is 1 cm. Calculate the capacity of the jar.

18 **a** Using only a ruler and pair of compasses, construct the parallelogram ABCD in which AB = 8 cm, BC = 12 cm and angle ABC = 120°.

b Calculate the length of the perpendicular from A to BC produced, giving your answer in square root form.

c Find the area of the parallelogram in square root form.

19 A hollow pipe is 2 metres long. The internal radius is 20 cm and the external radius is 22 cm.



a Calculate the area of the cross-section of the pipe.

b Calculate the capacity of this length of pipe.

c Calculate the volume of material needed to make a 20 m length of this pipe.

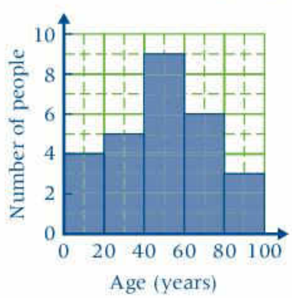
20 This histogram shows the ages of people living in a village.

a Construct a frequency table to show this information.

b Estimate the mean age of the people living in the village.

c Construct a cumulative frequency table for this information.

d Find an estimate for the median age.

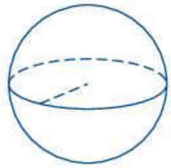


21 **a** Using only a ruler and pair of compasses, construct triangle ABC in which AB = 6 cm, AC = 8 cm and BC = 4 cm.
b Calculate the area of this triangle, giving the answer in square root form.

22 **a** Using only a ruler and a pair of compasses
i draw a line AB, 8 cm long
ii draw CD, the perpendicular bisector of AB to cut AB at E such that CE = ED = 3 cm.
b Join the ends of the two lines to form a quadrilateral and name the quadrilateral.
c Calculate the length of a side of the quadrilateral.

23 Using only a ruler and pair of compasses, construct a square of side 6 cm. Measure and record the length of a diagonal of the square.

24 The diagram shows a sphere of radius 4 cm.



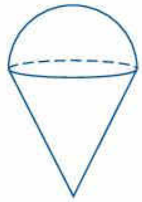
a The spheres are made from material that has a density of 5 g per cubic centimetre.
Find the mass of **i** one sphere **ii** 100 spheres.
b Fifty of these spheres are used on the tops of railings.
They are to be painted. One tin of paint covers $4\,m^2$.
Calculate the number of tins required.

25 The table shows the masses of a sample of babies under 2 years old.

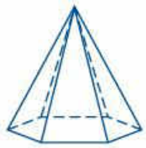
Mass (kg)	3–4	5–6	7–8	9–10
Number of babies	7	15	5	2

a Estimate the mean mass of these babies.
b Using a scale of 2 cm to represent a mass of 5 kg and 2 cm to represent 5 babies, draw a frequency polygon to represent the data in the table.
c Calculate the probability that one of these babies, chosen at random, has a mass of no more than 6 kg.

26 The diagram shows a hemisphere on top of a cone.
The radius of the hemisphere is 6 cm and the height of the cone is 12 cm.
Calculate
a the volume of the solid
b the surface area of the solid.



27 The diagram shows a pyramid whose base is a regular hexagon.
A side of the hexagon is 3 cm long and the height of the pyramid is 4 cm.
Calculate
a the surface area of the solid
b the volume of the solid.



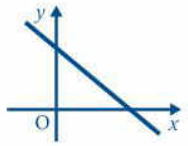
MULTIPLE CHOICE TESTS

In tests **1** to **3** several possible answers are given.
Write down the letter that corresponds to the correct answer.

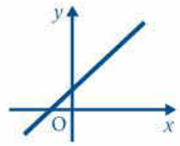
Multiple choice test 1

1 Which of the following diagrams best represents the line with equation $y = 2x + 3$?

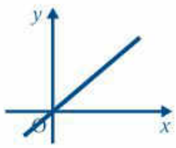
A



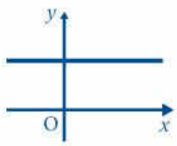
B



C



D



2 If $\frac{1}{2^x} = \frac{1}{32}$ then x is

A -5 **B** $-\frac{1}{5}$ **C** $\frac{1}{5}$ **D** 5

3 If f: $x \rightarrow \frac{2}{x}$, then $f^{-1}: x \rightarrow$

A $2x$ **B** $\frac{x}{2}$ **C** $-\frac{2}{x}$ **D** $\frac{2}{x}$

4 Ten cards are each marked with one of the numbers from 1 to 10.
A card is chosen at random.
The probability that the number is odd and prime is

A $\frac{1}{10}$ **B** $\frac{3}{10}$ **C** $\frac{2}{5}$ **D** $\frac{1}{2}$

5 If P and Q are two disjoint sets such that $n(P) = 6$ and $n(Q) = 4$, then $n(P \cup Q) =$

A 10 **B** 6 **C** 4 **D** 2

6 If $a * b$ denotes $\frac{a^b}{b^a}$, then $1 * 2$ is

A $\frac{12}{21}$ **B** $\frac{1}{2}$ **C** 1 **D** 2

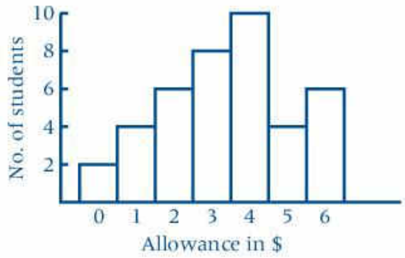
7 If $Q = \{x: 2 < x - 2 < 5, x \in \mathbb{Z}\}$, then Q is

A $\{3, 4\}$ **B** $\{5, 6\}$ **C** $\{4, 5, 6, 7\}$ **D** $\varnothing$

8 The solution set of $\frac{5}{2} > \frac{5}{x}$, $x > 0$, is given by

A $\{x: x > 2\}$ **B** $\{x: x = 2\}$ **C** $\{x: x < 5\}$ **D** $\{x: x > 1\}$

Questions **9** to **11** are based on the following diagram.



The above diagram shows the daily allowance, in dollars, of students in a class.

9 The mode of the distribution is
A \$3 **B** \$4 **C** \$6 **D** \$10

10 The number of students in the class is
A 40 **B** 36 **C** 30 **D** 21

11 The median of the distribution is
A \$3.00 **B** \$3.50 **C** \$4.00 **D** \$6.00

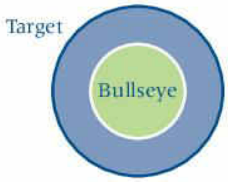
12 Given that y varies directly as the square root of x then
a $y \propto \sqrt{x}$ **B** $y \propto x^2$ **C** $y \propto x$ **D** $y \propto \frac{1}{\sqrt{x}}$

13 A 1 metre length of string is cut into three pieces such that the second piece is 10 cm longer than the first piece and the third piece is 5 cm longer than the second piece. The length of the smallest piece, in cm, is
A 15 **B** 25 **C** $28\frac{1}{3}$ **D** $33\frac{1}{3}$

14 If $\mathbf{a} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$, then the magnitude of $\mathbf{a} + \mathbf{b}$ is
A 23 **B** 14 **C** 13 **D** 10

15 If $T = 2\pi\sqrt{\frac{l}{g}}$, then $l =$
A $2\pi\sqrt{\frac{T}{g}}$ **B** $\frac{gT^2}{4\pi^2}$ **C** $\frac{gT}{2\pi}$ **D** $\frac{\sqrt{gT^2}}{2\pi}$

16



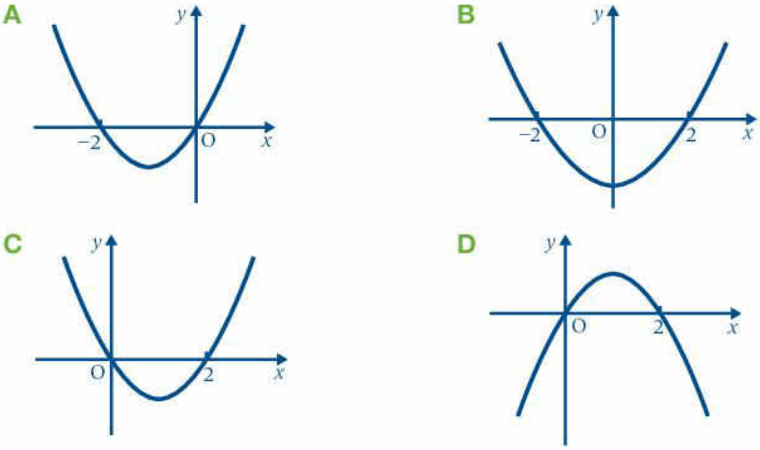
The diagram above illustrates a circular target of radius 6 cm with a circular 'bullseye' of radius 2 cm. A marksman always hits the target and is equally likely to hit any part. The probability that he hits the 'bullseye' with a single shot is
A $\frac{1}{9}$ **B** $\frac{1}{4}$ **C** $\frac{1}{3}$ **D** $\frac{1}{2}$

17 15 out of 20 students who did a test scored more than 75%. The probability that one student chosen at random from the 20 students scored 75% or less is
A $\frac{3}{4}$ **B** $\frac{1}{2}$ **C** $\frac{1}{4}$ **D** $\frac{1}{5}$

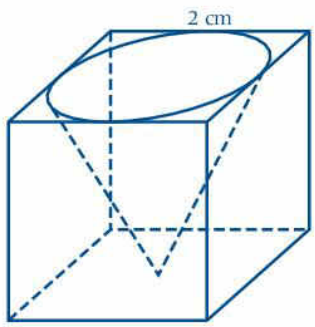
18 $3(2x - 1) + 2(3x + 1) =$
A $12x - 1$ **B** $12x - 5$ **C** 0 **D** $4x - 5$

19 Which of the following is a subset of {a, b, c, d}?
A {b, c} **B** {a, b, m} **C** {e, f, g, h} **D** {a, b, c, d, e}

20 Which of the following curves represents the graph with the equation $y = x^2 - 2x$?



21 The figure shows the largest possible cone that may be inscribed in a cube of side 2 cm.
The cone is removed. The volume of the remaining solid, in cm^3, is
A $6\frac{20}{21}$ **B** $5\frac{19}{21}$ **C** 4 **D** $2\frac{2}{3}$
(Use $\pi = \frac{22}{7}$)



22 If $1\frac{1}{3} : 2\frac{1}{3} = 3\frac{1}{3} : x$ then $x =$
A $\frac{1}{2}$ **B** $4\frac{1}{3}$ **C** $4\frac{2}{3}$ **D** $5\frac{5}{6}$

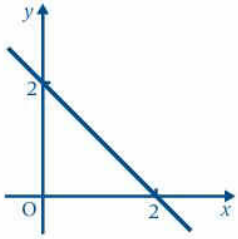
23 A man walks at the rate 2 m/s for 10 s and 1 m/s for 20 s.
His average speed, in m/s, is
A 3 **B** 2 **C** $1\frac{1}{2}$ **D** $1\frac{1}{3}$

24 The best suggested approximation to $\frac{\sqrt{24.8}}{9.8}$ is
A 5 **B** 1 **C** 0.5 **D** 0.1

25 The solution to the equation $x^3 - 10 = 0$ lies between
A −1 and 0 **B** 0 and 1 **C** 1 and 2 **D** 2 and 3

26 $(a - b)^2 - 1$ can be expressed as
A $(a - b - 1)(a - b + 1)$ **B** $(a - b - 1)(a - b - 1)$
C $(a - b + 1)(a - b + 1)$ **D** $(a + b + 1)(a - b - 1)$

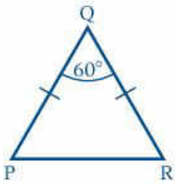
27



The diagram above illustrates the graph with equation

A $y = x + 2$ **B** $y = 2x + 2$ **C** $y = -x + 2$ **D** $y = 2x$

28



In the above diagram, the following statements are made:

i The triangle is equilateral

ii Angle $R = 60°$

The correct statement(s) is (are)

A i and ii **B** i only **C** ii only **D** Neither

29 A number, x, is increased by 1 and the result squared. This may be represented algebraically as

A $2(x + 1)$ **B** $x^2 + 1^2$ **C** $x + 1^2$ **D** $(x + 1)^2$

30 If p is a positive integer, then $p + 5$ is always

A an integer that is positive **B** an integer that is prime

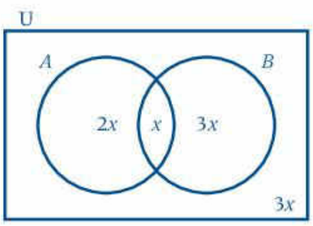
C an integer that is odd **D** an integer that is even

Multiple choice test 2

1 If $S = \{3\}$, then the number of subsets of S is

A 1 **B** 2 **C** 3 **D** 4

Questions **2** and **3** refer to the Venn diagram below, which shows the number of elements in each region.



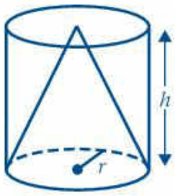
2 The probability that an element chosen at random from U does not belong to $A \cup B$ is

A $\frac{1}{6}$ **B** $\frac{1}{4}$ **C** $\frac{1}{3}$ **D** $\frac{1}{2}$

3 If $n(U) = 27$ then $n(A \cup B) =$

A 9 **B** 12 **C** 15 **D** 18

4



The ratio of the volume of the cylinder to the volume of the cone in the above diagram is

A 3 : 1 **B** 2 : 1 **C** 1 : 2 **D** 1 : 3

5 If the mean of x, $2x$ and $3x$ is 6, then x is

A 2 **B** 3 **C** 6 **D** 12

6 By mistakenly using π as 3, the area of a circle was found to be $147\,\text{cm}^2$. Taking π to be $\frac{22}{7}$, the true area, in cm^2, is

A 154 **B** 150 **C** 144 **D** 138

7 $5^{x+2} = 1$, then $x =$

A -2 **B** -1 **C** 0 **D** 1

8 The point P (h, k) is mapped onto P′ under a reflection in the line $y = x$. The coordinates of P′ will be

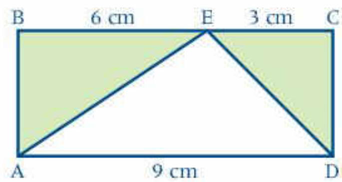
A $(h, -k)$ **B** (k, h) **C** $(-h, k)$ **D** $(-h, -k)$

9 The set of values of x that satisfy $3x - 2 < 2x - 1 < x + 3$ is

A $\{x: x < 4\}$ **B** $\{x: 1 < x < 4\}$

C $\{x: x < 1\}$ **D** $\{x: x < 1\} \cup \{x = 4\}$

10 ABCD is a rectangle with E, a point on BC, and BE = 6 cm, EC = 3 cm. The ratio of the area of the shaded regions to the area of the unshaded region is

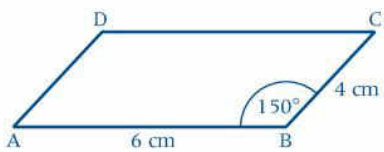


A 1 : 1 **B** 1 : 2 **C** 2 : 1 **D** 3 : 1

11 Given that $\frac{20x^3}{y^2} = N \times \frac{x^2}{y^3}$, then N is

A $\frac{4x}{y}$ **B** $20xy$ **C** $\frac{4y}{x}$ **D** $\frac{20x^2}{y^2}$

12 ABCD is a parallelogram with adjacent sides AB = 6 cm and BC = 4 cm.
If B = 150° then the area of triangle ABD, in cm^2, is



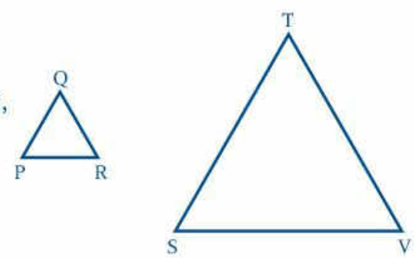
A 6 **B** 8 **C** 10 **D** 12

13 $(27)^{-\frac{2}{3}} =$

A -18 **B** $-\frac{1}{9}$ **C** $\frac{1}{9}$ **D** 18

14 Triangle STV is the image of triangle PQR under an enlargement with scale factor 3.

If the area of triangle STV is $36\,\text{cm}^2$, then the area of triangle PQR, in cm^2, is



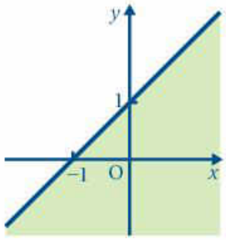
A 33 **B** 24 **C** 12 **D** 4

15 The exact value of sin 30° + cos 30° may be represented by

A $\frac{1}{2}$ **B** $\frac{\sqrt{3}+1}{4}$ **C** $\frac{\sqrt{3}}{2}$ **D** $\frac{\sqrt{3}+1}{2}$

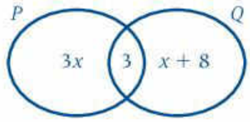
16 The shaded region shown in the diagram is represented by the inequation

A $y \leqslant x + 1$ **B** $y \geqslant x + 1$
C $y \leqslant -x + 1$ **D** $y \geqslant -x + 1$



17 P and Q are sets with the number of elements shown in the diagram. If $n(P) = n(Q)$, then the number of elements in Q only is

A 4 **B** 11 **C** 12 **D** 15



18 If $\frac{1}{a} + \frac{2}{b} = \frac{3}{c}$, then c is

A $\frac{2a+b}{3ab}$ **B** $\frac{2a+b}{ab}$ **C** $\frac{ab}{2a+b}$ **D** $\frac{3ab}{2a+b}$

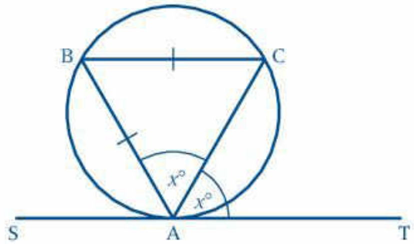
19 If $4^3 \times 4^x = 4^{12}$, then $x =$

A 0 **B** 1 **C** 4 **D** 9

20 $2(x - 3) - 3(x - 2) =$

A $x - 12$ **B** $-x - 12$ **C** $-x$ **D** 0

21

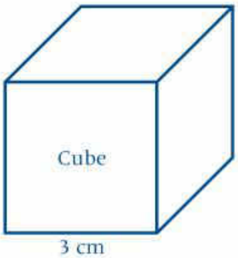


In the above diagram SAT is the tangent to the circle at A.

If ∠BAC = ∠CAT and BA = BC, then ∠BAC =

A 30° **B** 45° **C** 50° **D** 60°

22



In the figure above, the volume of the cube, in m^3, expressed in scientific notation is

A 3×10^{-5} **B** 2.7×10^{-5} **C** 3×10^{-6} **D** 2.7×10^{-6}

23 20% of 20 exceeds 10% of 10 by

A $\frac{1}{2}$ **B** 1 **C** 3 **D** 10

24 The next term in the sequence $\frac{1}{4}, \frac{2}{9}, \frac{3}{16}$ ..., is
A $\frac{4}{25}$ B $\frac{4}{21}$ C $\frac{6}{27}$ D $\frac{1}{4}$

25 $(0.1)^3$ written in scientific notation is
A 1×10^{-4} B 1×10^{-3} C 1×10^{-1} D 1×10^{3}

26 A man shares 36 pearls among his three daughters Amy, Beth and Carol. Beth gets one pearl more than Amy and Carol gets one pearl more than Beth. The number of pearls received by Amy is
A 13 B 12 C 11 D 10

27 0.1 of 1% as a fraction is
A $\frac{1}{1000}$ B $\frac{1}{100}$ C $\frac{1}{10}$ D $\frac{1}{2}$

28 The highest integer that satisfies the inequality $3x - 1 \leqslant x + 6$ is
A 2 B 3 C 4 D 7

29 The highest common factor of the numbers 70, 84, 98 and 112 is
A 2 B 7 C 14 D 28

30 Which of the following calculations does not have the same value as 19?
A $4 + 3 \times 5$ B $5 \times 3 + 4$ C $4 + (5 \times 3)$ D $(4 + 5) \times 3$

Multiple choice test 3

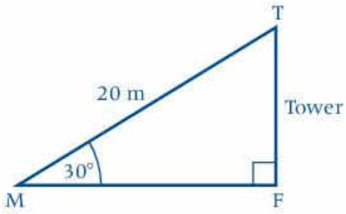
1 A man is three times as old as his son. Three years ago the man was 39 years old.
The son's age is
A 11 B 13 C 14 D 15

2 The next term in the sequence . . . 3, 5, 5, 7, 7, 9, 9, 11, . . . is
A 11 B 13 C 15 D 17

3 If $\frac{a}{4} > \frac{b}{2}$ then
A $a > 2b$ B $2a > b$ C $a < 2b$ D $2a < b$

4 If $x^2 - 5x + 6 = (x - 3)(x - a)$, then $a =$
A 3 B 2 C −2 D −3

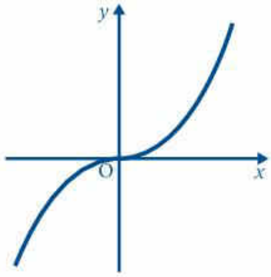
5



The figure above shows a vertical tower TF on level ground.
The angle of elevation of the top, T, of the tower is 30° from a point M, and MT is 20 m.
The height of the tower, in metres, is
A 20 B $\frac{20}{\sqrt{3}}$ C 2 D 10

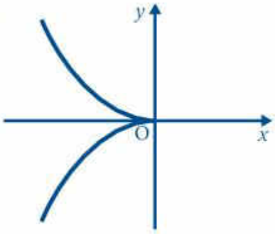
6 If $4^x = 2^5$ then $x =$
A 10 B 5 C $2\frac{1}{2}$ D $\frac{2}{5}$

7

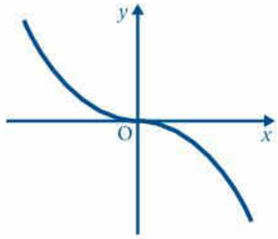


The curve shown in the above diagram is reflected in the *y*-axis. The resulting curve is

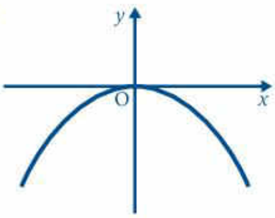
A



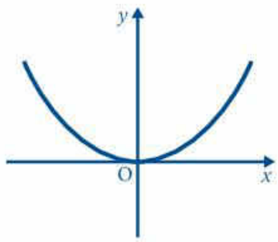
B



C



D



8 If $F = \{\text{Fat men}\}$ and $H = \{\text{Hungry men}\}$ then $F \cap H = \varnothing$ indicates

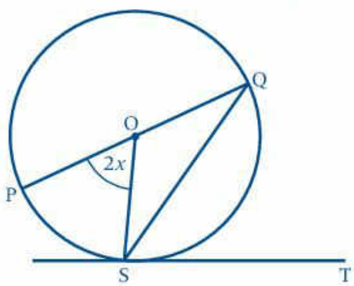
i Some fat men are hungry men.
ii No fat man is a hungry man.
iii Men who are fat are also hungry.
iv Some hungry men are fat.

A i and iii only **B** ii and iv only **C** i only **D** ii only

9 $(1 \quad -1)\begin{pmatrix}-1\\1\end{pmatrix} =$

A (2) **B** (-2) **C** $\begin{pmatrix}1 & -1\\-1 & 1\end{pmatrix}$ **D** $\begin{pmatrix}-1 & 1\\1 & -1\end{pmatrix}$

10



In the above diagram, PQ is a diameter to the circle centre O.
ST is the tangent at S and $\angle POS = 2x°$. $\angle QST =$

A $x°$ **B** $2x°$ **C** $(90 - x)°$ **D** $(90 - 2x)°$

11 Five equal discs, numbered 1, 2, 3, 4 and 5 respectively, are placed in a bag. A disc is drawn at random.
The probability that the disc is the 4 is

A $\frac{1}{5}$ **B** $\frac{1}{4}$ **C** $\frac{2}{5}$ **D** $\frac{4}{5}$

12 **a** and **b** are perpendicular vectors with $|\mathbf{a}| = 6$ units and $|\mathbf{b}| = 8$ units. $|\mathbf{a} + \mathbf{b}| =$

A 14 **B** 10 **C** 8 **D** 2

13 A pie chart consists of four sectors such that the second is 10° bigger than the first, the third is 10° bigger than the second and the largest is 10° more than the third.
The size of the smallest sector, is

A 75° **B** 80° **C** 90° **D** 120°

14 A bouquet of flowers consists of x roses and y lilies. If there are at least twice the number of roses as lilies, then

A $x \geqslant 2y$ **B** $2x \geqslant y$ **C** $2y \geqslant x$ **D** $y \geqslant 2x$

15 The point (a, b) under reflection in the x-axis is mapped onto the point

A $(a, -b)$ **B** $(-a, b)$ **C** (b, a) **D** $(-a, -b)$

16 The point $(11, -3)$ lies on the line $3x + 4y = 3r$. The value of r is

A 21 **B** 15 **C** 8 **D** 7

17 The lines $2y = x + 4$ and $3y = mx + 8$ are parallel. The value of m is

A $\frac{1}{3}$ **B** $\frac{1}{2}$ **C** $\frac{2}{3}$ **D** $\frac{3}{2}$

18 The lines $2y = x + 4$ and $y = mx + 2$ are perpendicular. The value of m is

A 2 **B** 1 **C** $\frac{1}{2}$ **D** -2

19 One angle of a triangle measures between 30° and 40° and the second angle measures between 40° and 50°. The third angle measures between

A 20° and 30° **B** 50° and 60°
C 90° and 110° **D** 120° and 150°

20 Two circles C and S have radii 2 cm and 3 cm respectively.
The ratio of the area of C to the area of S is

A 4 : 9 **B** 2 : 3 **C** 3 : 2 **D** 9 : 4

21 $\mathbf{M} = \begin{pmatrix} 2 & 4 \\ 3 & x \end{pmatrix}$. If **M** is singular then $x =$

A -12 **B** -6 **C** 6 **D** 12

22 A car travelling on a straight road passes a point P at a steady speed of 10 m/s.
Two seconds later a second car, travelling on the same road, passes P at a steady speed of 15 m/s. The second car meets the first car after

A 2 s **B** 4 s **C** 5 s **D** 10 s

23 If θ is acute and $\sin\theta \div \cos\theta = 1$ then $\theta =$

A 30° **B** 45° **C** 60° **D** 75°

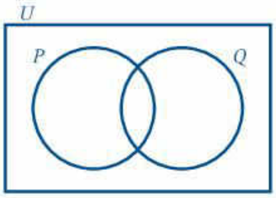
24 $2^4 + 2^2 + 2$ written as a binary number is

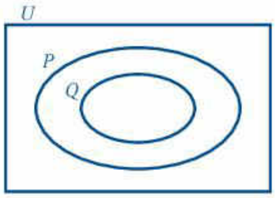
A 11101 **B** 10110 **C** 1011 **D** 1101

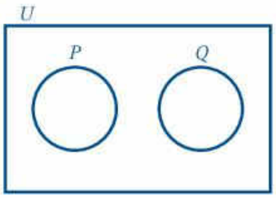
25 The value of $\frac{1190}{5.9 \times 405}$ estimated correct to one significant figure is

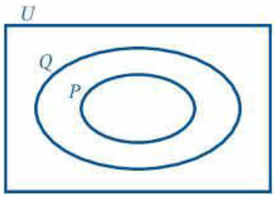
A 50 **B** 5 **C** 0.5 **D** 0.0526

26 Two sets P and Q are such that $P \subset Q$. Which one of the following Venn diagrams satisfies this condition?

A 

B 

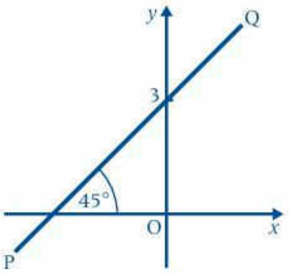
C 

D 

27 $\{x: 4 < x - 2 < 8, x \in \mathbb{Z}\}$ is
A $\{3, 4, 5\}$ **B** $\{7, 8, 9\}$ **C** $\{6, 7, 8, 9, 10\}$ **D** $\varnothing$

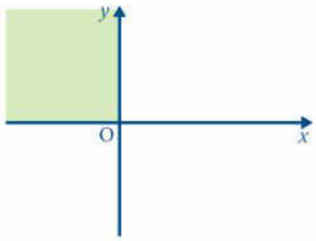
28 $(3 \times 10^{-2})^2$ written in standard form is
A 3×10^{-4} **B** 9×10^{-4} **C** 3×10^{4} **D** 9×10^{4}

29



The equation of the line PQ is
A $y = 3x + 1$ **B** $y = x + 1$ **C** $y = x + 3$ **D** $y = 3x + 3$

30



The shaded region represents
A $\{(x, y): x > 0 \text{ and } y < 0\}$
B $\{(x, y): x < 0 \text{ and } y > 0\}$
C $\{(x, y): x > 0 \text{ and } y > 0\}$
D $\{(x, y): x < 0 \text{ and } y < 0\}$

MODEL EXAM 1

Paper 2

Section 1

Answer all questions in this section.
All working must be clearly shown.

1 **a** Calculate the exact value of

$$\frac{2\frac{1}{4} - 1\frac{3}{4}}{\frac{2}{5} \times 1\frac{1}{3}}$$

b Find $\sqrt{0.0625}$, expressing the answer

i in exact form

ii to one decimal place

iii in scientific notation.

c Mrs Jones buys 1000 US dollars with Jamaican dollars at an exchange rate of USD 1 = JMD 1.98.

i How much, in Jamaican dollars does this transaction cost? Mr Shah buys 1000 US dollars at a different bank where the exchange rate is USD 1 = JMD 1.99 and pays a commission of 1.5%.

ii How much, in Jamaican dollars, does this cost Mr Shah?

iii Which is the better buy?

iv Mrs Jones spent half her US dollars and converts the rest into Jamaican dollars at the exchange rate of USD 1 = JMD 2.01. What percentage of her initial outlay did she get back?

2 **a** Solve for x in the equation

$$\frac{3}{x} - \frac{1}{2} = \frac{1}{x} + 2$$

b Factorise completely

i $2x^2 - 8x + 6$

ii $9 - (a - b)^2$

iii $6 - 3x + 4k - 2kx$

c Solve the equation

$$2x^2 - 3x - 2 = 0$$

3 **a** A class consists of 36 students with x students studying Art and Spanish, $2x$ students studying Art only and $3x$ students studying Spanish only.
The number of students studying Art or Spanish or both is the same as the number of students who do not study either of these subjects.

i Illustrate the data on a clearly labelled Venn diagram.

ii By writing a suitable equation in x, find the number of students who study Art.

iii If a student from the class is chosen at random, find the probability that the student studies Spanish only.

b Given that $\mathbf{A} = (1 \;\; -1)$ and $\mathbf{B} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ are two matrices, calculate

i **AB** **ii** **BA**

4 **a** The distance between two towns A and B is 8 km.
On a map this distance measures 1.6 cm. Calculate the scale used to draw the map.

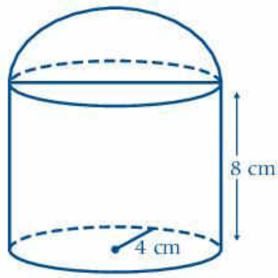
b This solid consists of a cylinder of base radius 4 cm and height 8 cm surmounted by a hemisphere, with no overlapping edges. Find

i the height, in cm, of the figure

ii its total surface area, in cm^2

iii its volume, in cm^3.

Write the volume in m^3, expressing your answer in scientific notation.



5 **a** Construct triangle ABC with $\angle A = 60°$, AB = 8 cm and AC = 6 cm.
Construct the perpendicular from C to AB, meeting AB at X.
Measure and state

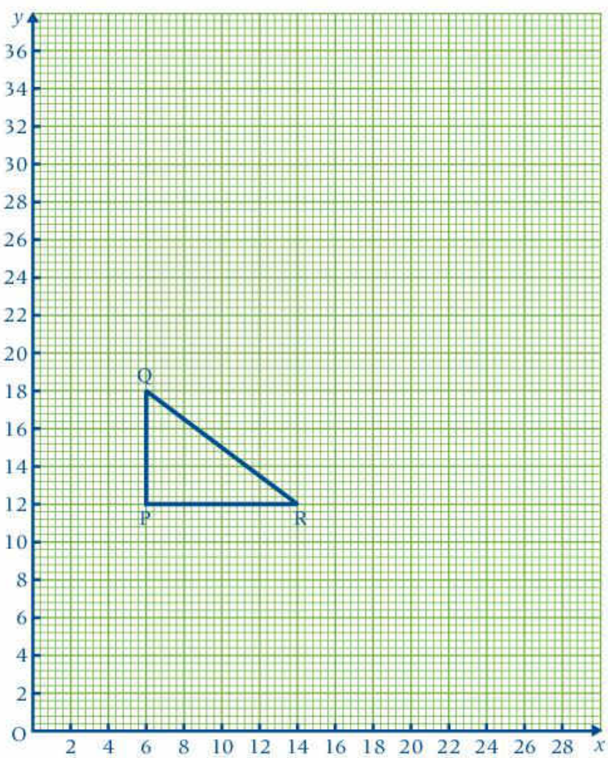
i the size of angle B

ii the length of AX.

b Triangle PQR is mapped onto triangle P′Q′R′ by an enlargement, centre O, and scale factor 2.
Draw a diagram to illustrate triangle PQR.
Obtain the coordinates of P′, Q′ and R′.

Find the ratio of the area of triangle P′Q′R′ : area of triangle PQR.

Find also the ratio of QR : Q′R′



6 **a** Three similar shirts are priced at three different stores.

STORE	OFFER
'Price-S-Right'	Marked price \$85.00 plus 15% value added tax.
'Shopper's-D'Lite'	Marked price \$100.00 and a $6\frac{1}{2}$% discount off the marked price.
'All-U-Need'	Marked price \$93.00

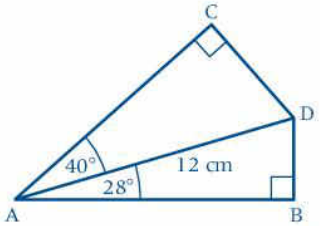
Show, with full working, which of the three stores offers the shirt at the lowest price.

b Three pens and four rulers cost a total of \$24. Twice the number of pens and half the number of rulers cost \$30. Calculate the cost of

i a pen

ii a ruler.

7 **a** In the adjacent diagram $\angle BAD = 28°$, $\angle DAC = 40°$.



If $\angle ABD = \angle ACD = 90°$ and AD = 12 cm, find

i DC

ii AB

b Given that $f(x) = \frac{2}{x}$

i State the value of x for which $f(x)$ is not defined.

ii If $f^2(x) = f(f(x))$, $f^3(x) = f(f^2(x))$, etc., find $f^2(x)$, and $f^3(x)$.

iii Hence, suggest expressions for $f^{20}(x)$ and $f^{21}(x)$.

Section 2

Answer all questions in this section.

8 A car, P, starts at rest from a point, X, and accelerates uniformly, reaching a speed of 25 m/s in 10 s. It then decelerates uniformly for 20 s before coming to rest.
At the same time that P starts, a second car, Q, passes X at 10 m/s and continues to travel at this constant speed.
On the same axes, illustrate the time, speed (t, v) graphs for both cars, using 1 cm ≡ 2 s on the horizontal axis and 1 cm ≡ 2 m/s on the vertical axis, for $0 \leqslant t \leqslant 30$.

a Find
- i the total distance covered by car P before coming to rest
- ii the distance covered by car Q in the same time.

b Determine
- i the speed of P at the instant when it overtakes Q
- ii the time taken by car P to overtake car Q.

c By completing the square of the function $f(x) = 2x^2 - 6x + 1$, find the minimum value of $f(x)$ and the value of x at which it occurs.
Hence show, with the aid of a sketch, that the equation $2x^2 - 6x + 1 = 0$ has two real and distinct roots.

9 (Solutions to this question by accurate drawing will not be accepted.)
In a game of soccer, a goal-keeper, G, is standing 8 m away from a defender, D, and on a bearing of 045° from D.
A forward, F, is standing 12 m away from G and on a bearing of 130° from G.

a Draw a clearly labelled diagram to illustrate the given information.

b Calculate
- i the distance of D from F
- ii the bearing of F from D.

c Find how far north F will have to move so that F is due east of D.

d If F and D run towards each other at the same speed from their original positions, find the distance covered by each when they meet.

10 a $\mathbf{A} = \begin{pmatrix} 1 & -1 \\ 2 & 2 \end{pmatrix}$ $\mathbf{B} = \begin{pmatrix} -1 & 1 \\ 2 & 2 \end{pmatrix}$ $\mathbf{C} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

If $\mathbf{AB} + \mathbf{C} = \begin{pmatrix} p & q \\ r & s \end{pmatrix}$ determine the value of p, q, r and s.

b $\mathbf{R}_x$ and $\mathbf{R}_N$ are both 2 × 2 matrices where R_x denotes a reflection in the x-axis and R_N denotes a rotation of 90° clockwise about O.
- i State $\mathbf{R}_x$
- ii State $\mathbf{R}_N$
- iii Determine the image of the point A(2, 3) under R_x followed by R_N
- iv State the single matrix, $\mathbf{R}_C$ that represents the transformations R_x followed by R_N
- v Verify that the image of A(2, 3) under R_C gives the same result obtained in **iii**.
- vi Verify that P(a, a) is invariant under R_C.

MODEL EXAM 2

Paper 2

Section 1

Answer all questions in this section.
All working must be clearly shown.

1 **a** Calculate the exact value of

$$\frac{2\frac{1}{2} - 1\frac{1}{3}}{2\frac{1}{2} + 1\frac{1}{3}}$$

b Simplify $2.5 \times 10^{-9} - 1.5 \times 10^{-4}$, expressing your answer in scientific notation.

c A banker wishes to deposit \$1000 into a bank account for a period of two years. Bank A offers simple interest at the rate of $9\frac{1}{2}$ per cent per annum. Bank B offers compound interest at the rate of 9 per cent per annum. Show, with full working, which bank offers the greater interest at the end of the two years.

2 **a** Solve for x in $3 + 2x \leqslant 5x - 9$

b Factorise completely

i $\frac{a^2}{9} - \frac{b^2}{9}$

ii $am - na + 2bm - 2bn$

iii $2x^2 - 3x - 2$

c $\frac{1}{u} + \frac{1}{v} = \frac{1}{f}$

i Express f in terms of u and v.

ii Find f when $u = 3$ and $v = -4$.

3 Study the number pattern

$3^2 - 1^2 = 9 - 1 = 8$
$5^2 - 3^2 = 25 - 9 = 16$
$7^2 - 5^2 = 49 - 25 = 24$

a Use this pattern to complete the following lines

i $9^2 - 7^2 =$

ii $13^2 - 11^2 =$

b One line in the pattern ends = 64. What comes in front of this?

c Write down the tenth line in the pattern.

d Complete the line in the pattern $(2n + 1)^2 - (2n - 1)^2 =$
Hence show that the right-hand side of every line is a multiple of 4.

4 **a** The sum of two numbers is 14 and their difference is 20.
Denoting the larger number by x and the smaller by y,

i write two equations in x and y to represent the information given

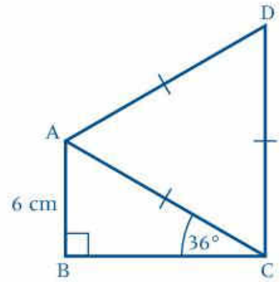
ii solve the equations to find x and y.

b In the diagram, AB = 6 cm, $\angle ABC = 90^\circ$, $\angle ACB = 36^\circ$ and the triangle ADC is equilateral, find

i BC

ii AC

iii the area of triangle ACD.



c Given $x * y = 2x - y$, calculate

i $4 * -1$

ii $4 * -1 * 2$

5 **a** Given that y varies as $\frac{x^2}{z}$. If $x = -2$ when $y = 8$ and $z = 1$, find

i y when $x = 3$ and $z = 9$

ii x when $y = 9$ and $z = 18$.

b Given that $f(x) = x + 1$ and $g(x) = \frac{2}{x}$, calculate

i $gf(x)$ and state the value of x for which $gf(x)$ is not valid

ii $g^{-1}f^{-1}(x)$ and hence or otherwise state $(fg)^{-1}$.

6 The table shown below gives the times that a bus is late over a period of 100 days.

Time in mins (to the nearest min.)	0–3	4–7	8–11	12–15	16–19
No. of times	6	18	34	40	2

a Find the mean of the distribution.

b Draw a cumulative frequency curve for the data and use it to estimate

i the number of days that the bus is less than 14 minutes late

ii the probability that on any given day, the bus is more than 14 minutes late.

7

1	2	3	4	5	6	7	8	9
10	11	12	13	14	15	16	17	18
19	20	21	22	23	24	25	26	27
28	29	30	31	32	33	34	35	36
37	38	39	40	41	42	43	44	45
46	47	48	49	50	51	52	53	54
55	56	57	58	59	60	61	62	63

Look at this pattern of numbers

L_{31} refers to the L which has 31 at the heel.

a Draw L_n and fill in the five squares with expressions in terms of n. Hence find the sum of these numbers in terms of n.

b Find the number in the heel of the L if the sum of the five numbers is 216.

Section 2

Answer all questions in this section.

8 A company wishes to employ both full-time workers and part-time workers. There are to be no less than 20 part-time workers and at least three times as many full-time workers as there are part-time workers. The company does not wish to employ more than 160 workers in all.

a Representing the number of full-time workers by f and the number of part-time workers by p, write down three inequalities in f and p, other than $f \geqslant 0$ and $p \geqslant 0$, that satisfy the above information.

b On the same axes draw the graphs of the inequalities and identify the region, R, that satisfies all three.

c The company realises a profit of \$60 per day from a part-time worker and \$100 per day from a full-time worker. Express in terms of f and p, the total profit, P, expected per day.

d Using the graph, identify the value of p and of f that would yield the maximum profit. Find the maximum profit.

9 In a game of cricket, a fieldsman tries to dismiss a batsman by catching a ball before it hits the ground. In one such game, a batsman, B, strikes a ball which is about to hit the ground 56 m away from B and on a bearing of 120° from B.

A fieldsman, S, is 28 m south of B and another fieldsman, T, is 49 m due south of the point where the ball is about to hit the ground.

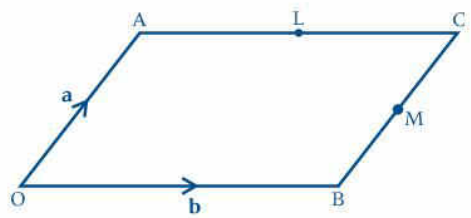
a Draw a clearly labelled diagram to illustrate the information given.

b Show that S has to run in the direction of east, in an effort to catch the ball just before it hits the ground.

c Assuming that S catches the ball just before it hits the ground, find the distance that has to be covered.

d If S and T run towards the ball at 8 m/s and 7 m/s respectively and the ball takes 6 seconds to hit the ground, show with full working, that neither S nor T will catch the ball before it hits the ground.

10 The diagram shows a parallelogram, OACB, with $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. L and M are the mid-points of AC and BC respectively.



If OL and OM are produced to P and Q respectively, so that OP = 2OL and OQ = 2OM, find, in terms of **a** and **b**,

a $\overrightarrow{OL}$

b $\overrightarrow{OM}$.

c Show that $\overrightarrow{AB} = \overrightarrow{PQ}$.

d Prove that LM is parallel to PQ.

e Find the ratio LM : PQ.

MODEL EXAM 3

Paper 2

Section 1

Answer all questions in this section.
All working must be clearly shown.

1 **a** Find the exact value of $\dfrac{1\frac{1}{2}\left(2\frac{1}{3} - 1\frac{1}{4}\right)}{3\frac{1}{4}}$

b Evaluate

$$12\tfrac{1}{2}\% \text{ of } 9.6 + \tfrac{2}{5} \text{ of } 0.225$$

expressing your answer

i exactly

ii to 2 significant figures

iii to 2 decimal places.

c Express $10^{-6} \div 2 \times 10^{-9}$ in scientific notation.

d A plot of land is expected to appreciate by P% each year. The land was bought for \$50 000 and is valued at \$52 500 after one year. Find

i the value of P

ii the expected value of the land after two years.

2 **a** Express $\dfrac{2y}{3} - \dfrac{3}{2y}$ as a single fraction.

b Given that $T = 2\pi\sqrt{\dfrac{l}{g}}$, find l, given that $T = 12\frac{4}{7}$ when $g = 10$ and $\pi = \frac{22}{7}$.

c Factorise completely,

i $x^4 - y^4$

ii $3(a^2 - b^2) + a + b$

iii $4x - y + 2xy - 2$

3 **a** The line $2y = 3x - 6$ cuts the x-axis at A and the y-axis at B. Find the area of triangle AOB.

b The two lines $3x + 4y = a$ and $2x - y = b$ intersect at $(11, -3)$. Find the value of a and of b.

c Construct triangle ABC with AB = 8 cm and AC = BC = 9 cm. Construct the perpendicular bisectors of AB and AC. The mid-points of AB and AC are M and N respectively. Join M to N. Measure and state the length of MN.

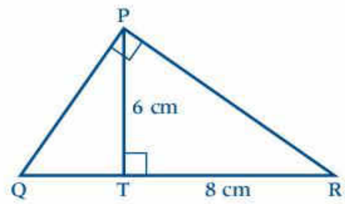
4 **a** A, B and C are three finite sets such that

i $n(A \cap B) > 0$

ii $A \cap C = \varnothing$

iii $C \subset B$.

Illustrate the data on a clearly labelled Venn diagram.



b T is the foot of the perpendicular from P to the side QR of the triangle PQR. Angle P is 90°, PT = 6 cm and TR = 8 cm.

  i Calculate PR.
  Show that triangle PQT is similar to triangle RPT.
  ii Hence, or otherwise, calculate QT and PQ.

5 The functions f and g are defined by

$$f(x) = ax + 2 \quad \text{and} \quad g(x) = bx + 3.$$

a Find
  i $fg(x)$
  ii $gf(x)$

  If $fg(x) = gf(x)$, find the relation between a and b.

b i If $a = 5$, find x for which $f^2(x) = f(x)$.
  ii Using the same value of a, find x for which $g^2(x) = g(x)$.

6

Marks	1–10	11–20	21–30	31–40	41–50	51–60
No. of candidates	15	30	40	60	35	20

The table shows the marks, out of 60, obtained by 200 candidates sitting a college entrance examination.

a i Construct a cumulative frequency table for the above data.
  ii Using a scale of 1 cm ≡ 10 candidates on the vertical axis and 1 cm ≡ 5 marks on the horizontal axis, draw the cumulative frequency graph for the information.

b From the graph, estimate
  i the pass mark if the topmost 40 candidates are to be chosen
  ii The probability that a candidate chosen at random scores less than 50%.

7 To get a Fibonacci sequence start with any two numbers, for example 2 and 4. The next number is found by adding the two previous numbers, so the next three numbers in this sequence are 6, 10 and 16.

a Write down the 6th and 7th numbers in this sequence.

b If the first two terms of a Fibonacci sequence are a and b write down the first six terms as expressions in a and b.

c i The 2nd and 5th terms of a Fibonacci sequence are respectively 7 and 27. Find the first six terms.
  ii Find the sum of the first six terms of this sequence and show that it is a multiple of 4.

d Use your answer to part **b** to show that the sum of the first six terms of any Fibonacci sequence is a multiple of 4.

Section 2

Answer all questions in this section.

8 **a** In the above diagram, P, R and S are the points (0, −2), (2, 1) and (0, 1) respectively. PQ is parallel to the x-axis and QR meets the y-axis at T (0, 4), find

i the coordinates of Q

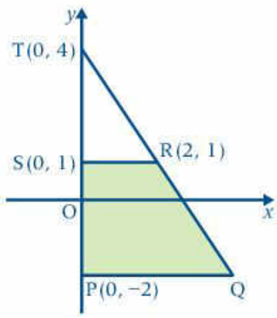
ii the area of PQRS

iii the ratio of the area of triangle TSR : the area of triangle TPQ.

b Write down four inequalities which define the shaded region PQRS.

c Solve the equations

$x - 2y = 1$
$x^2 - xy = 15$

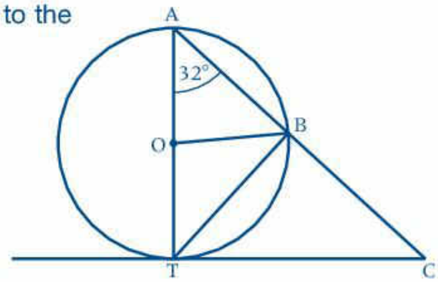


9 **a** In the figure, O is the centre of the circle ATB. TC is tangent to the circle at T and ABC is a straight line. If ∠OAB = 32°, find, giving reasons for your deductions, the size of

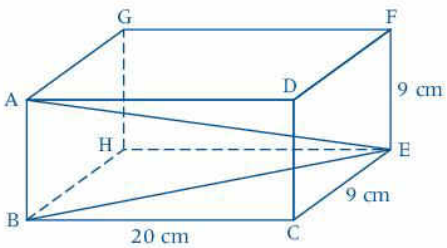
i angle BTC

ii angle OBT

iii angle ACT.



b



The figure shows a cuboid BC = 20 cm, CE = EF = 9 cm.
Calculate

i the length of the diagonal AE

ii the angle between AE and BE.

10 **a** **i** **M** is the matrix $\begin{pmatrix} 2 & 3 \\ p & -q \end{pmatrix}$. If **M** is singular, express p in terms of q.

ii If $\begin{pmatrix} 3 & 2 \\ 5 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 8 \\ 9 \end{pmatrix}$, find x and y.

b **i** O, A and B have coordinates (0, 0), (1, 2) and (2, 2) respectively. Triangle OAB is mapped onto OA′B′ under the transformation T. If A′ and B′ have coordinates (3, 2) and (4, 2), find the matrix **T**.

ii Triangle OA′B′ is mapped onto OA″B″ under the transformation S. If A″ and B″ have coordinates (3, −2) and (4, −2) respectively, find the matrix **S**.

c Determine the single matrix **R** that maps triangle OAB onto triangle OA″B″.

SAMPLE PAPER WITH SOLUTIONS

The solutions give an indication of how you should set out your work in the examination and what you should include by way of explanation. There are notes given to explain and assist your appreciation of what is required.

Remember that mathematics is a language and that you are communicating with the examiner. Your work should read logically in the same way as an essay on any other subject such as history.

Section I

Answer ALL the questions in this section.
All working must be clearly shown.

1 a Using a calculator or otherwise, evaluate

i 6.24 (5 − 1.18), expressing your answer in exact form.

ii $\dfrac{2.75}{(1.2)^2 - 1.24}$, expressing your answer correct to 3 significant figures.

b A number of marbles is shared between Bret and Charlie in the ratio 4:7. Charlie got 35 marbles. How many marbles were shared altogether?

c The cost of 5 litres of diesel is $10.25.

i Calculate the cost of 17 litres of diesel.

ii How many litres of diesel can be bought for $36.00, giving your answer to the nearest litre?

Solution

a i $(5 - 1.18) = 3.82$

and $6.24 \times 3.82 = 23.8368$ (exactly)

Alternative method

$6.24(5 - 1.18)$
$= (6.24 \times 5) - (6.24 \times 1.18)$
$= 31.2 - 7.3632$
$= 23.8368$ (exactly)

This can be done in one step on a calculator: press 6 . 2 4 (5 − 1 . 1 8) = and just write down the answer. If this does not work on your calculator, consult your manual.

This method is longer than the first and more prone to mistakes.

ii $\dfrac{2.75}{(1.2)^2 - 1.24} = \dfrac{2.75}{1.44 - 1.24}$
$= \dfrac{2.75}{0.20}$
$= 13.7\underline{5} = 13.8$ (to 3 s.f.)

This can also be done in one step on a calculator. Press: 2 . 7 5 ÷ (1 . 2 x^2 − 1 . 2 4) =

b Charlie's 7 shares = 35 marbles

$\therefore$ 1 share $= \frac{35}{7} = 5$ marbles

Total number of shares $= 4 + 7 = 11$

$\therefore$ Number of marbles to be shared altogether
$= 11 \times 5$ marbles
$= 55$ marbles

c i 5 litres of diesel cost \$10.25

$\therefore$ the cost of 1 litre of diesel $= \dfrac{\$10.25}{5}$

$= \$2.05$

Hence, cost of 17 litres of diesel $= \$2.05 \times 17$

$= \$34.85$

ii No. of litres that may be bought for \$36.00 $= \dfrac{\$36.00}{\$2.05}$

$= 17.\underline{6}$

17.6 litres to the nearest litre = 18 litres.

17.6 litres is 18 litres to the nearest litre but \$36 will not buy 18 litres.

Question

2 a If $x = 3$, $y = -2$ and $z = 4$, evaluate

i $xy - xz$

ii $y^x - y$

b Solve for x, where $x \in \mathbb{Z}$

i $\dfrac{x}{3} - \dfrac{x}{4} = 2$

ii $6 - x \leqslant 2$

c Three pens and two pencils cost \$24. If a pen costs as twice as much as a pencil which costs \$$P$, express, in terms of P, the cost of

i one pen.

ii three pens and two pencils.

Find the actual cost of one pen.

Solution

a i $xy - xz = (3) \times (-2) - (3) \times (4)$

$= -6 - 12$

$= -18$

Brackets round the numbers are not necessary but they help to avoid mistakes.

ii $y^x - y = (-2)^3 - (-2)$

$= -8 + 2$

$= -6$

b i $\dfrac{x}{3} - \dfrac{x}{4} = 2$

$\dfrac{x}{3} - \dfrac{x}{4} = \dfrac{2}{1}$

L.C.M. of 3, 4 and 1 is 12

$(\times 12) \quad 4x - 3x = 24$

$x = 24$

Alternative method

Taking L.H.S.

$\dfrac{x}{3} - \dfrac{x}{4} = \dfrac{4x - 3x}{12} = \dfrac{x}{12}$

L.H.S. = R.H.S.

$\therefore \quad \dfrac{x}{12} = \dfrac{2}{1}$

$x \times 1 = 2 \times 12$

$x = 24$

This method needs to be laid out carefully to avoid writing nonsense.

ii $6 - x \leqslant 2$

$-x \leqslant 2 - 6$

$$-x \leqslant -4$$

$\times(-1)$ $\quad x \geqslant 4, x \in \mathbb{Z}$

OR $\quad x = \{4, 5, 6...\}$

Alternative method

$6 - x \leqslant 2$

$6 \leqslant 2 + x$

$6 - 2 \leqslant x$

$4 \leqslant x$ so $x \geqslant 4, \quad x \in \mathbb{Z}$

c i Cost of 1 pencil = $\$P$

$\therefore$ Cost of 1 pen = $\$P \times 2 = \$2P$

ii Cost of 3 pens and 2 pencils = $\$(3 \times 2P) + \$(2 \times P)$

$= \$(6P + 2P)$

$= \$8P$

Total cost of 3 pens and 2 pencils = $24

$\therefore \$8P = \$24$

$\therefore P = 3$

$\therefore$ Cost of 1 pen = $\$2P$

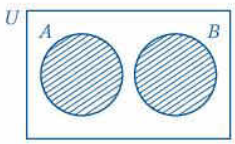
$= \$3 \times 2$

$= \$6$

Question

3 a Describe, using only set notation, the shaded regions in each of the Venn diagrams, shown below.

i



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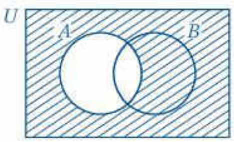

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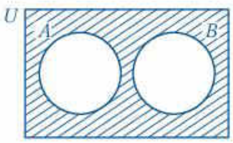


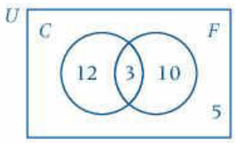
b $U = \{1, 2, 3, ..., 10\}$

B = {prime numbers}

C = {even numbers}

Draw a Venn diagram to represent the above information and list the elements of $(B \cup C)'$

c



The Venn diagram shows the number of members of a club who play cricket (C) and football (F).

Find the probability that a member, chosen at random, plays

i cricket only

ii both cricket and football

iii neither of the mentioned sports

iv only one of the two mentioned sports.

Solution

3 a i $A \cup B$

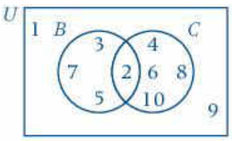
ii A'

iii $(A \cup B)'$

b $B = \{2, 3, 5, 7\}$

$C = \{2, 4, 6, 8, 10\}$

From the diagram $(B \cup C)' = \{1, 9\}$



c Number of members of the club $= 12 + 3 + 10 + 5$ (from the diagram)

$= 30$

i P(member plays cricket only)

$= \dfrac{\text{number of members who play cricket only}}{\text{total number of members}}$

$= \dfrac{12}{30} = \dfrac{2}{5}$

ii P(member plays both cricket and football)

$= \dfrac{\text{number of members who play both sports}}{\text{total number of members}}$

$= \dfrac{3}{30} = \dfrac{1}{10}$

iii P(member does not play either of the two sports)

$= \dfrac{\text{number of members who do not play either sport}}{\text{total number of members}}$

$= \dfrac{5}{30} = \dfrac{1}{6}$

iv P(member plays only one of these two sports)

$= \dfrac{\text{number of members playing only one sport}}{\text{total number of members}}$

$= \dfrac{12 + 10}{30} = \dfrac{22}{30} = \dfrac{11}{15}$

Question

4 a Use a ruler and pair of compasses only for this question.

i Construct $\triangle ABC$ with $\hat{A} = 90°$, $AC = 6$ cm and $AB = 8$ cm.

ii Construct the perpendicular bisector of BC, meeting BC at M.

iii With centre M, draw the circumscribed circle of $\triangle ABC$.

iv Measure and state the radius of the circle.

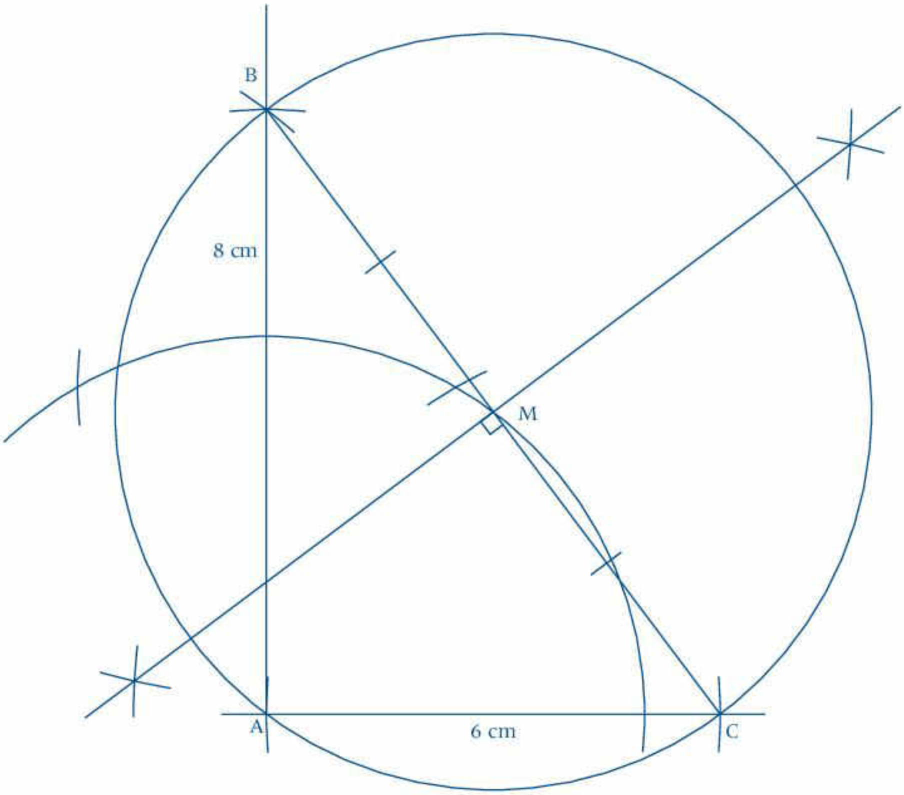
b Given $A = (3, -1)$ and $B = (6, 5)$, calculate

i the gradient of a line perpendicular to AB

ii the length of AB, in exact form.

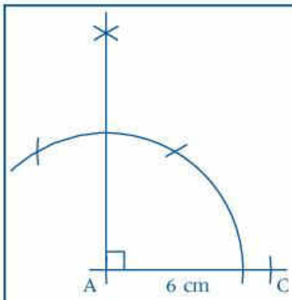
Solution

4 a

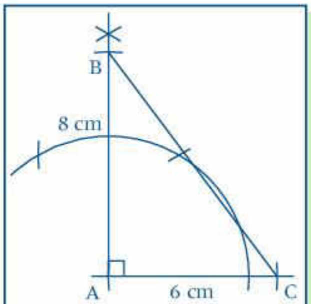


Length of radius = length of MC = 5 cm (by measurement)

Start with a line longer than AC then cut off line segment AC = 6 cm



$\angle A = 90°$ constructed at A



Cut off AB = 8 cm. Join B to C to complete $\triangle$ABC.

Construct perpendicular bisector of BC, meeting BC at M. Centre M and radius MA or MB or MC, the circumscribed circle is drawn.

b i Gradient of AB $= \dfrac{5 - (-1)}{6 - 3}$

$= \dfrac{6}{3}$

$= 2$

Hence, gradient of any line perpendicular to AB $= -\dfrac{1}{2}$

(Since product of the gradients of perpendicular lines = −1)

ii Length of AB $= \sqrt{(6 - 3)^2 + (5 - (-1))^2}$

$= \sqrt{(3)^2 + (6)^2}$

$= \sqrt{9 + 36}$

$= \sqrt{45}$ or $3\sqrt{5}$ or units

In exact form means you must leave the square root in the answer.

Question

5 **a** Functions f and g are defined as follows:

$f : x \to 3 - 2x, x \in \mathbb{R}$

$g : x \to \frac{2}{x}, x \neq 0$

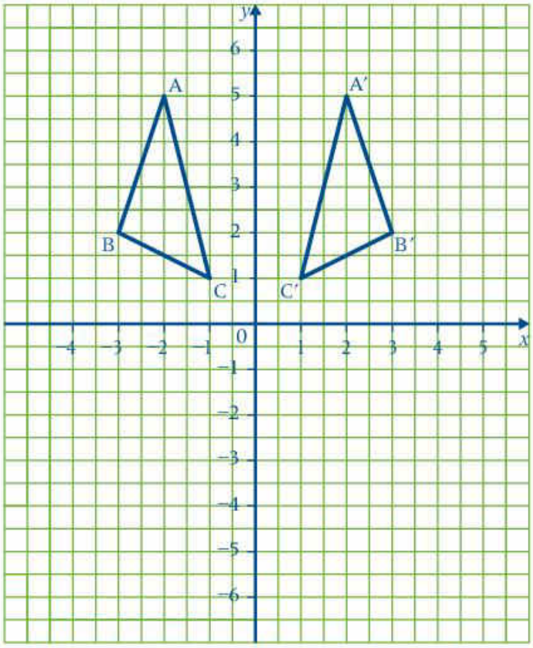
Find expressions for

i fg

ii gf

State the value of x for which gf does not exist.

b



i △ABC is mapped onto △A′B′C′ by a reflection in the line, l. Determine the equation of l.

ii △A′B′C′ is mapped onto △A″B″C″ under a reflection in the x axis. Determine the coordinates of A″, B″ and C″.

iii State the transformation that maps △ABC onto △A″B″C″.

Solution

a **i** $fg : x \to 3 - 2\left(\frac{2}{x}\right) = 3 - \frac{4}{x}$

ii $gf : x \to \frac{2}{3 - 2x}$

gf does not exist when the denominator, $3 - 2x$, is 0,

i.e. when $x = \frac{3}{2}$

b **i** l is the y axis with equation $x = 0$

ii Under a reflection in the x-axis we determine that

A″ is (2, −5). B″ is (3, −2). C″ is (1, −1).

iii △ABC → △A″B″C″ by a rotation of 180° about the origin.

Alternative method

b ii $\triangle A'B'C' \rightarrow \triangle A''B''C''$ by $\mathbf{M} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 3 & 2 \\ 1 & 2 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 2 \\ -1 & -2 & -5 \end{pmatrix}$

Hence $A'' = (1, -1)$, $B'' = (3, -2)$, $C'' = (2, -5)$

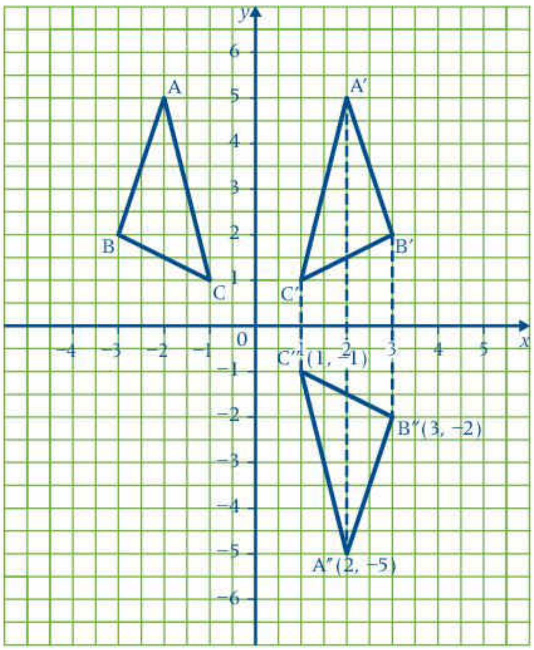
This method is longer than the first method given and is much more prone to mistakes.

iii $\triangle ABC \rightarrow \triangle A'B'C'$ by $\mathbf{N} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

so $\triangle ABC \rightarrow \triangle A''B''C''$ by $\mathbf{MN} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$

$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ represents a rotation of 180° about O.

$\therefore \triangle ABC \rightarrow \triangle A''B''C''$ by a rotation of 180° about the origin.



Question

6 The table shows the frequency distribution of 100 batsmen in a "Twenty20" competition.

Score in runs	No. of batsmen (frequency)	Cumulative frequency
1–10	3	3
11–20	7	7 + 3 = 10
21–30	22	
31–40	48	
41–50	15	
51–60	5	

i Complete the cumulative frequency table for the distribution.

ii Using a scale of 2 cm to a score of 10 runs on the horizontal axis and 2 cm to represent 10 batsmen on the vertical axis, draw the cumulative frequency curve for the distribution.

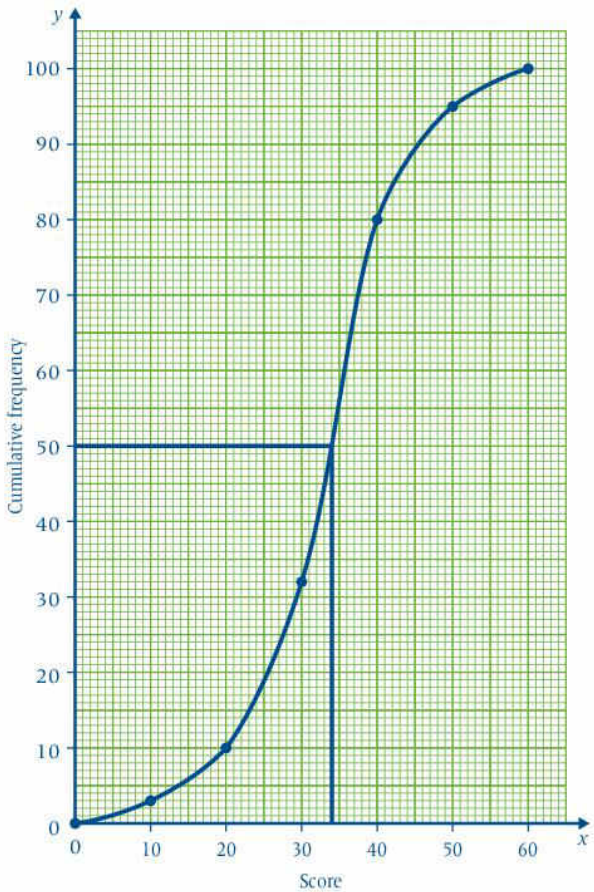
iii From the curve, determine the median score of the distribution.

iv Find the probability that a batsman chosen at random scores more than 30 runs.

Solution

i

Score in runs	No. of batsmen (frequency)	Cumulative frequency
1–10	3	3
11–20	7	10
21–30	22	22 + 10 = 32
31–40	48	48 + 32 = 80
41–50	15	80 + 15 = 95
51–60	5	95 + 5 = 100



Points to be plotted for cumulative frequency curve are (Upper Class Boundary, Cumulative Frequency) are

(0, 0) ← The curve must start from horizontal axis and from the Lower Class boundary of the first group.
(10, 3)
(20, 10)
(30, 32)
(40, 80)
(50, 95)
(60, 100)

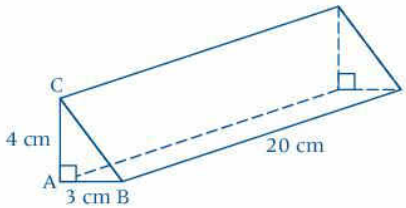
iii From the curve the median score is approximately 34.

iv P (Batsman's score is > 30) $= \dfrac{\text{number of scores} > 30}{\text{total number of batsmen}}$

$$= \frac{48 + 15 + 5}{\Sigma f = 100} = \frac{68}{100} = \frac{17}{25}$$

Question

7 a



The diagram shows a prism of length 20 cm. The cross-section is in the form of a right-angled triangle, ABC, with shorter sides of 3 cm and 4 cm.

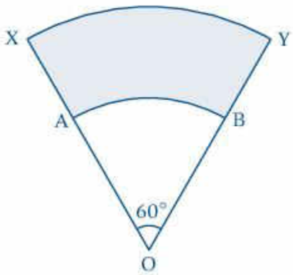
Calculate

i the length of BC

ii the total surface area of the prism

iii the volume of the prism.

b



The diagram shows XY and AB, the arcs of two circles, centre, O. Given that OB = 5 cm, OY = 8 cm, and $\angle AOB = \angle XOY = 60°$, calculate, giving your answers correct to two decimal places

i the length of arc AB

ii the length of arc XY

iii the perimeter of the shaded region XABY

iv the area of the shaded region XABY.

Solution

a **i** $BC^2 = (3)^2 + (4)^2 = 25$ (Pythagoras' Theorem)

$\therefore BC = 5\text{ cm}$

ii Area of front face ABC $= \dfrac{4 \times 3}{2} = 6\text{ cm}^2$

Area of back face opposite ABC $= 6\text{ cm}^2$

Area of vertical rectangular face $= 20 \times 4 = 80\text{ cm}^2$

Area of base rectangular face $= 20 \times 3 = 60\text{ cm}^2$

Area of slanting rectangular face $= 20 \times 5 = 100\text{ cm}^2$

Total area of prism $= (6 + 6 + 80 + 60 + 100)\text{ cm}^2$

$= 252\text{ cm}^2$

iii Volume of prism = (Area of cross-section) $\times$ length

$= 6 \times 20\text{ cm}$

$= 120\text{ cm}^3$

b **i** Length of arc AB $= \dfrac{60°}{360°} \times 2\pi(5)$

$= 5.236\text{ cm}$

$= 5.24\text{ cm}$ (to 2 d.p.)

ii Length of arc XY $= \frac{60^\circ}{360^\circ} \times 2\pi(8)$

$= 8.37\underline{8}\,\text{cm}$

$= 8.38\,\text{cm}$ (to 2 d.p.)

iii Perimeter of shaded region XABY

$=$ length of XA $+$ length of arc AB $+$ length of YB $+$ length of arc XY

$= 3 + 5.236 + 3 + 8.378$

$= 19.61\underline{4}\,\text{cm}$

$= 19.61\,\text{cm}$ (to 2 d.p.)

iv Area of shaded region XABY

$=$ Area of sector XOY $-$ Area of sector AOB

$= \pi(8)^2 \times \frac{60^\circ}{360^\circ} - \pi(5)^2 \times \frac{60^\circ}{360^\circ}$

$= 33.510 - 13.090$

$= 20.42\underline{0}\,\text{cm}^2$

$= 20.42\,\text{cm}^2$ (to 2 d.p.)

Section II

Answer all questions in this section

8 **a** Factorise completely

i $2y^2 + y - 6$

ii $2a + 2b + a^2 - b^2$

b Expand, expressing the result in ascending powers of a.

$(a - 3)^2(2 + 3a)$

c Given that $f(x) = x^2 + 2x - 15$

i Express $f(x)$ in the form $(x + a)^2 + b$ where a and $b \in \mathbb{Z}$.

ii Hence state the coordinates of the minimum point of the graph $y = f(x)$.

iii Sketch the graph of $y = f(x)$ showing the axis of symmetry.

Solution

a **i** $2y^2 + y - 6 = (2y - 3)(y + 2)$

ii $2a + 2b + a^2 - b^2 = 2(a + b) + (a - b)(a + b)$

($a^2 - b^2$ is the difference between two squares.)

$= (a + b)(2 + (a - b))$

$= (a + b)(2 + a - b)$

b $(a - 3)^2 = (a - 3)(a - 3) = a^2 - 6a + 9$

$(a - 3)^2(2 + 3a) = (a^2 - 6a + 9)(2 + 3a)$

$= 2a^2 - 12a + 18 + 3a^3 - 18a^2 + 27a$

$= 3a^3 - 16a^2 + 15a + 18$

$= 18 + 15a - 16a^2 + 3a^3$ written in ascending powers of a

c **i** $f(x) = x^2 + 2x - 15$

$= x^2 + 2x + 1 - 1 - 15$

$= (x + 1)^2 - 16$

which is of the form $(x + a)^2 + b$ where $a = 1$ and $b = -16$

($a, b \in \mathbb{Z}$)

Alternative method

$x^2 + 2x - 15 = (x + a)^2 + b$

$= x^2 + 2ax + a^2 + b$

$x^2 + 2x - 15 = x^2 + 2ax + (a^2 + b)$

Equating coefficients:

$2x = 2ax$

$2 = 2a$ and $a = 1$

$a^2 + b = -15$

$(1)^2 + b = -15$

$1 + b = -15$ and $b = -16$

Hence $x^2 + 2x - 15 = (x + 1)^2 - 16$ which is of the form $(x + a)^2 + b$ where $a = 1$ and $b = -16$ $(a, b \in \mathbb{Z})$

ii $x^2 + 2x - 15 = (x + 1)^2 - 16$

$(x + 1)^2 \geqslant 0$ for all values of x.

$\therefore$ minimum value of $f(x) = 0 - 16 = -16$

The minimum value occurs at

$(x + 1)^2 = 0$

$(x + 1) = 0$ i.e. $x = -1$

$\therefore$ minimum point on curve $y = f(x)$ is $(-1, -16)$

iii $y = f(x) = x^2 + 2x - 15$

$= (x + 5)(x - 3)$

when $f(x) = 0$, $x = -5$ or 3

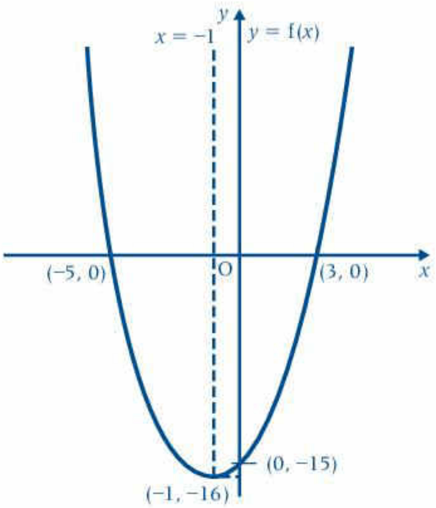
$\therefore$ the curve cuts the x-axis at $(3, 0)$ and $(-5, 0)$.

$f(0) = -15$

$\therefore f(x)$ cuts the vertical axis at $(0, -15)$.

Minimum point is $(-1, -16)$.

The line of symmetry goes through the minimum point so the equation of the line of symmetry is $x = -1$



Minimum point

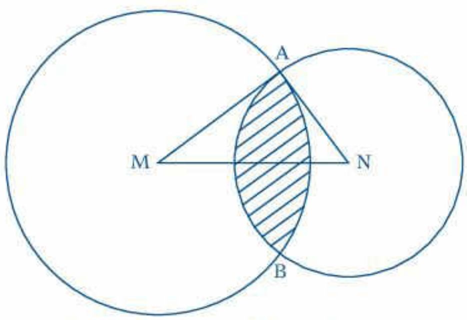
Alternative method

For $y = ax^2 + bx + c$, the axis of symmetry is $x = \frac{-b}{2a}$.

$\therefore$ The axis of symmetry for $y = f(x) = x^2 + 2x - 15$ is

$x = \frac{-(2)}{2(1)} = -1$

9 a



Two circles of radius 8 cm and 6 cm, and centres M and N respectively, intersect at A and B, as shown in the above diagram, which is not drawn to scale.

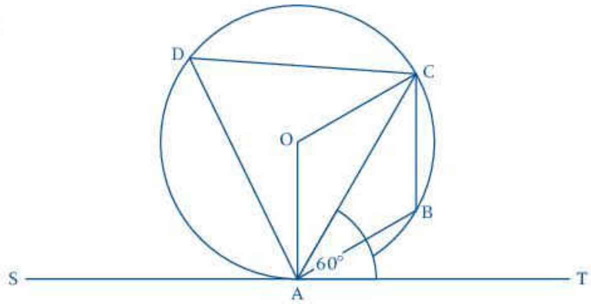
i If MN is 10 cm, prove that AN is the tangent to the circle, centre M, at the point A.

ii Calculate the length of the perpendicular from A to MN.

iii Calculate the area of **a** $\triangle$MAB **b** the minor sector MAB

iv Hence, or otherwise, find the area of the shaded region, bounded by the two circles.

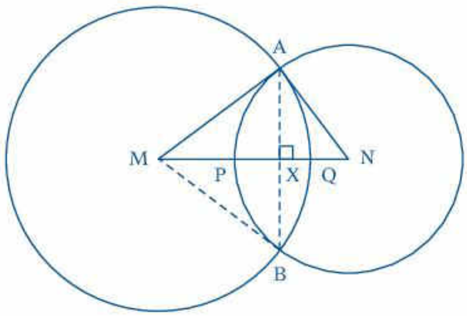
b



The straight line SAT is the tangent to the circle ABCD with centre O, at the point A. If $\angle CAT = 60°$ calculate, giving reasons for your answer, the size of the following angles

i $\angle CDA$
ii $\angle OAS$
iii $\angle COA$
iv $\angle OAC$
v $\angle ABC$

Solution



a i Consider $\triangle MAN$:
MA = 8 cm (radius) and AN = 6 cm (radius). MN = 10 cm (data)
Since $10^2 = 8^2 + 6^2$, $\angle MAN = 90°$ (Converse of Pythagoras' Theorem)
Hence AN is the tangent to the circle, centre M, at the point A. (The angle made by a tangent to a circle and a radius, at the point of contact, is 90°.)

ii Let the perpendicular from A meet MN at X and let AX = h.

Area of $\triangle MAN = \frac{1}{2} \times MA \times AN = \frac{6 \times 8}{2} = 24\,\text{cm}^2$

$\therefore \frac{1}{2} \times MN \times AX = 24\,\text{cm}^2$

Hence $\frac{10 \times h}{2} = 24\,\text{cm}^2$ so $h = \frac{24}{5} = 4.8\,\text{cm}$

$\therefore$ Length of the perpendicular from A to MN = 4.8 cm.

iii a $\sin A\hat{M}X = \frac{4.8}{8}$ so $\angle AMX = 36.87°$
By symmetry $\angle AMB = 36.87° \times 2 = 73.74°$

Area of $\triangle MAB = \frac{1}{2}(8)(8)\sin 73.74° = 30.72\,\text{cm}^2$

b Area of minor sector MAQB $= \pi(8)^2 \times \frac{73.74°}{360°} = 41.18\,\text{cm}^2$

iv P and Q are as shown in the diagram.
Area of segment AQBX = 41.18 − 30.72 = 10.46 cm²
$\angle ANB = 360° - (90° + 90° + 73.74°) = 106.26°$
(sum of angles in a quadrilateral = 360°)

Similarly, the area of the segment APB

$= \pi(6)^2 \times \frac{106.26°}{360°} - \frac{1}{2}(6)^2 \sin 106.26°$

$= 33.38... - 17.28\ ... = 16.10\,\text{cm}^2$

$\therefore$ Area shaded region = Area segment AQB + Area segment APB

$= 26.56\,\text{cm}^2$

This calculation can be done in one step on a calculator.

b **i** $\angle CDA = 60°$ (alternate segment theorem).

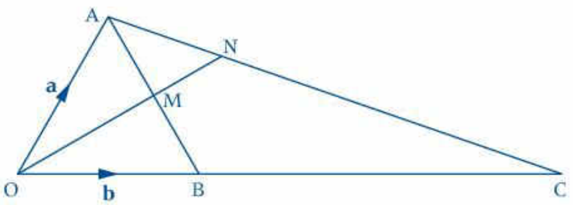
ii $\angle OAS = 90°$ (the angle made by a tangent to a circle and a radius at the point of contact is equal to 90°).

iii $\angle COA = 2 \times 60° = 120°$ (angle subtended by a chord at the centre of a circle is twice the angle it subtends at the circumference, standing on the same arc).

iv $\angle OAC = \frac{180° - 120°}{2} = 30°$ (base angles of an isosceles triangle are equal) and (sum of angles in a triangle = 180°)

v $\angle ABC = 180° - 60° = 120°$ (opposite angles in cyclic quadrilateral are supplementary)

10

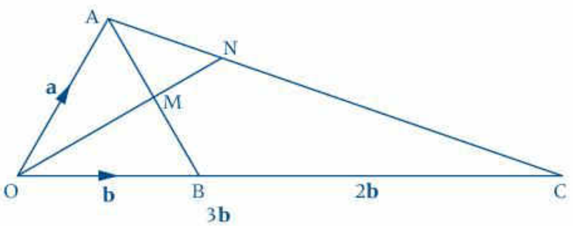


i In the above diagram, $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$. OB is extended to a point such that $\overrightarrow{OC} = 3\overrightarrow{OB}$. M is the midpoint of AB. Express $\overrightarrow{AM}$ in terms of **a** and **b**.

ii OM is produced to N where $MN = \frac{1}{2}OM$. Find $\overrightarrow{OM}$ and $\overrightarrow{ON}$ in terms of **a** and **b**.

iii Find $\overrightarrow{AC}$ in terms of **a** and **b** and prove that N lies on AC.

Solution



i $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$

$= -\mathbf{a} + \mathbf{b}$

$\overrightarrow{AM} = \frac{1}{2}(-\mathbf{a} + \mathbf{b})$

$= -\frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$

ii $\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM}$

$= \mathbf{a} - \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$

$= \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}$

$\overrightarrow{ON} = \overrightarrow{OM} + \overrightarrow{MN}$

$= \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b} + \frac{1}{2}\left(\frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}\right)$

$= \frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b} + \frac{1}{4}\mathbf{a} + \frac{1}{4}\mathbf{b} = \frac{3}{4}\mathbf{a} + \frac{3}{4}\mathbf{b}$

Alternative method

This method is more direct than the first one.

$\overrightarrow{ON} = \frac{3}{2}\overrightarrow{OM}$

$= \frac{3}{2}\left(\frac{1}{2}\mathbf{a} + \frac{1}{2}\mathbf{b}\right) = \frac{3}{4}\mathbf{a} + \frac{3}{4}\mathbf{b}$

iii $\overrightarrow{AC} = \overrightarrow{AO} + \overrightarrow{OC}$

$\therefore \overrightarrow{AC} = -\mathbf{a} + 3\mathbf{b}$ $\quad(\overrightarrow{BC} = 2\mathbf{b}$ and $\overrightarrow{OC} = 3\mathbf{b})$

$\overrightarrow{AN} = \overrightarrow{AO} + \overrightarrow{ON}$

$= -\mathbf{a} + \frac{3}{4}\mathbf{a} + \frac{3}{4}\mathbf{b}$

$= -\frac{1}{4}\mathbf{a} + \frac{3}{4}\mathbf{b}$

$= -\frac{1}{4}(\mathbf{a} - 3\mathbf{b})$

Since $\overrightarrow{AN}$ is a scalar multiple of $\overrightarrow{AC}$ then AN and AC are parallel.
A is a common point of both line segments.
$\therefore$ A, N and C are collinear.

ANSWERS

Chapter 1

Exercise 1a

1 1, 2, 3, 4, 6, 8, 12, 24 2 11, 13, 17

3 200, 225 4 53, 59 5 104, 117

6 1, 2, 3, 4, 6, 9, 12, 18, 36

7 a 2^4 b $3^2 \times 5^2$
c $3^2 \times 7^4$ d $2^4 \times 3^2 \times 13$
e $5^3 \times 7^2 \times 13$ f $2^2 \times 3^3 \times 5^3$

8 a 8 b 9 c 625 d 343
e 2592

9 a 5×7 b $2^2 \times 7$ c $2^2 \times 3^3$
d $2^3 \times 11 \times 13$ e $2^4 \times 11^2$
f $2 \times 3 \times 5 \times 7 \times 11 \times 13$

10 a 15 b 8 11 a 4 b 18

12 a 27 b 48 13 a 7 b 39

14 a 180 b 576 15 a 168 b 1352

16 a 1452 b 432 17 a 5 b 420

18 60 m 19 $10.50

20 25 teeth wheel 12 times, the other wheel 5 times

21 65 min 22 50 cm 23 120 24 210

25 1 min 26 a 120 b 3

27 72 min

Exercise 1b

1 a $1 \times 2^2 = 4$ b 1
c $2^3 = 8$ d $2^5 = 32$

2 a $3 \times 4^2 = 48$ b $2 \times 8^1 = 16$
c $2 \times 4^2 = 32$

3 a $4 \times 4^2 = 48$ b $2 \times 10^1 = 20$
c $2 \times 8^2 = 128$

4 a $3 \times 4^1 = 12$ b $2 \times 8^1 = 16$
c $7 \times 8^2 = 448$

5 a 65 b 33 c 5

6 a 110_4 b 132_4 c 302_4

7 a 9 b 25 c 56

8 3211_4 9 11011_4 10 271_8

11 93_{10} 12 2233_4 13 11100101_2

14 47_{10} 15 a 1000_2 b 101_2

16 a 64_8 b 5_8 17 a 121_4 b 31_4

18 a 111_4 b 13_4 19 a 57_8 b 20_8

20 a 103_8 b 104_8 21 a 101_2 b 10_2

22 a 10111_2 b 100_2

Exercise 1c

1 12 2 −9 3 7 4 33

5 −96 6 36 7 19 8 −1

9 1 10 15 11 40 12 9

13 −1 14 −6 15 2

Exercise 1d

1 $17\frac{13}{14}$ 2 $6\frac{7}{8}$ 3 $2\frac{71}{72}$ 4 $\frac{2}{9}$

5 $1\frac{1}{10}$ 6 $4\frac{5}{8}$ 7 $2\frac{5}{8}$ 8 $1\frac{5}{7}$

9 $\frac{10}{39}$ 10 2 11 $1\frac{5}{8}$ 12 $28\frac{1}{2}$

13 $\frac{3}{4}$ 14 $1\frac{1}{3}$ 15 $\frac{4}{9}$ 16 $\frac{7}{32}$

17 $\frac{4}{5}$ 18 $2\frac{1}{2}$ 19 $1\frac{3}{32}$ 20 2

21 $\frac{2}{3}$ 22 $1\frac{1}{6}$ 23 $2\frac{23}{24}$ 24 $1\frac{2}{5}$

25 $\frac{100}{413}$ 26 $\frac{173}{258}$ 27 $\frac{1}{4}$ 28 $3\frac{20}{63}$

29 $\frac{148}{213}$ 30 $4\frac{1}{3}$ 31 5 32 4

33 $\frac{1}{10}$ 34 $23\frac{5}{19}$ 35 $\frac{6}{77}$ 36 $\frac{20}{21}$

37 36 38 80 miles 39 500 litres

40 $6432 41 5 litres 42 $14 560

43 Anne $3752, Betty $3216, Cheryl $3752

44 336 45 $2\frac{2}{5}$ 46 $8\frac{1}{6}$ 47 56

48 $5\frac{1}{4}$ m 49 $8\frac{8}{9}$ min 50 $3\frac{1}{3}$

Exercise 1e

1 a 2^6 b 3^0 c 5^2

2 a $2^{\frac{7}{2}}$ b $3^{-\frac{3}{2}}$ c 3^6

3 a $\frac{1}{4}$ b $\frac{1}{10000}$ c $\frac{1}{2209}$

4 $\frac{81}{16}$ 5 100 6 16

7 $\frac{1}{4096}$ 8 128 9 49

10 $\frac{3}{5}$ 11 $3\frac{2}{5}$ 12 5

Exercise 1f

1 a 6 b 500 c 50 d 0.07
2 a 390 b 0.081 c 3.2 d 54
3 a 0.0678 b 1.00 c 45.7 d 765
4 a 1.55 b 0.71 c 3.14 d 65.89
5 a 0.153 b 1.26 c 3.745 d 67.2
6 a 5.4×10^4 b 2.35×10^1 c 7.2×10^{-4} d 1.005×10^{-2}
7 a 1.735 b 1.7 c 1.74
8 a 0.16 b 0.04 c 1.17 d 0.65
9 a 528 b 740 c 186
10 a 40.73 b 1.312 c 0.293
11 a 7.14 b 9.4
12 a 25.75 b 24
13 a 7.4 b 0.008
14 a 7.97 b 1.72 c 1.70
15 a 4.05 b 0.05 c 0.89
16 a 0.469 b 1.01 c 4.99
17 a 174.015 b 170 c $1.740\,15 \times 10^2$
18 a 7.5×10^4 b 1.25×10^9 c 2×10^0 d 5×10^{-1}
19 a 1.44×10^2 b 9×10^6
20 a 4.26×10^{-3} b 4.14×10^{-3} c 2.52×10^{-7} d 7×10^1
21 a i 0.09 mm ii 7.2 mm iii 45 mm
b i 9 mm ii 25.2 mm
c i 320 ii 38
d i 440 ii 160

Exercise 1g

1 Missing values are 0.75, 75%, $\frac{4}{5}$, 80%, $\frac{14}{25}$, 0.56, $\frac{3}{8}$, 0.375, 0.875, $87\frac{1}{2}$%, $\frac{5}{8}$, 62.5%
2 a $\frac{9}{20}$ b $1\frac{1}{4}$ c $\frac{7}{8}$ d $\frac{11}{16}$
3 a i 0.06 ii 6% b i 0.28 ii 28%
c i 2.375 ii $237\frac{1}{2}$%
d i 3.875 ii $387\frac{1}{2}$%
4 a i $\frac{7}{20}$ ii 35% b i $\frac{5}{8}$ ii $62\frac{1}{2}$%
c i $\frac{11}{20}$ ii 55% d i $\frac{3}{20}$ ii 15%
e i $\frac{7}{25}$ ii 28% f i 3 ii 300%
5 a 3.9 kg b $3.92 c 156 cm d 4.41 m
6 76%, 0.82, $\frac{5}{6}$, $\frac{6}{7}$
7 a 57% b 0.28 c 78% d $\frac{12}{25}$
8 $\frac{11}{16}$, $\frac{7}{11}$, 63%, 0.625
9 $\frac{14}{31}$, $\frac{6}{13}$, $\frac{9}{19}$, $\frac{18}{37}$
10 $\frac{5}{3}$, 1.55, 1.47, $\frac{15}{11}$, $1\frac{2}{7}$
11 0.064 (3 d.p.)
12 0.0771, 0.7071, 0.7107, 0.7701
13 a $\frac{12}{7}$, $\sqrt{5}$, $\frac{7}{3}$, 2.7, 3.4 b 5.2, $\frac{27}{5}$, $\frac{23}{4}$, 5.9, $\sqrt{40}$
c 4.05, 4.4, $\frac{40}{9}$, $\sqrt{20}$, 4.5 d π, 3.41, $\frac{17}{4}$, $\sqrt{24}$, 6.3
14 a 3.81, $\sqrt{8}$, $\frac{11}{4}$, $\frac{9}{4}$, $\sqrt{5}$ b $\frac{13}{3}$, $\frac{7}{2}$, $\sqrt{12}$, 2.8, 1.46
c $\frac{21}{5}$, 3.5, $\sqrt{10}$, π, $\frac{25}{8}$ d 7.3, $\frac{23}{4}$, $2\sqrt{8}$, 5.6, $\sqrt{30}$
15 a $\sqrt{50}$, 6.9, $\frac{33}{5}$, $\frac{17}{3}$, π b 3.72, $\sqrt{12}$, $\frac{10}{3}$, 3.27, $\frac{3}{10}$
c 9.23, 3.29, $\sqrt{10}$, 2.93, $\frac{13}{6}$
16 a 2.64, $\frac{18}{5}$, $\frac{9}{2}$, 4.52, 6.43
b $2\sqrt{12}$, $3\sqrt{8}$, 8.9, 9.8, $\frac{100}{3}$
c 0.59, $\frac{1}{\sqrt{3}}$, 7.07, $\frac{1}{\sqrt{2}}$, $\frac{15}{3}$
17 a 2.4 m b 3.25 cm c 2.125 km d $245\frac{1}{3}$ kg
18 65% 19 $3422.50 20 36
21 $33\frac{1}{3}$% 22 a 63 b 243
23 $83\frac{1}{3}$% 24 64%

Exercise 1h

1 a 4 : 13 b 20 : 1 c 1 : 2 d 3 : 4
2 a 49 : 108 b 3 : 1 c 13 : 30
3 a 5 : 7 b 26 : 17 c 7 : 3
4 a 3 : 4 b 4 : 7 c 4 : 3
5 13 : 15 6 a 1 : 2 b 2 : 3
7 9 : 14 8 5 : 4 9 9 : 217
10 650 : 37 11 11 : 9
12 a $24, $18 b 21 cm, 33 cm
13 a 22 c, 33 c b 1.6 m, 2 m
14 81, 63 15 60, 45
16 65 mm, 52 mm, 39 mm
17 90 g, 126 g, 234 g 18 $14, $35, $49
19 16 cm, 28 cm, 36 cm 20 48 c, 24 c, 16 c
21 17.5 cm, 21 cm, 24.5 cm

22 a \$108 000 **b** \$36 000

23 \$4680, \$6840

24 a 3 : 5 **b** 1 : 4 **c** 12 : 5

25 392 cm by 224 cm

26 A \$90, B \$45, C \$30 **27** \$1044

28 zinc 10 g, copper 35 g, tin 20 g

Mixed exercise 1

1 C	**2** A	**3** D	**4** C
5 C	**6** D	**7** C	**8** B
9 C	**10** C	**11** C	**12** B
13 B	**14** C	**15** A	**16** B
17 D	**18** A	**19** C	**20** D
21 B	**22** C	**23** B	**24** B

Chapter 2

Exercise 2a

1 $A = \{11, 13, 17, 19, 23, 29, 31, 37\}$, $B = \{12, 24, 36\}$, $C = \{12, 24\}$

2 a $A = \{1, 2, 3, 4, 6, 8, 12, 16, 24, 48\}$, $B = \{11, 13, 17, 19, 23, 29\}$, $C = \{a, e, i, o, u\}$
b $n(A) = 10$, $n(B) = 6$, $n(C) = 5$

3 $P = \{-2, -1, 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$, $Q = \{-2, -1, 0, 1\}$, $R = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$

4 a e.g. $\{2, 4, 6, 8, 10, 12\}$ **b** infinite

5 $\{(-2, -4), (-1, -2), (0, 0), (1, 2)\}$ **6** 5

7 a $A = \{2, 4, 6, 8, 10, 12, 14\}$, $B = \{3, 6, 9, 12, 15\}$, $C = \{2, 3, 5, 7, 11, 13\}$, $D = \{\ \}$
b $n(A) = 7$, $n(B) = 5$, $n(C) = 6$, $n(D) = 0$

8 A and C are infinite, B and D finite.

9 a 13 **b** 4

10 a 7 **b** $\{3, 5, 7, 11, 13, 17, 19\}$, no
c $\{\ \}$ or $\varnothing$, the empty set

11 a $A' = \{0, 1, 3, 5, 7, 9\}$
b $B' =$ {Mon, Tues, Wed, Thurs}
c $P' =$ {pupils in my school but not in my maths class}
d $Q' = \{\ \}$ or $\varnothing$

12 a $P' = \{0, 2, 4, 6, 8, 9\}$
b $Q' = \{6, 7, 8, 9\}$
c $R' = \{0, 1, 2, 4, 5, 7, 8\}$
d $S' = U$ *or* $\{\ \}$

13 a $A = \{1, 2, 5, 10\}$ **b** $B' = \{1, 2, 4, 5, 7, 8\}$
c 6 **d** e.g. $\{1, 2\}$

14 a {Barbados, Jamaica, Grenada}
b e.g. {West Indies, India, Australia}
c {spider, mantis, fly}
d {your own answer}

15 a i {natural numbers up to 10} **ii** 5
b i {homes} **ii** finite
c $\{r, s, t, u, v, w, x, y, z\}$

16 **a** and **c**

17 $\{\ \}$, $\{2\}$, $\{4\}$, $\{9\}$, $\{16\}$, $\{2, 4\}$, $\{2, 9\}$, $\{2, 16\}$, $\{4, 9\}$, $\{4, 16\}$, $\{9, 16\}$, $\{4, 9, 16\}$, $\{2, 9, 16\}$, $\{2, 4, 16\}$, $\{2, 4, 9\}$, $\{2, 4, 9, 16\}$

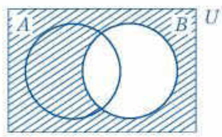
18 A and C **19** A and B

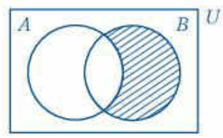
Exercise 2b

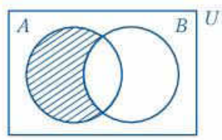
1 $A \cup B = \{2, 3, 4, 5\}$ $A \cap B = \{3, 4\}$

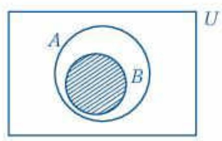
2 $P \cup Q =$ {A, B, E, G, L, M, O, T, R, Y}
$P \cap Q =$ {E, G, R}

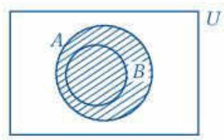
3 $n(L \cup M) = 5$, $n(L \cap M) = 2$

4 a 

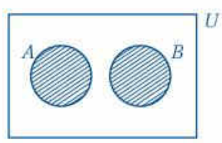
b 

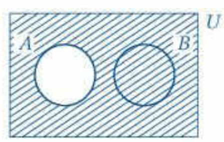
c 

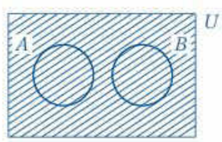
5 a 

b 

c 

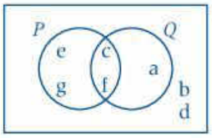
6 a 

b 

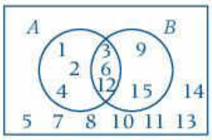
c 

7 **a** $A' \cap B$ **b** $P \cap Q'$ **c** $(R \cup S)'$

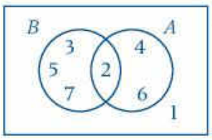
8 **a** 6 **b** 1, 3, 5, 7, 9
c 2, 3, 4, 6, 8, 9, 10

9 **a**


a {a, b, d}
b {b, d, e, g}
c {a, c, e, f, g}
d {e, g}
e {b, d}
f {b, d}

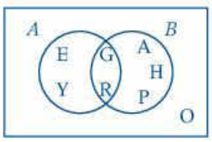
10 **a**


b **i** {3, 6, 12}
ii {5, 7, 8, 10, 11, 13, 14}
c **i** 3 **ii** 7

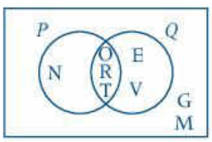
11 **a**


b **i** 2
ii 2, 3, 4, 5, 6, 7

12 **a** Those in my class who do not play cricket
b Those in my class who do not play football
c Those in my class who do not play cricket or football
d Those in my class who do not play cricket or football.

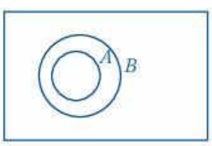
13


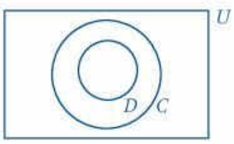
a {A, O, H, P}
b {E, O, Y}
c {A, G, H, E, P, R, Y}
d {O}
e {A, E, H, O, P, Y}

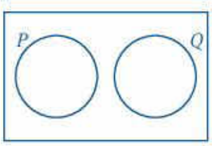
14


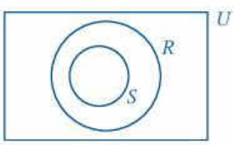
a {E, G, M, V}
b {G, M, N}
c {E, N, O, R, T, V}
d {G, M}
e {E, G, M, N, V}

15 **a** $P' = \{0, 1, 4, 6\}$
$Q' = \{0, 2, 4, 6\}$
$P' \cup Q' = \{0, 1, 2, 4, 6\}$
$P' \cap Q' = \{0, 4, 6\}$
$(P \cup Q)' = \{0, 4, 6\}$
b no, $2 \in P$ but $2 \notin Q$

16 **a**


b


c


d


	17	18	19	20
$n(A)$	5	6	7	7
$n(B)$	8	9	5	8
$n(A')$	9	8	8	11
$n(B')$	6	5	10	10
$n(A \cup B)$	10	9	7	15
$n(A \cap B)$	3	6	5	0
$n(A' \cup B')$	11	8	10	18
$n(A \cap B)'$	11	8	10	18

21 **a** $x = 3$ **b** 9 **c** 15

22 **a** $n(P) = 42 - x$, $n(Q) = 31 - x$ **b** 13

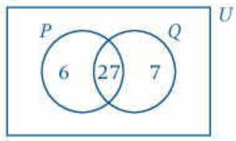
23 **a** 7 **b** 18

24 **a** 21 **b** 18 **c** 9 **d** 12

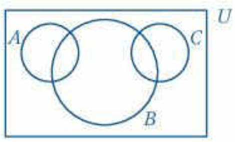
25 $n(P) = 12$, $n(Q) = 9$, $n(P \cup Q) = 18$,
$n(U) = 20$, $n(P' \cap Q) = 6$, $n(P' \cup Q) = 11$,
$n(P' \cup Q') = 17$

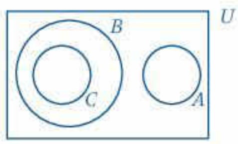
26 12 **27** 19 **28** 6

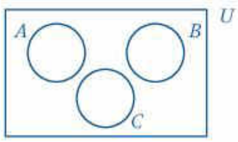
29 **a** 31 **b** 17 **c** 15 **d** 19

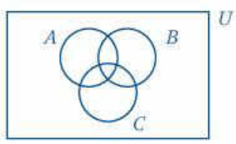
30 **a** 33 **b** 6
c 13 **d**


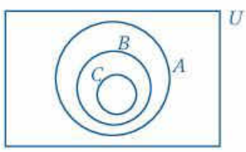
Exercise 2c

1


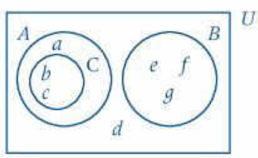
2


3


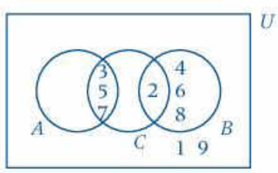
4


5


6 **a** $A \cap B$
b $A \cap C$
c $(A \cup B) \cap C'$
d $(A \cup B)'$

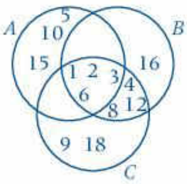
7 **a**


b *d, e, f, g*
c *a, b, c, e, f, g*

8 **a**


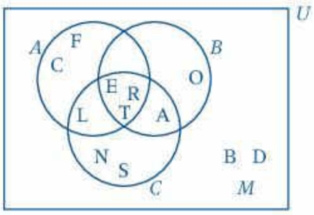
b 4 **c** 2, 3, 5, 7

9 a



b 1, 2, 3, 6

10 a



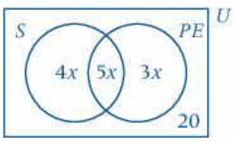
b i A, E, L, N, O, R, S, T **ii** A, E, R, T
iii E, L, R, T
c i 5 **ii** 9

11 a 15 **b** 36 **c** 33

12 a Athletes who participate in track events but not in hurdling or long distance running.
b No member takes part in both. **c** 6

13 a $10x + 88$ **b** 148

14 a



b 60

15 a 16 **b** 55 **c** 74 **d** 20
e bicycle, motorcycle.

16 30

17 a 3 **b** 10 **c** 19

18 a 11 **b** 5 **c** 3

19 a 6 **b** 10 **c** 6

Mixed exercise 2

1 C	**2** B	**3** B	**4** C
5 B	**6** D	**7** D	**8** A
9 A	**10** A	**11** C	**12** D
13 C	**14** C	**15** C	**16** D
17 B	**18** A	**19** C	**20** C
21 B	**22** A	**23** B	

Chapter 3

Exercise 3a

1 $7x + 5$	**2** $3x - 9$
3 $a + 7b$	**4** $x + 4y$
5 $6p - 4q + 5$	**6** $3a^2 + 4b - 4$
7 $p + 8$	**8** $7x^2 + 5x + 3$
9 $x^2 + 36$	**10** $x^3 - 3x^2y$
11 $3a^3b + ab^2$	**12** $a^4 + a^3b$
13 $3x^4 + x^2y^2$	**14** $a^5b + a^2b^4$
15 $6a^3b - 9a^2b^2$	**16** $10p^2q^3 - 15p^3q$
17 $7x - 9$	**18** $6x - 6$
19 $20a + 3$	**20** $3a + 16$
21 $x + 8$	**22** $6x - 1$
23 $14x + 16$	**24** $10a - 4b - 9$
25 4	**26** $6x + y - 2z$
27 $17a + 3$	**28** $y - 5$
29 $9b - 15c$	**30** $11m - 13n$
31 $3x^2 - xy + 4y^2t$	**32** $10p^2 - 12pq - 3q^2$
33 $2x^2 + x$	**34** $a^2 + 2a$

Exercise 3b

1 $5(x + y)$	**2** $3(3a - 4b)$
3 $7(t + 2)$	**4** $5(2 - a)$
5 $3(x - 2y - z)$	**6** $4(3a - 2b + 4c)$
7 $x(x + 6)$	**8** $3(a^2 + 2)$
9 $2p(2p - 1)$	**10** $a(5a - 1)$
11 $6x(2x - 1)$	**12** $2a(5 - a)$
13 $4(x^2 + 2x - 1)$	**14** $x(x - 2y)$
15 $2a(2a - b)$	**16** $a^2(a + 2)$
17 $3x^2(4x - 3)$	**18** $5p^2(2 - 3p^2)$
19 $8x(y + 2z + 1)$	**20** $3a(3 - 4b + b^2)$
21 $5xy(x - 2y + 3)$	**22** $3p(1 - 2p + 3q)$
23 $2xy(2 - 3x + 2y)$	**24** $x(1 + x - 2x^2)$
25 $x(4y - 2z + 3)$	**26** $3a(a^2 - 2ab + 3b^2)$
27 $r(\pi r - \pi + r^2)$	**28** $\pi ab(a - b)$
29 $\pi r(r - 2h)$	**30** $\frac{1}{2}h(a - b)$

Exercise 3c

1 $a + 3$	**2** $2 - x$	**3** $a^2 + ab$
4 $\dfrac{3}{b - b^2}$	**5** $2a + 2b - 1$	**6** $\dfrac{3x}{1 - 2y}$
7 $a + b$	**8** $\dfrac{x + 2}{2}$	**9** $\dfrac{1 + pq}{3}$

10 $\frac{4a-3b}{5}$
11 $3-2a+b$
12 $3x+4-2y$
13 a^2-2ab
15 $\frac{15a}{28}$
16 $\frac{x^2}{2}$
17 $\frac{8x^2}{21}$
18 $\frac{a^3}{8}$
19 $\frac{4p^3}{3}$
20 $\frac{15a^2b}{2}$
21 $\frac{18x^2y^3}{5}$
22 $\frac{ad}{bc}$
23 $\frac{a^2}{c^2}$
24 $\frac{5a}{9b}$
25 $\frac{6pq}{5r}$
26 $\frac{25b}{8a}$
27 $\frac{7x}{12y}$
28 $\frac{25q^2}{3p^2}$
29 $\frac{2(x+2)}{3(x+3)}$
30 $\frac{3a-1}{4(2a+3)}$
31 $\frac{4x}{3(x-3)}$
32 $\frac{x}{2(x-1)}$
33 $\frac{2(x-1)}{2x+1}$
34 $\frac{3(4x+1)}{2(4x-1)}$
35 $\frac{p^5}{q^7}$
36 $\frac{a^3b^3}{10c^3}$
37 $\frac{q}{p}$
38 $\frac{b}{a}$
39 $\frac{c^2}{15b}$
40 $\frac{6}{xy}$
42 $\frac{5a^4b^2}{3}$
43 $\frac{3c}{4b}$
44 $4xz^3$
45 $\frac{5x}{6}$
46 $\frac{17x}{12}$
47 $\frac{13a}{15b}$
48 $\frac{p}{4}$
49 $\frac{x}{12y}$
50 $\frac{x}{3}$
51 $\frac{17a}{12}$
52 $\frac{3x+5}{4}$
53 $\frac{11a+15}{6}$
54 $\frac{19a-14}{30}$
55 $\frac{8b-11}{6}$
56 $\frac{-13y-72}{21}$
57 $\frac{5x-1}{8}$
58 $\frac{11-2x}{24}$
59 $\frac{10p-23}{10}$
60 $\frac{22a+41b}{21}$
61 $\frac{10x+7y}{10}$
62 $\frac{3a+23b}{30}$
63 $\frac{50q-7p}{45}$
64 $\frac{11a-6}{4}$
65 $\frac{4a^2-9a-6}{6a}$
66 $\frac{10p-4-21p^2}{6p}$
67 $\frac{7a^2-3a+6b^2-8b}{4ab}$
68 $\frac{3x+5}{(x+1)(x+2)}$
69 $\frac{6x+23}{(x+3)(x+4)}$
70 $\frac{a+6}{(a+2)(a+3)}$
71 $\frac{4x+31}{(x+1)(x+4)}$
72 $\frac{8p+7}{(p-1)(p+2)}$
73 $\frac{9x-22}{(x-2)(x-3)}$
74 $\frac{9b-22}{(b-2)(b-3)}$

Exercise 3d

1 a 8 b −22 c −3
2 a 20 b 14 c $\frac{11}{9}$ d $\frac{9}{8}$
3 a 2 b 272 c −84 d −28
4 a 0 b −4 c 4
5 a −24 b 12 c 8
6 a 3 b 9
7 a 5 b −1 c $6\frac{1}{2}$
8 a 4 b 6
9 a −3 b 4
10 a 12 b −5 c 6
11 a $\frac{5}{6}$ b $\frac{1}{6}$ c 13 d 5
12 a $\frac{12}{7}$ b 6 c 1
13 a 8 b 18 c $\sqrt{2}=1.414...$
14 a 9 b 24 c 8
15 a −128 b 57 c 4
16 a $\frac{25}{36}$ b 2 c −16
17 a 5 b 5 c 3
18 a 49 b −8 c 0

Exercise 3e

1 $x=4$
2 $x=3$
3 $x=0$
4 $x=-3$
5 $x=-1$
6 $x=2$
7 $x=2\frac{3}{5}$
8 $x=-4$
9 $p=-3$
10 $x=4$
11 $x=4$
12 $y=4$
13 $x=7$
14 $x=4$
15 $x=4$
16 $x=-3$
17 $x=-2$
18 $x=-2$
19 $x=1$
20 $x=4$
21 $x=-6$
22 $x=16$
23 $x=\frac{6}{7}$
24 $y=5$
25 $x=\frac{12}{19}$
26 $x=10$
27 $x=5$
28 $x=5$
29 $x=\frac{5}{2}$
30 $x=8$
31 $x=6$
32 $x=-6$
33 $x=-6$
34 $x=\frac{5}{9}$
35 $x=25$
36 $x=3$
37 $x=-\frac{4}{7}$
38 $x=4$
39 $x=5$
40 $x=5$

Exercise 3f

1 15
2 7
3 5
4 26
5 a $\$(x-25)$ b i \$150 ii \$125
6 3 kg and 9 kg
7 31 kg and 43 kg
8 25
9 5 days
10 12
11 a \$3.50 b \$2

12 15, 16, 17 **13** 44 cm, 45 cm, 46 cm

14 9, 10, 11 **15** 2 kg

16 **a** $(19 − x) **b** Lance $10 Clarrie $9

17 24 cm by 12 cm **18** 18, 19, 20

19 3 m **20** 6 years

21 John 20 yr, Emily 10 yr

22 $\frac{60}{7}$ or $8\frac{4}{7}$ **23** 23 **24** 12 cm, 8 cm, 9 cm

25 When 5 is subtracted from half a number the result is 7.

26 The area of a rectangle is equal to its length multiplied by its width.

27 The area of a triangle is equal to half of its base multiplied by its perpendicular height.

28 The cost of a number of oranges is equal to the price of one orange multiplied by the number of oranges.

29 The result of multiplying the square of one number by another number is 40.

30 The square root of the quotient of two numbers is 5.

31 The perimeter of a rectangle is twice its length added to twice its width.

32 The area of a circle is equal to the square of its radius multiplied by π.

33 When 40 is subtracted from three times a number is result is half the original number.

34 When the square of one number is subtracted from half of a second number the result is the same as when the second number is subtracted from 3.

Exercise 3g

1 $x = 3, y = -1$ **2** $x = 2, y = -2$

3 $x = 4, y = -1$ **4** $x = 2, y = 7$

5 $x = 1, y = 2$ **6** $x = 4, y = 8$

7 $x = 1, y = 4$ **8** $x = 2, y = -1$

9 $x = 4, y = -1$ **10** $a = 2, b = -3$

11 $x = 2, y = 3$ **12** $x = 0, y = -1$

13 $x = -1, y = -1$ **14** $x = 4, y = 1$

15 $x = \frac{1}{2}, y = \frac{3}{4}$ **16** $x = 2, y = 5$

17 $x = 3, y = -2$ **18** $x = 2, y = 3$

19 $x = 3, y = -2$ **20** $x = 3, y = 2$

21 $x = 4, y = 3$ **22** $x = -2, y = 2$

23 $x = 2, y = -4$ **24** $x = 3, y = -3$

25 $x = 6, y = 2$ **26** $p = 5, q = -3$

27 $m = 6, n = -3$ **28** $x = 1, y = -1$

29 $x = -3, y = 2$ **30** $x = \frac{1}{2}, y = \frac{1}{3}$

31 $a = 2\frac{1}{2}, b = -\frac{1}{2}$ **32** $p = \frac{1}{4}, q = \frac{2}{3}$

33 $x = -2, y = -6$ **34** $x = -\frac{6}{5}, y = -\frac{8}{5}$

35 $x = -1\frac{1}{2}, y = -2\frac{1}{3}$ **36** $p = 1\frac{3}{5}, q = 1\frac{2}{5}$

37 $x = 1\frac{3}{10}, y = -1\frac{1}{5}$ **38** $x = -\frac{8}{5}, y = \frac{12}{5}$

39 $x = -2, y = 11$ **40** $x = \frac{6}{5}, y = \frac{12}{5}$

41 $a = \frac{4}{3}, b = 3$ **42** $x = \frac{7}{3}, y = \frac{3}{2}$

43 $x = \frac{5}{2}, y = -2$ **44** $p = \frac{5}{2}, q = -\frac{13}{4}$

45 $x = -5, y = 3$ **46** $x = 7, y = -2$

47 $x = 4, y = 3$ **48** $x = 3, y = -4$

49 8 and 17

50 small 64 c, large 112 c

51 4 @ $8.50 and 3 @ $12.50

52 **a** $3.40 **b** $3.20 **c** $13.20

53 5 @ 25 c, 8 @ 50 c

54 **a** $1.25 **b** $1.50

55 **a** 37 yr **b** 17 yr

56 5 @ $5, 9 @ $10

57 3 @ $6.50, 4 @ $8.75

58 **a** $1 **b** $3

59 **a** $4.50 **b** $2 **60** 83

Exercise 3h

1 **a** $x \geq 1$


b $x \leq 3$


2 **a** $x \geq 2$


b $x < 2.5$


3 **a** $x > \frac{4}{5}$


b $x > -1$


4 **a** $x \leq 4.5$


b $x \geq -\frac{4}{3}$

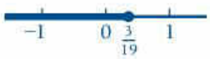

5 **a** $x > 3$


b $x > -4$


6 a $x > 1$ 

b $x > -1.5$ 

7 a $x \leqslant -\frac{7}{2}$ 

b $x \leqslant \frac{3}{19}$ 

8 a $x < -1$ 

b $x > \frac{6}{5}$ 

9 $1 < x < 5$ 

10 $-9 \leqslant x \leqslant -5$ 

11 $-3 < x \leqslant 2$ 

12 $x < -\frac{3}{2}, x \geqslant 1$ 

13 $x \geqslant 7$ 

14 $x \geqslant 2$ 

15 $-1 \leqslant x < -\frac{1}{3}$ 

16 $-2 < x < 2$ 

17 $-3 \leqslant x < 2$ 

18 $x < -2$ 

19 $-1 < x < 12$ **20** $2 < x < 3$

21 $1 < x < 2$ **22** $1.5 < x \leqslant 3$

23 a -3 **b** 0 **24 a** 4 **b** 5

25 $x + y \leqslant 20$ **26** $5a + 2b \leqslant 15$

Mixed exercise 3

1 A **2** C **3** D **4** D
5 C **6** C **7** B **8** B
9 D **10** A **11** C **12** A
13 C **14** B **15** C **16** D
17 B **18** A **19** C **20** C

Chapter 4

Exercise 4a

1 a m **b** cm **c** m **d** km
e mm

2 a kg **b** tonnes **c** mg **d** mg
e kg

3 a minutes **b** seconds
c years **d** months **e** millennia

4 a 13 000 m **b** 4320 m
c 670 m **d** 5.29 m
e 47.6 m **f** 0.26 m
g 0.736 m **h** 0.038 m

5 a 3400 cm **b** 47 cm **c** 2.9 cm
d 4.4 cm **e** 53.4 cm **f** 0.005 cm
g 173 200 cm **h** 2740 cm

6 a 340 mm **b** 81.2 mm
c 4.6 mm **d** 26 000 mm
e 770 mm **f** 34 mm
g 1 243 000 mm **h** 92 000 mm

7 a 6 km **b** 10 km
c 0.75 km **d** 7.36 km
e 0.1783 km **f** 0.4935 km
g 0.007 63 km **h** 0.1834 km

8 a 0.25 g **b** 5.7 g
c 73.4 g **d** 4.73 g
e 5420 g **f** 20 400 g
g 730 g **h** 49.3 g

9 a 7.6 kg **b** 0.491 kg
c 0.0972 kg **d** 6.04 kg
e 0.9264 kg **f** 0.055 45 kg
g 0.000 374 kg **h** 0.008 497 kg

10 a 54 000 mg **b** 429 000 mg
c 1240 mg **d** 460 mg
e 1 200 000 mg **f** 44 000 mg
g 217 000 mg **h** 840 mg

11 a 3.42 t **b** 1740 kg
c 47 kg **d** 83.44 t
e 504.6 t **f** 25 km
g 150 mm **h** 25.6 cm

12 The missing values are:
9.25 p.m., twenty-five past nine in the evening
15.15, a quarter past three in the afternoon
10.55, 10.55 a.m.
10.30 a.m., half past ten in the morning
02.15, a quarter past two in the morning
22.30, 10.30 p.m.
6.40 p.m., twenty to seven in the evening
23.45, 11.45 p.m.

13 a 4 h 30 min **b** 4 h 22 min
c 11 h 5 min **d** 10 h 54 min
e 3 h 6 min **f** 19 h 24 min

14 a 14.55 or 2.55 p.m. **b** 21.57 or 9.57 p.m.

15 a 104 °F **b** 77 °F **c** 82.4 °F **d** 14 °F

16 a 37.8 °C **b** −5 °C **c** $26\frac{2}{3}$ °C **d** 18.9 °C

17 a 13.68 m **b** 3.76 m
c 11.186 m **d** 3090.4 m

18 **a** 10 m **b** 10.7 m **c** 0.288 m **d** 1.574 m

19 **a** 9 kg 430 g **b** 1 kg 691 g
c 3 kg 450 g **d** 13 kg 740 g
e 28 kg 400 g

20 **a** 9 h 40 min **b** 16 min 40 s
c 83 h 20 min **d** 694 h 26 min 40 s

21 556 **22** 56 **23** 250

24 16.56 kg **25** 21.792 kg

26 1.25 kg **27** 12.736 kg

28 14.806 kg **29** $150 **30** 436 g

31 8.188 t **32** 42 **33** 0.2 kg

34 **a** $\left(5 - \frac{x}{100}\right)$m

b **i** $\frac{x}{100} = \frac{1}{4}\left(5 - \frac{x}{100}\right)$ **ii** 1 m

35 75 g

36 **a** $3175 Jamaican **b** £260
c $928 Canadian **d** BD$198

37 **a** $117 **b** $673
c $16 **d** $556

38 **a** $7937.50 Jamaican **b** $1421.43 TT
c $1931.03 EC **d** $27.40 Canadian

39 $74.80 **40** $22.85 Canadian

41 **a** £594.29 **b** BD$582.69

42 $985

43 **a** $140 **b** $6666.67 Jamaican

44 $9\frac{1}{3}$ km/h **45** **a** 50 km **b** 60 km/h

46 **a** 5 p.m. **b** 6 a.m.

47 **a** 21.30 (9.30 p.m.) **b** 08.15 (8.15 a.m.)

48 9 p.m. and 10 p.m.

49 **a** 9 a.m. **b** 8.45 p.m.

50 0830 (8.30 a.m.)

51 2 p.m. the following day

52 **a** **i** $\frac{x}{10}h$ **ii** $\frac{20-x}{30}h$

b **i** $\frac{x}{10} + \frac{20-x}{30} = \frac{4}{3}$ **ii** 20 min

53 655 km/h

Exercise 4b

1 **a** feet and inches **b** miles
c inches **d** yards

2 **a** tons **b** ounces **c** pounds

3 **a** 35 in **b** 28 ft **c** 28 in
d 3 ft **e** 4 yd 1 ft **f** 3 yd
g 2 ft 5 in **h** 6 ft 3 in **i** 10 ft 0 in

4 **a** 38 oz **b** 1 lb 8 oz **c** 28 oz
d 67 oz **e** 1 lb 2 oz **f** 31 lb
g 2 lb 4 oz **h** 2 tons **i** 5 lb

5 **a** 2640 yd **b** 440 yd **c** 220 yd

6 960 yd

7 **a** 6.6 lb **b** 6.5 ft **c** 1.8 kg
d 2.7 m **e** 3.3 lb **f** 16 ft
g 7.7 lb **h** 1.1 lb **i** 3.5 oz
j 26.4 lb **k** 80 km **l** 160 km
m 13.6 kg **n** 120 g **o** 188 miles
p 150 miles

8 4 ft by 8 ft **9** Calais and Paris

10 8 oz **11** 15 cm **12** 4 in **13** 1.3 m

14 13 stone 12 lb

15 0.04 kg, 50 g, 2 oz, $\frac{1}{4}$ lb

16 8 in, 25 cm, 1 in, 25 mm

Exercise 4c

1 32 cm **2** 44 cm **3** 46 cm **4** 34 cm

5 27 cm **6** 35 cm

7 **a** $125\frac{5}{7}$ cm **b** 220 mm

8 **a** **i** 22 cm **ii** 36 cm
b **i** 66 mm **ii** 150 mm

9 **a** **i** 9.78 cm **ii** 25.8 cm
b **i** 32.7 cm **ii** 56.7 cm
c **i** 25.7 cm **ii** 39.7 cm

10 158 cm **11** 114 m **12** 17.1 cm

13 65.4 cm **14** **a** 66 cm **b** 150

15 453 cm **16** 146° (3 s.f.) **17** $\frac{80}{\pi}$ cm

18 76.4° **19** 7.64 cm **20** 80.6 cm

Exercise 4d

1 20 cm² **2** 13 cm² **3** 7.99 cm²

4 4.9 cm² **5** 84 cm² **6** 48 cm²

7 96 cm² **8** 45 cm² **9** 47.25 cm²

10 168 cm² **11** 88 cm² **12** 41.25 cm²

13 24 cm² **14** 18 cm² **15** 12 cm²

16 20.25 cm² **17** 82.5 cm² **18** 68.9 m²

19 36 m² **20** 15.68 cm² **21** 171 cm²

22 a 150 cm b 814 cm^2 c 536 cm^2

23 17.4 m^2

24 a 290.25 cm^2 b 117.75 cm^2 c 476 010 cm^2

25 a 452 cm^2 b 91.6 cm^2 c 133 cm^2

26 a 11.4 cm^2 b 155 cm^2 c 180 cm^2

27 a 5.64 cm b 2.65 cm

28 146 cm^2 29 141 cm^2 30 114 cm^2

31 74.7 cm^2 32 83.0 cm^2 33 64.1 cm^2

34 42.7 cm^2 35 10.5 cm^2

36 a 451.2 m b 501.44 m
c 16 835 m^2 d 3811 m^2

37 a 34.4 cm b 73.9 cm^2 c 61.1 cm^2

Exercise 4e

1 a About 28 cm^2 b About 20 cm^2
c About 22 cm^2

2 a A b B

3 The irregular edges do not fit recognisable fractions of the squares.

4 All three are probably underestimates.

5 About 49 cm^2 6 About 50 cm^2

7 a 65 b 616

Exercise 4f

1 a 8.768 cm^3 b 8624 cm^3

2 325 000 cm^3 (0.325 m^3)

3 12 m^3 4 $3\frac{1}{3}$ m 5 480

6 0.0418 m^3

7 a 5832 cm^3 b 6264 cm^3

8 0.2025 m^3 9 42.5 cm^2, 8500 cm^3

10 a 90 cm^2 b 31 500 cm^3 c 258 kg

11 a 191 cm^3 b 813 cm^3

12 35 862 cm^3 13 125 cm^3

14 a 20 cm^2 b 4540 cm^2 c 5000 cm^3

15 a 6 ℓ b 0.7 m^2

16 a 48 cm^3 b 108 cm^2

17 a 148.2 cm^2 b 81 cm^3

18 a 80 cm^3 b 1

19 a 105 m^3 b 71 m^2

Mixed exercise 4

1 B 2 C 3 C 4 C

5 D 6 D 7 A 8 A

9 B 10 C 11 C 12 A

13 A 14 B 15 C 16 B

17 D 18 C 19 A 20 C

21 C 22 B 23 D 24 B

25 B 26 D 27 B

Chapter 5

Exercise 5a

1 \$2575 2 \$7560 3 TT\$225 4 3.3 kg

5 a \$16.20 b \$30.78

6 42 7 442 8 36 m^2 9 \$270

10 11 147 11 15% 12 \$840

13 \$962.55 14 \$3827.40

15 a \$198 b \$162

16 20% 17 \$1862 less

18 \$1517.40 19 \$789.60

20 a \$2400
b 50% on the cheaper, 30% on the dearer

21 \$288.91 22 66 240 litres

23 1587 24 \$6800 25 \$65 160

26 161 27 9016 28 \$278.21

29 .383 000 (3 s.f.) 30 \$15 200 (3 s.f.)

31 \$28 700 (3 s.f.) 32 \$2500

33 \$6500 34 \$4310.34

35 \$1362.50 36 \$55

37 \$75 38 \$46.80, 13% 39 \$150

40 \$648 41 3000 42 \$5015

43 a \$219.14 b \$245.44

44 18.2% 45 1200 46 50

47 a reduction of 1%

Exercise 5b

1 A = \$3.00, B = \$4.26, C = \$2.70,
D = \$9.60, E = \$19.56

2 A = \$5.10, B = \$17.25, C = \$4.74,
D = \$10.40, E = \$37.49

3 A = \$19.05, B = \$32.13, C = \$17.85, D = \$3.80, E = \$72.83, F = \$10.92, G = \$83.75

4 A = \$58.40, B = \$8.04, C = \$26.50, D = \$22.72, E = \$115.66, F = \$13.88, G = \$129.54

5 A = \$194.97, B = \$49.95, C = 5, D = \$579.87, E = \$666.85

6 A = \$7.65, B = \$6.88, C = $3\frac{1}{2}$, D = \$64.61, E = \$9.69, F = \$74.30

7 A = 740, B = 145, C = \$3584, D = \$448, E = \$4032

8 A = \$394.14, B = \$394.14, C = \$2782.84, D = 2697.84

9 \$32.50 10 \$1269.60 11 \$6.05 too little

12 a \$93.13 b \$31.37 13 \$1565.78

Exercise 5c

1 a i 10 ii $83\frac{1}{3}$ iii $\frac{1}{2}$ iv 0.4
b i 1.5 ii 10 iii 0.2 iv 3
c i 72 ii 9

2 a \$232.94 b \$229.29

3 \$17.24 4 1324

5 \$37.84 6 14.72 c

7 a 5553 b \$2221.20
c \$1443.78 d \$3664.98

8 a 8824 b \$3706.08
c \$2117.76 d \$5823.84

9 \$110.87 10 \$213.75

11 \$9975.60

12 a \$1182.72 b \$189
c \$256.27 d \$1679.99

13 \$519 14 \$1673.68

15 a \$2890 b \$4760 c \$3740

16 a i \$13 000 000 ii \$111 800 000
iii \$169 000 000
b i \$0.45 ii \$0.84

Exercise 5d

1 \$89 818 2 \$432 3 \$88.20 4 \$105

5 \$1140 6 \$16.32 7 a \$20.16 b $5\frac{1}{3}$%

8 \$287.28 9 \$171.15 10 \$1251.72, \$216.72

11 \$1418.64, 19.3% 12 Option 2 by \$1532

13 \$7296, 32% 14 \$3304

15 a \$79.69 b \$102.46 c 8%

16 Buying on H.P. by \$550.20

17 a Option (i) bOption (ii)

18 \$3222.80 19 \$22 500

20 \$42 500

21 a \$180 000 b \$191 250

22 a \$212 500 b \$206 250

23 \$596 160, 248%

24 a \$48 750 b \$276 250 c \$1 020 900

25 a \$18 000 b \$132 000 c \$398 880

26 a \$29 250 b \$165 750 c \$563 280
d \$592 530

27 a \$162 000 b \$24 824.88
c \$514 497.60

28 \$217 249.80

Exercise 5e

1 \$234.38 2 \$7.40 3 \$272.80
4 \$646 5 \$263.63, \$4.63 6 \$7083

7 Mr Gedge by \$2790 8 \$2242.50

9 \$38 592 10 \$32 500.06

11 \$736 12 \$880.42

13 a \$42 200 b \$8770 c \$29 470

14 \$2666

15 a \$4250 b \$33 250
c \$5637.50 d \$25 882.50

16 \$232.08

17 a \$19 970 b \$22 030 c \$1904.50
d \$25 275.50 after tax, mortgage and NI deductions

18 a \$63 600 b \$246.67 c \$2605

19 a \$135.83 b \$1754.51

20 \$1496.60

Exercise 5f

1 \$3600 2 \$324.80

3 a \$26 250 b \$1750

4 \$2800 5 $4\frac{1}{2}$% 6 5

7 a \$26 000 b 7

8 \$560

9 a $3315 b $11 115 c $185.25

10 a $210.79 b $5685.79 c $947.63

11 $4\frac{1}{2}\%$ 12 $900

13 Yes, she earns $50 more

14 $1800 15 $753.48

16 $2426.98 17 $1018.12

18 $4442.13 19 $18 277.62

20 $12 531.74 21 $865 926.58

22 Bank $116 000, finance company $124 000

23 $10 637.50

24 If he sells the old car and if he buys the car, $57 600. If he does not he has a car valued at $6400 + $93 312 in a building society.

25 $614 400 26 $61 965

27 Mr Khan by $487.18

28 5 yrs at 5.5% compound interest increases each $ to $1.307 whereas 5 yrs at 6.2% simple interest increases each $ to $1.31. The simple interest rate is marginally better.

29 $750 30 $3.15 31 $310 32 $520

33 $610 34 $360 35 $555, $671.55

36 a i $1170 ii £22.50 b $1415.70

37 a $490 b $280

38 $960 39 a $540 b $148.50

40 $260

Mixed exercise 5

1 D	2 C	3 B	4 D
5 A	6 C	7 C	8 B
9 D	10 B	11 B	12 D
13 B	14 C	15 A	16 C
17 B	18 C	19 B	20 C

Review test 1

Multiple choice questions

1 D	2 A	3 B	4 C
5 C	6 A	7 D	8 C
9 C	10 B	11 A	12 B
13 A	14 D	15 C	16 B
17 A	18 A	19 A	20 D
21 A	22 B	23 C	24 D
25 A	26 C	27 A	28 B
29 B	30 D		

General proficiency questions

1 665 kg, 1596 kg 2 11 : 5 : 3

3 $4xy(3x - 4y)$

4 a 4 b 159

5 a $x = 4.8$ b $x = 4$

6 25 7 24 8 7.5 km

9 a 60 km/h b 240 km

10 50 km 11 a $\frac{25}{27}$ b $\frac{15}{14}$

12 a $p = \frac{1}{3}, q = \frac{1}{2}$ b $x = \frac{1}{2}, y = \frac{5}{6}$

13 38 h normal and 14 h overtime

14 9 and 16 15 7 and 12

16 a $-5 \leqslant x \leqslant 2$ b $\frac{9}{5} < x < 3$

17 a i $4x + 3y$ ii $7a + 3b$
b $2x(1 - 2y + 4xy)$

18 a $\frac{3a^5}{2}$ b $\frac{6x}{y}$
c $\frac{7p}{8q}$ d $\frac{8a - 11b}{30}$

19 a 22 b 2 c 5

20 a $x = 2$ b $x = 2$
c $x = -16$ d $x = 3$

21 10.5 cm

22 a $x = 2, y = -3$ b $x = \frac{1}{2}, y = \frac{3}{4}$

23 a $1.50 b $2.50

24 a $x \geqslant \frac{4}{5}$

b $-2\frac{1}{2} < x \leqslant 3$

25 a i 60 240 m ii 0.497 m
b i 360 cm ii 0.731 cm
c i 135 mm ii 9400 mm
d i 0.83 km ii 9.23 km

26 a 2.29 m b 5.43 m
c 6.01 m d 0

27 8 28 23.692 kg

29 2.195 kg 30 $30

31 The blanks in order are \$1, 24 c, 8, \$6.75, \$16.85, \$1.69, \$18.54

32 \$9227.50 **33** \$276.46

34 \$400 **35** \$85.36

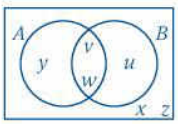
36 \$693.75 **37** 6

38 \$24 412.50 **39** \$10 980

40 **a** {12, 24, 36, 48}
b {1, 2, 5, 10, 25, 50}
c {−1, 0, 1, 2, 3}
d {(−1, 2), (0, 1), (1, 0), (2, −1), (3, −2)}

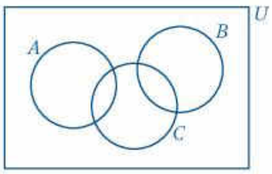
41 **a** 2 **b** {}, {25}, {36} **c** 19

42 **a**



b 2

43 **a** 12 **b** 8 **44**



Chapter 6

Exercise 6a

1 **a** Domain = {4}, Range = {1, 2, 3}
b Domain = {5, 6, 7, 8},
Range = {6, 7, 8, 9}
c Domain = {1, 2, 3, 4},
Range = {1, 2, 3, 4}
d Domain = {−6, −3, −2, −1, 1, 2, 3, 6},
Range = {−6, −3, −2, −1, 1, 2, 3, 6}

2 **a** $(1, \frac{1}{3})$ **b** All three **c** (0, 2) and $(1, \frac{2}{3})$

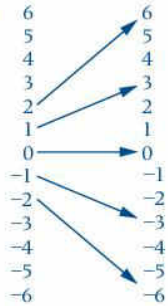
3 **a** (−4, 16), (−2, 4), (0, 0), (2, 4), (4, 16), $\mathbb{R}^+$
b (−4, −7), (−2, −3), (0, 1), (2, 5), (4, 9), $\mathbb{R}$
c (−4, −4), (−2, −2), (0, 0), (2, 2), (4, 4), $\mathbb{R}$

Exercise 6b

1 A = $\{(0, 5), (3, 0), (4, -\frac{5}{3}), (5, -\frac{10}{3})\}$
B = {(0, 5), (0, −5), (3, 4), (3, −4), (4, 3), (4, −3), (5, 0)}
A is a function, B is not.

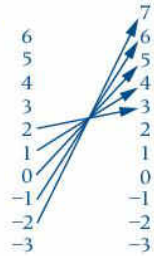
2 **a** (0, −5), (1, −5), (2, −2), (3, −2), (4, −1); function
b (0, −10), (2, −8), (4, −6), (6, −4), (8, −2), (10, 0); function
c (3, 3), (3, 4), (4, 4), (5, 4), (5, 5), (7, 6), (8, 6); not a function

3 **a**



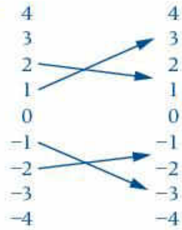
{−6, −3, 0, 3, 6}

b



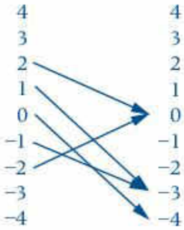
{3, 4, 5, 6, 7}

c



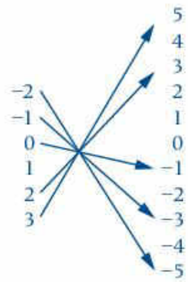
$\{-3, -1\frac{1}{2}, 1\frac{1}{2}, 3\}$

d



{0, −3, −4}

4



5 C = {(0, 0), (1, 2), (1, −2), (4, 4), (4, −4), (9, 6), (9, −6)}
D = $\{(0, \frac{1}{3}), (1, 1), (3, \frac{7}{3}), (5, \frac{11}{3})\}$
C is not a function, D is.

6 *b* and *c*

7 **a** 4 **b** −2 **c** −8 **d** 34
e $3x + 1$ **f** $3x^2 - 2$ **g** $3x + 3h - 2$

8 **a** 2 **b** −2 **c** 1 **d** 2 **e** −2

9 **a** 9 **b** $-\frac{1}{3}$ **c** 1 **d** 2

10 **a** 1 **b** −3 **c** 2 **d** $\frac{3}{5}$
e $\frac{3}{x+2}$ **f** $\frac{3}{x^2+1}$

11 **a** 3 **b** 7 **c** −3 **d** $35\frac{1}{2}$

12 **a** $\frac{1}{3}$ **b** 7 **c** $-\frac{1}{2}$ **d** $\frac{5}{3}$

13 **a** 1 **b** **i** −4 **ii** −9

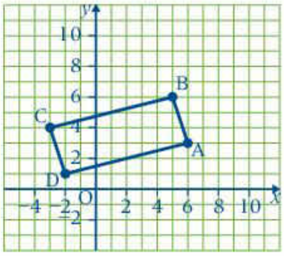
14 **a** **i** 2 **ii** 7 **b** −3

15 **a** 2 **b** −5 **c** 7

16 **a** **i** $\frac{3}{2}$ **ii** 0 **iii** 4
b 3 **c** Yes, $x = 0$

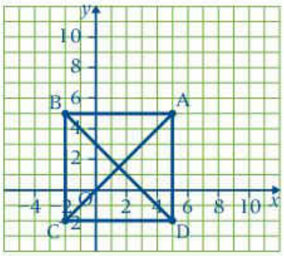
Exercise 6c

1 a



b parallelogram

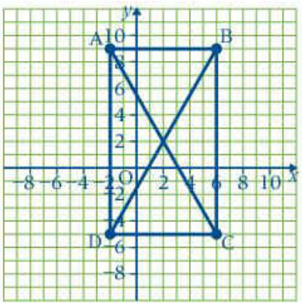
2 a



b ii (5, −2) **iii** $(1\frac{1}{2}, 1\frac{1}{2})$

c i 1 **ii** −1

3 a



b ii (−2, −5) **iii** (2, 2)

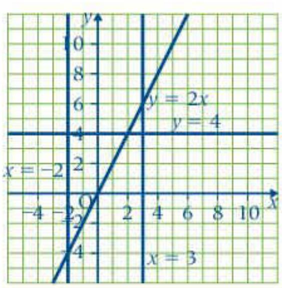
c i $1\frac{3}{4}$ **ii** $-1\frac{3}{4}$

4 a 2, (0, 6) **b** 3, (0, −7) **c** −5, (0, 2)
d $-\frac{1}{2}$, (0, 3) **e** −3, (0, 4)

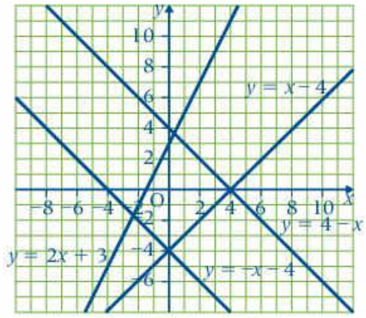
5 a $y = 4x + 5$ **b** $y = -3x + 2$
c $y = \frac{1}{2}x - 4$ **d** $y = -\frac{3}{2}x + \frac{1}{2}$

6 a $y = 2x + 2$ **b** $y = -3x - 10$
c $2y + x = 11$ **d** $3y = 5x + 24$

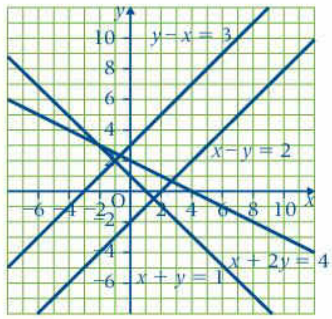
7 a, b, c, d



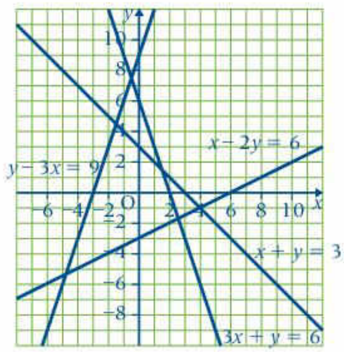
e, f, g, h



8 a, b, c, d



e, f, g, h



9 a $-\frac{1}{3}$ **b** $-\frac{3}{2}$ **c** $\frac{4}{5}$ **d** $\frac{2}{7}$

10 a $\frac{4}{3}$ **b** −2 **c** $\frac{5}{2}$ **d** $\frac{2}{3}$

11 a $5x - 2y = 6$ **b** $y = 3x + 19$
c $2x + 5y - 9 = 0$ **d** $5x - 3y + 8 = 0$

12 a i $\frac{2}{3}$ **ii** $y = \frac{2}{3}x + 4$ **b** $3x + 2y = 0$

13 a $-\frac{3}{2}$ **b** $3x + 2y - 6 = 0$
c $-\frac{3}{2}$ **d** $3x + 2y = 0$

14 a i 2 **ii** −10 **b i** 4 **ii** 3
c i 6 **ii** 6 **d i** 9 **ii** −6

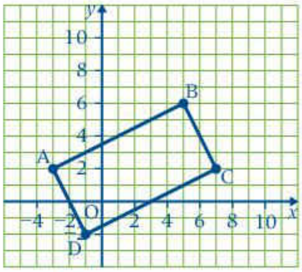
15 $y = \frac{5}{3}x - 5$

16 a −2 **b** $2x + y = 13$
c $\frac{1}{2}$ **d** $x - 2y + 12 = 0$

17 a 2 **b** $2x - y + 11 = 0$
c $x + 2y - 2 = 0$

18 **a** $5x - 6y - 19 = 0$
b $6x + 5y + 26 = 0$, $(0, -\frac{26}{5})$ and $\left(-\frac{13}{3}, 0\right)$

19

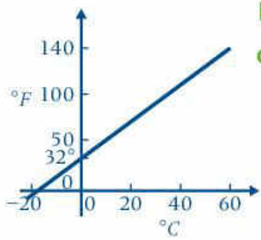


a AB; $x - 2y + 7 = 0$, DC: $x - 2y - 3 = 0$
BC: $2x + y - 16 = 0$, AD: $2x + y + 4 = 0$, rectangle
b **i** $2x + y - 6 = 0$ **ii** $x - 2y + 2 = 0$
iii $x = 3, y = 6; x = -2, y = 1$

20 **a** C **b** B **c** D **d** A

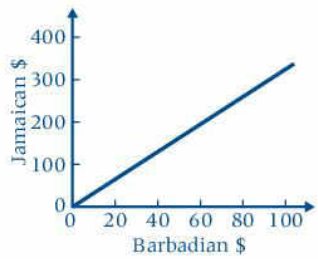
21 **a** \$100 **b** \$120 **c** \$240
d C = 100 + 20 × the number of days

22 **a**



b **i** 2°C **ii** 70°F
c $F = \frac{9C}{5} + 32$

23 **a**



b **i** JM\$187 **ii** BB\$220 **c** J = 3.4 B

24 **a** −115 000 000, the water is decreasing by 115 000 000 litres per week
b $l = -115w + 3200$
c that there is no rainfall and that water will be consumed at the same rate as during the previous 20 weeks

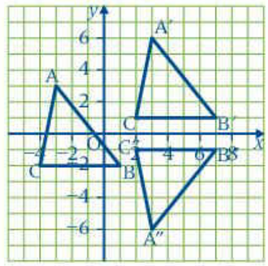
Mixed exercise 6

1 D	**2** A	**3** A	**4** B
5 C	**6** A	**7** B	**8** B
9 A	**10** D	**11** B	**12** B
13 C	**14** C	**15** D	**16** B
17 A			

Chapter 7

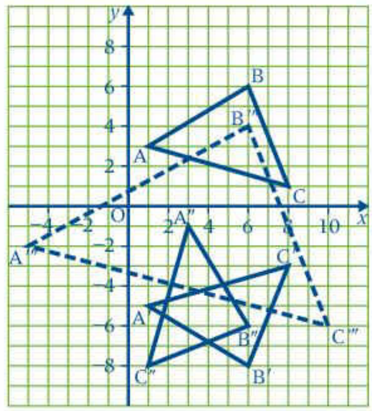
Exercise 7a

1

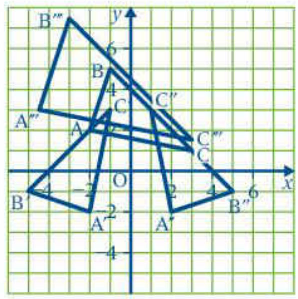


A′(3, −6), B′(7, −1), C′(2, −1)

2

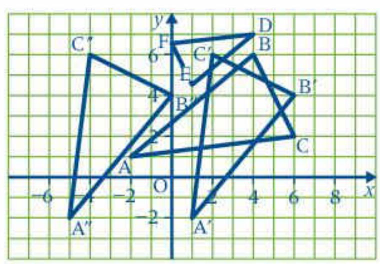


3



a A′(−2, −2), B′(−5, −1), C′(−1, 3)
b A″(2, −2), B″(5, −1), C″(1, 3)

4



A″(−5, −2), B″(0, 4), C″(−4, 6)

5 **a** (−1, 0)
b 180°; no, 180° clockwise or anticlockwise gives the same result.

6 **a** Translation defined by the vector $\begin{pmatrix} 2 \\ -5 \end{pmatrix}$
b Reflection in $y = -x$

7 **a** $\frac{1}{2}$ **b** (11, 2)

8 **a** $\begin{pmatrix}3\\0\end{pmatrix}$ **b** 180° about $\left(2, 3\frac{1}{2}\right)$

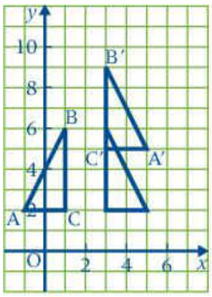
c $x = 2$

9 A′(2, 1), B′(4, 3), C′(1, 5)

10 **a** 90° clockwise **b** (5, −6)

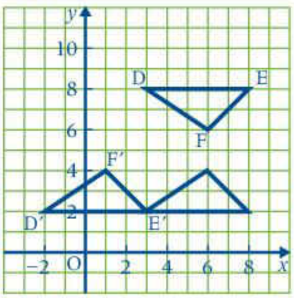
11 A(−2, −1), B(1, 1), C(5, 0), D(2, −2)

12 **a** and **b**



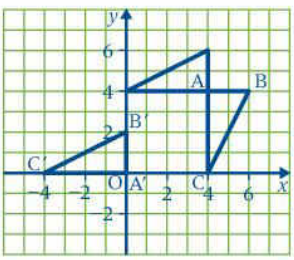
A′(5, 5), B′(3, 9), C′(3, 5)

13 **a** and **b**



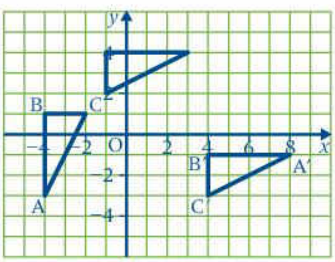
D′(−2, 2), E′(3, 2), F′(1, 4)

14 **a** and **b**



A′(0, 0), B′(0, 2), C′(−4, 0)

15 **a** and **b**



A′(8, −1), B′(4, −1), C′(4, −3)

16 **a** **i** Rotation 90° clockwise about O followed by reflection in the y-axis or rotation of 90° anticlockwise about O followed by reflection in the x-axis.

b Reflection in $y = -x$

17 **a** **i** Reflection in the y-axis

ii Reflection in $y = 1$

b Rotation 180° about (0, 1)

c **i** Rotation 90° clockwise about O

ii Rotation 90° clockwise about (−2, −2)

d Reflection in the y-axis.

18 **a** **i** Enlargement, scale factor 3, centre of enlargement (1, 1)

ii Rotation 90° anticlockwise about (4, 1)

b No, it is an enlargement followed by a reflection.

19 **a** **i** Reflection in $y = x$

ii Reflection in $y = -x$

iii Reflection in y-axis

b Reflection in the x-axis

20 **a** **i** Enlargement, centre of enlargement (4, 3), scale factor −2

ii Translation defined by the vector $\begin{pmatrix}-3\\-4\end{pmatrix}$

b **i** (−1, −1), (−1, −5), (−3, −2)

ii Enlargement, centre (−9, −5), scale factor 2

21 Glide reflection. Reflection in $y = -1$ followed by a translation parallel to this line defined by the vector $\begin{pmatrix}5\\0\end{pmatrix}$

22 Glide reflection. Reflection in $x = 2$ followed by a translation parallel to this line defined by the vector $\begin{pmatrix}0\\5\end{pmatrix}$

Exercise 7b

1 $p = 84°, q = 12°$

2 $a = 31°, b = 66°$

3 $c = 67°, d = 46°$

4 $a = 42°, b = 96°$

5 $x = 55°, y = 45°$

6 $a = 57°, b = 27°, c = 33°$

7 $e = 36°, f = 132°, g = 81°$

8 $p = 84°, q = 102°, r = 78°$

9 $a = 51°, b = 35°, c = 71°$

10 $d = 117°, e = 90°, f = 90°$

11 $g = 110°$

12 $h = 45°$

13 $k = 89°$

14 $i = 112°, j = 112°$

15 **a** 124° **b** 130°, a kite **c** a rectangle

16 **a** 150° **b** 15°

17 **a** ∠ABE = 136°, ∠BAE = ∠BEA = 22°

b ∠EAD = 126°, ∠AEB = 22°, ∠EDA = 32°

18 **a** 30° **b** a trapezium

19 a 45° b 135°

20 a 36° b 10

21 a 120°, 30°, 30° b 90°, 60°, 30°

22 $a = 137°$ **23** $b = 109°$

24 $x = 76$ **25** 300°

26 a 54°, 54°, 72° b 36°, 36°, 108°

27 8 **28** 8 **29** 18°

Exercise 7c

1 a similar
b congruent
c neither similar nor congruent
d congruent

2 b i BC = 10.8 cm ii 8.4 cm

3 a BD = 2.4 cm b 2 cm c 6.6 cm

4 a $\frac{AB}{DF} = \frac{BC}{FE} = \frac{AC}{DE}$
b DE = 2.8 cm, DP = 3.2 cm

5 a $\frac{AC}{AB} = \frac{DF}{DE}$ b 25 cm c $\frac{5}{2}$

6 a 7.5 cm b 5 cm

7 a $\frac{AB}{DE} = \frac{AC}{DF} = \frac{BC}{EF}$
b AB = 6 cm, AC = 4.5 cm c 9 : 4

8 a $\frac{AB}{DE} = \frac{BC}{EF} = \frac{AC}{DF}$
b DE = 3.2 cm, DF = 2.8 cm

9 a AX = 18 cm, AY = 21 cm
b AB = 3.6 cm, AC = 2.7 cm
c AB = 11.2 cm, AX = 6 cm, $\frac{\triangle ABC}{\triangle AXY} = \frac{49}{25}$

10 a $\frac{BC}{EF} = \frac{AC}{DF} = \frac{AB}{DE}$
b i 9.6 cm ii 10.4 cm c $\frac{25}{16}$

11 a 3 b $\frac{4}{25}$ c $\frac{4}{21}$ d $\frac{2}{3}$

21 rectangle

Exercise 7d

1 136 cm **2** 38.1 cm^2

3 a 19.0 cm b 362 cm^2

4 13.3 cm **5** 13.9 cm

6 20 m **7** 16.0 cm

8 a yes, ∠A b no c yes, ∠Z

9 a 8.54 m b 7.55 m

10 587 m

11 a 67.2 cm b 405 cm^3 c 16 200 cm^3

12 383 mm **13** 121 cm^2

14 a 62.2 ft b 0.345 s

Exercise 7e

1 a 8.92 km b 8.03 km

2 80.0 m **3** 140 m

4 a 40.4 m b 86.7 m

5 a 5.18 km b 19.3 km

6 20.1 m **7** 6.10 m **8** 130 m

9 a 16.8 cm b 24.9 cm c 56° d 11.3 cm

10 a 8.29 m b 6.80 m

11 a 14.5 cm b 14.2 cm c 50°

12 a i 130° ii 155° iii 335° iv 310°
b ∠A = 99°, ∠B = 25°, ∠C = 56°
c i 3.06 km ii 2.57 km

13 a 7.63 cm b 4.77 cm
c 62° d 8.62 cm

14 a 16.9 m b 20.2 m

15 a 81.0 m b 476 m

16 395 m

17 a i 3.42 n. miles ii 10.4 n. miles
b i 9.40 n. miles ii 6 n. miles
c i 15.4 n. miles ii 6.97 n. miles
d 16.9 n. miles on a bearing of 294°

18 a 52.0 m b 64.2 m c 1760 cm^2

19 a 6.43 cm b 7.66 cm c 6.11 cm
d 1.55 cm e 5.57 cm f 15.6°

20 a 11.5 cm b 18.5 cm

21 a i 10.6 cm ii 5.63 cm iii 14.4 cm
b 36°

22 a 13.2 cm b 22.4 cm

23 a 13.0 cm b 21.3 cm c 37°

24 a 8 cm
b AM = 4 cm, AD = 7.34 cm, MD = 6.16 cm
c 40.0 cm^2

25 a 7.13 cm b 7.86 cm c 49.9 cm^2

26 a i 6 cm ii 18.7 cm iii 26.3 cm
b i 120 cm^2 ii 245 cm^2

27 8 cm **28** a i 7.51 km ii 14.13 km
b i 18.02 km ii 12.62 km
c 28.74 km d 291°

29 **a** 48.0 m **b** 76.0 m

30 **a** **i** 43.9 cm **ii** 28.7°
b 57.0 cm **c** 27.4 cm **d** 0.137 m^3

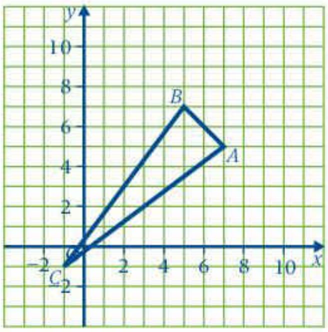
31 **a** 14.9 m **b** 31.7 m

Exercise 7f

1 **a** (4, 5) **b** (−2, 3)
c (−3, 1) **d** $(3, -6\frac{1}{2})$

2 **a** $\sqrt{29}$ **b** $\sqrt{68}$ **c** $\sqrt{116}$ **d** $\sqrt{122}$

3 **a**

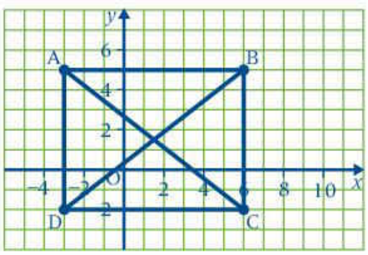


b AC = BC = 10 units **c** AB = $2\sqrt{2}$ units
d **i** 36.9° **ii** 53.1°

4 (3, 3) **5** $(-\frac{1}{2}, 1)$ **6** (2, 0)

7 **a** The distances from (3, 2) to A, B, C and D are all equal.
b 5 units **c** 81.9°

8 **a**



b **i** 9 **ii** 7 **iii** 9 **iv** 7
c rectangle **d** **i** $\left(1\frac{1}{2}, 1\frac{1}{2}\right)$ **ii** $\left(1\frac{1}{2}, 1\frac{1}{2}\right)$

9 **a** **i** 5 **ii** 5
b gradient of AB = $\frac{1}{3}$ = gradient of DC
c **i** (1, 4) **ii** $\left(2\frac{1}{2}, -\frac{1}{2}\right)$
d isosceles trapezium

10 **a** **i** $\sqrt{116}$ **ii** $\sqrt{58}$ **iii** $\sqrt{58}$
b (2, 2) **c** **i** $\frac{2}{5}$ **ii** $\frac{7}{3}$
d **i** $\sqrt{29}$ **ii** $\sqrt{29}$ **iii** $\sqrt{29}$

11 **a** **i** $\sqrt{13}$ **ii** $\sqrt{13}$ **iii** $\sqrt{65}$ **iv** $\sqrt{65}$
b **i** $(2\frac{1}{2}, 2\frac{1}{2})$ **ii** (3, 0)
c kite **d** They are all equal.

Mixed exercise 7

1 D	**2** B	**3** C	**4** B
5 A	**6** C	**7** B	**8** D
9 C	**10** C	**11** B	**12** D
13 D	**14** D	**15** C	**16** D
17 B	**18** B	**19** B	**20** C

Chapter 8

Exercise 8a

1

	mean	mode	median
a	6.43	4	6
b	2.86	2.6	2.65
c	13.8	14	14
d	0.613	0.7	0.65

2 **a** 164 cm **b** 158 cm and 164 cm **c** 164 cm

3 **a** **i** 152 cm **ii** 153 cm
b **i** no **ii** Yes, new mean is 152.8 cm.

4 **a** 20 **b** 38 **c** 1.9 **d** 1
e 2

5 **a** 13 **b** 80.8 **c** 84 and 88
d 53.8% **e** 82

6 **a** 7 **b** \$9336 **c** \$934 **d** \$768
e The modal wage since it is this that the friend is likely to be offered.

7 **a** The median is nearest to most of his scores.
b 76, no the mode is still 72

8 **a** e.g. 1, 2, 3, 4, 5, 20 **b** 1, 11, 12, 13, 14

9 **a** **i** 10 **ii** 94
b **i** 9.4 **ii** 0 **iii** 3.5
c **i** lower it to 8.82 **ii** lower it to 3
iii unchanged

10 **a** **i** 13 yr 5 m. (nearest month)
ii 10 yr 2 m. **iii** 13 yr 8 m.
b **i** mean 13 yr 3 m. **ii** mode 11 yr 2 m.
iii median 13 yr 11 m. (They are all 1 year older.)

11 **a** **i** 13.3 **ii** 16 **iii** 13.5
b any score below 13 and any score above 14
c up, new mean is $\frac{94}{7}$ i.e. 13.4

12 50

13 **a** 1092 kg **b** 546 kg **c** 546 kg **d** 78 kg

14 **a** **i** 288 **ii** 105 **iii** 393
b 5.95 **c** 399 = (12 − 2.5) × 7 × 6

15 **a** \$34 800 **b** \$48 996 **c** \$14 196

16 62.8 **17** 58 m

18 **a** 26.6 **b** It would be 27.9, no.

Exercise 8b

1 **a** 0 **b** 0 **c** 0.5

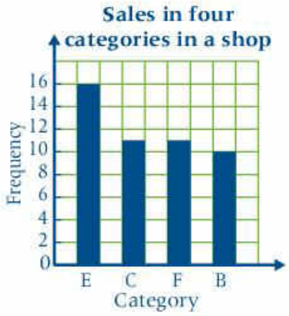
2 **a** 70 **b** mean 6, median 6, mode 6

3 **a** 92 **b** mean 4.13, mode 2, median 3

4 **a**

Category	E	C	F	B
Frequency	16	11	11	10

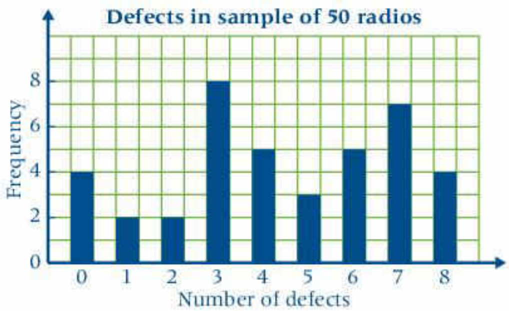
b



c 6 **d** 0 **e** 48

5 **a** 11

b

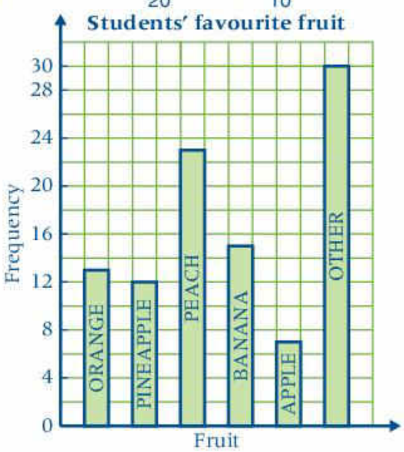


c mean 4.4, mode 3, median 4

6 **a** **i** 47 **ii** 2 **b** 69
c 46 **d** mean $\frac{2}{3}$, mode 0, median 0

7 **a** 55 **b** mean 2.49, median 2, mode 2

8 **a** 100 **b** **i** $\frac{3}{20}$ **ii** $\frac{3}{10}$



9 **a**

b **i** 38 **ii** 73
c mean 1.92, median 1.5, mode 1

10 **a** 60 **b** 23

c

d mean 70.5, median 71, mode 72

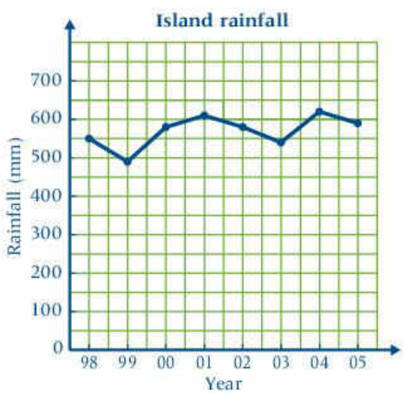
Exercise 8c

1 **a** **i** 36 °C **ii** 41 °C
b No, temperature between times is not recorded.
c normal temperature

2 **a** **i** 10 cm **ii** 25 cm
b **i** 15 cm **ii** 30 cm
c **i** 3rd week **ii** 7th week

3 **a** **i** 15 000 **ii** 22 500
b **i** 21 000 **ii** 24 000
c **i** 10000 **ii** 9000
d 2006 **e** seasonal, 2nd quarter the worst, 4th quarter the best
f anything associated with Christmas

4



The trend is slightly increasing as indicated by the straight line through the data.

5 **a** **i** 20 **ii** 0
b No, there is no indication when or how the number changed from 20 to 2 between 11 a.m. and noon.
c Residents went to another room for lunch.

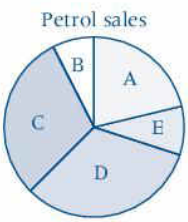
6 **a** **i** $\frac{1}{12}$ **ii** $\frac{7}{36}$ **iii** $\frac{5}{9}$
b **i** $16\frac{2}{3}$ g **ii** 6 g
c **i** 55.6 g **ii** 20 g

7 **a** **i** $\frac{7}{20}$ **ii** $\frac{3}{10}$ **iii** $\frac{13}{20}$
b **i** 27 **ii** 144 **c** **i** 30% **ii** 35%

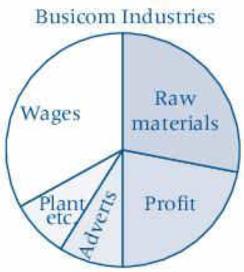
8 **a** 62.2 million sq. km **b** 20 million sq. km

9 **a** **i** $\frac{2}{9}$ **ii** $\frac{47}{72}$ **b** 9 : 47
c $11.1 million **d** 43 : 32
e A bar chart would not be a good way of illustrating this data because it would show the amounts in dollars and not the proportion of the whole in each category.

10



11 **a**



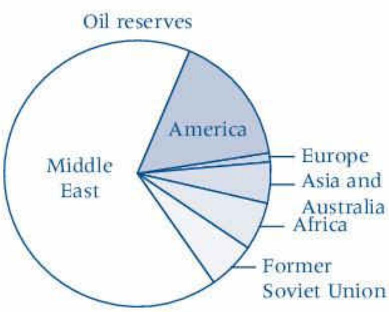
b It is easier to compare relative sizes by comparing the sizes of the pieces.

12 **a**

b

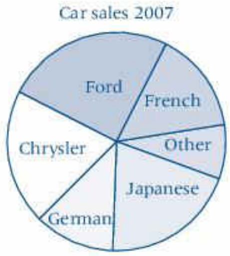
c Yes, it's easier to compare heights than angles.

13 **a**



b probably not

14 **a** **i** $\frac{3}{10}$ **ii** $\frac{13}{90}$ **iii** $\frac{19}{180}$
b **i** 670 **ii** 465
c **i** 8% **ii**



d **i** Chrysler and German (within 1%)
ii French, Japanese and Other
e Ford

15 **a** 3 **b** **i** 120° **ii** 30° **c**

Exercise 8d

1 **a** $\frac{1}{6}$ **b** $\frac{1}{2}$ **c** 1 **d** 0
2 $\frac{1}{4}$ **3** $\frac{1}{6}$
4 **a** $\frac{2}{9}$ **b** $\frac{1}{9}$ **c** $\frac{4}{9}$ **d** $\frac{5}{9}$ **e** 0 **f** $\frac{2}{3}$
5 **a** $\frac{1}{2}$ **b** $\frac{1}{2}$ **c** 0 **d** 0 **e** $\frac{2}{3}$ **f** $\frac{1}{6}$
6 **a** $\frac{1}{13}$ **b** $\frac{1}{2}$ **c** $\frac{3}{13}$ **d** $\frac{5}{26}$
7 **a** $\frac{1}{7}$ **b** $\frac{5}{21}$ **c** $\frac{1}{21}$
8 **a** $\frac{1}{4}$ **b** $\frac{1}{10}$ **c** $\frac{3}{25}$ **d** $\frac{87}{100}$
9 **a** $\frac{3}{5}$ **b** $\frac{1}{5}$ **c** $\frac{4}{5}$ **d** $\frac{1}{5}$
10 **a** $\frac{1}{40}$ **b** $\frac{3}{40}$ **c** **i** $\frac{9}{20}$ **ii** $\frac{1}{2}$
11 **a** $\frac{15}{22}$ **b** $\frac{7}{22}$ **c** $\frac{1}{22}$ **d** $\frac{3}{11}$
12 **a** $\frac{5}{14}$ **b** $\frac{1}{7}$ **c** 0 **d** $\frac{5}{14}$ **e** $\frac{9}{14}$
13 **a** $\frac{4}{5}$ **b** $\frac{3}{20}$ **c** $\frac{17}{20}$
14 **a** yes **b** **i** $\frac{1}{4}$ **ii** $\frac{1}{26}$ **iii** $\frac{6}{13}$
15 **a** $\frac{1}{3}$ **b** $\frac{2}{3}$
16 $\frac{7}{19}$ **17** **a** 15 **b** 45 **c** 30
18 **a** **i** $\frac{4}{5}$ **ii** theoretical **b** 2
19 **a** **i** $\frac{11}{31}$ **ii** $\frac{9}{124}$ **iii** $\frac{17}{31}$ **b** 30
20 **a** $\frac{5}{18}$ **b** $\frac{1}{3}$ **c** $\frac{19}{36}$ **d** $\frac{17}{36}$
21 **a** calculate for **a**, experiment for **b**, **c** and **d**

Exercise 8e

1 exclusive
2 **a** exclusive **b** independent

3 a independent b exclusive

4 a exclusive b independent

5 a exclusive b independent

6 a independent b neither

7 a $\frac{1}{26}$ b $\frac{1}{26}$ c $\frac{1}{13}$

8 a $\frac{1}{6}$ b $\frac{1}{3}$ c $\frac{1}{2}$

9 a $\frac{1}{6}$ b $\frac{1}{6}$ c $\frac{1}{3}$

10 a $\frac{74}{117}$

b $\frac{43}{117}$, ignoring the chance that it is a pocket in a garment in the car!

11 a $\frac{4}{7}$ b $\frac{11}{14}$ c $\frac{1}{14}$

12 a $\frac{32}{63}$ b $\frac{17}{36}$ c $\frac{191}{252}$ d $\frac{61}{252}$

13 $\frac{19}{45}$ 14 a $\frac{1}{2}$ b $\frac{1}{2}$

c $\frac{5}{6}$, 2 is both even and prime.

15 $\frac{1}{36}$ 16 a $\frac{1}{4}$ b $\frac{3}{4}$ c $\frac{9}{16}$

17 a $\frac{6}{25}$ b $\frac{6}{25}$

18 a $\frac{1}{2}$ b $\frac{1}{4}$ c $\frac{1}{4}$

19 a $\frac{2}{3}$ b $\frac{3}{4}$ c $\frac{1}{6}$

20 not independent (you cannot eat the same sweet twice)

21 a $\frac{1}{3}$ b $\frac{1}{3}$ c $\frac{1}{36}$ d $\frac{1}{4}$

22 a $\frac{1}{13}$ b $\frac{1}{26}$ c $\frac{3}{26}$

23 a $\frac{1}{3}$ b $\frac{5}{6}$

24 0.54 25 a $\frac{2}{13}$ b $\frac{2}{13}$ c $\frac{1}{169}$

Exercise 8f

1 a i $\frac{1}{30}$ ii $\frac{29}{30}$ b $\frac{25}{26}$ c $\frac{1}{21}$

2 a i $\frac{42}{73}$ ii $\frac{31}{73}$ b $\frac{46}{73}$ c $\frac{1}{7}$

d $\frac{3}{73}$

3 a $a = 25, b = 15, c = 90$

b $\frac{3}{8}$ c $\frac{7}{10}$ d $\frac{1}{3}$ e $\frac{7}{16}$

4 a $a = 240, b = 160, c = 200, d = 400, e = 360, f = 640$

b $\frac{2}{5}$ c $\frac{1}{2}$ d $\frac{7}{16}$

5 a $a = 550, b = 650, c = 1200$

b $\frac{42}{65}$ c $\frac{8}{11}$ d $\frac{19}{60}$ e $\frac{41}{60}$

6 a 145, 13, 27, 205, 245

b $\frac{1}{29}$ c $\frac{13}{245}$ d $\frac{8}{49}$ e $\frac{3}{25}$

f yes, 0.86 compared with 0.8

g $\frac{1}{5}$ h $\frac{20}{29}$

7 a $a = 50, b = 40, c = 15, d = 25, e = 100$

b $\frac{13}{24}$ c $\frac{2}{5}$ d $\frac{2}{3}$ e $\frac{9}{10}$

f $\frac{4}{11}$ g $\frac{21}{44}$

8 a Missing values for A 90 and 60, for B 80, for Total 105, 450

b i 140 ii 55 iii 150

c i $\frac{9}{20}$ ii $\frac{7}{10}$

d i $\frac{5}{9}$ ii $\frac{23}{90}$ iii $\frac{1}{9}$ iv $\frac{31}{45}$

9 a Missing values: line 1, 50, 10; line 2, 40, 12 ; line 3, 90, 150

b i 42 ii 90 iii 18

c i $\frac{1}{9}$ ii $\frac{5}{9}$

d i $\frac{2}{3}$ ii $\frac{1}{5}$ e $\frac{2}{135}$

Mixed exercise 8

1 C	2 C	3 C	4 C
5 A	6 C	7 D	8 D
9 A	10 B	11 B	12 B
13 A	14 B		

Chapter 9

Exercise 9a

1 a 3 b 2 c −2

2 a 5 b 7 c 5

3 a 17 b $\frac{17}{16}$ c 24

4 a $\frac{4}{3}$ b −12 c $\frac{3}{10}$

5 a 7 b 18 c 15

6 a 36 b 20736 c 36864

7 a 12 b 12 c 12

8 a $\frac{5}{4}$ b $-\frac{5}{8}$ c $-\frac{5}{2}$

9 a 14 b 46 c i 48 ii 42

10 a i 4 ii 4 iii 4 b i yes ii no

11 a i 1 ii 1 iii 3 b i yes ii no

12 a i 9 ii 9 iii 11 b i yes ii yes

13 a i 0 ii −6 iii −6 iv 12 b i no ii no

14 a i 11 ii 11 iii 14 b i yes ii yes

15 a i $\frac{5}{2}$ ii $\frac{8}{5}$ iii $\frac{5}{2}$ iv $\frac{5}{2}$ b i no ii no

16 a 24 b ±3

17 a 7 b 6 18 a 1 b 3 or 7

19 a $\frac{5}{6}$ b 4 20 a $\frac{16}{15}$ b 4

21 a 40 b 4 c ±5

22 a 6 b 4 c 1

23 a 30 b yes

24 $3 \wedge 2 = \frac{6}{5}$, $2 \wedge 3 = -\frac{6}{5}$ so not commutative

25 non-commutative; 2 * 3 = 16, 3 * 2 = 1

Exercise 9b

1 $x^2 + 3xz + xy + 3yz$
2 $x^2 + 4xz + 2xy + 8yz$
3 $x^2 + 5xz + 3xy + 15yz$
4 $x^2 + 5xz + 4xy + 20yz$
5 $a^2 + 7ac + 5ab + 35bc$
6 $a^2 + ac + 4ab + 4bc$
7 $a^2 + 6ac + 3ab + 18bc$
8 $a^2 + 9ac + 2ab + 18bc$
9 $x^2 - 3xz - 2xy + 6yz$
10 $x^2 - 5xz - 4xy + 20yz$
11 $x^2 - xz - 3xy + 3yz$
12 $x^2 - 6xz - 7xy + 42yz$
13 $x^2 - 7xz + 2xy - 14yz$
14 $a^2 + 2ab - 15b^2$
15 $a^2 - ac + 10ab - 10bc$
16 $a^2 - 5ac + 6ab - 30bc$
17 $abx^2 + 3adx - 2bcx - 3cd$
18 $pr - 4ps + 2qr - 8qs$
19 $x^2 - 2x - 35$
20 $a^2 + 3a - 4$
21 $x^2 + 2x - 15$
22 $a^2 + 5a - 14$
23 $x^2 + x - 6$
24 $x^2 - 9x + 20$
25 $x^2 + 2x - 3$
26 $x^2 - 13x + 42$
27 $x^2 - 9x + 14$
28 $a^2 - 2a - 15$
29 $a^2 - 11a + 10$
30 $a^2 - a - 30$
31 $x^2 + 10x + 9$
32 $x^2 + 2x - 63$
33 $x^2 + 17x + 72$
34 $2x^2 - x - 15$
35 $3x^2 + 34x - 24$
36 $2p^2 - 7p - 22$
37 $20n^2 + 2n - 42$
38 $14a^2 - 3a - 5$
39 $8x^2 + 117x - 45$
40 $15q^2 - 41q - 30$
41 $7c^2 - c - 8$
42 $25p^2 + 30p - 7$
43 $12a^2 - 31a - 30$
44 $20b^2 - 104b + 63$
45 $20q^2 - 37q - 18$
46 $5p^2 - 57p - 36$
47 $14p^2 - 65p + 56$
48 $x^2 - 5x - 66$
49 $x^2 - 12x + 35$
50 $15 + 2a - a^2$
51 $10 - 25a + 10a^2$
52 $30 + 19a - 4a^2$
53 $2x^2 + 7x - 9$
54 $-x^2 + 13x - 42$
55 $72 - 7x - 2x^2$
56 $21 + 32m - 5m^2$
57 $5a^2 - 38ab + 48b^2$
58 $4x^2 + 20x + 25$
59 $4x^2 - 20x + 25$
60 $4x^2 - 25$
61 $25n^2 + 60n + 36$
62 $49a^2 - 70a + 25$
63 $64x^2 - 9$
64 $25q^2 - 9$
65 $49c^2 - 112c + 64$
66 $9x^2 - 42x + 49$
67 $9x^2 + 42x + 49$
68 $9x^2 - 49$
69 $36n^2 + 60n + 25$
70 $49p^2 - 56p + 16$
71 $16x^2 - 49$
72 $25q^2 - 16$
73 $25q^2 - 110q + 121$
74 $4x^2 + 9$
75 $4x^2 + 1$
76 $4x^2$
77 $25n^2 + 70n + 16$
78 $18a^2 - 24a + 20$
79 $40x^2 - 33x - 9$
80 $6x^2 - 6x - 27$
81 $14x^2 - 33x + 32$
82 $3x^2 - 12x + 12$
83 $5x^2 - 20$
84 $-5x^2 + 20x - 16$
85 $8x^2 + 12x - 8$
86 $3x^2 - 26x + 54$
87 $5x^2 + 7x - 80$
88 $8x^2 - 22x + 17$
89 $17x^2 + 5x - 4$

Exercise 9c

1 $(a + 3)(b + 3)$
2 $(a + 1)(b + c)$
3 $(x + y)(x + z)$
4 $(b - c)(a + d)$
5 $(a + 2)(b - 5)$
6 $(p - 3)(q + 4)$
7 $(a - d)(b - c)$
8 $(x + 2y)(x - 2)$
9 $(a + b)(1 - b)$
10 $(2a - b)(b + 1)$
11 $(2s - 5t)(1 - t)$
12 $(3y - x)(2y - 3)$
13 $(a + 3)(b - c)$
14 $(p - 3)(2 - q)$
15 $(x - 4)(3 - y)$
16 $(x - y)(x + 1)$
17 $(3p + q)(p - 1)$
18 $(a - 1)(b - 1)$

Exercise 9d

1 $(x + 5)(x + 7)$
2 $(x - 4)(x - 6)$
3 $(x + 3)(x + 7)$
4 $(x - 3)(x - 4)$
5 $(x + 4)(x + 8)$
6 $(x - 3)(x + 5)$
7 $(x - 4)(x + 8)$
8 $(x - 5)(x - 7)$
9 $(x + 15)(x - 2)$
10 $(x - 10)(x - 12)$
11 $(x - 2)(x - 5)$
12 $(x - 9)(x + 7)$
13 $(x + 13)(x - 2)$
14 $(x - 7)(x + 5)$
15 $(x - 7)(x + 13)$
16 $(x + 3)(x - 17)$

17 $(x-4)(x-9)$
18 $(x-4)(x+9)$
19 $(x-4)(x-6)$
20 $(x-3)^2$
21 $(x-2)(x-5)$
22 $(x+3)(x+5)$
23 $(x-7)(x-9)$
24 $(x+5)(x+8)$
25 $(x-1)(x-10)$
26 $(x-4)(x-7)$
27 $(1+x)(1-2x)$
28 $(7+x)(2-x)$
29 $(2-x)(3+x)$
30 $(7-x)(5+x)$
31 $(7+x)(9-x)$
32 $(10+x)(1-x)$
33 $(12-x)(7+x)$
34 $(8-x)(9+x)$
35 $(6-x)(5+x)$
36 $(7+x)(10-x)$
37 $(x+6)(x-6)$
38 $(x+7)(x-7)$
39 $(x+10)(x-10)$
40 $(x+1)(x-1)$
41 $(x+11)(x-11)$
42 $(x+8)(x-8)$
43 $(x+2)(x-2)$
44 $(a+1)(a-1)$
45 $(p+9)(p-9)$
46 $(x+12)(x-12)$
47 $(2x+7)(2x-7)$
48 $(3x+5)(3x-5)$
49 $(5x+6)(5x-6)$
50 $(3x+10)(3x-10)$
51 $(2x+5y)(2x-5y)$
52 $(3x+4y)(3x-4y)$
53 $(2x+7y)(2x-7y)$
54 $(6p+7q)(6p-7q)$
55 $(10a+9b)(10a-9b)$
56 $(11x+8y)(11x-8y)$
57 $\left(\frac{x}{2}+\frac{y}{3}\right)\left(\frac{x}{2}-\frac{y}{3}\right)$
58 $\left(2x+\frac{y}{2}\right)\left(2x-\frac{y}{2}\right)$
59 $\left(\frac{a}{5}+\frac{b}{4}\right)\left(\frac{a}{5}-\frac{b}{4}\right)$
60 $\left(\frac{2x}{5}+1\right)\left(\frac{2x}{5}-1\right)$
61 $2(x-3)^2$
62 $2(3x-2)(2x+3)$
63 $5(2x-3)(2x-5)$
64 no factors
65 $(3x-1)(3x+2)$
66 $2(5x-2)(3x-4)$
67 $5(x+5)(x-5)$
68 $4(x+3)(x-3)$
69 $2(5+x)(3-x)$
70 $6(3+x)(2-x)$
71 $2(x^2-8x+63)$
72 $5(x-1)(x+2)$
73 $(2x-3)(x+5)$
74 $(3x+1)(2x-5)$
75 $(2x-1)(x-4)$
76 $2x+1)(x-3)$
77 $(3x-2)(x-4)$
78 $(2x-5)(2x+3)$
79 $(3x+2)(3x-5)$
80 $(3x-7)(x-7)$
81 $(5x-3)(7x+2)$
82 $(4x+3)(x-1)$
83 $(3x-1)(2x+7)$
84 $(2x+3)(6x-1)$
85 $(5x-4)(8x+3)$
86 $(5x+4)(4x-3)$
87 $(4x-3)(2x+9)$
88 $(2x-3)(2x+9)$
89 $(4x-3)(4x-7)$
90 $(9x-2)(9x-1)$
91 $(7x-1)(2x+3)$
92 $(4x-3)(2x+5)$
93 $(8x+1)(8x+7)$
94 $-x(x+4)$
95 $-4x(x+1)$
96 $4(x+1)(x-2)$
97 $(x+1)(x-7)$
98 $(3x+1)(3x+5)$
99 $-4(3x+1)(3x-4)$
100 $(x+2)(3x+7)$
101 $(3x-1)(3x+1)$
102 $(5x+1)(5x+2)$
103 $(x-1)(x-3)$
104 $(3x-4)(3x-14)$
105 $(5x-1)(5-x)$

Exercise 9e

1 $\frac{1}{x-y}$
2 $\frac{1}{x+3}$
3 $\frac{1}{x+6}$
4 $\frac{1}{x+7}$
5 $\frac{1}{a+2}$
6 $\frac{1}{x+2}$
7 $\frac{1}{3x-2}$
8 $\frac{3}{x-2}$
9 $(x-3)$
10 $(3x-1)$
11 $(7x+2)$
12 $(5x-3)$
13 $(3x+5)$
14 $(x-4)$
15 $(2x+1)$
16 $\frac{2x-3}{x-4}$
17 $\frac{2x-5}{x+7}$
18 $\frac{3x+1}{4x+1}$

Exercise 9f

1 $5, -9$
2 $7, 12$
3 $\frac{1}{2}, 4$
4 $-\frac{1}{2}, 3$
5 $\frac{2}{3}, 4$
6 $-2, -\frac{1}{3}$
7 $\frac{1}{2}, -\frac{2}{3}$
8 $-\frac{1}{2}, \frac{5}{3}$
9 $-\frac{1}{5}, \frac{3}{2}$
10 $-\frac{5}{2}, \frac{3}{4}$
11 $-\frac{3}{2}, \frac{1}{7}$
12 $\frac{5}{6}, 3$
13 $\frac{4}{3}, 3$
14 $3, \frac{9}{2}$
15 $\frac{7}{3}, 7$
16 $-\frac{1}{2}, \frac{7}{3}$
17 $-\frac{3}{2}, \frac{5}{2}$
18 $-\frac{2}{3}, \frac{5}{3}$
19 $-\frac{4}{3}, \frac{1}{2}$
20 $-\frac{5}{6}, 3$
21 $\frac{1}{2}, \frac{4}{3}$

Exercise 9g

1 $-0.83, 4.83$
2 $-4.24, 0.24$
3 $0.28, 2.72$
4 $-0.73, 2.73$
5 $0.46, 1.21$
6 $-6.24, -1.76$
7 $2(x+3)^2-13$
8 $4(x+2)^2-3$
9 $2(x-3)^2+5$
10 $5(x-5)^2-105$
11 $7(x+5)^2-225$
12 $4(x-\frac{1}{2})^2-4$

13 0.88, 5.12
14 0.42, 5.58
15 −5.39, 1.39
16 0.55, −12.55
17 −0.47, 1.97
18 −2.34, 0.34
19 −5.61, 1.61
20 −0.74, 6.74
21 −2.87, 0.87

Exercise 9h

1 −3.14, 0.64
2 −0.39, 3.89
3 −1.43, – 0.23
4 −0.36, 0.79
5 −1.82, 0.22
6 −1.45, 1.20
7 −0.87, 1.54
8 −0.82, 0.12
9 −4.64, −0.86
10 −1.27, −0.39
11 −2.39, 1.89
12 −1.84, −0.41
13 0.67, 2
14 0.31, 3.19
15 −0.89, 0.32
16 −0.59, 2.26
17 −0.67, 2.07
18 −0.41, 0.91
19 −1.80, −0.37
20 0.73, −1.06
21 −2.10, 1.43

22 **a** no, $b^2 - 4ac < 0$
b yes, $b^2 - 4ac$ is positive
c no, $b^2 - 4ac < 0$ **d** no, $b^2 - 4ac < 0$

23 Parallel sides 5 cm and 7 cm are 2.5 cm apart.
24 24
25 9, 10, 11
26 4.57 cm, 6.57 cm
27 7.08 m, 12.08 m
28 12
29 9 cm, 12 cm
30 4 cm
31 **a** 5 cm **b** 2 cm, 10 cm
32 5.96 cm, 18.04 cm
33 2 cm
34 \$20, \$25
35 12 cm by 18 cm
36 15 cm by 20 cm
37 12
38 7.24 m by 5.52 m or 2.76 m by 14.48 m
39 1.44 m
40 0.27 or 3.73
41 **a** After 2 s going up and 4 s coming down.
b 6 s
42 6 cm
43 20.49 m
44 0.92 m
45 5.62 cm, 7.12 cm; 4.54 cm
46 5.59 cm
47 2.91 m
48 40 km/h

Exercise 9i

1 $t = \frac{v - u}{a}$
2 $c = \frac{N - a}{b}$
3 $m = \frac{y - c}{x}$
4 $h = \frac{1}{2}\left(\frac{A}{\pi r} - 3r\right)$
5 $\mu' = \frac{W - S}{\mu S}$
6 $h = \frac{A - \pi r^2}{2r}$
7 $\frac{c = a + 2b + 3ab}{3a \ + \varphi\, 1}$
8 $q = \frac{p - r + 3pr}{3(p + 1)}$
9 $A = B + PC$
10 $R = \frac{s - r}{Q}$
11 $r = \frac{1}{p + q}$
12 $q = 3r - p$
13 $b = \frac{a}{c - 1}$
14 $m = \frac{a}{b - a}$
15 $p = \frac{1}{r} - q$
16 $T = \frac{100\, I}{PR}$
17 $z = y - \frac{1}{x}$
18 $a = \frac{bcr}{bc - cr - br}$
19 $c = b - \frac{2aP}{W}$
20 $b = \frac{a(2h - 3x)}{3x - h}$
21 $r = \frac{S - a}{S}$
22 $h = \frac{V}{\pi r^2} - \frac{r}{3}$
23 $x = -(a + b)$
24 $x = \frac{2z^2 - y^2}{2y - z}$
25 $w = \frac{2W(T_2 - T_1)}{T_1 - 3T_2}$
26 $c = -\frac{(a + b)}{2}$
27 $r = \sqrt{\frac{V}{\pi h}}$
28 $n = \sqrt{\frac{A - b}{c}}$
29 $R = \sqrt{\frac{A}{\pi} + r^2}$
30 $u = \sqrt{v^2 + 2as}$
31 $M = \frac{m(g\sqrt{3} - f)}{4f}$
32 $h = \sqrt{\frac{A^2}{\pi^2 r^2} + r^2}$
33 $g = \frac{4\pi^2 l}{T^2}$
34 $c = s - \frac{A^2}{s(s - a)(s - b)}$
35 $p = C^2 + q$
36 $b = \sqrt{a^2 - L^2}$
37 $p = q\sqrt{\frac{n^2 + 4}{3n^2 - 1}}$
38 $\mu = \frac{\sqrt{F^2 - P^2}}{P}$
39 $\lambda = \frac{\pi^2 am}{4T^2}$
40 $g = \frac{4\pi^2 (h^2 + k^2)}{T^2 h}$
41 $c = \sqrt{\frac{b - a^2}{a}}$
42 $y = \frac{\sqrt{b^2 x^2 - a^2}}{b}$
43 $z = \pm x$
44 $a = b\sqrt{2}\sqrt{\frac{A^2 + 2B^2}{5A^2 + 3B^2}}$

Mixed exercise 9

1 C 2 D 3 D 4 B 5 B
6 C 7 C 8 B 9 D 10 C
11 D 12 B 13 C 14 D 15 D
16 C 17 B 18 D 19 C 20 C

Chapter 10

Exercise 10a

1 $a = 63°$
2 $b = 62°, c = 124°$
3 $d = 112°, e = 68°$
4 $f = 154°$

5 $g = 228°$ 6 94° 7 47°
8 a 130° b 25° 9 52°
10 a 65° b 90°
11 a 152° b 28°
12 a 36° b 72°
13 a 90° b 66°
14 57° 15 28°
16 a i 37° ii 37° b They are parallel.
18 71° 20 a i 19° ii 19°
22 b i 46° ii 44° iii 112°

Exercise 10b

1 $a = 107°, b = 27°$ 2 $c = 98°$
3 $d = 35°, e = 105°$ 4 a 60° b 75°
5 a 68° b 84° 6 a 79° b 113°
7 a 21° b 55° 8 a 35° b 20°
9 a i 68° ii 68° 10 57°
11 a 72° b 144° c 55°
12 a i 128° ii 38°
13 a 39° b 63° c 117° d 12°

Exercise 10c

1 a 33° b 114° 2 4.84 cm
3 138° 4 9.95 cm
5 a 7.48 cm b 67.5° 6 a 76.0° b 10.2 cm
7 a 11.1 cm b 13.4 cm
8 a 8.15 cm b 9.84 cm
9 a 4.97 cm b i 56.4° ii 113° c 8.28 cm
10 a i 63° ii 27°
11 a 77.4° b 6.41 cm c 51.2 m^2
12 a 8 cm b 73.7° c 24 m^2
13 a 60° b 4.62 cm c i 18.5 m^2 ii 37.0 cm^2
14 19.6 cm, 21.0 cm, 22.2 cm 15 28.6 cm

Exercise 10d

1 $a = 16°$ 2 $b = 62°, c = 40°$
3 $d = 34°, e = 56°$
4 a 90° (∠ in semicircle) b 37° (alt. seg. theo.)
c 53° (∠ s of a triangle)
5 a ∠BAC = ∠ACS = 75° (alt. seg. theo.)
b ∠ACB = 180° − 75° − 75° = 30°
6 a 38° (∠s of a triangle) b 68° (alt. seg. theo.)
c 74° (alt. seg. theo.)
7 ∠A = 38°, ∠B = 48°, ∠C = 94°
8 a 52° b 90° c 52°
9 a 40° b 50°
10 a 62° b 34° c 62°
11 a 12° b 12°
12 a 25° b 50° c 25°
13 a 134° b 67° c 23°
14 a 110° b 50° c 30°
15 a 32° b 32° c 90° d 58°
16 a 49° b 78° c 78°
17 a 68° b 112° c 34°
18 a 74° b 53° c 37°
d 53° e 48° f 79°
19 a 42° b 42° c 42°
d 84° e 54° f 54°
20 a 100° b 34° c 66°

Mixed exercise 10

1 C 2 B 3 D 4 B
5 D 6 D 7 A 8 B
9 B 10 B 11 A 12 C
13 C 14 D 15 A 16 B

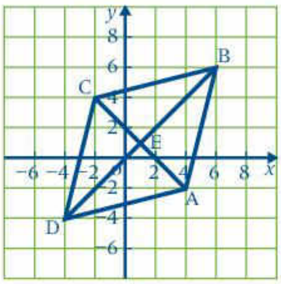
Review Test 2

1 A 2 B 3 C 4 C 5 B
6 B 7 C 8 C 9 D 10 D
11 D 12 A 13 B 14 A 15 C
16 C 17 B 18 B 19 D 20 C
21 D 22 D 23 A 24 A 25 B
26 D 27 A 28 D 29 B 30 C

General proficiency questions

1 31 360 kg 2 5778 kg 3 30 cm
4 a i 9 ii 9 iii 77
b i commutative because $4 \wedge 5 = 5 \wedge 4$
ii not associative because $(4 \wedge 5) \wedge 2$ is not equal to $4 \wedge (5 \wedge 2)$
5 52.7 cm^2

6 a, b, c



c ii (−4, −4) **iii** E(1, 1)
d i 2 **ii** rhombus
e i 4 **ii** $\frac{1}{4}$
f i $\sqrt{68}$ **ii** $\sqrt{68}$

7 a A′(1, 2), B′(6, 3), C′(4, 7)
b Reflection in $y = x$

8 a i Reflection in $y = -x$
ii Reflection in x-axis
iii Rotation 180° about O
b Rotation 90° clockwise about O

9 a i 2 **ii** 18 **b** no
c Shows 2 early risers, builds up in numbers during the morning, all but 2 leave for lunch, builds up again in the afternoon to a maximum of 24, all leave for the evening meal, some return, all in bed by 11 p.m.
d i yes because the pattern of use will remain much the same each day
ii no because hotel residents usually stay out all day

10 7 cm

11 a $(a - b)^2$ **b i** 9 **ii** 9 **iii** 49 **iv** 1

12 a $wx + 5xz - 3wy - 15yz$
b $acx^2 - adx + bcx - bd$
c $m^2 - 6m - 27$ **d** $35s^2 - 97s + 44$
e $(x - 10)(x + 11)$ **f** $(x + 3)(x - 13)$
g $(x + 7)(x + 9)$

13 a $(x + 3)(x - 4)$ **b** $(2x + 3)(6x + 5)$
c $(x + 6)(x + 9)$

14 a $2x + 1$ **b** $\frac{x + 7}{x + 3}$

15 4.58 m **16 a** $\frac{mgx}{(2mg - T)}$ **b** $p = \pm\frac{q}{\sqrt{5}}$

17 6 cm **18 a** 120° **b** 16°

19 a −6 **b** $y = -6x + 17$
c $x - 6y - 16 = 0$

20 a i 20 **ii** 20 **iii** 2 **b** cannot divide by 0

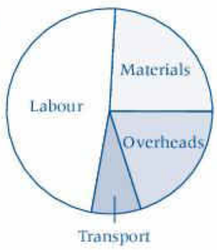
21 a \$10 **b** \$0.01
c $C = 0.01n + 10$ **d** \$75

22 a $a = 80°$, $b = 28°$ **b** 94°

25 a 4 **b** 3 **c** 2.81 **d** $\frac{6}{31}$

26 a 10 cm **b** 53.1° **27** 2.4 mm

28 a



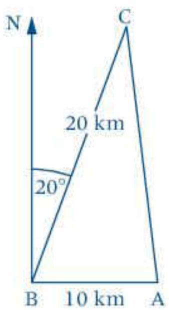
b A bar chart does not show clearly the proportions that the categories are of the whole cost.

29 a $\frac{1}{2}$ **b** $\frac{2}{5}$ **c** $\frac{3}{10}$

30 a $x = 3$, $y = 4\frac{1}{2}$ **b** −0.87 and 1.54

31 $5\left(x + \frac{7}{10}\right)^2 - \frac{29}{20}$, −1.24, −0.16

32 a 130° **c** 10.7 cm **33**



34 $d = e = 50°$

35 a $\frac{2 - 3x}{x^2}$ **b** $\frac{x}{x - 2}$

36 a 36 **b** −2 or 7

37 a $-\frac{5}{2}$ **b** $5x + 2y - 12 = 0$ **c** (1.2, 3)

38 a $(x + y + 1)(x + y - 1)$ **b** $(x + 1)(x + 2)$

39 a i 6 **ii** 6 **iii** 6 **b** $\frac{7}{20}$

40 No, (longest side)^2 is not equal to the sum of the squares of the other two sides.

42 2 **43 a** $\frac{3x + 1}{2x + 1}$ **b** $\frac{2}{3x - 2}$

44 4.5 cm **45** 78%

Chapter 11

Exercise 11a

1 $\begin{pmatrix} 12 \\ -3 \end{pmatrix}$ **2** $\begin{pmatrix} 6 & 4 \\ -1 & 3 \end{pmatrix}$ **3** $\begin{pmatrix} -1 & 9 \\ 7 & -7 \end{pmatrix}$

4 $\begin{pmatrix} 2 & -2 \\ -2 & 0 \end{pmatrix}$ **5** $\begin{pmatrix} 7 & 0 \\ 13 & 0 \end{pmatrix}$ **6** not possible

7 not possible **8** $\begin{pmatrix} 4 & 3 & 10 \\ -2 & 4 & -2 \end{pmatrix}$

9 $\begin{pmatrix}26\\19\end{pmatrix}$ **10** $\begin{pmatrix}10\\-12\end{pmatrix}$ **11** $\begin{pmatrix}8\\-11\end{pmatrix}$

12 $\begin{pmatrix}9&12\\24&33\end{pmatrix}$ **13** $\begin{pmatrix}16&17\\28&33\end{pmatrix}$ **14** $\begin{pmatrix}18\\14\end{pmatrix}$

15 $\begin{pmatrix}9&1\\2&-12\end{pmatrix}$ **16** $\begin{pmatrix}11&-4\\18&-8\end{pmatrix}$ **17** $\begin{pmatrix}14&-22\\-12&21\end{pmatrix}$

18 $\begin{pmatrix}10&-12\\29&-6\end{pmatrix}$ **19** $\begin{pmatrix}-21&12\\21&-12\end{pmatrix}$ **20** $\begin{pmatrix}2&-20\\-6&0\end{pmatrix}$

21 $\begin{pmatrix}-20&-80\\36&0\end{pmatrix}$ **22** $\begin{pmatrix}8&10\\8&7\end{pmatrix}$ **23** $\begin{pmatrix}3&6\\6&-3\end{pmatrix}$

24 $\begin{pmatrix}-4&2\\7&6\end{pmatrix}$ **25** $\begin{pmatrix}10&1\\-10&0\end{pmatrix}$ **26** $\begin{pmatrix}33&47\\-33&-47\end{pmatrix}$

27 $\begin{pmatrix}-10&6\\22&-34\end{pmatrix}$ **28** $\begin{pmatrix}4&7\\0&-1\end{pmatrix}$ **29** 79

30 $a = -2, b = 2$ **31** $a = -2, b = 4$

32 $a = 3, b = 2$ **33** $a = 4, b = 2$

34 $a = -1, b = 1$ **35** $a = -3, b = -4$

36 $a = 5, b = 7$ **37** $a = -3, b = 2$

38 $a = -3, b = 8$ **39** $a = 1, b = 2$

40 $a = 5, b = 9$ **41** $a = \frac{8}{3}, b = -\frac{9}{2}$

42 $a = 5, b = 3$

Exercise 11b

1 $\begin{pmatrix}1&-2\\-1&\frac{5}{2}\end{pmatrix}$ **2** $\begin{pmatrix}4&-9\\-3&7\end{pmatrix}$ **3** $\begin{pmatrix}-3&8\\2&-5\end{pmatrix}$

4 no inverse, determinant = 0

5 $\begin{pmatrix}-2&-3\\3&4\end{pmatrix}$ **6** $\begin{pmatrix}-1&\frac{-5}{2}\\2&\frac{11}{2}\end{pmatrix}$ **7** $\begin{pmatrix}-1&-2\\2&\frac{7}{2}\end{pmatrix}$

8 $\begin{pmatrix}3&-2\\-\frac{7}{3}&\frac{5}{3}\end{pmatrix}$ **9** $\begin{pmatrix}-\frac{1}{5}&-\frac{1}{5}\\-\frac{3}{20}&\frac{1}{10}\end{pmatrix}$ **10** $\begin{pmatrix}-\frac{1}{3}&\frac{2}{3}\\\frac{2}{3}&-\frac{5}{6}\end{pmatrix}$

11 $\begin{pmatrix}\frac{1}{2}&0\\-\frac{1}{6}&-\frac{1}{3}\end{pmatrix}$ **12** $\begin{pmatrix}2&3\\5&7\end{pmatrix}$ **13** 1

14 2 **15** $\frac{1}{2}$ **16** 4

17 $-\frac{1}{2}$ or 3 **18** 1 or 3

Exercise 11c

1 $x = 6, y = 2$ **2** $p = 5, q = -3$

3 $m = 6, n = -3$ **4** $x = 1, y = -1$

5 $x = -3, y = 2$ **6** $x = \frac{1}{2}, y = \frac{1}{3}$

7 $a = 2\frac{1}{2}, b = -\frac{1}{2}$ **8** $p = \frac{1}{4}, q = \frac{2}{3}$

9 $x = -2, y = -6$ **10** $x = -\frac{6}{5}, y = -\frac{8}{5}$

11 $x = -1\frac{1}{2}, y = -2\frac{1}{3}$ **12** $p = 1\frac{3}{5}, q = 1\frac{2}{5}$

13 $x = 1\frac{3}{10}, y = -1\frac{1}{5}$ **14** $x = -\frac{8}{5}, y = \frac{12}{5}$

15 $x = -2, y = 11$ **16** $x = \frac{6}{5}, y = \frac{12}{5}$

17 $a = \frac{4}{3}, b = 3$ **18** $x = \frac{7}{3}, y = \frac{3}{2}$

19 $x = 2, y = 5$ **20** $x = 3, y = -2$

21 $x = 2, y = 3$ **22** $x = 3, y = -2$

23 $x = 3, y = 2$ **24** $x = 4, y = 3$

25 $x = -2, y = 2$ **26** $x = 2, y = -4$

27 $x = 3, y = -3$

Exercise 11d

1 $\begin{pmatrix}12&8\\8&15\end{pmatrix}\begin{pmatrix}50\\25\end{pmatrix} = \begin{pmatrix}800\\775\end{pmatrix}$

a 800 c or $8 **b** 775 c or $7.75

2 **a** $\mathbf{A} = \begin{pmatrix}2&3\\3&1\end{pmatrix}$ **b** $\mathbf{B} = \begin{pmatrix}130\\80\end{pmatrix}$

c $\mathbf{AB} = \begin{pmatrix}2&3\\3&1\end{pmatrix}\begin{pmatrix}130\\80\end{pmatrix} = \begin{pmatrix}500\\470\end{pmatrix}$

so Nikki spent 500 c and Joe spent 470 c

3 **a** $\begin{pmatrix}7&1&2\\4&1&5\\5&4&1\\6&2&2\end{pmatrix}\begin{pmatrix}3\\1\\0\end{pmatrix} = \begin{pmatrix}22\\13\\19\\20\end{pmatrix}$

b The column matrix gives the number of points scored by each team.

c **i** Abbott **ii** Berwick

4 **a** $\mathbf{P} = \begin{pmatrix}13&5&2\\11&2&7\\14&4&2\end{pmatrix}$ $\mathbf{Q} = \begin{pmatrix}3\\-1\\0\end{pmatrix}$

$\mathbf{PQ} = \begin{pmatrix}13&5&2\\11&2&7\\14&4&2\end{pmatrix}\begin{pmatrix}3\\-1\\0\end{pmatrix} = \begin{pmatrix}34\\31\\38\end{pmatrix}$

Colin won.

5 $\mathbf{A} = \begin{pmatrix}3&8&5\\6&4&2\end{pmatrix}$ $\mathbf{B} = \begin{pmatrix}50\\25\\10\end{pmatrix}$

$\mathbf{AB} = \begin{pmatrix}3&8&5\\6&4&2\end{pmatrix}\begin{pmatrix}50\\25\\10\end{pmatrix} = \begin{pmatrix}400\\420\end{pmatrix}$

Difference is 20 c.

6 **a** $\begin{pmatrix}10&21&14\\11&20&16\\12&18&15\end{pmatrix}$ **b** $\begin{pmatrix}3\\5\\2\end{pmatrix}$

c $\begin{pmatrix}10&21&14\\11&20&16\\12&18&15\end{pmatrix}\begin{pmatrix}3\\5\\2\end{pmatrix} = \begin{pmatrix}163\\165\\156\end{pmatrix}$

Shop Z is the cheapest.

7 **a** $\begin{pmatrix}16&2\\24&5\\30&3\\20&4\end{pmatrix}$ **b** $\begin{pmatrix}12\\15\end{pmatrix}$ **c** $\begin{pmatrix}16&2\\24&5\\30&3\\20&4\end{pmatrix}\begin{pmatrix}12\\15\end{pmatrix} = \begin{pmatrix}222\\363\\405\\300\end{pmatrix}$

gives the pay for each person **d** $183

8 **a** $\begin{pmatrix} 240 & 180 & 260 \\ 140 & 180 & 150 \\ 230 & 200 & 140 \\ 200 & 190 & 190 \\ 310 & 260 & 280 \end{pmatrix}$ **b** $\begin{pmatrix} 8 \\ 7 \\ 6.50 \end{pmatrix}$

c $\begin{pmatrix} 240 & 180 & 260 \\ 140 & 180 & 150 \\ 230 & 200 & 140 \\ 200 & 190 & 190 \\ 310 & 260 & 280 \end{pmatrix}\begin{pmatrix} 8 \\ 7 \\ 6.50 \end{pmatrix} = \begin{pmatrix} 4870 \\ 3355 \\ 4150 \\ 4165 \\ 6120 \end{pmatrix}$

\$6120 − \$3355 = \$2765

9 $\begin{pmatrix} 5 & 3 & 2 \\ 10 & 4 & 3 \end{pmatrix}\begin{pmatrix} 90 & 65 \\ 120 & 110 \\ 200 & 165 \end{pmatrix} = \begin{pmatrix} 1210 & 985 \\ 1980 & 1585 \end{pmatrix}$

This shows that she spent \$12.10 if she used first class mail and \$9.85 if she used second class mail on Friday. She also spent \$19.80 if she used first class mail and \$15.85 if she used second class mail on Saturday.

10 **A** **B** **C** **HB** **PB**

$\begin{pmatrix} 35 & 30 & 40 \\ 20 & 15 & 25 \end{pmatrix}\begin{pmatrix} 4 & 5 & 4 \\ 7 & 6 & 6 \\ 3 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 470 & 260 \\ 515 & 290 \\ 440 & 245 \end{pmatrix}$

a The hardback costs for the three groups are, respectively, \$470, \$515 and \$440.

b The paperback costs for the three books are, respectively, \$260, \$290 and \$245.

Mixed exercise 11

1 D **2** D **3** B **4** C **5** C
6 C **7** B **8** B **9** C **10** D

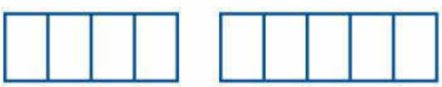
Chapter 12

Exercise 12a

1 **a**

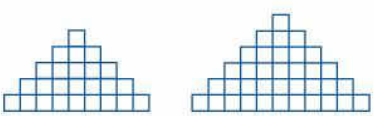
b **i** 4 cm^2, 5 cm^2 **ii** 16 cm, 20 cm
c 40 cm **d** 15 m^2

2 **a**



b **i** 8 cm^2, 10 m^2 **ii** 12 cm, 14 cm
c 40 cm **d** 44 cm^2

3 **a**



b **i** 25 sq. units, 36 sq. units
ii 28 units, 34 units
c 40 units **d** 49 sq. units

4 **a**

b **i** 4 sq. units, 5 sq. units
ii 17, 21
c 29 **d** 6 sq. units

5 **a**

b **i** 4, 5 **ii** 13, 16
c 28 **d** 12

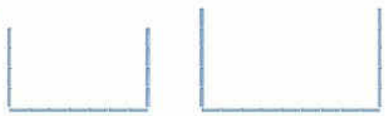
6 **a**

b **i** 5, 6 **ii** 12, 15
c 27 **d** 12

7 **a**

b **i** 5, 6 **ii** 20, 25 **iii** 25, 31
c 49 **d** 8 **e** 30

8 **a**

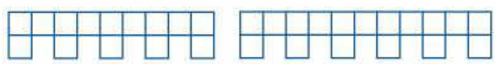


b **i** 4, 5 **ii** 15, 19
c 7 **d** 47

9 **a**

b **i** 8, 10 **ii** 9, 11 **iii** 17, 21
c 12 **d** 20 sq. units **e** 53

10 **a**



b **i** 14 sq. units, 17 sq. units
ii 30 units, 36 units
c 54 units **d** 32 sq. units

11 **a** 6, 11, 16, 21, 26 **b** **i** $5n + 1$ **ii** 151
c 8 with 12 left over

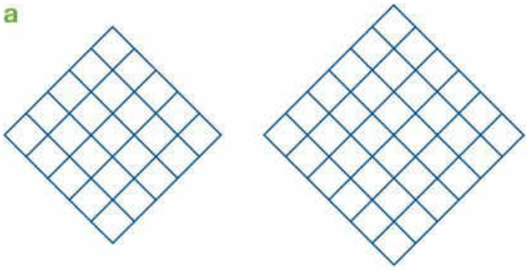
12 **a** 1 6 15 20 15 6 1
1 7 21 35 35 21 7 1
b 1, 2, 4, 8, 16, ... **c** 2^{n-1}

13 **a** 1, 3, 6, 10, 15 **b** 210
c $u_{n-1} = \frac{1}{2}(n-1)n$ **d** n
e 2, 3, 4, 5 **f** 19

14 **a** 1, 5, 14, 30, 55 **b** 2870
c $\dfrac{(n-1)\,n(2n-1)}{6}$ **d** n^2
e 4, 9, 16, 25 **f** 441

15 a (5, 26), (6, 37) **b** (10, 101)
c $(n, n^2 + 1)$

16 a

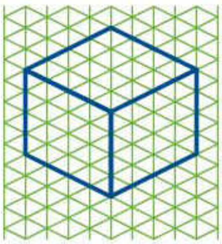


b

	Shape	1	2	3	4	5
i	Area (sq. units)	4	9	16	25	36
ii	Perimeter (units)	8	12	16	20	24

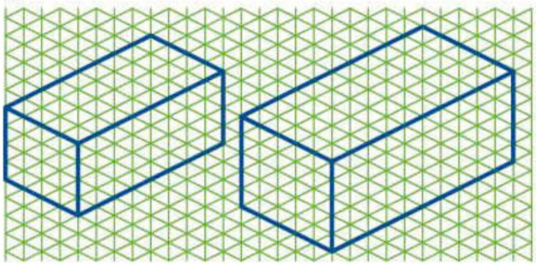
c 32 units **d** 49 sq. units

17 a



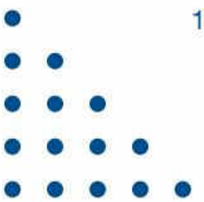
b i 64 cubic units **ii** 96 sq. units
c i 512 cubic units **ii** 384 sq. units
d i 15 **ii** 10
e i n^3 **ii** $6n^2$

18 a



b i 128 cubic units **ii** 64 units
c i 686 cubic units **ii** 112 units
d i 6 **ii** 9
e i $2n^3$ **ii** $16n$

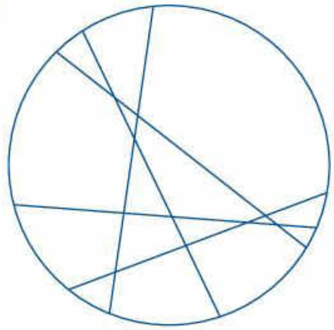
19 a



15

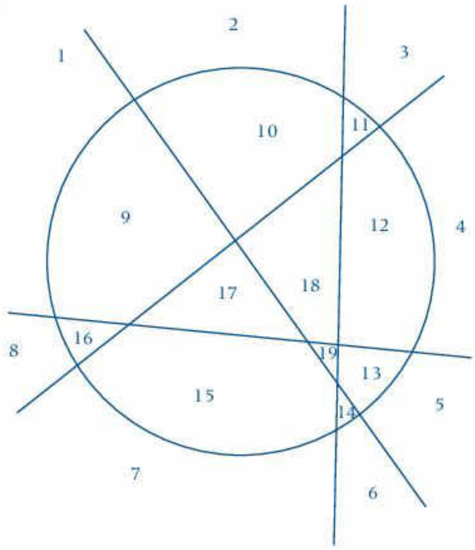
b 15 21 28
25 36 49
c $(n + 1)$ **d** 120 **e** 22

20 a



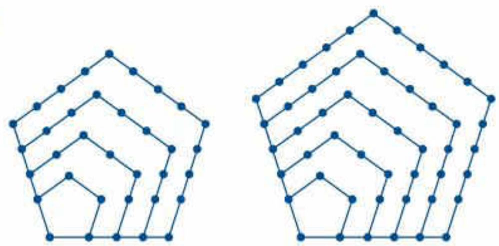
b 16 **c** 29 **d** $a = n$ **e** 10

21 a



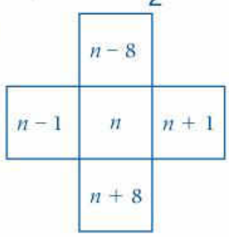
b 19 **c** 26 **d** 53
e $a = 2$ **f** 11

22 a



b 35, 51 **c** $\frac{n}{2}$ **d** 145

23 a



b $(9n - 7)$

24 a $a = 21, b = 441$ **b** $c = 1296$
c $d = 3025$ **d** $e = 78$

25 a i $11 = 6 + 5$ **ii** $15 = 8 + 7$
iii $23 = 12 + 11$

b $10^2 - 9^2 = 19$
c $14^2 - 13^2 = 27 = 14 + 13$
d $n^2 - (n-1)^2 = 2n - 1 = n + (n-1)$

26 a $7^2 = 49 \quad 6 \times 8 = 48$
$8^2 = 64 \quad 7 \times 9 = 63$
$9^2 = 81 \quad 8 \times 10 = 80$
b i $14^2 = 196 \quad 13 \times 15 = 195$
ii $17^2 = 289 \quad 16 \times 18 = 288$
c 42 d 55×57

27 a 100 b 8
c i 8100 ii 2500 iii 5600

28 a 0.142 857, 0.285 714, 0.428 571, 0.571 428, 0.714 285, 0.857 142
The digits are in the same cyclic order.
b 1.142 857, 1.857 142
c 0.571 428 571 428 571
d $\frac{20}{7}$

29 a 0.272 727 272 727
b i 0.454 545 45 ii 0.545 454 54

30 a i $\frac{1}{72}$ ii $\frac{1}{420}$ b 13 c $\frac{1}{10}$

31 a 192 b $(3x - 1)2^{n-1}$ c $x = \frac{1}{3}$
d No, coefficient of x must be -3 times the number.

32 a $10(x - 21)$
b Number in brackets is always odd.
c $(n + 2)(x + 11 - 4n)$

33 a $5x^2 + 6x$, $6x^2 + 7x$ b $nx^2 + (n+1)x$
c i $10x^2 + 11x$ ii $20x^2 + 21x$

34 a $\frac{5}{6}(5x + 1)(10x - 1)$, $(6x + 1)(12x - 1)$
b $\frac{n}{6}(nx + 1)(2nx - 1)$
c i $2(12x + 1)(24x - 1)$
ii $\frac{10}{3}(20x + 1)(40x - 1)$

35 a $5x - 6x^6$, $6x + 7x^7$ b $10x + 11x^{11}$
c 2nd term always + for an odd index
d $nx + (-1)^n(n + 1)x^{n+1}$
e i $15x - 16x^{16}$ ii $23x - 24x^{24}$

36 a $\frac{5x^6}{26}$, $\frac{-6x^7}{37}$ b $\frac{n(-x)^{n+1}}{n^2 + 1}$
c Index of x is 1 more than the number in front of it.
d $\frac{-12x^{13}}{145}$

37 a $(5x + 1)^2 - (10x - 1)^2$, $(6x + 1)^2 - (12x - 1)^2$
b $(nx + 1)^2 - (2nx - 1)^2$
c 14 should be inside the second bracket.
d 2nd bracket is same as $(20x - 1)^2$ which fits the pattern.

38 $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}$, for ever

Chapter 13

Exercise 13a

1 a i $-2, 6$ ii $x^2 - 4x - 12 = 0$
b 16 when $x = 2$

2 a $-1, 5$; $x^2 - 4x - 5 = 0$
b 9 when $x = 2$

3 a i 4 ii 4
b -1.8 and 3.8, $x^2 - 2x - 7 = 0$
c 8 when $x = 1$ d $-1.8 < x < 3.8$

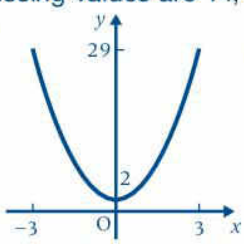
4 a 0.8 and 5.2; $x^2 - 6x + 4 = 0$
b -5 when $x = 3$
c i 0.4 and 5.6 ii 1.6 and 4.4

5 a i -21 ii 28.5 b 0.2 and 6.8
c -21.5 d 3.5 e $0.4 < x < 6.6$

6 a i -4.75 ii 13.25 b -0.2 and 4.2
c 14 d 2 e 0.8 or 3.2
f $3x^2 - 12x + 8 = 0$ g $0.3 < x < 3.7$

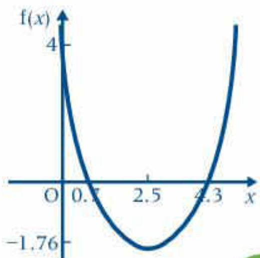
7 Missing values are line 2: 2, line 3: 3, line 5: -3 and 0.
a i $-2\frac{1}{2}, 1$ ii $2x^2 + 3x - 5 = 0$
b $-6\frac{1}{8}$ c 5.9
d -1

8 Missing values are line 3: 4 and 0; line 4: 0 and -12; line 5: 4, 12, -4
a i -2.36, 1.69 b $12\frac{1}{3}$ when $x = -\frac{1}{3}$
c 10

9 Missing values are 14, 2, 29
b


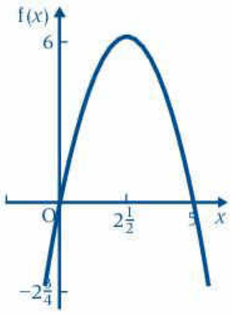
c $x = 0$
d 2 when $x = 0$
e ± 1.29

10 Missing values are 0, -2
a 0.70 and 4.30
b -1.76
c -4

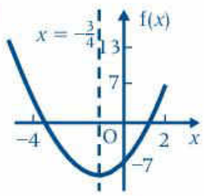


11 Missing values are 4, 4, 0

a 4.81 **b** 0.70, 4.30 **c** −0.37, 5.37



12 **a and b i**

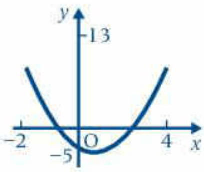


b ii $-8\frac{1}{8}$

c i −2.8, 1.3

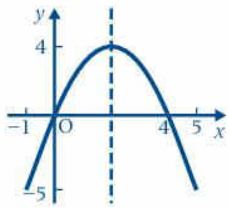
ii −3.3, 1.8

13 **a**



b i −0.87, 2.87 **ii** −0.22, 2.22

14 **a**



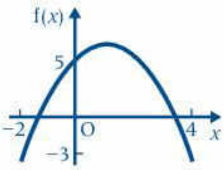
b $x = 2$

c 4 when $x = 2$

d i 0.59, 3.41

ii 0.27, 3.73

15 **a**



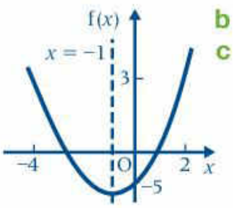
b 6 when $x = 1$

c i −1.45, 3.45

ii −0.73, 2.73

iii −1.65, 3.65

16 **a**

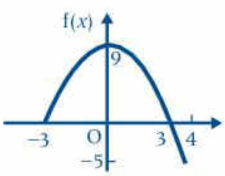


b $x = -1$

c i −3.45, 1.45

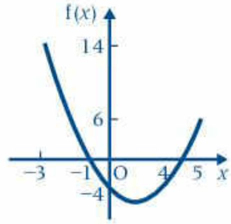
ii −2.73, 0.73

17



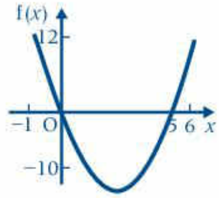
−1.73, 1.73

18



−1.19, 4.19

19



−12.5 when $x = 2.5$

20 **a** (−3, 0), (4, 0)

b $a = 1, b = -1, c = -12$

c $-12\frac{1}{4}$ when $x = \frac{1}{2}$

21 **a** $a = 1, b = -2, c = -8$

b −9 when $x = 1$

c $x = 1$

d $-4 < x < 6, -9 < f(x) < 15$

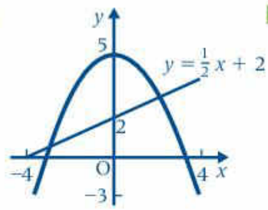
22 **a** $a = -\frac{7}{8}, b = \frac{7}{8}, c = 9$

b $-5 < x < 6, -20 < f(x) < 9$

c 9 when $x = 0.2$

d i 7.5 **ii** 0

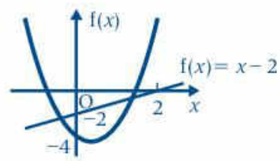
23 **a**



b i −3, 2

ii $x^2 - x - 6 = 0$

24 **a**

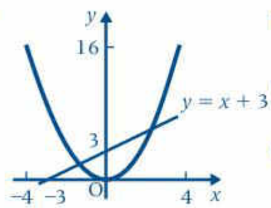


b i −0.87, 1.54

ii −0.55, 1.22

c −0.46, 1.46; $3x^2 - 3x - 2 = 0$

25 **a**



b i 2.89 **ii** 5.76

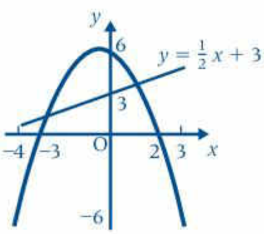
iii 3.46 **iv** 2.65

c ii −1.30, 2.30

iii $x^2 - x - 3 = 0$

d −2.56, 1.56

26 **a**

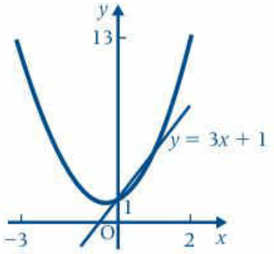


b $x = -\frac{1}{2}$

c Domain $-4 < x < 3$, $-6 < y < 6\frac{1}{4}$

d −2.64, 1.14; $2x^2 + 3x - 6 = 0$

27 a



b The equation $2x^2 + 2x + 1 = 0$ has no solution.

c $(0, 1)$, $\left(\frac{1}{2}, 2\frac{1}{2}\right)$; $2x^2 - x = 0$

Exercise 13b

1 minimum -13

2 minimum $-12\frac{1}{4}$

3 maximum 4

4 minimum $-9\frac{1}{2}$

5 maximum $-1\frac{5}{8}$

6 maximum $4\frac{5}{8}$

7 minimum -28

8 minimum 8

9 minimum -29

10 minimum $3\frac{3}{4}$

11 minimum $4\frac{3}{4}$

12 maximum 22

13 maximum 114

14 maximum 14

15 minimum 4

16 minimum -15

17 maximum -2

18 maximum 14

19 maximum 12.45

20 maximum $5\frac{2}{3}$

21 maximum, $3\frac{1}{2}$

22 a $5625\,\text{m}^2$ **b** $75\,\text{m} \times 75\,\text{m}$

23 81

24 $75\,\text{m} \times 37.5\,\text{m}$

25 12 500 m after 50 s

26 $73.5\,\text{m}^2$

27 Area $= \frac{x}{2}(5 - x)$; $x = \frac{5}{2}$, $3\frac{1}{8}\,\text{cm}^2$

Exercise 13c

1 a least value $\frac{11}{4}$, $f(x) > \frac{11}{4}$

b least value 3, $f(x) > 3$

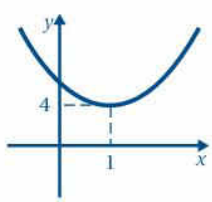
c greatest value 4, $f(x) < 4$

d greatest value $\frac{29}{4}$, $f(x) < \frac{29}{4}$

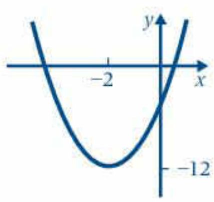
e least value -2, $f(x) > -2$

f greatest value 1, $f(x) < 1$

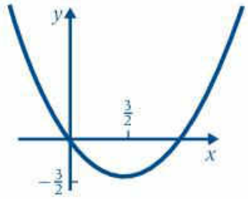
2 a



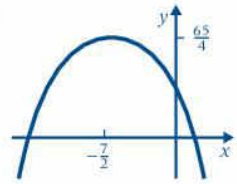
b



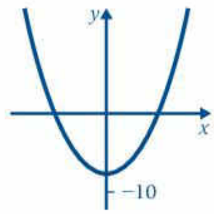
c



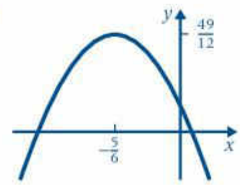
d



e

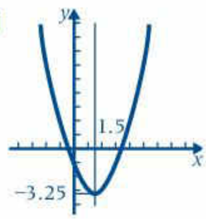


f



3 a $(x - 1.5)^2 - 3.25$

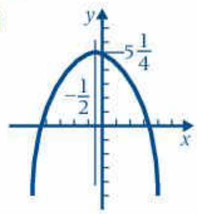
i



ii 2

b $5.25 - (x + 0.5)^2$

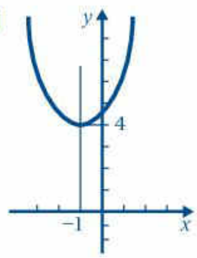
i



ii 2

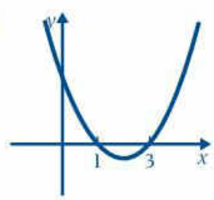
c $(x + 1)^2 + 4$

i



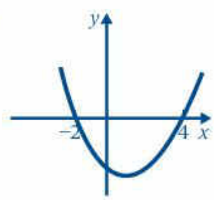
ii none

4 a



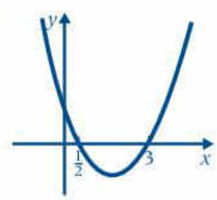
$1 < x < 3$

b



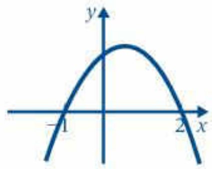
$x < -2$, $x > 4$

c



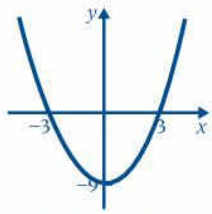
$\frac{1}{2} < x < 3$

5 **a**



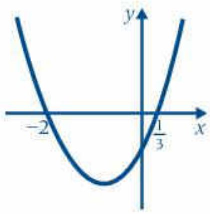
$-1 < x < 2$

b



$-3 < x < 3$

c



$x < -2, x > \frac{1}{3}$

6 **a** $-\frac{2}{3} < x < 1$ **b** $x < -\frac{1}{2}, x > 1$
c $-2 < x < \frac{5}{3}$

7 **a** $x < -\frac{1}{3}, x > 2$ **b** $2.5 < x < 3$
c $-\frac{1}{4} < x < 2$

8 **a** $x < -\frac{1}{2}, x > 3$ **b** $-\frac{3}{2} < x < 4$
c $x < -\frac{5}{2}, x > \frac{2}{3}$

9 **a** $-\frac{2}{5} < x < 1$ **b** $x < -3, x > \frac{7}{4}$
c $x < -\frac{2}{3}, x > 6$

10 **a** $-4 < x < \frac{5}{2}$ **b** $-\frac{3}{2} < x < 2$
c $x < -\frac{2}{3}, x > 1$

Exercise 13d

1 $x = 2, y = 4; x = -4, y = -2$
2 $x = 3, y = 4; x = -4, y = -3$
3 $x = 2, y = 3; x = 3, y = 2$
4 $x = 1, y = 2; x = 2, y = 1$
5 $x = 1, y = -3; x = 3, y = -1$
6 $x = 3, y = 5; x = -5, y = -3$
7 $x = 4, y = 3; x = -3, y = -4$
8 $x = 2, y = 5; x = 5, y = 2$
9 $x = 0, y = 5; x = -4, y = -3$
10 $x = -\frac{2}{5}, y = -\frac{11}{5}; x = 2, y = -1$
11 $x = \frac{2}{11}, y = \frac{19}{11}; x = -2, y = 1$
12 $x = \frac{27}{11}, y = \frac{29}{11}; x = 3, y = 1$
13 $x = -2, y = -7; x = 1, y = 2$
14 $x = 3, y = 6$
15 $x = \frac{17}{3}, y = -\frac{7}{3}; x = 3, y = -1$
16 $x = 2, y = 4$
17 $x = -\frac{4}{3}, y = -\frac{3}{2}; x = 2, y = 1$
18 $x = 3, y = -2; x = -15, y = -14$
19 $x = 2, y = -12; x = 12, y = 2$
20 $x = -\frac{3}{2}, y = 6; x = 2, y = -1$
21 $x = 2, y = 2; x = \frac{3}{2}, y = \frac{5}{2}$
22 $x = 4, y = -2; x = -\frac{8}{3}, y = \frac{14}{3}$
23 $x = 0, y = 3; x = 3, y = 0$
24 $x = 3, y = 4; x = 4, y = 3$
25 $x = 1, y = 1; x = 5, y = -5$
26 $x = 3, y = 2; x = -\frac{3}{2}, y = -4$
27 $x = 3, y = 1; x = -1, y = 5$
28 $x = 3, y = -2; x = 1, y = 2$
29 $x = -1, y = 2; x = \frac{15}{2}, y = -\frac{7}{5}$
30 $x = 1, y = -2; x = -\frac{1}{3}, y = -\frac{26}{9}$
31 $x = 3, y = 4; x = 4, y = 6$
32 $x = 1, y = -6; x = -8, y = -15$
33 $x = \frac{8}{3}, y = 3; x = -2, y = -4$
34 $x = -\frac{2}{5}, y = 3\frac{2}{5}; x = \frac{1}{2}, y = -2$
35 $x = 2, y = 4; x = -4, y = -2$
36 $x = 3, y = 4; x = -4, y = -3$
37 $x = -1, y = -1$ (twice)
38 $x = 5, y = 2; x = -2, y = -5$
39 $x = \frac{3}{5}, y = -\frac{19}{5}; x = -\frac{1}{2}, y = \frac{1}{2}$
40 $x = \frac{3}{5}, y = 4\frac{1}{5}; x = -\frac{1}{2}, y = 2$
41 $x = 2, y = \frac{3}{4}; x = 1\frac{1}{3}, y = \frac{7}{12}$

Mixed exercise 13

1 A	**2** B	**3** C	**4** C
5 D	**6** A	**7** B	**8** C
9 D	**10** A	**11** C	**12** B
13 A	**14** B	**15** B	**16** B
17 C	**18** D	**19** A	**20** D

Chapter 14

Exercise 14a

1 **a** 2.7 **b** −2.5
2 **b** $x = 2$ **c** $x^3 - x - 6 = 0$

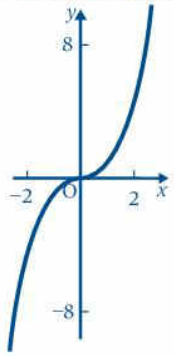
3 **a** **i** −3 **ii** 8 **b** 0.3, 1.5
c yes, because the curve is going up so it will cut the line again

4 $x = -3$

5 **a** 1.7, only 1 solution
b 1; the line $y = -1$ cuts the curve once only; 2.0
c **i** $c < 0.85$, $c > 1$ **ii** $0.85 < c < 1$

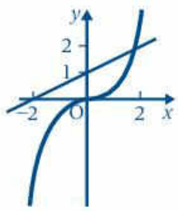
6 **a** When $y = 0$, $(x - 2)$ or $(x - 3)$ or $(x - 4)$ is zero, i.e. $x = 2$, 3 or 4.
b $2 < x < 3$, $x > 4$

7 **a** Missing values are −1, 0, 0.125, 8
b



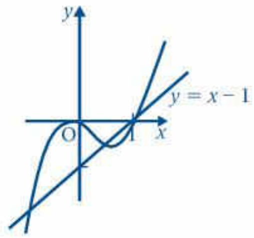
c **i** 1.59
ii −1, 0, 1

8 **a**



b $x < 1.59$
c **ii** $x^3 - 2x - 4 = 0$

9 **a**



b $x < 0$ and $0 < x < 1$
c **ii** $x^3 - x^2 - x + 1 = 0$

Exercise 14b

1 **a** 0.4 **b** 0.4 **c** 5
d gets larger very fast
e cannot divide by zero
f rotation order 2, 2 lines of symmetry

2 **a** cannot divide by zero
b **i** 0.8 **ii** −1.1

3 **a** approaches zero
b no, $\frac{1}{x}$ is never 0 however large x becomes
c no, $x = 0$, $\frac{1}{x}$ is indeterminate

4 **a** no, x = 0, $\frac{1}{x^2}$ is indeterminate
b $\frac{1}{4}$ **c** $\frac{1}{9} < f(x) < 9$

Exercise 14c

1 D **2** B **3** A **4** C

5 No scales given so it is not possible to check other points

Exercise 14e

1 **b** 0, 180, 360

2 **b** 90, 270

3 **a** $-1 \leqslant \sin x° \leqslant 1$ **b** $-1 < \cos x° \leqslant 1$

5 $y = \cos x°$ is a translation of $y = \sin x°$ by 90 units to the left.

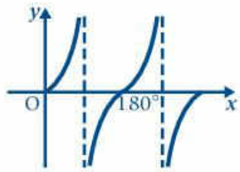
6 **b** reduces width by factor of 2
c reduces width by factor of 3

7 **b** $-1 \leqslant \sin x° \leqslant 1$

8 $-1 \leqslant \cos x° \leqslant 1$

Exercise 14f

1



2 2; 62, 208 **3** 2; 38, 142

4 A(0, 2), B(90, 1), C(180, 0), D(270, 1), E(360, 2)

5 **a** from 4 p.m. to 8 p.m.
b 4 hours 48 minutes

6 **a** 0, 180, 360 **b** 90, 270

8 1, the curves only intersect once.

9 **a** $y = \cos x°$, $y = \frac{(x - 20)}{50}$
b e.g. $y = 15 \sin x°$, $y = 10 - x$
c $(x - 1)^2 = 14 \tan x°$

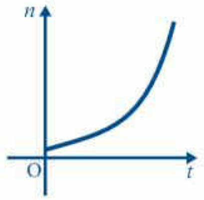
10 **a** −66, −294, 66, 294
b −226, −134, 134, 226
c −264, −96, 96, 264

11 **a** −348, −12, 192, 168
b −168, −12, 192, 348
c −53, −127, 233, 307

12 **a** 1.4, −1.4
b stretch by a factor of 1.4 parallel to y-axis and a translation by 45 units to the left; $y = 1.4 \sin(x° + 45°)$

Exercise 14g

1 b



c $n = 2^t$
d $n = 10 \times 2^t$

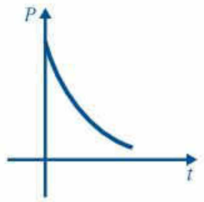
2 a $P = 1000 \times 1.1^n$ **c** $\simeq 7.5$, $\simeq 7.25$
d $P = 5000 \times 1.1^n$

Exercise 14h

1 a

P	9000	6000	4000	2670	1780	1190	700

b



c 1.7 years, 1.7 years; no, the curve never crosses the horizontal axis.

2 a

m	5	2.5	1.25	0.625	0.313	0.156	0.078

d m approaches O but never gets there.

Exercise 14i

1

−3	−2	−1	0	1	2	3	4	5	6
0.125	0.25	0.5	1	2	4	8	16	32	64

2

−3	−2	−1	0	1	2	3
8	4	2	1	0.5	0.25	0.125

3 One is the reflection of the other in the y-axis; yes because if $f(x) = 2^x$ then $f(-x) = 2^{-x}$.

4 a (0, 1)
b No. There is no value of x for which $3^x = 0$.

5 a $1.5^0 = 1 = 3^0$; (0, 1), $a^0 = 1$ for all values of a except $a = 0$
c (0, 500) **d i** (0, 10) **ii** (0, 10)

6 a 2 **b** 1; $\frac{1}{2}$

7 2, 1.8; $y = 2 \times 1.8^{-x}$

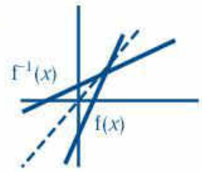
8 4, 1.6 **9** 1000, 1.3

10 b $A = P \times \left(1 + \frac{r}{100}\right)^T$

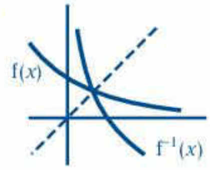
11 a i 80 mg **ii** Thursday **12** 6

Exercise 14j

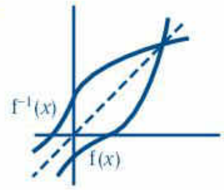
1 a



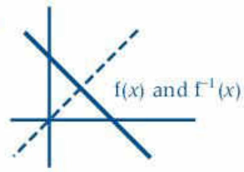
b



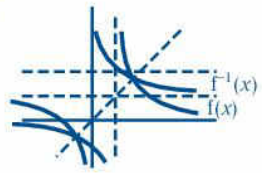
c



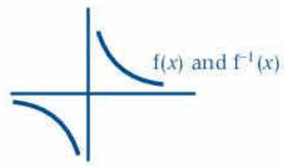
d



e



f



2 **d** and **f**

3 a $f^{-1} = (x - 1)$ **b** no
c $f^{-1}(x) = \sqrt[3]{(x - 1)}$
d $f^{-1}(x) = \sqrt{(x + 4)}$, $x \geqslant -4$
e $f^{-1}(x) = \sqrt[4]{(x - 1)}$, $x \geqslant 0$

4 a $-\frac{1}{3}$ **b** $\frac{1}{2}$ **c** there isn't one

5 a 9 **b** 2 **c** −1

Exercise 14k

1 a $\frac{1}{x^2}$ **b** $(1 - x)^2$ **c** $1 - \frac{1}{x}$
d x^4 **e** $\frac{1}{x^2}$ **f** $(1 - x^2)$

2 a 125 **b** 15 **c** −1 **d** −1

3 a $(1 + x)^2$ **b** $2(1 + x)^2$ **c** $1 + 4x^2$

4 $g(x) = x^2$, $h(x) = 2 - x$

5 $g(x) = x^4$, $h(x) = (x + 1)$

6 a $f(x) = gh(x)$, $g(x) = 10^x$, $h(x) = x + 1$
b $f(x) = gh(x)$, $g(x) = \frac{1}{x^2}$, $h(x) = 3x - 2$
c $f(x) = g(x) + h(x)$, $g(x) = 2^x$, $h(x) = x^2$
d $f(x) = \frac{g(x)}{h(x)}$, $g(x) = 2x + 1$, $h(x) = x$
e $f(x) = gh(x)$, $g(x) = x^4$, $h(x) = 5x - 6$
f $f(x) = g(x)h(x)$, $g(x) = x - 1$, $h(x) = x^2 - 2$

7 a i -8 ii 7 iii $f(0) = -1$
iv $g(-4) = 19$
b i 6 ii -49 c i 5 ii -3

8 a i -19 ii -1 iii 31 iv 23
b i 7 ii 4

9 a i 0 ii 2 iii 9
b i $3 - 6x$ ii $27 - 6x$

10 a i -7 ii 8 iii 11 iv 26
b i $\frac{5 + x}{2}$ ii $\frac{5 - x}{3}$ iii $\frac{20 - x}{6}$

11 a i $\frac{3}{5}$ ii 0 b $\frac{3 - x}{4 - x}$

12 a i $-\frac{1}{13}$ ii 7 b $\frac{1 + 2x}{7 - 6x}$ d 3

13 a i $2 - 6x$ ii $-19 - 6x$
iii $3 - 6x$
b i -16 ii -31 iii -9

14 a $\frac{3}{2}$ b 1 c $\frac{3x - 2}{x + 1}$
d $\frac{4}{3}$ e $-\frac{1}{3}$

15 a i $\frac{3}{x + 5}$ ii $\frac{3x + 9}{x + 2}$
iii $\frac{3}{7}$ iv $\frac{18}{5}$
b i $\frac{3 - 2x}{x}$ ii $x - 3$
iii $\frac{9 - 2x}{x - 3}$ iv $\frac{3 - 5x}{x}$

16 a i $\frac{1}{4}$ ii 2
b i $\frac{4x + 2}{1 - x}$ ii $\frac{x - 5}{x + 1}$
iii $x + 3$ iv $\frac{-2x - 14}{x + 4}$

17 a i $\frac{3}{5}$ ii $\frac{3}{5}$ b i 7 ii $\frac{11}{23}$
c $\frac{-7}{4x - 5}$

18 a i $\frac{1}{2}$ ii 10
b i $\frac{3x}{1 - x}$ ii $\frac{2x + 4}{2x + 7}$
iii $\frac{x - 4}{2}$ iv $\frac{6x + 12}{x + 3}$

19 a i $\frac{5}{3}$ ii $\frac{7}{4}$ b i $-\frac{1}{2}$ ii $-\frac{2}{3}$
c i $\frac{x - 3}{3 - 2x}$ ii $\frac{2x + 3}{2 - 3x}$

20 a 9 b $\frac{1}{3}$ c $\frac{1}{32}$ d $\frac{1}{4}$

21 a i 16 ii 3 iii 0 iv $\frac{20}{3}$
b i $3x - 12$ ii $\frac{1}{3}x^3 + 4$
iii $\sqrt[3]{x}$ iv $\frac{(x + 12)^3}{27}$

22 a -5 c $\frac{x + 2}{x - 1}$

23 a i 6 ii $\frac{2}{3}$ c $\frac{13}{3}$

24 a $(3 - x)^2$ b $3 - x^2$ c 0 d -6
e $3 - \sqrt{x}$ f $\sqrt{3 - x}$ g $\sqrt{3 - x}$ h $3 - \sqrt{x}$

25 a i $\frac{5}{7}$ ii $\frac{3}{5}$ iii 0
iv $\frac{3}{4}$ v $\frac{4}{3}$
b i $\frac{3 - 5x}{x - 1}$ ii $\frac{x}{1 - x}$
iii $\frac{4x + 3}{6x + 5}$ iv $\frac{x + 3}{2x + 8}$

26 a $x^2 - 6x + 11$ b 3 c $\frac{57}{4}$

27 a $6x - 17$ b $6x - 6$ c -11 d -3

28 a $64x^6$ b $8x^6$ c $2x^6$ d 64
e 512 f 128

29 a i $2(3 - x)$ ii $3 - 2x$
iii $\frac{6 - x}{2}$ iv $\frac{6 - x}{2}$
b $(fg)^{-1}(4) = 1 = g^{-1}f^{-1}(4)$

30 a $\frac{x - 2}{2}$, $\frac{x - 2}{3}$ b 1, 1

31 a i $\frac{5}{2}$ ii 5 c $\frac{8}{7}$

Mixed exercise 14

1 B	2 B	3 D	4 A
5 B	6 C	7 C	8 D
9 B	10 B	11 D	12 B
13 B	14 C	15 C	16 D
17 C	18 B	19 C	20 C

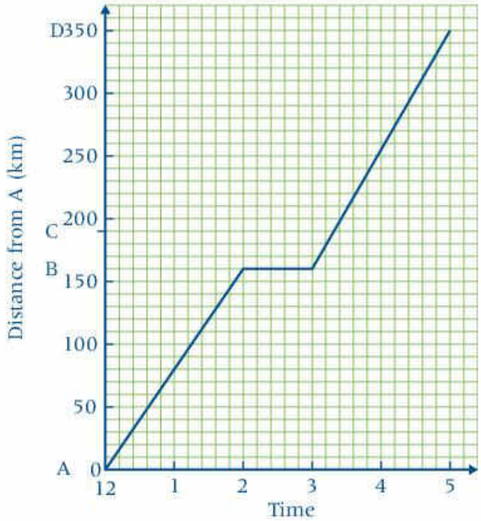
Chapter 15

Exercise 15a

1 a $1\frac{1}{2}$ h b outward
c i 20 km/h ii 16 km/h iii $17\frac{7}{9}$ km/h

2 **a** 50 km
b **i** 30 km **ii** 1 h **iii** 20 km/h
c **i** A **ii** 30 min.

3



5.00 p.m.

4 **a** 128 min. after he started **b** 15 km

5 **a** 60 km/h **b** 32 km/h **c** 44.4 km/h
d 12.45, 2.00 **e** 80 km/h **f** 1.53, 87 miles

6 **a** 80 km/h **b** 12 min. **c** 75 km/h
d 2.24 p.m., 3.48 p.m. **e** 86 km/h
f 3 p.m., 65 km from A

7 **a** Dick and 21 min.
b at 12.37, 2.1 km from Axeter
c 47 min

8 **a** 11.57 a.m., 60 km from A
b 25 km **c** $33\frac{1}{3}$ km/h

9 **a** 1.44 p.m., 83 km from A
b 56 km/h **c** 2.24 p.m.

10 **a** 11.36 a.m. **b** 50 km/h
c 12.28, 67 km from A **d** 58 km

11 **a** 1.09 p.m. **b** 1.40 p.m., 64 km from A
c 38 km/h

12 **a** 11.39 **b** 12.30 **c** 4.7 km/h

13 **a** 4.41 p.m. **b** 104 km/h
c 3.41 p.m., 69 km from A

14 **a** 1.54 p.m., 50 km from A **b** the same
c 20 km

15 **a** 1.47 p.m., $10\frac{1}{2}$ km from A
b 2.45 p.m. **c** 6 km

Exercise 15b

1 **a** **i** 4.9 cm
ii 0.107 cm/s; this is the gradient of CD
b **i** 3.3 cm **ii** 0.136 cm/s
c **i** 0.95 cm **ii** 0.19 cm/s
d **i** 0.2 cm **ii** 0.2 cm/s
e estimate 0.2 as gradient at C
f move the point closer to C

2 **a** y increases to 3.1, then decreases to -3
b gradient $= -3$

Exercise 15c

1 **a** $\frac{1}{2}$ **b** $\frac{3}{7}$ **c** $\frac{5}{6}$
d $-\frac{11}{3}$ **e** 0
grad AE $= \frac{6-1}{1-1} = \frac{5}{0}$ which is indeterminate

2

x	0	1	2	3	4	5	6	7	8
y	0	0.1	0.4	0.9	1.6	2.5	3.6	4.9	6.4

a 0.6 **c** 0.2, 0.8, 1.2

3

x	1	1.5	2	3	4	5	6	7	8
y	10	6.7	5	3.3	2.5	2	1.7	1.4	1.3

a -2.5 **b** A: -10 B: -0.6
c -1.1 to 2 d.p.

4 **a** 3 **b** 27
5 **a** **i** 29 **ii** 33 **b** 34
c 5.25; number of ripe strawberries increasing by $5\frac{1}{4}$ a day on average from Monday to Friday
d -6; number of ripe strawberries decreasing by 6 a day on average from Monday to Friday

6 **a** -7.2; the population decreased by an average of 7.2 people per year
b -11; in 1910 the population was decreasing at the rate of 11 people per year

7 **a** 800; in month 2 sales were increasing at a rate of 800 jars per month
b 1800; in month 4 sales were increasing at a rate of 1800 jars per month

Exercise 15d

1 **a** The second graph shows acceleration/deceleration
b 1.7 m/s. Either graph gives this information. (Total distance/total time)

2 **a** No. The dip shows where the driver slows down and then speeds up so he goes round the roundabout.
b B attempts to show the direction of travel as well as the speed – the velocity. Negative speed shows travel in the opposite direction; but the car does not travel only in two directions in a line.

Exercise 15e

1 **a** 12.5 m **b** 6 m/s **c** 10 m/s
2 **a** 450 m **b** 80 m/s **c** 185 m/s
3 **a** 31 m **b** 12.4 m/s
c 12.4 m/s **e i** 62 m **ii** 5 seconds
f 12.4 m/s **g** gradient is zero
h **f** gives the average speed, **g** gives the average velocity
i The object did not change direction.

Exercise 15f

1 **a** speed **b** velocity **c** velocity
d speed **e** speed **f** velocity

2 **a** 

b 

c 

d 

3 **a** 10 m/s **b** +10 m/s **c** 7.5 m/s
d −7.5 m/s **e** 6 m/s **f** $1\frac{1}{2}$ s

4 +0.8 m/s, then −0.4 m/s, (i.e. speed 4 m/s in the opposite direction), then stationary

5 **a** false **b** true **c** false
d true **e** true

Exercise 15g

1 **a** 140 m/s **b** 95 m/s
c i 60 m/s **ii** 94 m/s

2 **a** after 6 seconds
b 15 m/s
c 17.5 m/s
d i 20 m/s **ii** −10 m/s **iii** −20 m/s

3 **a** 15 m/s **b** 20 m/s **c** 5 m/s
d 31 m **e** −15 m/s

4

t	0	1	2	3	4
d	0	6	8	6	0

c $t = 2$, 0 m/s $t = 3$, −4 m/s **d** 8 m

Exercise 15h

1 30 m/s, 60 seconds
2 **a** 0.4 m/s **b** 6 m/s **c** 12 m/s
3 15 seconds
4 2 m/s²
5 **a** 90 km/h **b** 300 km/h **c** 22.5 km/h
6 14 km/h/s (= 3.89 m/s²)
7 2.78 m/s²

Exercise 15i

1 **a** 2.5 m/s² **b** less acceleration
c 2 seconds **d** 2 seconds
e 7.5 m/s² **f** 12 seconds

2 **a** 10 km/minute²
b 15 km/minute
c 27.5 km/minute
d 2.5 km/minute

3 **a** 300 km/h² **b** 15 km/hour
c 3 minutes **d** zero
e −300 km/h²

Exercise 15j

1 **a i** 3/s² **ii** 1.5 m/s²
b 5 seconds **c** 75 m **d** 337.5 m

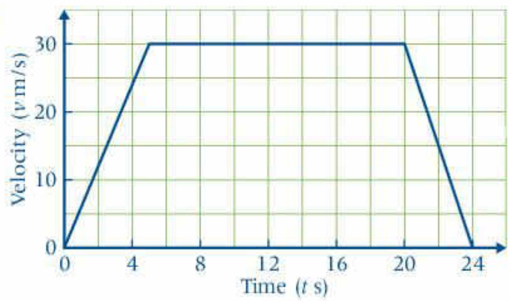
2 **a** 115 m/s² **b** 35 m
c 167 m to nearest whole metre

3 **a** 600 m/s² **b** 3.0625 m
c 1300 m/s² **d** 4.125 m
e 4000 m/s² **f** 0.3125 m

4 **a** 0.185 m/s² **b** 1000 m

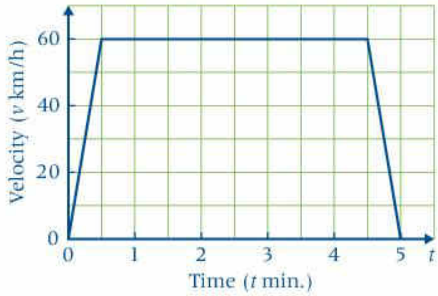
5 **a** 0.017 m/s² **b** 5 m/s **c** 0.025 m/s²
d 300 m **e** 420 m

6 **a** 0.67 m/s² to 2 d.p. **b** 1 m/s²
c 75 m **d** 50 m **e** 475 m

7 **a** 

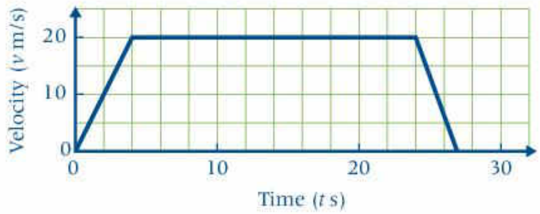
b i 6 m/s² **ii** $7\frac{1}{2}$ m/s² **iii** 585 m
iv $24\frac{3}{8}$ m/s

8



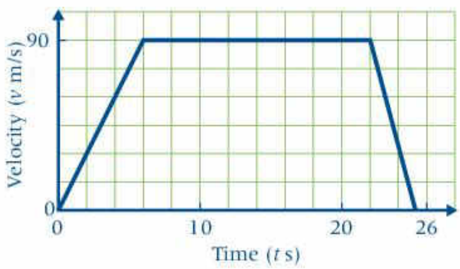
a $\frac{5}{9}$ m/s² **b** $\frac{5}{9}$ m/s² **a** 4.5 km

9 a



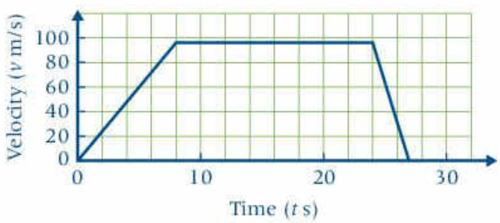
b i 5 m/s² **ii** $6\frac{2}{3}$ m/s²
iii 470 m **iv** 17.4 m/s²

10 a



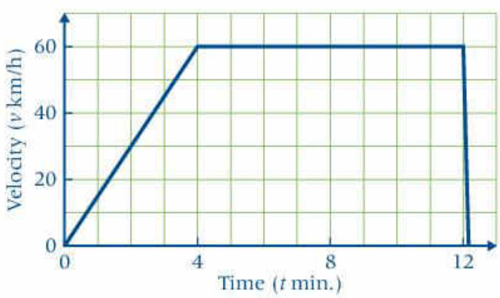
b i 15 m/s² **ii** 30 m/s²
iii 1845 m **iv** 73.8 m/s

11 a



b i 384 m **ii** $76\frac{4}{9}$ m/s

12 a



b i 900 km/h² **ii** 21 600 km/h²
iii 10.1 km **iv** 50 km/h

13 a 28 m/s **b** $\frac{1}{5}$ s
14 a 32 m/s **b** $\frac{1}{4}$ s
15 a 32 m/s **b** $\frac{1}{3}$ s
16 a 24 m/s **b** $\frac{6}{19}$ s

Mixed exercise 15

1 B	**2** C	**3** A	**4** A
5 D	**6** A	**7** C	**8** A
9 A	**10** D	**11** A	**12** B
13 B	**14** C	**15** D	

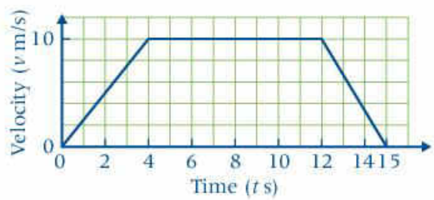
Review test 3

1 B	**2** B	**3** B	**4** D
5 B	**6** A	**7** B	**8** D
9 A	**10** A	**11** C	**12** C
13 A	**14** D	**15** C	**16** C
17 D	**18** C	**19** C	**20** D
21 B	**22** A	**23** C	**24** B
25 C	**26** D	**27** A	**28** C
29 D	**30** D		

General proficiency questions

1 $x = 1, y = 1$ or $x = -\frac{7}{5}, y = -\frac{1}{5}$

2 a



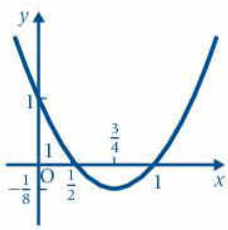
ii 2.5 m/s² **iii** 115 m **iv** $3\frac{1}{3}$ m/s²
v $7\frac{2}{3}$ m/s

b 23 s **c** $\frac{10}{23}$ m/s²

3 a $2\left(x - \frac{3}{4}\right)^2 - \frac{1}{8}$; $\frac{1}{2}$ or 1

b i c **ii** $-b$

c

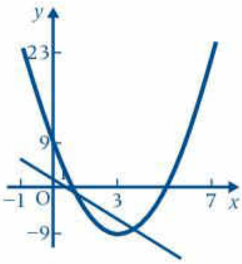


$f(x) > 0$ for $x < \frac{1}{2}$ or $x > 1$, $f(x) < 0$ for $\frac{1}{2} < x < 1$

4 $x = \frac{1}{3}$ or 1

5 $x = -\frac{1}{4}$ or $\frac{2}{3}$

6 a



b 0.88, 5.12

c 1.22, 3.28;
$2x^2 - 9x + 8 = 0$

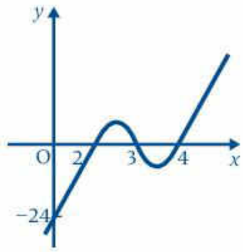
7 $x = -\frac{1}{6}, y = \frac{5}{3}$

8 a $10^4, \frac{1}{9}, \frac{1}{100}$ **b** $10^{\frac{1}{x^2}}, 10^{\frac{2}{x}}$

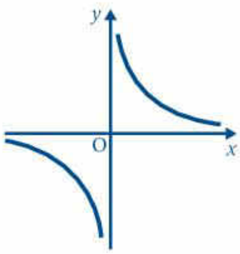
c $g^{-1}(x) = \sqrt{x}, x \geqslant 0, h^{-1}(x) = \frac{1}{x} (x \neq 0)$

d $\pm\frac{1}{3}$ **e** yes $(gh)^{-1} = \frac{1}{\sqrt{x}}$

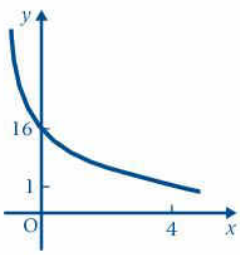
9 a



b

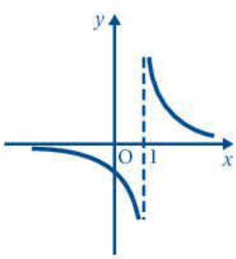


c



10 a $\frac{1}{0}$ is indeterminate **b** $\frac{1}{4}$

c



d $f^{-1}(x) = \frac{x-1}{x} \neq 0$
Domain all x except $x = 0$

11 a least value of $2\frac{3}{4}$ at $x = 1\frac{1}{2}$

b least value of $-\frac{41}{8}$ at $x = 1\frac{3}{4}$

c greatest value of 9 at $x = -2$

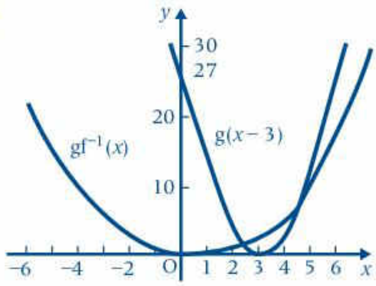
12 a $g(x) = 2^x, h(x) = 3x - 2$
b $ff(x) = 2^{46}, f^{-1}(2) = 1$

13 $x = -3, y = 6$ or $x = 7, y = -4$

14 a $2(x - \frac{5}{4})^2 + \frac{23}{8}$

b $2x^2 - 5x + 6$ is always greater than 0.
Minimum value $\frac{23}{8}$

15 a



b 0 or $\frac{1}{6}$

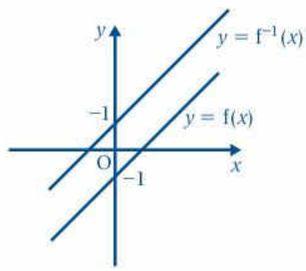
16 a $fg(x) = \frac{3}{x}$ **b** $gfh(x) = \frac{1}{3(x^2 - 1)}$

c $g^{-1}f^{-1}(x) = \frac{3}{x}$ **d** $(gf)^{-1}(x) = \frac{1}{3x}$

17 a −4 **b** 0 **c** 21

18 30°, 150° **19** 120°, 240°

20



$f(x) = x - 1$
$g(x) = x^2$
$f^{-1}(x) = x + 1$

21 $f^{-1}(x) = 2 + \sqrt{x + 3}$ Range $\geqslant -3$

22 a $\mathbf{AB} = \begin{pmatrix} 2 & 0 \\ 5 & k \end{pmatrix}$ $\mathbf{BA} = \begin{pmatrix} 2 & 0 \\ 4 + 3k & k \end{pmatrix}$

b $k = \frac{1}{3}$

c because the determinant is 0

23 $\begin{pmatrix} 16 & -4 \\ -6 & 2 \end{pmatrix}$ **24** $k = 9, a = 1.2$

25 a C **b** B **c** E **d** A **e** D

26 **a** $\begin{pmatrix} 10 & -14 \\ 8 & -11 \end{pmatrix}$ **b** $\begin{pmatrix} -3 & -2 \\ 4 & 2 \end{pmatrix}$

c $\begin{pmatrix} 1 & -1 \\ -\frac{5}{2} & 3 \end{pmatrix}$ **d** $\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}$

e $\begin{pmatrix} 46 & 16 \\ 40 & 14 \end{pmatrix}$ **f** $\begin{pmatrix} 26 & -45 \\ -15 & 26 \end{pmatrix}$

g $\begin{pmatrix} 12 & 4 \\ 10 & 4 \end{pmatrix}$ **h** $\begin{pmatrix} 6 & -9 \\ -3 & 6 \end{pmatrix}$

27 **a** $\begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$ **b** $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ **c** $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

d $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ **e** $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ **f** $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

g -1 **h** 1 **i** $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ **j** $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

28 $a = 23$, $b = -16$, $c = 19$, $d = -6$

29 **a** 17, 20, 23 **b** $u_n = 3n - 1$

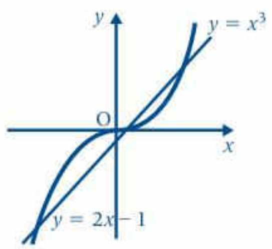
30 **a** 20 **b** **i** $5\frac{1}{2}$ m **ii** 244

31 **a** 3 999 996

b Any number of fours × 9, begins with 3, ends with 6, has one fewer 9 between the 3 and the 6 as there are fours in the given number.

32 $-2 < x < 2$

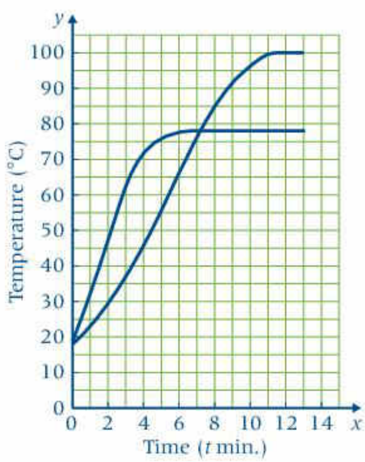
33



34 **a** $\frac{5}{4} + \frac{4}{5} = \frac{41}{20}$

b $\frac{n+1}{n} + \frac{n}{n+1} = \frac{2n^2 + 2n + 1}{n(n+1)}$

35 **a** and **b**



c **i** 8.7°/min **ii** 14.5°/min

d No, e.g. from 4 min. to 6 min. gradient of tangent to the curve for water is steeper than the tangent to the curve for alcohol.

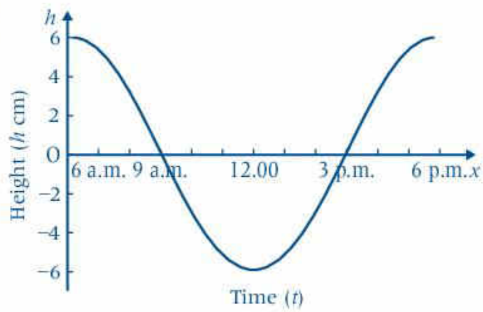
36 **a** 90 s

b No, there are no lines with zero gradient.

c $\frac{4}{15}$ m/s^2 **d** Cyclist was decelerating.

e 347.5 m **f** 3.86 m/s

37 **a**



b **i** 7.36 a.m. and 4.24 p.m.

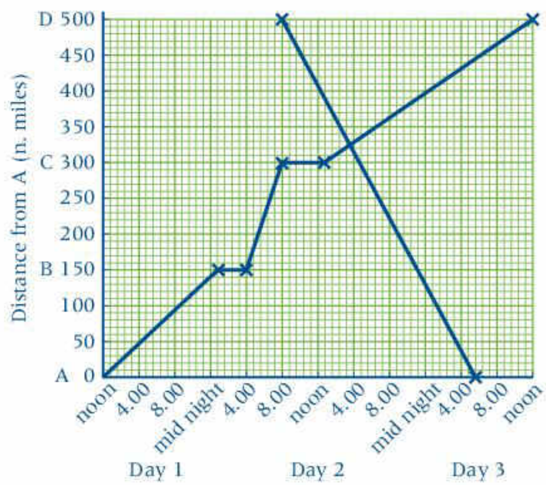
ii 3 m below the marked rung

38 **a** 37.4 mph **b** 1.15 p.m., 61.8 mph

c **i** 44.5 mph

ii 53 miles from A at 1.25 p.m. and 35 miles from A at 1.50 p.m.

39



(The graph is shown half-scale.)

a 30 knots between B and C; 10 knots between C and D

b 8 a.m. on the second day

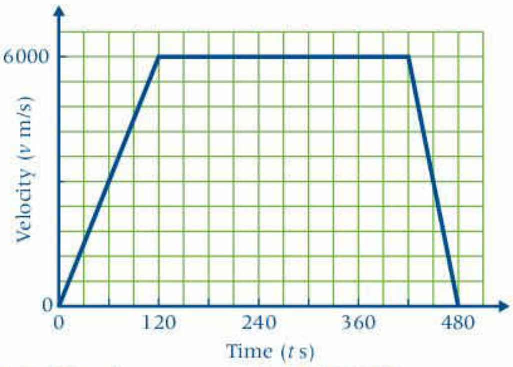
c 320 n. miles from A at 3 p.m. on the second day

40 **a** 84 km **b** 161 km from Bodley at 05.04

c 25 km

41 **a** 12.56 **b** 137 km/h
c 100 km from Caxton at 11.56
d 100 km/h

42 **a**

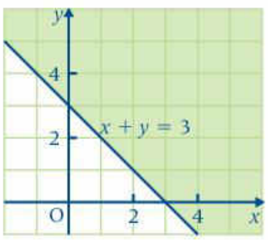


b **i** 6000 m/s **ii** 2 340 000 m

Chapter 16

Exercise 16a

1



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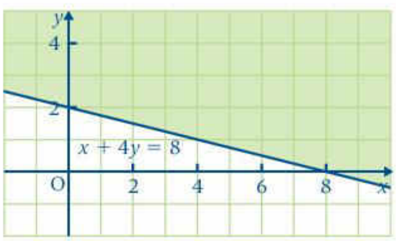

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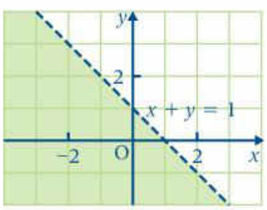
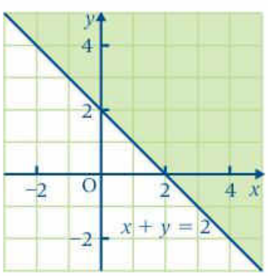
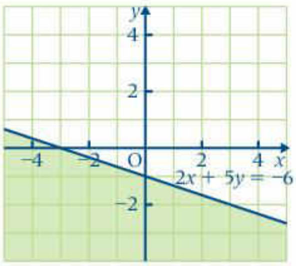


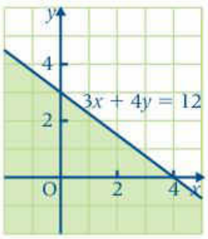
4



5



6



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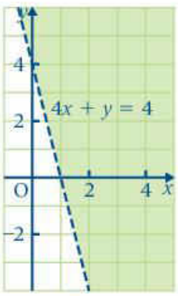

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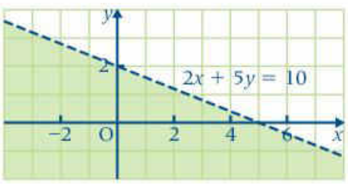

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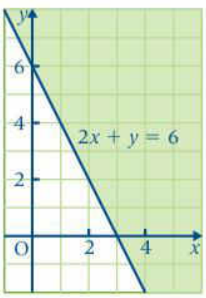
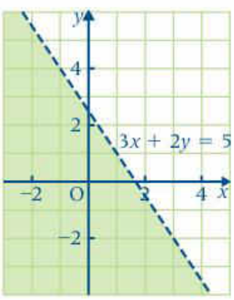
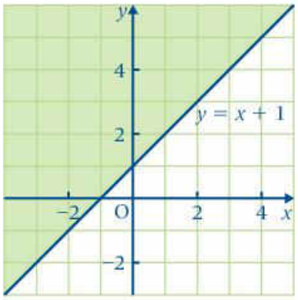


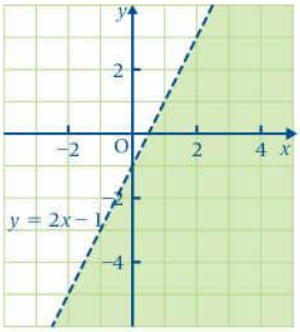
10



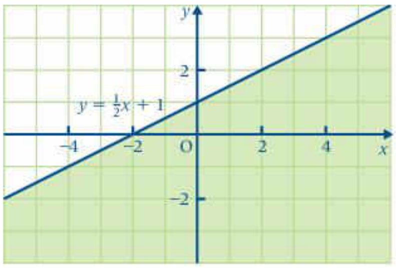
11



12



13



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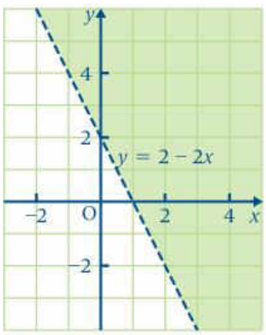
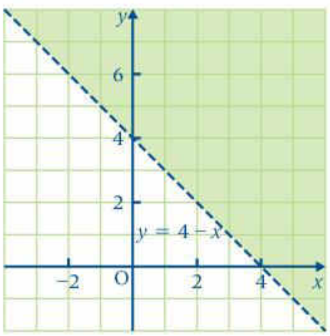
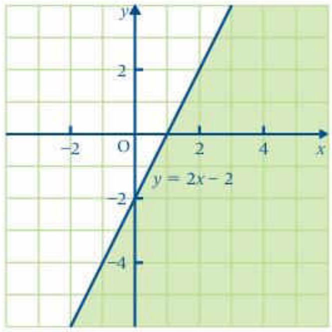


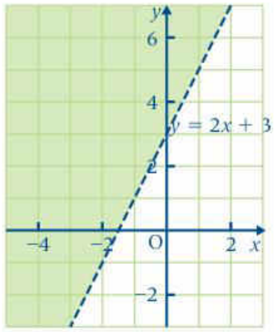
15



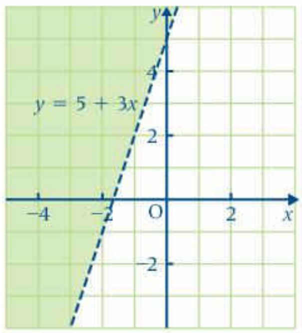
16



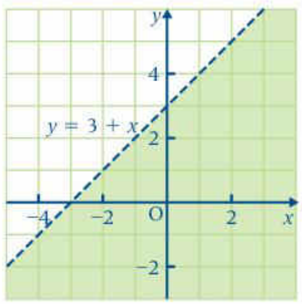
17



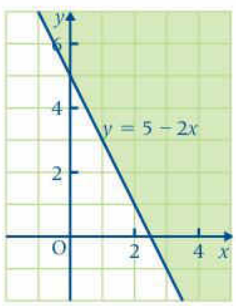
18



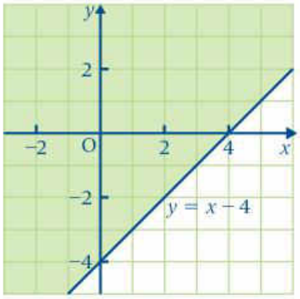
19



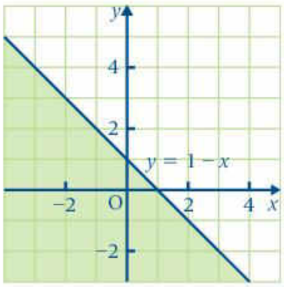
20



21



22

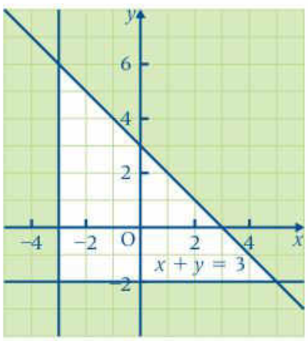


Exercise 16b

1 $x + y \leqslant 3$
2 $2x + y \geqslant 2$
3 $x + 2y < 2$
4 $x + y < 2$
5 $3x - y \leqslant 3$
6 $2y - 3x \leqslant 6$
7 $y \leqslant x + 1$
8 $y > -2x - 4$
9 $y \geqslant \frac{1}{2}x + 2$
10 $y > -x + 2$ or $x + y > 2$
11 $y \geqslant 2x - 2$
12 $y < -\frac{1}{2}x + 2$ or $x + 2y < 4$
13 $y \leqslant 2x + 2$

Exercise 16c

1



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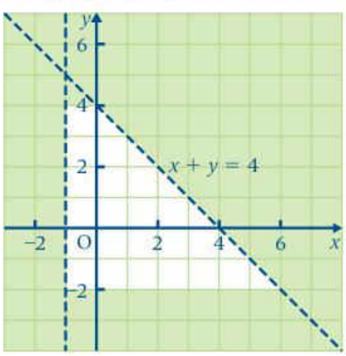

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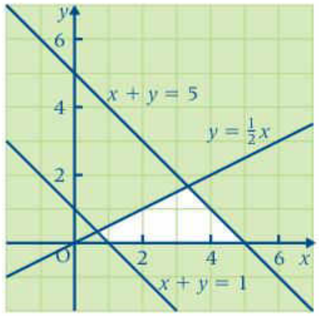

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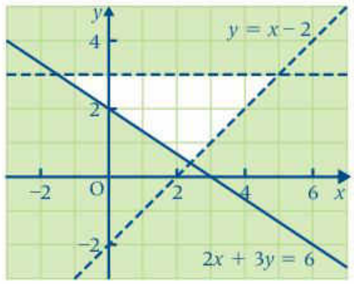

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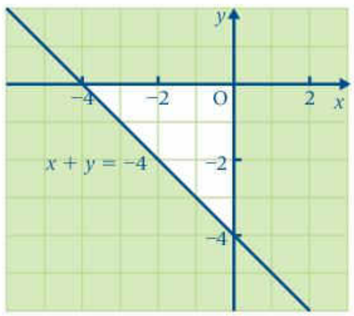

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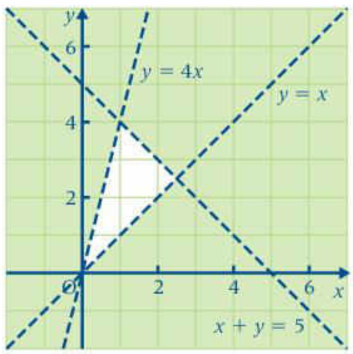

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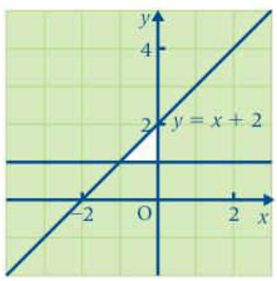

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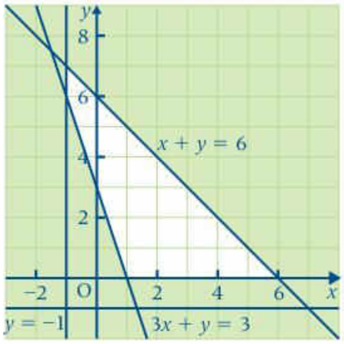

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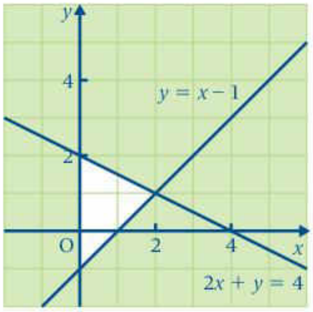

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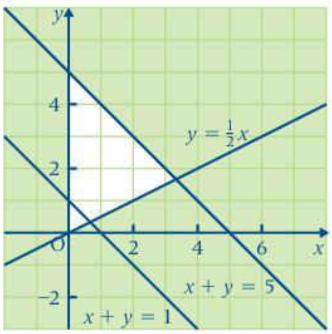

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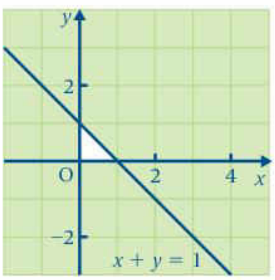
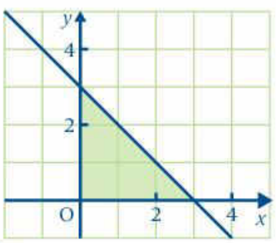


12 It does not exist.

13 a Region consists of 1 point (1, 2).
b Region does not exist.

Exercise 16d

1


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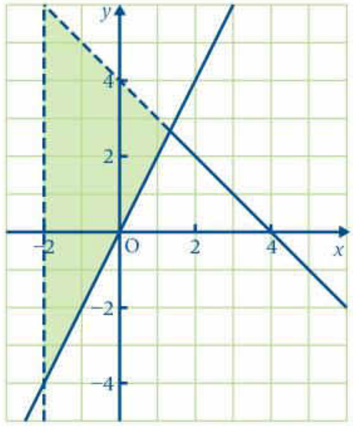

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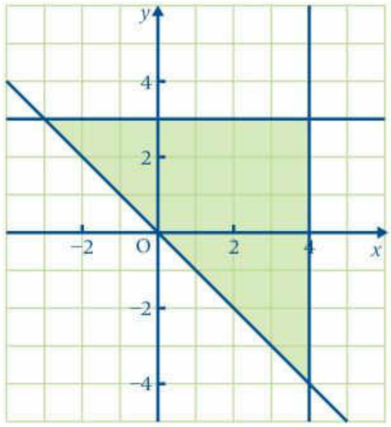

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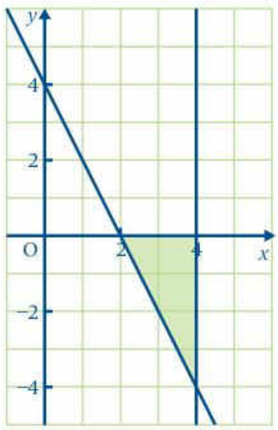
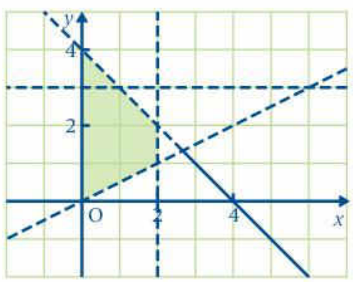


5


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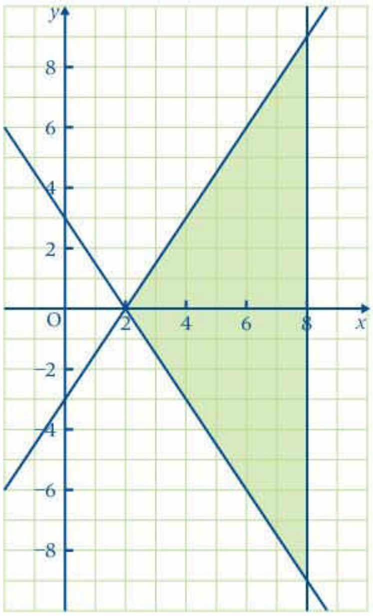
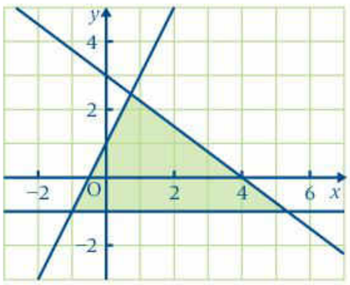


7


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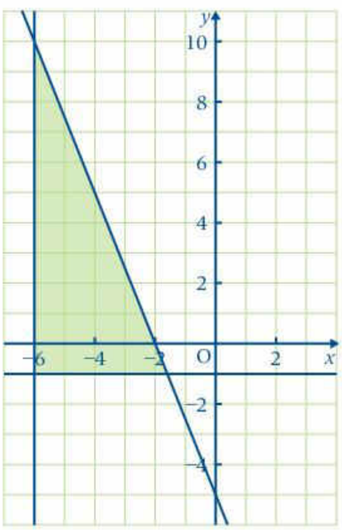

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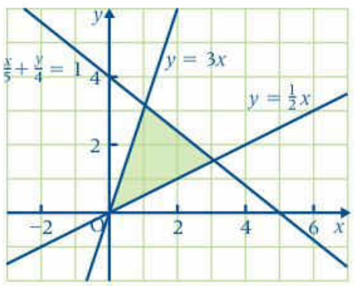
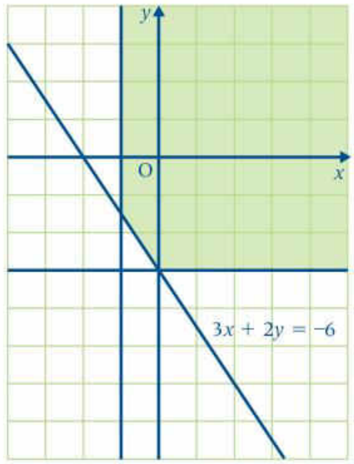


10


Exercise 16e

1 $x \geqslant -1, y \geqslant -2, x + y \leqslant 3$

2 $y \geqslant 0, 2y \leqslant x + 2, x + y \leqslant 4$

3 $y \leqslant x - 3, 2y \geqslant x - 6$

4 $x \leqslant 1, y \leqslant x + 1, 3x + y > -3$

5 $y > -1, x + y < 3, y \leqslant 2x + 2$

6 $y \geqslant 0, x \geqslant -1, y \leqslant x + 2$

7 $y < 3x + 3, y > 3x - 3$

8 $y \leqslant \frac{1}{3}x + 1, y \geqslant -\frac{1}{3}x - 1, y \geqslant \frac{5}{3}x - 7$

9 a $x + y \leqslant 3, 4y \geqslant x, y \leqslant x + 3$
b $4y \leqslant x, x + y \geqslant 3$
c $y \leqslant x + 3, x + y \geqslant 3, 4y \geqslant x$
d $4y \leqslant x, x + y \leqslant 3, y \leqslant x + 3$
e $y \geqslant x + 3, x + y \geqslant 3$
f $x + y \leqslant 3, 4y \geqslant x, y \geqslant x + 3$

10 a $x + y \geqslant 1, y \leqslant 2x + 4$
b $y \leqslant 2, x + y \geqslant 1$
c $y \geqslant 2, y \leqslant 2x + 4$
d $y \geqslant 2x + 4, x + y \geqslant 1$
e $x + y \leqslant 1, y \geqslant 2x + 4$
f $y \leqslant 2, y \leqslant 2x + 4$

11 a C b A c B

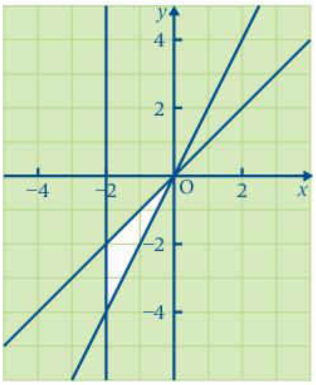
Exercise 16f

1 (2, 2), (−2, 4), (−2, −2)

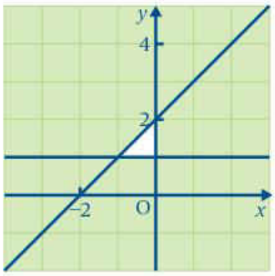
2 (2, 3), (−1, 0), (0, −2)

3 (1, −2), (1, 1.5), (−6, −2)

4 (1, 1), (7, 3), (4, 6), (−4, 6)

5 

(−2,−4), (−2,−2), (0, 0)

6 

(0, 1), (0, 2) and (−1, 1)

7 1. 19 points (−2, −2), (−2, −1), (−2, 0), (−2, 1), (−2, 2), (−2, 3), (−2, 4), (−1, −1), (−1, 0), (−1, 1), (−1, 2), (−1, 3), (0, 0), (0, 1), (0, 2), (0, 3), (1, 1), (1, 2), (2, 2)
2. 4 points (0, 0), (1, 1), (0, −1), (0, −2)
3. 20 points (−6, −2), (−5, −2), (−4, −2), (−4, −1), (−3, −2), (−3, −1), (−2, −2), (−2, −1), (−2, 0), (−1, −2), (−1, −1), (−1, 0), (0, −2), (0, −1), (0, 0), (0, 1), (1, −2), (1, −1), (1, 0), (1, 1)

8 13 points (−1, 0), (−1, 1), (−1, 2), (−1, 3), (0, −1), (0, 0), (0, 1), (0, 2), (0, 3), (1, 0), (1, 1), (1, 2), (2, 1)

9 3 points (1, 1), (2, 1), (1, 2)

10 9 points (2, −1), (2, 0), (2, 1), (2, 2), (3, 1), (4, 0), (5, −1), (4, −1), (3, −1)

Exercise 16g

1 5, 2, −4 2 4, −3, 7

3 11, −14, 0 4 5, 22; At (2, −8)

5 10, −1; At (3, 8)

6 a (−2, −1), (3, −1), (3, 3), (0, 3) b (3, 3)
c (−2, −1) d 20 e No

7 a (6, 0), (0, 3), (−2, −3) b (6, 0)

8 a (4, 1), (−2, 2), (−2, 5)
b i (4, 1) ii (−2, 5)
c (−2, 3), (−2, 4), (−1, 2), (−1, 3), (−1, 4), (0, 2), (0, 3), (1, 2), (1, 3), (2, 2); 10 points
d No

9 b (5, −3), (−2, 4), (−2, −3)
c 33 points
d greatest at (5, −3), least at (−2, 4)

Exercise 16h

1 (1, 2) 2 (2, −2) 3 (2, −1)

4 (0, 4), (1, 2), (2, 0), (3, −2)

5 (0, 3) 6 (0, 3); No

Exercise 16i

1 $3000 2 18

3 $1 \leqslant x < 5$; $7 \leqslant y \leqslant 10$; $x + y \leqslant 12$

4 $2x + y \geqslant 8$; $x + 9y \geqslant 18$; $x + 3y \geqslant 12$

Mixed exercise 16

1 C 2 A 3 A 4 B

5 B 6 C 7 A

Chapter 17

Exercise 17a

1 $y = 3x$ 2 $q = p^2$ 3 $V = x^3$

4 $r = \sqrt{A}$ 5 $y = \frac{24}{x}$ 6 $s = \frac{r}{10}$

7 $y = 4x^2$ 8 $q = -\frac{36}{p}$

9

2	4	5	6	8	10
2	8	12.5	18	32	50

$A = \frac{1}{2}L^2$

10

3	6	9	12
3	12	27	48

$A = \frac{1}{3}b^2$

11

2	3	4	5
8	27	64	125

$y = x^3$

Exercise 17b

1 $y = 10x$;

x	2	4	7	8	9.5
y	20	40	70	80	95

2 $C = 6r$;

r	1	3	5	6	8
C	6	18	30	36	48

3 $C = 6n$;

n	100	120	142	260	312	460
C	600	720	852	1560	1872	2760

4 $Y = 10X$;

X	2	4	7	9	11	15
Y	20	40	70	90	110	150

5 a 9 b 16

6 a $\frac{3}{2}$ b 20

7 a 21 b 7

8 a 21 b 40

9 a 24 b 15

10 a 15 b 8

11 a 6 b 3

Exercise 17c

1

0	2	3	4	5	8
0	12	27	48	75	192

$y = 3x^2$

2

2	4	5	6	10
20	80	125	180	500

$S = 5t^2$

3

−3	−1	0	2	4	7
36	4	0	16	64	196

$y = 4x^2$

4 a 32 b ±1

5 a $\frac{3}{4}$ b $\pm\frac{1}{3}$

6 a 108 b 8

7

2	4	6	8	10
2	16	54	128	250

$V = \frac{1}{4}H^3$

8

3	6	9	12	15
9	72	243	576	1125

$y = \frac{1}{3}x^3$

9 a 24 b 6

10 a 216 b 1

11 a 108 b 2

12

0	1	4	9	25
0	4	8	12	20

$V = 4\sqrt{R}$

13 a 2 b 900

14 no; yes, $y = \sqrt{x}$

15 no; $y = \frac{1}{2}x^3$

Exercise 17d

1

25	50	100	125
20	10	5	4

$N = \frac{500}{C}$

2

12	9	8	6
60	80	90	120

$C = \frac{720}{N}$

3

4	5	6	8	12
30	24	20	15	10

$V = \frac{120}{P}$

4 $y = \frac{72}{x}$ 5 $y = \frac{2.16}{x}$

6 a $y = \frac{1}{x}$

b

$\frac{1}{x}$	0.1	0.2	1	2	4
y	0.1	0.2	1	2	4

c 1; constant of variation

7 a $y = \frac{7.2}{x}$

b

$\frac{1}{x}$	0.42	0.56	0.83	1.11	1.25
y	3	4	6	8	9

c 7.2; constant of variation

Exercise 17e

1

2	4	6	9	12	18
18	9	6	4	3	2

$y = \frac{36}{x}$

2

0.5	1	2	3	6	10
144	36	9	4	1	0.36

$y = \frac{36}{x^2}$

3

0.25	1	4	9	16	25
120	60	30	20	15	12

$q = \frac{60}{\sqrt{p}}$

4 a 4 b 20 c −10

5 a $\frac{4}{3}$ b 16

6 a 10 b 40

7 a 6 b 0

8 a 25 b ±10

9 a 4 b 12

10 a 56 b 8

11 a 2 b 2 c −1
d $\frac{1}{2}$ e 1 f −1

Exercise 17f

1 a 1 b 1 c 6.25

2 a 21 b 6

3 a $y = \frac{3}{4}x^3$ b 6 c 2

4 a 14 b 3

5 a 8 b ±3

6 a i 2 ii 1.25 b i 16 ii 49

7

0	1	2	4	8
0	0.25	1	4	16

8

0	4	9	16	64
0	0.5	0.75	1	2

9 a i doubled ii halved
b i multiplied by 9 ii divided by 9

10 a i divided by 4 ii multiplied by 4
b i divided by 3 ii multiplied by 3

11 a 2 b 3 c 1 d −1

12 a 4 kg b 25 cm

13 64 m 14 a 1.6 b 56.25

15 a 25 cm b 4.8 newtons

16 a $320 b 4.5 cm

17 a multiplied by 2 b multiplied by 5
c increased by 20%

18 a 0.216 litres b 20 cm

19 a 3 b 120

20 a 40 mph b 12 mph

21 a i 250 ii 40 iii 444
b $x = \sqrt{\frac{k}{n}}$
c 2.23 cm, round down because rounding up gives less than 800 squares

22 a i 1.26 units ii 2.45 units
b i 1.6 m ii 1.2 m

23 D

24 a doubled b halved
c quadrupled d multiplied by 8

25 a increased by 25% b decrease by 20%
c increase by 56.25%

Mixed exercise 17

1 C 2 A 3 A 4 C
5 B 6 D 7 A

Chapter 18

Exercise 18a

1 **b** and **c**

3 a $\sqrt{3}$ b $\sqrt{3}$ c $\sqrt{3}$ d $\sqrt{3}$
e $2\sqrt{5}$ f $\frac{3}{2}\sqrt{5}$ g $2\sqrt{5}$ h $3\sqrt{5}$

4 a 3 b 5 c 5 d 6
e 5 f 6 g 5
h b and c i b and g or c and g

Exercise 18c

1 a $\overrightarrow{PR}$ b $\overrightarrow{PQ}$ c $\overrightarrow{RP}$ d $\overrightarrow{QR}$

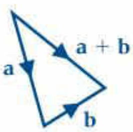
2 a $\overrightarrow{AC} + \overrightarrow{CB}$
b $\overrightarrow{AC} + \overrightarrow{CB}$ (or $\overrightarrow{AD} + \overrightarrow{DB}$)
c e.g. $\overrightarrow{AC} + \overrightarrow{CB}$

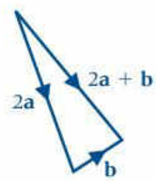
3 a $\overrightarrow{AC}$ b $\overrightarrow{BD}$ c $\overrightarrow{AD}$
d $\overrightarrow{AD}$ e $\overrightarrow{DC}$ f $\overrightarrow{AC}$

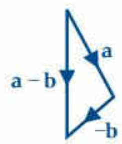
4 a $\overrightarrow{AD}$ b $\overrightarrow{BA}$ c $\overrightarrow{AD}$ d $\overrightarrow{DC}$

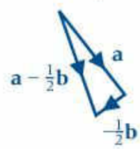
5 a $\overrightarrow{AB}$ b $\overrightarrow{BC}$ c $\overrightarrow{AC}$

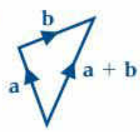
6 a $\overrightarrow{DC}$ b $\overrightarrow{DB}$ c $\overrightarrow{AB}$ d $\overrightarrow{AD}$

7 a


b


c


d


8 a


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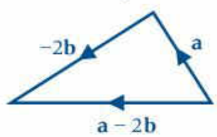

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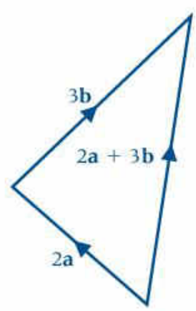
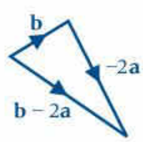
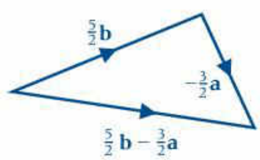


d


e


Exercise 18d

1 a i $\begin{pmatrix}0\\1\end{pmatrix}$ ii $\begin{pmatrix}6\\1\end{pmatrix}$ iii $\begin{pmatrix}4\\4\end{pmatrix}$
iv $\begin{pmatrix}2\\4\end{pmatrix}$ v $\begin{pmatrix}4\\3\end{pmatrix}$ vi $\begin{pmatrix}-4\\3\end{pmatrix}$

2 a i $\begin{pmatrix}4\\2\end{pmatrix}$ ii $\begin{pmatrix}1\\4\end{pmatrix}$
b AD ∥ BC, AD = 2BC

3 a i $\begin{pmatrix}1\\1\end{pmatrix}$ ii $\begin{pmatrix}1\\1\end{pmatrix}$

4 a i $\begin{pmatrix}4\\4\end{pmatrix}$ ii $\begin{pmatrix}-3\\3\end{pmatrix}$ iii $\begin{pmatrix}1\\7\end{pmatrix}$

5 b parallelogram

6 a i $\begin{pmatrix}2.5\\0.5\end{pmatrix}$ ii $\begin{pmatrix}-2.5\\-0.5\end{pmatrix}$

Exercise 18e

1 a Yes; PQ and QR are parallel and have Q in common.
b i $\mathbf{b}-\mathbf{a}$ ii $2\mathbf{a}-\mathbf{b}$ iii $\mathbf{b}-2\mathbf{a}$

2 a $\mathbf{q}-\mathbf{p}$ b $\frac{1}{2}(\mathbf{q}-\mathbf{p})$
c $\frac{1}{2}(\mathbf{q}-\mathbf{p})$ d $\frac{1}{2}(\mathbf{q}+\mathbf{p})$

3 a trapezium
b i $\mathbf{a}-\mathbf{b}$ ii $\mathbf{a}-2\mathbf{b}$ iii $\mathbf{a}+\mathbf{b}$

4 a $4\mathbf{b}-6\mathbf{a}$ b $\mathbf{b}-\frac{3}{2}\mathbf{a}$ c $\mathbf{b}+\frac{9}{2}\mathbf{a}$

5 a $\frac{1}{2}\mathbf{b}$ b $\frac{1}{2}\mathbf{c}$ c $\mathbf{c}-\mathbf{b}$
d $\frac{1}{2}(\mathbf{c}-\mathbf{b})$ f $\text{PQ}=\frac{1}{2}\text{BC};\ \frac{1}{2}(\mathbf{c}-\mathbf{a})=\frac{1}{2}\overrightarrow{\text{BC}}$

6 a $\frac{1}{2}\mathbf{b}$ b $\mathbf{b}-\mathbf{a}$ c $\frac{1}{2}(\mathbf{b}-\mathbf{a})$
d $\frac{1}{2}\mathbf{b}$ e $\frac{1}{2}\mathbf{b}$
g parallelogram (PQ and SR are // and =)

7 a $\mathbf{b}-\mathbf{a}$ b $\frac{1}{3}(\mathbf{b}-\mathbf{a})$ c $\frac{2}{3}(\mathbf{b}-\mathbf{a})$
d $\frac{1}{3}\mathbf{b}+\frac{2}{3}\mathbf{a}$ e $\frac{1}{3}\mathbf{b}+\frac{2}{3}\mathbf{a}$
f $\overrightarrow{\text{AP}}=\overrightarrow{\text{QC}}$ so AP and QC are equal and parallel, i.e. APCQ is a parallelogram

8 a $\mathbf{b}$ b $\frac{2}{3}\mathbf{b}$ c $\mathbf{a}+\frac{2}{3}\mathbf{b}$
d $\frac{1}{2}\mathbf{b}$ e $\frac{3}{4}\mathbf{b}$ f $\frac{1}{2}\mathbf{b}+\frac{3}{4}\mathbf{a}$
g $\overrightarrow{\text{RQ}}=\frac{3}{4}\overrightarrow{\text{OP}}$ h $\text{RQ}=\frac{3}{4}\text{OP}$

9 a They all lie on the same straight line.
b parallelogram
c trapezium d ECBF

10 a $6\mathbf{a}-2\mathbf{b}$ b $3\mathbf{a}-\mathbf{b}$
c $\overrightarrow{\text{BP}}=6\mathbf{a}-2\mathbf{b}=2(3\mathbf{a}-\mathbf{b})=2\overrightarrow{\text{BQ}}$ so $\overrightarrow{\text{BD}}$ and $\overrightarrow{\text{BQ}}$ are in the same direction and both start at B, i.e. B, P and Q lie on a straight line.
d 1 : 2

11 a i $\mathbf{b}-\mathbf{a}$ ii $-\mathbf{a}$ iii $-\mathbf{b}$ iv $\mathbf{a}-\mathbf{b}$
b i $\mathbf{b}-\mathbf{a}$ ii $-\mathbf{a}$ iii $-\mathbf{b}$ iv $\mathbf{a}-\mathbf{b}$
v $\mathbf{a}$ vi $\mathbf{b}$

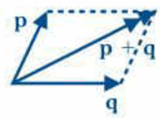
12 a $2\mathbf{x}+2\mathbf{y}$ b $3\mathbf{x}-\mathbf{y}$
c $-2\mathbf{x}-\mathbf{y}$ d $-2\mathbf{x}+3\mathbf{y}$

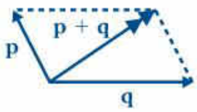
13 a $\mathbf{a}-\mathbf{b}$ b $\mathbf{b}-\mathbf{a}$ c $\frac{1}{2}(\mathbf{a}-\mathbf{b})$
d $\frac{1}{2}(\mathbf{b}-\mathbf{a})$ e Yes, both $\frac{1}{2}(\mathbf{b}+\mathbf{a})$

14 a $\mathbf{b}-\mathbf{a}$ b $\mathbf{a}-\mathbf{b}$ c $\frac{1}{4}(\mathbf{b}-\mathbf{a})$
d $\frac{3}{4}(\mathbf{a}-\mathbf{b})$ e $\frac{1}{4}(\mathbf{b}+3\mathbf{a})$
f $\frac{1}{6}(\mathbf{b}+3\mathbf{a})$ g $\frac{1}{12}(\mathbf{b}+3\mathbf{a})$

15 a $\mathbf{q}-\mathbf{p}$ b $k(\mathbf{q}-\mathbf{p})$
c $(1-k)\mathbf{q}-(1-k)\mathbf{p}$ d $k\mathbf{q}+(1-k)\mathbf{p}$

Exercise 18f

1 a 

b 

2 b 15 m/s 3 b 59°

4 b 084° c 337 mph

5 20.4 m/s, 169°

6 6.6 N at 104° to the 8 N force

7 b 65.4 km/h, from 276.5°

8 10 N 9 c 97 km/h on a bearing of 112°

10 a 63° to the bank c 1.52 m/s
b 60° to the upstream bank

11 No; 60° to the bank upstream; increases speed but lengthens distance when pointed at an angle to the bank upstream and vice versa; no

Exercise 18g

1 a $\frac{2}{3}k\mathbf{a}+\frac{1}{3}k\mathbf{b},\ \frac{2}{3}k\mathbf{a}+\left(\frac{1}{3}k-1\right)\mathbf{b}$
b 3 c 1 : 2

2 a $k\mathbf{a}-\frac{1}{4}\mathbf{b}$ b $\frac{1}{2}$ c $\frac{1}{2}$

3 a $h\mathbf{p}$ b $\mathbf{r}-\mathbf{p}$
c $(h-1)\mathbf{p}+\mathbf{r}$ d $k(\mathbf{r}-\mathbf{p})$
e $k\mathbf{r}+(1-k)\mathbf{p}$ f $h=\frac{1}{k}$
g $k=\frac{1}{2},\ h=2$

4 a $\mathbf{q}-\mathbf{p}$ b $\frac{1}{3}(\mathbf{q}-\mathbf{p})$ c $\frac{1}{3}\mathbf{q}+\frac{2}{3}\mathbf{p}$
d $\frac{1}{2}\mathbf{p}$ e $\frac{1}{2}\mathbf{p}-\mathbf{q}$ f $\frac{1}{2}h\mathbf{p}-h\mathbf{q}$

g $\frac{1}{2}h\mathbf{p} + (1 - h)\mathbf{q}$ h $\frac{1}{3}k\mathbf{q} + \frac{2}{3}k\mathbf{p}$

i $h = \frac{4}{5}, k = \frac{3}{5}$

5 a i $2\mathbf{b} - 2\mathbf{a}$ ii $2\mathbf{b} - 4\mathbf{a}$

iii $3\mathbf{a}$ iv $2\mathbf{b} - 2\mathbf{a}$

b $\overrightarrow{HG} = 2\mathbf{b} - 2\mathbf{a}$ and $\overrightarrow{EF} = \mathbf{b} - \mathbf{a}$

so $\overrightarrow{HG} = 2\overrightarrow{EF}$

$\therefore$ HG is parallel to EF

c rhombus

6 a $2\mathbf{c}, 2\mathbf{c} - \mathbf{a}$ b $\mathbf{b} = \frac{1}{2}\mathbf{a} - \mathbf{c}$ c $\frac{3}{2}$

7 (1, 10)

Mixed exercise 18

1 D	2 A	3 B	4 C
5 D	6 B	7 B	8 D

Chapter 19

Exercise 19a

1 $\begin{pmatrix}4\\5\end{pmatrix}$	2 $\begin{pmatrix}3\\-2\end{pmatrix}$	3 $\begin{pmatrix}-7\\5\end{pmatrix}$
4 $\begin{pmatrix}-3\\-5\end{pmatrix}$	5 $\begin{pmatrix}5\\3\end{pmatrix}$	6 $\begin{pmatrix}-5\\2\end{pmatrix}$
7 (5, 2)	8 (1, −3)	9 (−2, −4)
10 (2, −3)	11 (−6, 2)	12 (2, −6)

Exercise 19b

1 $\begin{pmatrix}2\\-5\end{pmatrix}$	2 $\begin{pmatrix}-4\\-3\end{pmatrix}$	3 $\begin{pmatrix}-4\\2\end{pmatrix}$
4 $\begin{pmatrix}5\\3\end{pmatrix}$	5 $\begin{pmatrix}3\\5\end{pmatrix}$	6 $\begin{pmatrix}4\\2\end{pmatrix}$

Exercise 19c

1 $\begin{pmatrix}5\\11\end{pmatrix}$	2 $\begin{pmatrix}9\\6\end{pmatrix}$	3 $\begin{pmatrix}4\\1\end{pmatrix}$
4 $\begin{pmatrix}-2\\-5\end{pmatrix}$	5 $\begin{pmatrix}5\\4\end{pmatrix}$	6 $\begin{pmatrix}-1\\-1\end{pmatrix}$

7 A′(−1, 1), B′(−3, 2)

8 A′(−1, −1), B′(2, 4)

9 A′(10, 3), B′(−5, −2)

10 A′(3, 3), B′(−6, 0)

11 A′(−3, 2), B′(3, −7)

12 A′(7, 4), B′(1, −8)

Exercise 19d

1 A′(−2, −1), B′(2, −1), C′(3, −2) D′(−1, −2); Reflection in *x*-axis

2 A′(1, 1), B′(1, 4), C′(2, 4); Reflection in line $y = x$

3 A′(−2, −3), B′(−5, −3), C′(−3, 2); Reflection in *y*-axis

4 A′(−1, −4), B′(−3, −3), C′(0, −2); Reflection in line $y = -x$

5 A′(1, 1), B′(1, 3), C′(2, 3), D′(2, 1); Reflection in $y = x$

6 A′(0, 2), B′(0, 4), C′(2, 4), D′(2, 2); Reflection in line $y = x$

7 A′(−1, −1), B′(−1, −2), C′(−2, −2), D′(−2, −1); Reflection in line $y = -x$

8 A′(−1, 0), B′(−4, 0), C′(−4, 2); Reflection in *y*-axis

9 A′(2, −1), B′(3, −1), C′(3, −4), D′(2, −4); Reflection in *x*-axis

10 A′(1, 1), B′(1, 3), C′(3, 4), D′(3, 3); Reflection in line $y = x$

11 A′(−2, 4), B′(−4, 5), C′(−3, 2); Reflection in *y*-axis

Exercise 19e

1 A′(1, −1), B′(1, −4), C′(3, −4), D′(3, −1); Rotation of 90° clockwise about O

2 A′(−1, −1), B′(−4, −1), C′(−4, −2), D′(−1, −2); Rotation of 180° about O

3 A′(0, 1), B′(0, 3), C′(−4, 4); Rotation of 90° anticlockwise about O

4 A′(1, −1), B′(1, −4), C′(4, −4); Rotation of 90° clockwise about O

5 A′(−3, −2), B′(−4, −3), C′(−1, −4); Rotation of 180° about O

Exercise 19f

1 A′(2, 0), B′(6, 0), C′(6, 6); Enlargement centre O, scale factor 2

2 A′(0, 3), B′(−6, 3), C′(−6, 0), O′(0, 0); Enlargement centre O, scale factor 3

3 A′(3, 3), B′(3, 6), C′(6, 6), D′(6, 3); Enlargement centre O, scale factor $1\frac{1}{2}$

4 A′(10, 5), B′(10, 10), C′(−10, 10); Enlargement centre O, scale factor $2\frac{1}{2}$

5 O′(0, 0), A′(0, −2), B′(2, −2), C′(2, 0); Enlargement centre O, scale factor −2

6 A′(0, −2), B′(−3, −2), C′(−3, −5), D′(0, −5); Enlargement centre O, scale factor −1

Exercise 19g

1 A′(−1, −3), B′(1, 3), C′(5, 5), D′(3, −1); (Parallelogram)

2 A′(−1, 3), B′(1, −3), C′(−3, −1), D′(−5, 5); (Parallelogram)

3 A′(−4, −3), B′(2, −3), C′(2, 6), D′(−4, 6); (Rectangle)

4 A′(−8, −4), B′(−2, −1), C′(6, 3), D′(0, 0); (Straight line)

5 A′(1, −2), B′(−3, 6), C′(−1, 2), D′(3, −6); (Straight line)

6 All points → the origin

Exercise 19h

1 O′(0, 0), A′(2, 0), B′(2, −1), C′(0, −1); Reflection in x-axis

2 O′(0, 0), A′(2, 0), B′$\left(3\frac{1}{2}, 1\right)$, C′$\left(1\frac{1}{2}, 1\right)$

3 O′(0, 0), A′(0, −2), B′(−1, −2), C′(−1, 0): Reflection in line $y = -x$

4 O′(0, 0), A′(−6, 0), B′(−6, 3), C′(0, 3)

5 O′(0, 0), A′(4, 6), B′(1, 8), C′(−3, 2)

6 O′(0, 0), A′(4, 0), B′(4, 1), C′(0, 1)

7 O′(0, 0), A′(8, 0), B′(8, 4), C′(0, 4); Enlargement centre O, scale factor 4

8 O′(0, 0), A′(2, 2), B′(4, 6), C′(2, 4)

9 O′(0, 0), A′(2, 0), B′(2, 3), C′(0, 3)

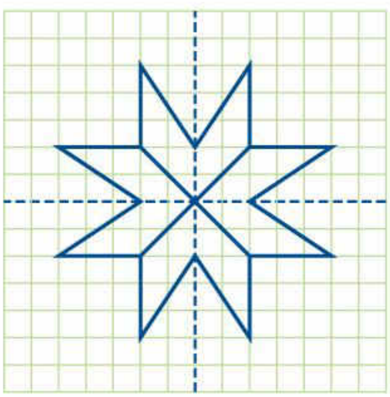
10 O′(0, 0), A′(1, 0), B′$\left(1, \frac{1}{2}\right)$, C′$\left(0, \frac{1}{2}\right)$; Enlargement centre O, scale factor $\frac{1}{2}$

11 O′(0, 0), A′(4, 2), B′(6, 5) C′(2, 3)

12 O′(0, 0), A′(2, 4), B′(2, 5), C′(0, 1)

13 The unit square OABC; A (1, 0), B (1, 1), C (0, 1); or the unit triangle OAB

14



15 The image is the same as the object in each case

16 **a** A′(1, 0), B′(0, −1)
b A′(2, 0), B′(0, 2)
c A′(1, 0), B′(2, 1)
d A′(2, 4), B′(5, −1)

The columns of the matrices give the position vectors of A′ and B′

Exercise 19i

1 Rotation of 90° anticlockwise about the origin

2 Enlargement centre O and scale factor $\frac{1}{3}$

3 Reflection in the x-axis

4 Rotation of 45° clockwise about the origin

Exercise 19j

1 **a** A′(0, 1), B′(0, 3), C′(−3, 3)
b Rotation of 90° anticlockwise about the origin
c $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ **d** The image is ABC
e Rotation of 90° clockwise about the origin

2 **a** A′(3, 3), B′(6, 3), C′(6, 6), D′(3, 6)
b Enlargement centre O, with scale factor 3
c $\begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{3} \end{pmatrix}$ **d** The image is ABCD
e Enlargement, scale factor $\frac{1}{3}$, centre O

3 **a** A′(0, −1), B′(1, −3), C′(2, −3), D′(1, −1)
b Rotation of 90° clockwise about the origin
c $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ **d** The image is ABCD
e Rotation of 90° anticlockwise about the origin; Yes

4 **a** A′(3, 1), B′(9, 3), C′(7, 3)
b $\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$ **c** The image is ABC

5 **a** A′(1, 3), B′(3, 9), C′(5, 13), D′(3, 7)
b $\begin{pmatrix} -2 & 1 \\ 3 & -1 \end{pmatrix}$ **c** A′B′C′D′ → ABCD

6 **a** O′(0, 0), A′(2, 1), B′(10, 5), C′(8, 4); Image is a straight line
b No inverse
c Transformation has no inverse either

Exercise 19k

1 O′(0, 0), A′(2, 0), B′(6, 2), C′(4, 2); O and A

2 O′(0, 0), A′(4, 0), B′(4, 4), C′(0, 4); O

3 O′(0, 0), A′(2, 0), B′(2, −2), C′(0, −2); O and A

4 O′(0, 0), A′(4, 2), B′(6, 6), C′(2, 4); O

5 The origin; Yes

Exercise 19I

1 Translation defined by the vector $\begin{pmatrix}5\\1\end{pmatrix}$

2 Translation defined by the vector $\begin{pmatrix}-5\\1\end{pmatrix}$

3 Translation defined by the vector $\begin{pmatrix}3\\-4\end{pmatrix}$

4 Translation defined by the vector $\begin{pmatrix}-4\\0\end{pmatrix}$

5 Reflection in y-axis followed by a translation defined by the vector $\begin{pmatrix}0\\2\end{pmatrix}$

6 Reflection in x-axis followed by a translation defined by the vector $\begin{pmatrix}-5\\0\end{pmatrix}$

7 Reflection in $x = -1$ followed by a translation defined by the vector $\begin{pmatrix}0\\-3\end{pmatrix}$

8 Rotation 90° clockwise about (2, 0) followed by a translation defined by the vector $\begin{pmatrix}-5\\0\end{pmatrix}$

9 O′(3, 1), A′(4, 1), B′(4, 3), C′(3, 3)

10 A′(l, −1), B′(2, −l), C′(2, 1)

11 A′(−l, −2), B′(−l, 0), C′(−2, 0)

12 A′(0, 1), B′(1, 1), C′(1, 3), D′(0, 3)

13 Translation defined by the vectors:

1 $\begin{pmatrix}-5\\-1\end{pmatrix}$ 2 $\begin{pmatrix}5\\-1\end{pmatrix}$ 3 $\begin{pmatrix}-3\\4\end{pmatrix}$ 4 $\begin{pmatrix}4\\0\end{pmatrix}$

Mixed exercise 19

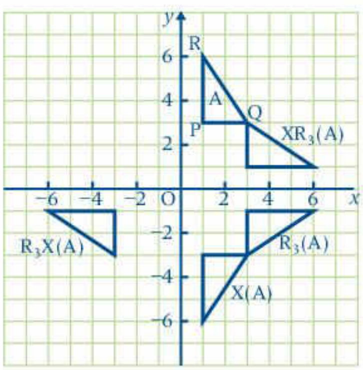
1 B 2 C 3 B 4 C

5 B 6 D 7 B

Chapter 20

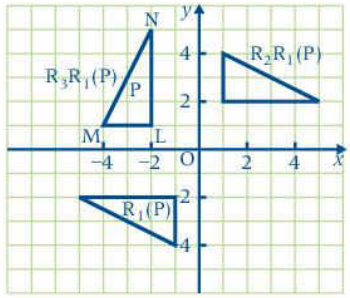
Exercise 20a

1



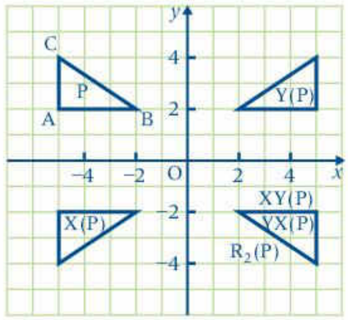
e Reflection in line $y = -x$

2



d R$_3$ e R$_2$

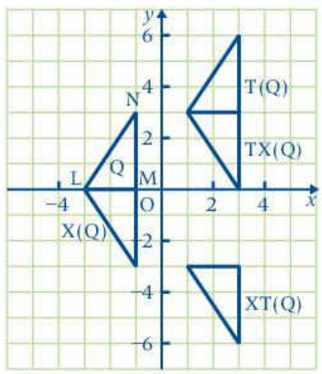
3



f Yes g Yes

h Rotation of 180° about O, i.e. R$_2$

4

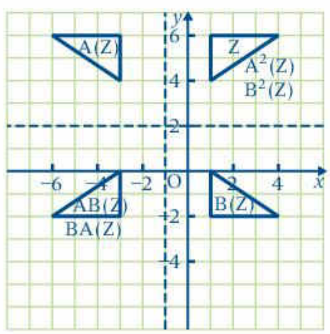


e Translation defined by the vector $\begin{pmatrix}4\\-3\end{pmatrix}$

f Translation defined by the vector $\begin{pmatrix}0\\6\end{pmatrix}$

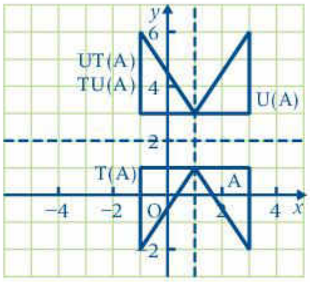
Exercise 20b

1



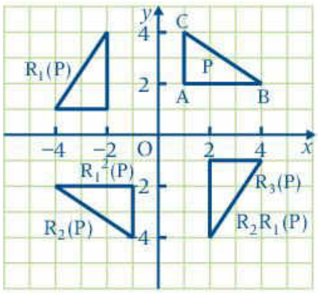
b Rotation of 180° about O: AB = BA

2



b Yes **c** Yes

3

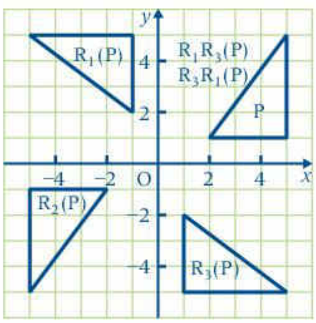


a $R_1^{\,2} = R_2$, $R_2R_1 = R_3$

b $R_3^{\,2} = R_2$, $R_2R_3 = R_1$, $R_3R_2 = R_1$

Exercise 20c

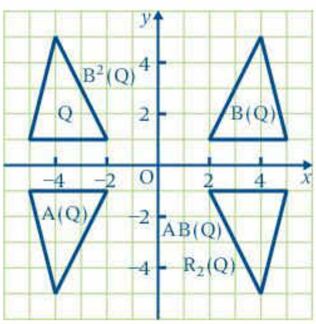
1



a $R_1R_2 = R_3R_1 = I$

b $R_2^{\,2} = I$, $R_2R_3 = R_1$, $R_1R_2 = R_3$

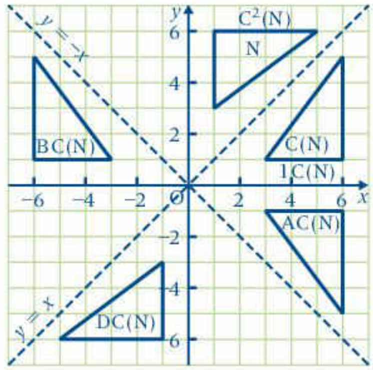
2



b **i** $B^2 = I$

ii $AB = R_2$

3



b **i** $DC = R_2$

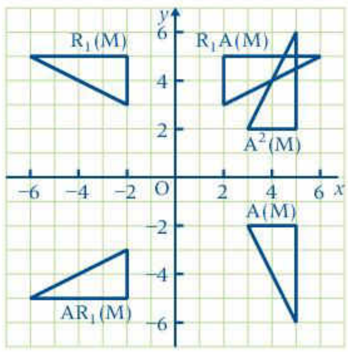
ii $C^2 = I$

iii $AC = R_3$

iv $BC = R_1$

v $IC = C$

4



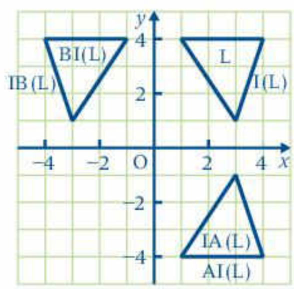
b **i** $A^2 = I$

ii $AR_1 = D$

iii $R_1A = C$

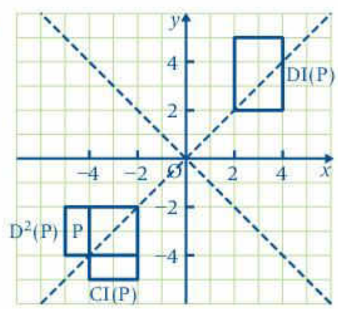
c False

5



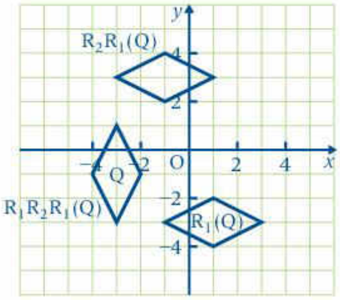
b AI = A, BI = B, IA = A, IB = B

6 **a**



b $CI = C$, $DI = D$, $D^2 = I$

7 a



b $R_2R_1 = R_3$, $R_1R_2R_1 = I$

c True

Exercise 20d

1 Rotation of 90° anticlockwise about O

2 (0, 0), (0, 1), (−1, 1), (−2, 1)

3 Rotation of 180° about O

4 Enlargement centre O, scale factor $2\frac{1}{2}$

5 (0, 0), (0, 1), (3, 1), (2, 1)

6 The square becomes a line segment

7 **1** 1 **2** 1 **3** 1

4 $\frac{25}{4}$ **5** 1 **6** 0

8 a A line segment **b** 0, 0 **c** 0

9 a 5.5 **b** 30

10 a 6 **b** 18

Exercise 20e

1 $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ **2** $\begin{pmatrix} 1 & 2 \\ 2 & 0 \end{pmatrix}$

3 $\begin{pmatrix} 3 & 0 \\ 1 & 1 \end{pmatrix}$ **4** $\begin{pmatrix} 1 & -1 \\ 0 & 5 \end{pmatrix}$

5 $\begin{pmatrix} 1 & -4 \\ 0 & 3 \end{pmatrix}$ **6** $\begin{pmatrix} 9 & -4 \\ -2 & 1 \end{pmatrix}$

7 $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ **8** $\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$

9 $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ **10** $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

11 $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ **12** $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Exercise 20f

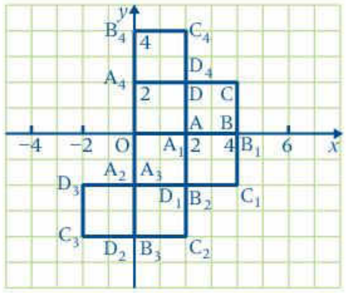
1 a P gives a reflection in the *x*-axis. Q a rotation of 90° clockwise about O.

c Reflection in the line $y = -x$

d $\mathbf{R} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ $\mathbf{S} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

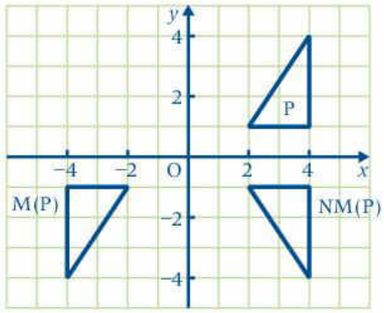
e $A_3B_3C_3D_3$ **f** $vA_4B_4C_4D_4$

g **b** and **e** are the same. If we used P on $A_2B_2C_2D_2$ we would get **f**.



2 a $L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

b



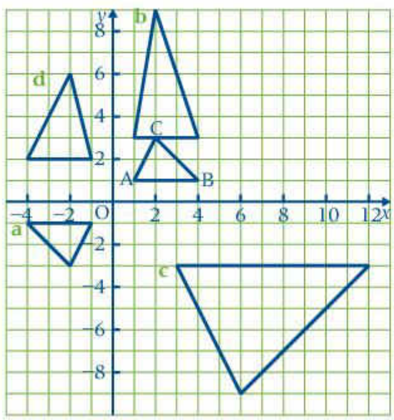
c M is a rotation of 180° about O. NM is a reflection in the *x*-axis.

3 a $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ **b** $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ **c** $\begin{pmatrix} 0 & -1 \\ -1 & 1 \end{pmatrix}$

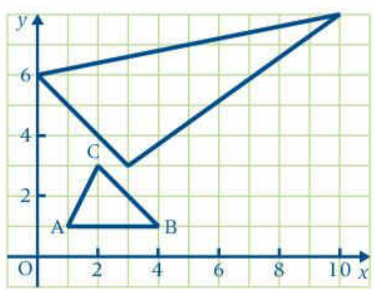
d R = QP

Exercise 20g

1



2



3 An enlargement by factor 2, centre O, followed by a translation of 3 units in the direction of the x-axis and 1 unit in the negative direction of the y-axis.

4 An enlargement by a factor $\frac{1}{2}$, centre O, and a reflection in the y-axis followed by a translation of 3 units in the negative direction of the x-axis and 2 units in the direction of the y-axis.

5 a Rotation of 90° clockwise about O.

b $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$

6 a Reflection in the y-axis followed by a translation given by $\begin{pmatrix} 6 \\ -4 \end{pmatrix}$

b $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 6 \\ -4 \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$

7 a Rotation of 90° clockwise about O and an enlargement by factor 2, centre O.

b $\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$

8 a Reflection in the x-axis followed by a translation of 2 units in the direction of the y-axis

b $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ 2 \end{pmatrix} = \begin{pmatrix} x' \\ y' \end{pmatrix}$

9 $(-4, 9)$;

$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\left[\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 4 \end{pmatrix}\right] = \begin{pmatrix} x' \\ y' \end{pmatrix}$

Mixed exercise 20

1 B	2 A	3 A	4 D
5 B	6 C	7 C	8 B

Review test 4

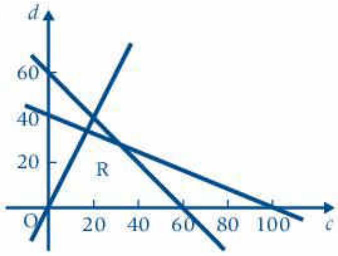
1 B	2 B	3 B	4 A
5 B	6 B	7 B	8 C
9 D	10 C	11 A	12 C
13 C	14 D	15 B	16 C
17 D	18 D	19 B	20 B
21 B	22 C	23 D	24 C
25 D	26 D	27 C	

General proficiency questions

1 a 12 b 2 or −2

2 a $c + d \leqslant 60$, $d \leqslant 2c$, $2c + 5d \leqslant 210$

b



c $P = \$(12c + 20d)$

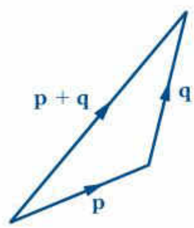
d $c = 30$, $d = 30$, \$960

3 $2 < x < 6$

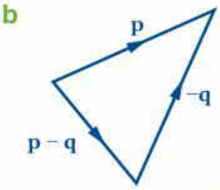
4 P : $y < 4 + 2x \geqslant y > 4 - 2x$
O : $y > 4 - 2x \geqslant y > 4 + 2x$
R : $y > 4 + 2x \geqslant y < 4 - 2x$
S : $y < 4 + 2x \geqslant y < 4 - 2x$

Region R, $\left(-1\frac{1}{2}, 3\right)$ is satisfied by the two inequalities for R but not for any of the others.

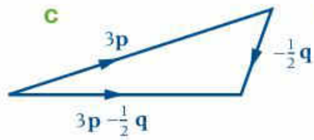
5 a



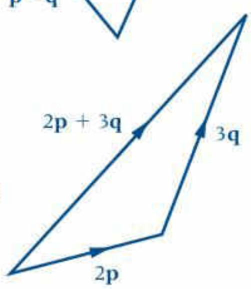
b



c



d



6 a $\overrightarrow{AD}$ b $\overrightarrow{AE}$ c $\overrightarrow{AB}$ d $\overrightarrow{AC}$

7 a $\mathbf{y} - \mathbf{x}$ b $\frac{1}{3}(\mathbf{y} - \mathbf{x})$ c $\mathbf{x} - \mathbf{y}$

d $\frac{2}{3}(\mathbf{x} - \mathbf{y})$ e $\frac{2}{3}\mathbf{x} + \frac{1}{3}\mathbf{y}$

8 a 284° b 25.4 km/h

9 a 2 b $a = 5$, scale factor 3

10 a $\frac{4}{5}\mathbf{p}$, $\frac{1}{5}\mathbf{p}$, $\mathbf{q} - \mathbf{p}$ b $\frac{1}{5}(\mathbf{q} - \mathbf{p})$, $\frac{1}{5}\mathbf{q}$

c parallel and RS $= \frac{1}{5}$OQ

d trapezium e $120\,\text{m}^2$

11 a (9, 5) b (−1, 1) c (2, 1)

12 a $\begin{pmatrix} 2 & -1 \\ -3\frac{1}{2} & 2 \end{pmatrix}$ b (1, 2)

13 $\begin{pmatrix} y \\ x \end{pmatrix}$ reflection in the line $y = x$

14 a $\begin{pmatrix} x \\ 3x + y \end{pmatrix}$ b $(p, 4p)$ c 4

15 a i $R_y = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ **ii** $R_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$

b A′(5, 1), B′(−3, 6), C′(−1, 1)

c A″(1, −5), B″(6, 3), C″(1, 1)

d $R_c = R_0 R_y = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

16 $\overrightarrow{BC} = k(\mathbf{b} - \mathbf{a}),\ k = 2$

18 a i 6 **ii** $\sqrt{41}$ **b** **c** and **h**
c **a** and **g**, **b** and **d** **d** **g**

19 a Missing values are 7 and 2

b $y = \dfrac{8}{x + 1}$

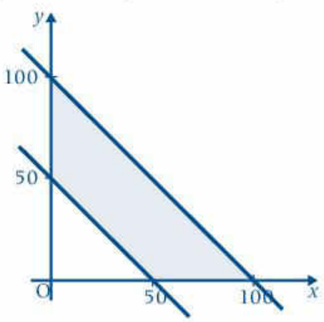
20 a increases 50% **b** decreases $33\frac{1}{3}$%

21 $p = \frac{1}{2}(q + r)$

22 $a = 2$ or -2 $b = 128$

23 a $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ **b** $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ **c** $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

24 $x \geqslant 0, y \geqslant 0, x + y \geqslant 50, x + y \leqslant 100$



25 $y \leqslant 2x, x + y \leqslant 5, y \geqslant -5$

27 b $\begin{pmatrix} 3 & 1 \\ 6 & 2 \end{pmatrix}$ is singular so the transformation has no inverse

28 $\overrightarrow{AC} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \overrightarrow{CB} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

29 a 5**p** + **q**

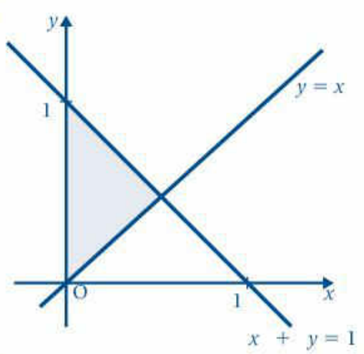
30 $a = \frac{2}{3}, b = 8$

31 increases by 12 400%

32 7 **33** (2, 0)

34 a 3 **b** 3

36



Chapter 21

Exercise 21a

1 c $x = 90$ **d i** 53° and 127°
ii 37° and 143° **iii** 24° and 156°
The two angles add up to 180°.

2 a sin 30° = 0.5, sin 150° = 0.5,
150° = 180° − 30°

b sin 40° = 0.64, sin 140° = 0.64,
140° = 180° − 40°

c sin 72° = 0.95, sin 108° = 0.95,
108° = 180° − 72°

Exercise 21b

1 $\sin x° = \frac{5}{13}$ **2** $\sin x° = \frac{1}{2}$ **3** $\sin x° = \frac{2}{5}$

4 $\sin x° = \frac{3}{10}$ **5** $\sin x° = \frac{7}{25}$ **6** $\sin x° = \frac{40}{41}$

7 $x° = 15°$ **8** $x° = 40°$ **9** $x° = 28°$

10 $x° = 80°$ **11** $x° = 5°$ **12** $x° = 89°$

Exercise 21c

1 c $x = 90$
d It is negative.
e i 37° **ii** 143°
They add up to 180°

2 a cos 30° = 0.87, cos 150° = −0.87,
150° = 180° − 30°

b cos 50° = 0.64, cos 130° = −0.64,
130° = 180° − 50°

c cos 84° = 0.10, cos 96° = −0.10,
96° = 180° − 84°

Exercise 21d

1 $\cos x° = -\frac{3}{5}$

2 $\cos x° = -\frac{2}{3}$

3 $\cos x° = -\frac{2}{5}$

4 $\cos x° = -\frac{7}{8}$

5 a 160° **b** 130°

Exercise 21e

1 $\cos A = \frac{24}{25}$ $\tan A = \frac{7}{24}$

2 $\sin A = \frac{12}{13}$ $\tan A = \frac{12}{5}$

3 $\sin P = \frac{3}{5}$ $\cos A = \frac{4}{5}$

4 $\tan D = \frac{4}{3}$ $\sin D = \frac{4}{5}$

5 $\cos X = \frac{40}{41}$ $\tan X = \frac{9}{40}$

6 $\sin A = \frac{1}{\sqrt{2}}$ $\cos A = \frac{1}{\sqrt{2}}$

7 b i $\sin 60° = \frac{\sqrt{3}}{2}$ ii $\cos 60° = \frac{1}{2}$

iii $\tan 60° = \sqrt{3}$

c i $\sin 30° = \frac{1}{2}$ ii $\cos 30° = \frac{\sqrt{3}}{2}$

iii $\tan 30° = \frac{1}{\sqrt{3}}$

8 b i $\sin 45° = \frac{1}{\sqrt{2}}$ ii $\cos 45° = \frac{1}{\sqrt{2}}$

iii $\tan 45° = 1$

Exercise 21f

1 $\cos A = \frac{\sqrt{3}}{2}$ a $\frac{\sqrt{3}}{2}$ b $\frac{1}{2}$

2 a i 1 ii $\frac{1}{2}$ b 90° c 30°

3 a $-\frac{24}{25}$ b $\frac{7}{25}$

4 30° and 150° 5 −0.515

6 108° 7 a $\frac{4}{5}$ b $\frac{3}{5}$ c $-\frac{3}{5}$

Exercise 21g

1 4.92 cm 2 13.0 cm 3 9.74 cm

4 3.68 cm 5 14.6 cm 6 66.0 cm

7 5.60 cm 8 413 cm 9 64.1 cm

10 19.7 cm 11 18.4 cm 12 77.0 cm

13 224 cm 14 18.4 cm 15 $\frac{20}{\sqrt{3}\sqrt{2}}$ cm

16 $\frac{20}{3}$

Exercise 21h

1 61.7° 2 59.7° or 120.3° 3 42.7°

4 37.2° 5 76.5° or 103.5° 6 28.2°

7 47.9°

8 a angle R

b ∠R > ∠P so if ∠P is obtuse so is ∠R but a triangle cannot have 2 obtuse angles.

c no, no angles given

Exercise 21i

1 5.73 cm 2 7.95 cm

3 7.68 cm 4 18.4 cm

5 7.03 cm 6 14.2 cm

7 15.9 cm 8 21.6 cm

9 43.0 cm 10 28.9 cm

11 5.05 cm 12 16.9 cm

13 24.1° 14 108.2°

15 92.9° 16 29.0° (angle A)

17 95.7° (angle Y) 18 48.2° (angle E)

19 a 13.1 cm b 72.3°

20 a $\frac{1}{16}$ b it is acute

21 40.7°

Exercise 21j

a	*a*	*b*	*c*	$\hat{A}$	$\hat{B}$	$\hat{C}$
1		17.0				
2	117				76.3°	
3		77.9				
4			23.3			
5	308		346			
6		17.4	12.4			
7	29.7				38°	
8	20.8					

Exercise 21k

1 2610 cm^2 2 572 cm^2

3 81.1 cm^2 4 18 900 mm^2

5 126 square units 6 22 800 cm^2

7 35.5 square units 8 6.82 m^2

9 23.5 sq. units 10 19.1 sq. units

11 21.3 cm^2 12 7.06 cm^2

13 186 cm^2

14 C = 83.3°, area = 27.8 cm^2

15 PR = 9.91 cm, area = 53.8 cm^2

16 ∠M = 48.9°, LN = 14.7 cm

17 AC = 4.46 cm

18 a 24 cm^2 b 7.44 cm

20 $\frac{3\sqrt{7}}{2}$ cm

Exercise 21l

1 a 3.97 cm^2 b 10.7 cm^2 c 224 cm^2

d 16.9 cm^2 e 2.53 cm^2

2 a 5.46 cm b 7.98 cm^2 c 46.2 cm^2

d 84.4 cm^2

3 a 77.4° b 7.81 cm^2 c 3.00 cm^2

4 a 91.2° b 24.5 cm^2 c 14.5 cm^2

5 a 7.07 cm b 25 cm^2 c 14.3 cm^2

d 65.7 cm^2

6 **a** 11.5 cm **b** 57.7 cm^2 **c** 81.7 cm^2
d 118 cm^2

7 **a** 8 cm **b** 27.7 cm^2 **c** 5.80 cm^2
d 26.2 cm^2

Exercise 21m

1 62.3 km

2 **a** 18.1° **b** 42.7 m

3 88°

4 **a** 42 cm^2 **b** 44.4° **c** 8.52 m

5 **a** 40° **b** 320° **c** 70°

6 AC = 5.76 km
BC = 4.29 km
69.1 km/h

7 **a** 0.2588 **b** 97.6 m

8 **a** AC = 4.26 cm **b** 75.1° **c** 16.0 cm^2

9 **a** ∠TAB = 39°, ∠ATB = 9°
b AT = 1430 m
c CT = 1080 m

10 **a** **i** 8.94 cm **ii** 6.40 cm **iii** 9.43 cm
b 73.8°

11 **a** 5 m **b** 116.7°

12 **a** ∠AOB = 47.2°
b ∠BOC = 17.3°
c AC = 10.7 cm
d area = 51.6 cm^2

13 **a** 26.8 km on a bearing of 321°
b **i** 17.0 km **ii** 20.7 km

14 **a** 57.0° **b** 131 cm^2

15 **a** 60° **b** 43.3 cm^2
c 18.1 cm^2 **d** 5920 cm^3

16 **a** 22.6°
b AE = 9.17 cm, EC = 3.83 cm
c 5 cm
d 6.48 cm

17 **a** 185.0 m **b** 95.3 m **c** 61.4 m

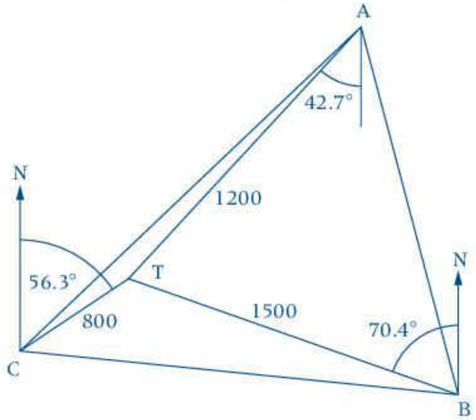
18 **a** 69.8° **b** 150.2° **c** 5584 m^2

19 **a** 36 n. miles
b 111 n. miles
c 109.5 n. miles

20 via B, 0.315 km

21 **a** 8 m **b** 106° **c** 12 m^2
d 67.4 m^2 **e** 1350 m^3

22 **a** 1509 on a bearing of 156.6°
b 1986 on a bearing of 228.1°



23 8.867 km on a bearing of 291.0°

24 38.05 km on a bearing of 174.6°

25 **a** 10 cm **b** 16 cm **c** 44.7 cm^2
d 260 cm^2 **e** 85.2%

Mixed exercise 21

1 D	**2** A	**3** B	**4** A
5 B	**6** C	**7** A	**8** A

Chapter 22

Exercise 22a

1 5 cm **2** 6.2 cm **3** 7 cm

4 10 cm **5** 9.0 cm

6 **b** AB and CD are parallel – ∠ABC and ∠BCD add up to 180°. CD is 6 cm – ∩DCB is a 3, 4, 5 triangle. ∠BAD and ∠BDA are equal – ∩ABD is an isosceles triangle.

7 **b** A rhombus – diagonals bisect each other at right angles.

8 **a** 30°, 30°, 120° **b** Isosceles

9 4.9 cm, 16.0 cm

10 4.7cm, 8.2 cm **11** 4.3 cm, 10.3 cm

12 9.7 cm **13** 7.5 cm **14** 9.1 cm

17 **c** Kite: diagonals cross at right angles, two adjacent sides are equal and the other two adjacent sides are equal.

Exercise 22b

1 b 1 : 100 c i 6.9 m ii 16 m

2 a i PQ as 9.2 cm, QR as 7.6 cm
ii PQ as 11.5 cm, QR as 9.5 cm
b 68 m c 57.6 m
d 1 cm to 10 m is easier than 1 cm to 8 m as multiplication and division by 10 is easier than by 8. 1 cm to 5 m is easier than 1 cm to 8 m as multiplication and division by 5 is easier than by 8.

3 6.40 km

4 c 217 m
d Calculated answers will be as accurate as the original measurements allow. Measurements from scale drawings will depend on accuracy and scale.
e 108 m

5 901 m, 337°

6 AC = 15.7 m, BC = 12.6 m

7 65.7 m 8 a 169 m b 106°

9 a 81 m, 50 m b 31 m

10 b 310° c 22 min.

11 9.3 km on a bearing of 173°

12 a 25 m b 32 m

13 a 38 km b 22 km on a bearing of 097°
c 78 km on a bearing of 015°
d 42 km on a bearing of 040°
e 76 km on a bearing of 137°

14 5.6 hectares

15 a 6 km b 4 cm^2

16 a 2.1 km b 2.8 km^2

17 a 1 : 3000 b 216 m c 1 cm^2

Exercise 22c

2 $x + 2y = 8$ 3 $5x + 3y = 13$

Exercise 22d

1 Angle of rotation 92°

2 Angle of rotation 93°

3 Angle of rotation 137°

4 Angle of rotation 55°

5 Enlargement by −3, centre (−1, 6)

6 Rotation 90° clockwise about (3, 4)

7 Translation by $\begin{pmatrix}-3\\-7\end{pmatrix}$

8 Reflection in the line $y = 3$.

9 d Rotation 180° about O
e Rotation 90° clockwise about (0.5, −1.5)

10 LMN itself

11 a i Rotation 90° clockwise about O.
ii Reflection in the line $x = -1$
iii Translation by $\begin{pmatrix}-8\\6\end{pmatrix}$
b (−7, −1)

Exercise 22e

1 5.8 cm 2 10.4 cm 3 3.0 cm

4 8.9 cm 5 3 cm 6 4.35 cm

Exercise 22f

1 4.6 cm 2 14.7 cm 3 2.4 cm

4 6.1 cm 5 1.7 cm 6 1.6 cm

7 4.2 cm 8 10.1 cm

9 AD = 6.3 cm, AC = 10.6 cm, BD = 13.7 cm

Chapter 23

Exercise 23a

1 a

Age (yrs)	7	8	9	10	11
Frequency	7	9	11	14	9

b 1 year

2 a

Cents	Frequency
0–99	9
100–199	10
200–299	16
300–399	12
400–499	5
500–599	4

b i 100c ii 399.5–499.5 ml

3 (suggested groups)

Weight (g)	Frequency
60–64	3
65–69	13
70–74	6
75–79	3
80–84	3
85–89	2

b i 74.5–79.5 g ii 5 g, 74.5 g, 79.5 g

Exercise 23b

1 **a** 50
b **i** (13–17) cm **ii** (13–17) cm
c 14.5 cm

2 **a** **i** 32 **ii** 21–30 **iii** 21–30
b 27.1 **c** $\frac{3}{8}$

3 **a** (160–164) cm **b** 163.4 cm

4 **a** (60–64) kg **b** 58.8 kg

5 20.3 yrs

6 **a** 50 **b** 117.1 kg **c** $\frac{3}{5}$

7 **a** 60 **b** 71 mm **c** $\frac{2}{3}$

8 **a** 25 min **b** $\frac{2}{5}$

9 **a** 100 **b** 44

10 **a** 60 **b** 9.3 **c** $\frac{17}{60}$

11 **a** 14 min. **b** $\frac{5}{14}$

12 **a** 40 **b** 207 mm

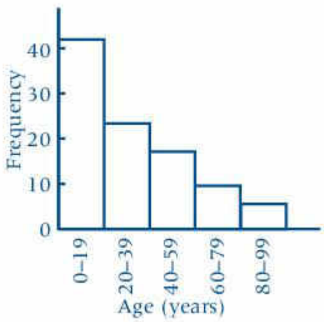
13 56.2

14 **a** 60 **b** 5 **c** $\frac{2}{5}$
d No, don't know how many 3s of 4s there are.

15 **a** 20 **b** 5 m **c** 6.68 m
d No, some could have jumped exactly 6.6 m.

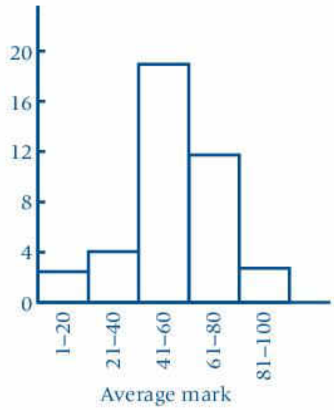
Exercise 23c

1

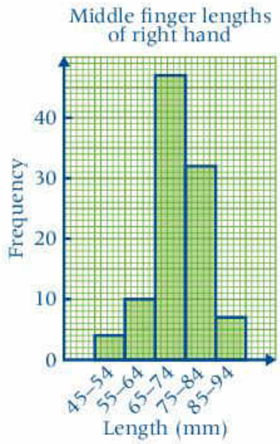


2

3



4



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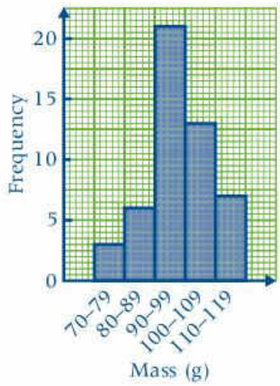
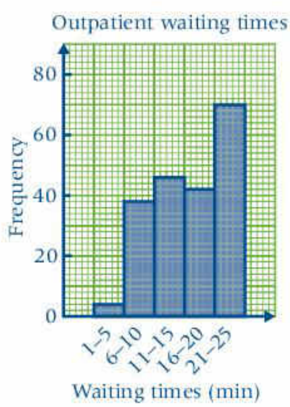


6



7 a

Time taken (minutes)	Frequency
1–10	8
11–20	16
21–30	13
31–40	9
41–50	2
51–60	2

b 10.5 min **c** 0.16

8 a

No. of hours spent watching T.V.	Frequency
up to 1	2
1 up to 2	12
2 up to 3	8
3 up to 4	6
4 up to 5	2

b 4 h **c** 2.3 h **d** $\frac{4}{15}$

9 a

Mass (m g)	Frequency
$0 < m < 10$	8
$10 < m < 20$	15
$20 < m < 30$	22
$30 < m < 40$	17
$40 < m < 50$	10
$50 < m < 60$	2

b 26.6 g

10 a 88 **b** 19.1

c No, there are no spaces between the bars

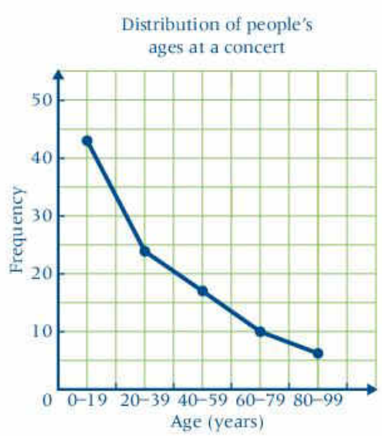
11 a The data are discrete so bars should not touch.

b 121 **c** 7–9 **d** 11.3

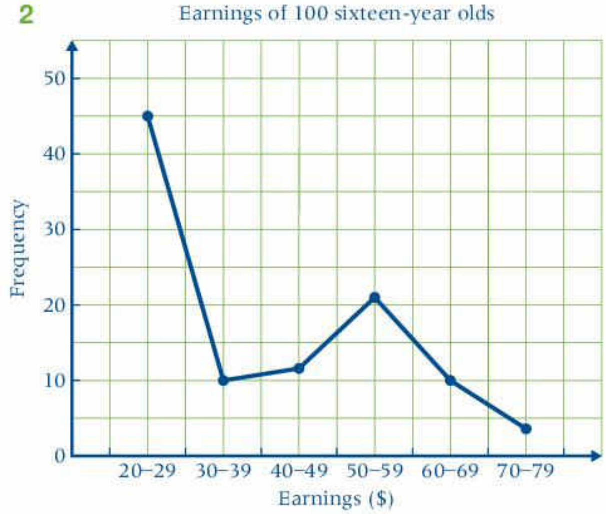
c No, probability is $\frac{52}{121}$ which is $< \frac{1}{2}$

Exercise 23d

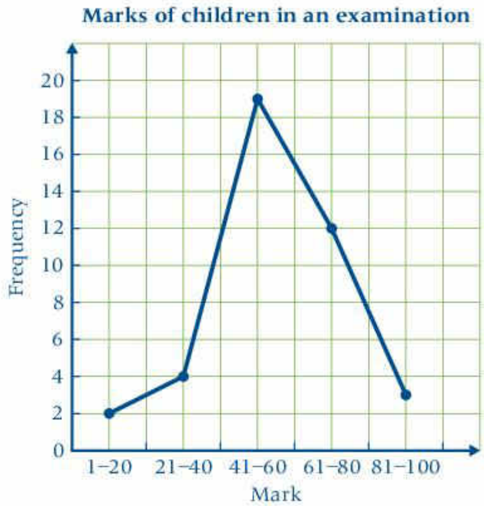
1



2



3

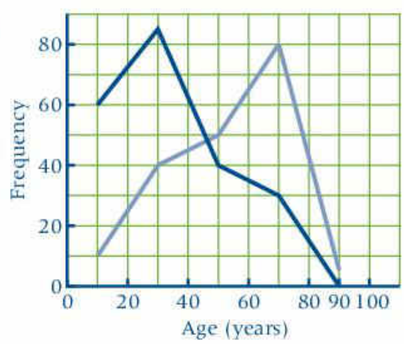


4 a

Frequency	10	40	50	80	5

b 185 **c** 53

5 a

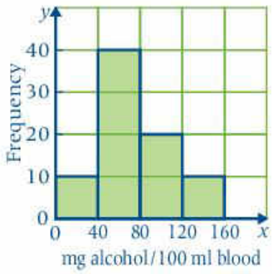


b The mean age is lower and the range of ages is less for the distribution in this question. Camberley may be a residential area and **Bramberdown** may be in a town centre near schools and offices.

c 34

6 a $37\frac{1}{2}\%$

b



c 75 mg/100 ml

Exercise 23e

1 Histograms are based on grouped data and show the data spread evenly within each group and this may not be the case.

2 a 46 b 191 c i 3 ii 6

3 a The last group does not have an upper end.
b The median and the quartiles are not in the open ended group; 33 g, 6 g
c Yes, if we make the last group end at, say, 100 g.

Mixed exercise 23

1 C	2 A	3 B	4 A
5 C	6 B	7 B	8 A
9 B	10 B	11 C	12 C

Chapter 24

Exercise 24a

1 4	2 4p	3 4 yrs
4 5	5 6	6 10
7 a 38	b 9	
8 a 66	b 250	
9 $18 000	10 20 miles	

Exercise 24b

1 a Missing values are 60, 89, 108, 136, 159
b 159

2 a Missing values are 10, 15, 18, 20
b 20 c 5

3 a Missing values are 5, 16, 34, 40
b 40 c 35

4 a Missing values are 6, 25, 49, 66, 84, 88, 91
b 84 c 7

5 a Missing values are 2, 8, 20, 27, 30
b 30 c 27 d 10

6 a Missing values are 2, 9, 28, 39
b 39 c 30
d No, 1st row gives 2 × 5 l; = 10 l;
2nd row gives 7 × 10 l; = 70 l;
3rd row gives 19 × 15 l; = 285 l;
4th row gives 11 × 20 l; = 220 l
i.e. altogether the largest possible amount is the sum of these i.e. 585 l which is less than 600 l.

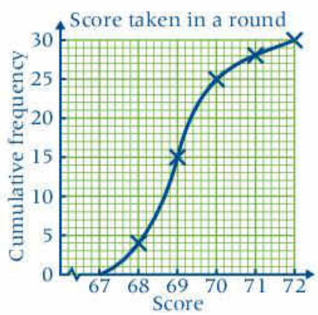
7 a Missing values are 12, 28, 36, 41, 43
b 43 c 15

Exercise 24c

1 a 56	b 51	c 19	2 14
3 a 43	b 20	c 33	d 3

Exercise 24d

1 a 25 b



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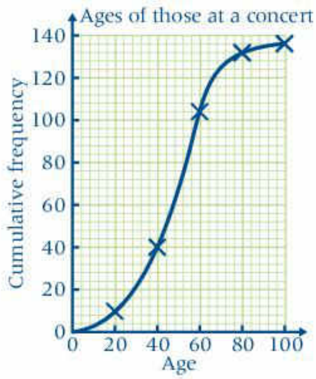
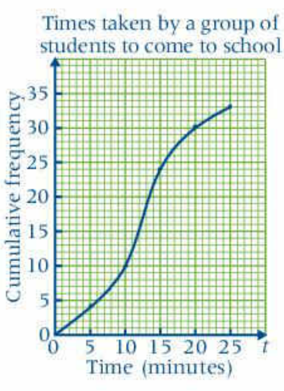


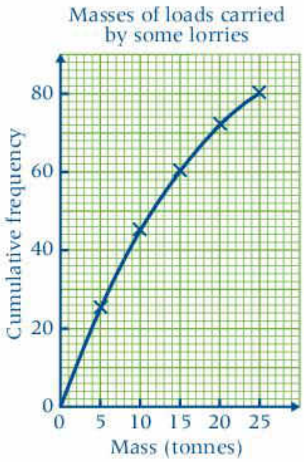
3 a 14

b



4 **a** Missing values; <20, <25; 25, 45, 60, 72, 80

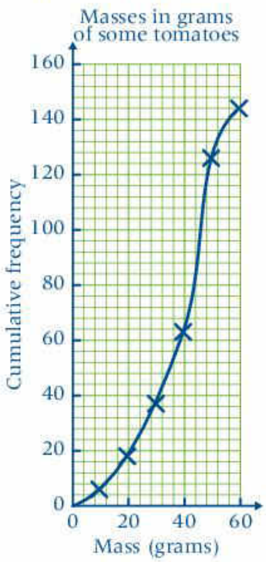
b



5 **a** Missing values are: <39.5, <49.5, <59.5; 6, 18, 37, 63, 126, 144

b 144 **c** 63

d



Exercise 24e

1 **a** $10 000 **b** $3800 **c** $6200 **d** $6500

2 **a** 75 **b** 24 cm **c** 15 cm

d 9 cm **e** 4.5 cm

3 **a** 20 **b** 30 **c** 16

d Either range, it shows that there are excessive speeds, or interquartile range, because this does not include the few cars going very fast or very slow.

4 **a** 50 **b** 4 **c** 14 **d** 15 min

5 **a** 107.5 cm, 9.5 cm

b

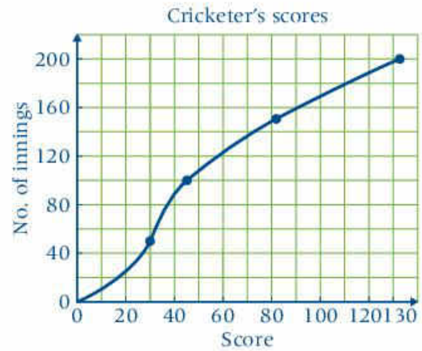
Height	90–95	95–100	100–105	105–110	110–115	115–120
Frequency	10	5	15	30	20	10

d Equal: the median is the value of the middle item so there are equal numbers of items above and below the median and the area of the bars in a histogram represents the number of items.

e 102.5 cm, 112 cm, 4.75 cm

6 **a** **i** 51 **ii** 100 **b** 50

c



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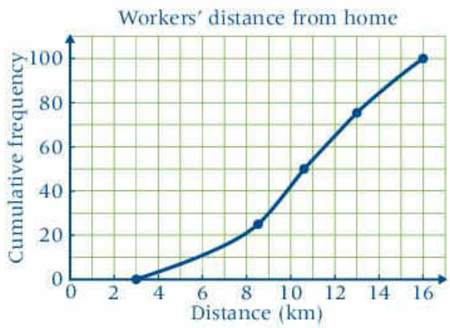
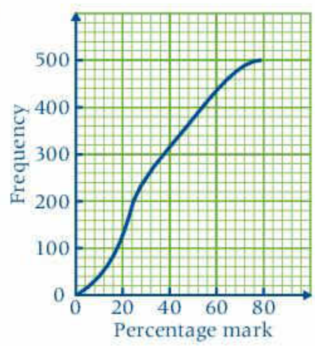


8 **a**



b 320

Exercise 24f

1 There are not equal numbers of men and women among the employees (50% more women than men).

2 **a** The proportion of left-handed people and right-handed people in the general public and how many of the top 250 tennis players are right-handed or left-handed.

b Use a search engine on the internet.

c no

3 mean

4 a two b one

5 72 (mean + 2 SDs)

6 Mean + 2 SDs for Radio Taxis is 35 + 16 = 51 min while mean + 2 SDs for Queen's Taxis is 40 + 8 = 48 min so Queen's is more likely to get Malik to Town B by 10.50.

7 a mode b mean + SD c mean

8 a It includes very expensive phones and says nothing about the popularity of any.
b Half the buyers spend $250 or less.
c It shows the most popular price for a smart phone.
d How spread out they are about the mean.

Mixed exercise 24

1 C 2 A 3 C 4 B 5 A
6 A 7 D 8 C 9 A

Chapter 25

Exercise 25a

1 $118\,cm^3$ 2 $72\,cm^3$

3 $640\,cm^3$ 4 $107\,cm^3$

5 a 7.2 cm b 7.08 cm

6 a $200\,cm^3$ b 7.41 cm

7 a i 6.25 cm and 6.15 cm
ii $20.94\,cm^2$ and $19.99\,cm^2$
iii $40.83\,cm^3$ and $38.31\,cm^3$
b 7%

8 a $110.2\,cm^2$ b $268.4\,cm^3$ c 37

Exercise 25b

1 $402\,cm^3$ 2 $245\,cm^3$

3 $0.785\,m^3$ 4 $196\,000\,m^3$

5 2.38 cm 6 2.69 cm

7 1.73 m 8 2.11 m

9 16.8 cm 10 1.22 cm

11 3.37 cm

Exercise 25c

1 $151\,cm^2$ 2 $377\,cm^2$

3 $226\,cm^2$ 4 $10.3\,m^2$

5 $1210\,cm^2$ 6 $255\,cm^2$

7 $13\,700\,cm^2$ 8 $25.9\,m^2$

9 a $377\,cm^2$ b $113\,cm^2$ c $603\,cm^2$

10 a $96.5\,cm^2$ b $161\,cm^2$

11 $209\,cm^2$ 12 $928\,cm^2$

13 $4.40\,m^2$ 14 $16\,400\,cm^2$

15 $33\,900\,cm^2$ 16 127 cm

17 a $1\,m^3$ b 1.03 m 18 2.57 cm

19 54 cm 20 $42\,400\,cm^3$

Exercise 25d

1 $1700\,cm^3$ 2 $29.4\,cm^3$

3 $19\,600\,cm^3$ 4 $0.528\,cm^3$

5 $27.2\,cm^3$ 6 $0.107\,m^3$

7 $1150\,cm^3$ 8 $330\,cm^3$

9 a $228\,cm^3$ b $5.50\,g/cm^3$

Exercise 25e

1 $126\,cm^2$ 2 $4.15\,m^2$

3 $434\,cm^2$ 4 $15\,200\,mm^2$

5 $163\,cm^2$

6 a $302\,cm^2$ b 10 cm c $188\,cm^2$

7 a $\frac{\pi x l}{180}$ b $2\pi r$ d sector area = $\frac{x\pi l^2}{360}$

Exercise 25f

1 $113\,cm^3$ 2 $1560\,cm^3$

3 $24.4\,m^3$ 4 $0.998\,cm^3$

5 $230\,000\,cm^3$ 6 $9200\,mm^3$

7 $262\,cm^3$

8 a $145\,cm^3$ b 4.52 cm

9 $\frac{9\pi}{2}\,cm^3$ 10 23 000

Exercise 25g

1 $1202\,cm^2$ to 3 s.f. 2 $254\,mm^2$

3 $21\,100\,cm^2$ to 3 s.f. 4 $10.2\,m^2$

5 $3320\,cm^2$ 6 6

7 a 2.82 cm b $94.0\,cm^3$

Exercise 25h

1 596 2 $572\,cm^3$

3 sphere – $25.7\,cm^3$ bigger 4 $239\,000\,cm^3$

5 **a** 15 cm **b** 3020 cm^3

6 **a** $\frac{64}{3}\pi$ cm^3 **b** 64π cm^3
c $\frac{128}{3}\pi$ cm^3

7 **a** $\frac{32\pi}{3}$ **b** $\frac{2048\pi}{3}$ **c** 64

8 sphere by 16.5 cm^2 **9** 462 cm^2

10 **a** 15 cm **b** 933 cm^2

11 enough paint to cover 2.4 m^2 still needed

12 **a** $2\pi\sqrt{13}$ square units

b 4π cubic units

13 **a** 2.3 cm (1.4 + 0.9) **b** 1.70 cm^3/min

14 511 cm^2

Mixed exercise 25

1 B **2** D **3** A **4** B
5 D

Review test 5

1 D **2** D **3** B **4** D
5 B **6** D **7** C **8** B
9 B **10** B **11** B **12** C
13 A **14** C **15** B **16** D
17 C **18** B **19** B **20** C
21 C **22** A **23** C **24** B
25 B **26** D **27** B **28** D
29 C **30** D

General proficiency questions

1 **a** **i** 0.7 m **ii** 0.2 m
b Interquartile range, 1.9 m is very different from the others.

2 20000 **3** **a** 5.32 cm **b** 3.32 cm

4 **a** $\frac{12}{13}$ **b** $\frac{5}{12}$ **5** **a** $\frac{12}{13}$ **b** $\frac{12}{5}$

6 21.5°, 158.5° **7** −0.43

8 196 cm (3 s.f.) **9** 10.9 cm

10 38.0 cm^2 **11** 13.5 km

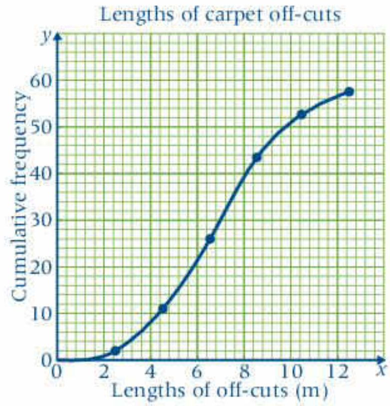
12 **a** $\frac{5}{8}$ **b** **i** $\frac{\sqrt{39}}{8}$ m **ii** $\frac{3\sqrt{39}}{4}$ m^2

13 **a** 11.33 cm **b** 8.82 cm **c** 49.78 cm^2

14 56.3°

15 **a** 32.0° **b** 13.7 m, 15.9 m **c** 345.2°

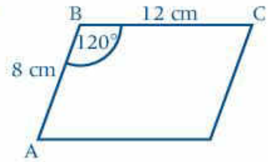
16



ii 3.4 m **iii** $\frac{7}{57}$ **b** 6.76 m

17 337 cm³ $\left(107\frac{1}{3}\pi \text{ cm}^3\right)$

18 **a**



b $4\sqrt{3}$ cm **c** $48\sqrt{3}$ cm^2

19 **a** 264 cm² (3 s.f.) (84π cm²)
b 0.251 m³ $\left(\frac{2\pi}{25}\text{m}^3\right)$ **c** 0.528 m³

20 **a**

Age (x years)	Frequency
$0 \leqslant x < 20$	4
$20 \leqslant x < 40$	5
$40 \leqslant x < 60$	9
$60 \leqslant x < 80$	6
$80 \leqslant x < 100$	3

b 49.3 years (3 s.f.)

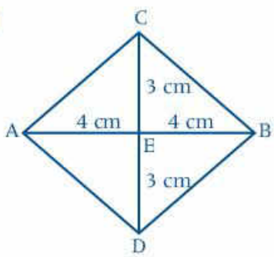
c

Age (x years)	Frequency
$x < 20$	4
$x < 40$	9
$x < 60$	18
$x < 80$	24
$x < 100$	27

d 50 years

21 **b** $3\sqrt{15}$ cm²

22 **a** **i** and **ii**



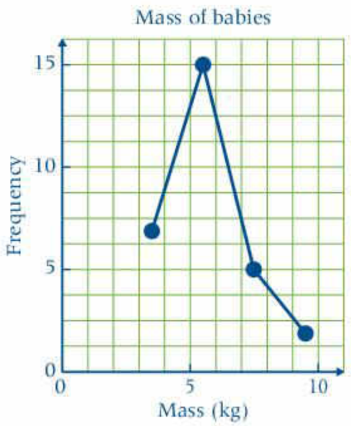
b rhombus **c** 5 cm

23 8.49 cm

24 a i 1340 g **ii** 134 kg **b** 2

25 a 5.64 kg

b



c $\frac{22}{29}$

26 a 905 cm³ (288π cm³)

b 479 cm² ($36\pi(2 + \sqrt{5})$ cm²)

27 a 66.3 cm² **b** 31.2 cm³

Multiple choice test 1

1 B	2 D	3 D	4 B	5 A
6 B	7 B	8 A	9 B	10 A
11 C	12 A	13 B	14 D	15 B
16 A	17 C	18 A	19 A	20 C
21 B	22 D	23 D	24 C	25 D
26 A	27 C	28 A	29 D	30 A

Multiple choice test 2

1 B	2 C	3 D	4 A	5 B
6 A	7 A	8 B	9 C	10 A
11 B	12 A	13 C	14 D	15 D
16 A	17 C	18 D	19 D	20 C
21 D	22 B	23 C	24 A	25 B
26 C	27 A	28 B	29 C	30 D

Multiple choice test 3

1 C	2 A	3 A	4 B	5 D
6 C	7 B	8 D	9 B	10 C
11 A	12 B	13 A	14 A	15 A
16 D	17 D	18 D	19 C	20 A
21 C	22 B	23 B	24 B	25 C
26 D	27 B	28 B	29 C	30 B

Model exam 1

1 a $\frac{15}{16}$

b i 0.25 **ii** 0.3 **iii** 2.5×10^{-1}

c i JHD 1980

ii JHD 1989

iii Mrs Jones's transaction

iv 49.2%

2 a $\frac{4}{5}$

b i $2(x - 1)(x - 3)$

ii $(3 - a + b)(3 + a - b)$

iii $(2 - x)(3 + 2k)$

c $x = -\frac{1}{2}$ or 2

3 a ii 9 **iii** $\frac{1}{4}$

b i (0) **ii** $\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$

4 a 1 : 500 000

b i 12 cm **ii** 351.9 cm²

iii 536.2 cm³

5.36×10^{-4} m³

5 a i 46° **ii** 3 cm

b P(12, 24), Q(12, 36), R(28, 24) 4 : 1, 1 : 2

6 a Store: 'All–U–Need'

b i One pen costs $4.00

ii One ruler costs $3.00

7 a i 12 sin 40° **ii** 12 cos 28°

b i $x = 0$ **ii** $f^2(x) = x$, $f^3(x) = \frac{2}{x}$

iii $f^{20}(x) = x$, $f^{21}(x) = \frac{2}{x}$

8 a i 375 m **ii** 300 m

b i 20 m/s **ii** 8 s

c Minimum $= -3\frac{1}{2}$ at $x = \frac{3}{2}$

9 b i 15 m **ii** 098°

c 2.06 m due North **d** 7.5 m

10 a $p = -2$, $q = -2$, $r = 2$, $s = 7$

b i $\mathbf{R}_X = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ **ii** $\mathbf{R}_N = \begin{pmatrix} 1 & -1 \\ 1 & 0 \end{pmatrix}$

iii (3, 2) **iv** $\mathbf{R}_C = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Model exam 2

1 a $\frac{7}{23}$ **b** 2.35×10^{-3}

c Bank A offers $1.90 more.

2 a $x \geqslant 4$

b i $\left(\frac{a}{3} - \frac{b}{2}\right)\left(\frac{a}{3} + \frac{b}{2}\right)$

ii $(m-n)(a + 2b)$

iii $(x - 2)(2x + 1)$

c i $f = \frac{uv}{u + v}$ **ii** 12

3 a i 32
ii 48
b $17^2 - 15^2$
c $21^2 - 19^2 = 20$
d $8n$ which is a multiple of 4

4 a i $x + y = 14, x - y = 20$
ii $x = 17, y = -3$
b i 8.26 cm ii 10.21 cm iii 45.1 cm^2
c i 9 ii 16

5 a i 2 ii ± 9
b i not valid for $x = 0$
ii $f^2(x) = x, f^3(x) = \frac{2}{x}$
iii $f^{20}(x) = x, f^{21}(x) = \frac{2}{x}$

6 a 10.1 min
b i 91–92 ii $\frac{2}{25}$

7 a

$n - 18$		
$n - 9$		
n	$n + 1$	$n + 2$

$5n - 24$
b 48

8 a $p \geqslant 20; f \geqslant 3p; f + p \leqslant 160.$
c $P = 100f + 60p$
d \$15 200 when $f = 140$ and $p = 20$

9 c 48.5 m

10 a $\mathbf{a} + \frac{1}{2}\mathbf{b}$ b $\frac{1}{2}\mathbf{a} + \mathbf{b}$ e 1 : 2

Model exam 3

1 a $\frac{1}{2}$
b i 1.29 ii 1.3 iii 1.29
c 5×10^2
d i 5 ii \$55 125

2 a $\frac{4y^2 - 9}{6y}$ b 40
c i $(x - y)(x + y)(x^2 + y^2)$
ii $(a + b)(3a - 3b + 1)$
iii $(2x - 1)(y + 2)$

3 a 3 sq. units b $a = 21, b = 25$
c 4.5 cm

4 b i 10 cm ii $4\frac{1}{2}$ cm, $7\frac{1}{2}$ cm

5 a i $abx + 3a + 2$
ii $abx + 2b + 3$ $3a = 2b + 1$
b i $-\frac{1}{2}$ ii $-\frac{1}{2}$

6 b i 45 ii $\frac{21}{50}$

7 a 26, 42
b $a, b, a + b, a + 2b, 2a + 3b,$ $3a + 5b$
c i 3, 7, 10, 17, 27, 44 ii $108 = 4 \times 27$
d $8a + 12b = 4(2a + 3b)$

8 a i $(4, -2)$ ii 9 sq. units iii 1 : 4
b $x > 0, y < 1, y > -2, 2y < 8 - 3x$
c $x = 5, y = 2;$
$x = -6, y = -\frac{7}{2}$

9 a 32°, 58°, 58°
b i 23.7 cm ii 22.3°

10 a i $\frac{-2q}{3}$ ii $x = 2, y = 1$
b i $\mathbf{M} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ ii $\mathbf{S} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
c $\mathbf{R} = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}$

INDEX

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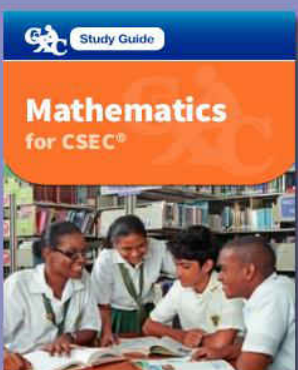

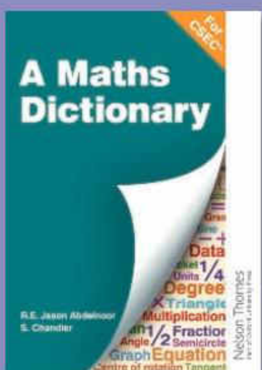

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